SOLUTION-ANSWER PAMPHLET

THIRTY SECOND ANNUAL
HIGH SCHOOL
MATHEMATICS EXAMINATION

1981
32

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1. This key is prepared for the convenience of teachers.
2. Some of the solutions may be intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, a more elementary procedure is also shown.
5. This solution-answer key validates our statement that nothing beyond precalculus mathematics is needed to solve the problems posed.

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SOLUTIONS

1. (E) $x + 2 = 4; (x + 2)^2 = 16.$

2. (C) $1^2 + (BC)^2 = 2^2$; Area $= (BC)^2 = 3.$

3. (D) $\frac{1}{x} + \frac{1}{2x} + \frac{1}{3x} = \frac{6}{6x} + \frac{3}{6x} + \frac{2}{6x} = \frac{11}{6x}.$

4. (C) Let x be the larger number. Then $x - 8$ is the smaller number and $3x = 4(x - 8)$, so that $x = 32$.

5. (C) In $\triangle BDC$, $\measuredangle BDC = 40^\circ$. Since DC is parallel to AB, $\measuredangle DBA = 40^\circ$. Also, $\measuredangle BAD = 40^\circ$ since base angles of an isosceles triangle are equal. Therefore $\measuredangle ADB = 100^\circ$.

6. (A) $(y^2 + 2y - 2)x = (y^2 + 2y - 1)x - (y^2 + 2y - 1)$

$$[(y^2 + 2y - 2) - (y^2 + 2y - 1)]\,x = -(y^2 + 2y - 1)$$

$$x = y^2 + 2y - 1.$$

OR

Rewrite the right member of the given equality as $\dfrac{(y^2 + 2y - 1)}{(y^2 + 2y - 1) - 1}$ and note by inspection that $x = y^2 + 2y - 1$.

7. (B) The numbers 5, 10, 15, . . . , 100 are the only positive integers not exceeding 100 which are divisible by 5. Of these only 20, 40, 60, 80 and 100 are also divisible by 4. From this last set of numbers only 60 is divisible by 3; and 60 is also divisible by 2.

OR

The least common multiple of 2, 3, 4 and 5 is 60. The numbers divisible by 2, 3, 4 and 5 are integer multiples of 60.

8. (A) The given expression equals

$$\frac{1}{x+y+z}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\left(\frac{1}{xy+yz+zx}\right)\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)$$

$$= \frac{1}{x+y+z}\left(\frac{xy+yz+zx}{xyz}\right)\left(\frac{1}{xy+yz+zx}\right)\left(\frac{x+y+z}{xyz}\right)$$

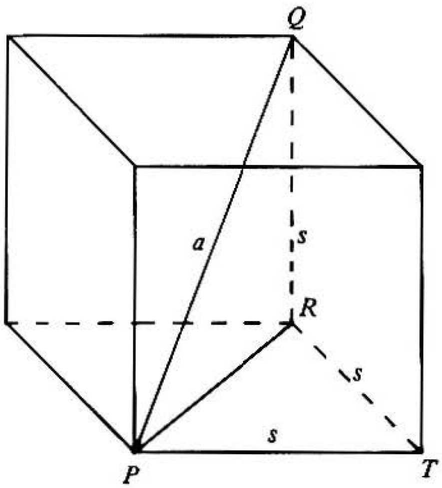
$$= \frac{1}{(xyz)^2} = x^{-2}\,y^{-2}\,z^{-2}.$$

9. (A) Let s be the length of an edge of the cube, and let R and T be vertices of the cube as shown in the adjoining figure. Then applying the Pythagorean theorem to ΔPQR and ΔPRT yields

$$a^2 - s^2 = (PR)^2 = s^2 + s^2$$

$$a^2 = 3s^2.$$

The surface area is $6s^2 = 2a^2$.



10. (E) If (p, q) is a point on line L, then by symmetry (q, p) must be a point on K. Therefore, the points on K satisfy

$$x = ay + b.$$

Solving for y yields

$$y = \frac{x}{a} - \frac{b}{a}.$$

11. (C) Let the sides of the triangle have lengths $s - d$, s, $s + d$. Then by the Pythagorean theorem

$$(s-d)^2 + s^2 = (s+d)^2.$$

Squaring and rearranging the terms yields

$$s(s - 4d) = 0.$$

Since s must be positive, $s = 4d$. Thus the sides have lengths $3d, 4d, 5d$. Since the sides must have lengths divisible by 3, 4 or 5, only choice (C) could be the length of a side.

12. (E) The following inequalities are equivalent to the inequality stated in words in the problem:

$$M\left(1 + \frac{p}{100}\right)\left(1 - \frac{q}{100}\right) > M$$

$$\left(1 + \frac{p}{100}\right)\left(1 - \frac{q}{100}\right) > 1$$

$$1 + \frac{p}{100} > \frac{1}{1 - \frac{q}{100}} = \frac{100}{100 - q}$$

$$\frac{p}{100} > \frac{100}{100 - q} - 1 = \frac{q}{100 - q}$$

$$p > \frac{100q}{100 - q}.$$

13. (E) If A denotes the value of the unit of money at a given time, then $.9A$ denotes its value a year later and $(.9)^n A$ denotes its value n years later. We seek the smallest integer n such that n satisfies these equivalent inequalities:

$$(.9)^n A \leqslant .1A$$

$$\left(\frac{9}{10}\right)^n \leqslant \frac{1}{10}$$

$$\log_{10}\left(\frac{9}{10}\right)^n \leqslant \log_{10}\frac{1}{10}$$

$$n(2\log_{10} 3 - 1) \leqslant -1$$

$$n \geqslant \frac{1}{1 - 2\log_{10} 3} \approx 21.7.$$

14. (A) Let a and r be the first term and the common ratio of successive terms in the geometric sequence, respectively. Then

$$a + ar = 7$$

$$\frac{a(r^6 - 1)}{r - 1} = 91.$$

Dividing the first equation into the second yields

$$\frac{(r^6 - 1)}{(r^2 - 1)} = 13$$

$$r^4 + r^2 - 12 = 0$$

$$(r^2 + 4)(r^2 - 3) = 0.$$

Thus $r^2 = 3$ and

$$a + ar + ar^2 + ar^3 = (a + ar)(1 + r^2)$$
$$= 7(4) = 28.$$

15. (B) For this solution write log for $\log_b$. The given equation is equivalent to

$$(2x)^{\log 2} = (3x)^{\log 3}$$

$$\frac{2^{\log 2}}{3^{\log 3}} = \frac{x^{\log 3}}{x^{\log 2}}$$

$$\frac{2^{\log 2}}{3^{\log 3}} = x^{\log 3 - \log 2}$$

Equating the logarithm of the left and right members of the last equality above yields

$$(\log 2)^2 - (\log 3)^2 = (\log 3 - \log 2)\log x$$
$$-(\log 2 + \log 3) = \log x$$
$$\log \frac{1}{6} = \log x$$
$$\frac{1}{6} = x.$$

16. (E) Grouping the base three digits of x in pairs yields

$$x = (1\cdot 3^{19} + 2\cdot 3^{18}) + (1\cdot 3^{17} + 1\cdot 3^{16}) + \ldots + (2\cdot 3 + 2)$$
$$= (1\cdot 3 + 2)(3^2)^9 + (1\cdot 3 + 1)(3^2)^8 + \ldots + (2\cdot 3 + 2).$$

Therefore, the first base nine digit of x is $1\cdot 3 + 2 = 5$.

17. (B) Replacing x by $\frac{1}{x}$ in the given equation, $f(x) + 2f(\frac{1}{x}) = 3x$, yields

$$f(\tfrac{1}{x}) + 2f(x) = \frac{3}{x}.$$

Eliminating $f(\frac{1}{x})$ from the two equations yields

$$f(x) = \frac{2-x^2}{x}.$$

Then $f(x) = f(-x)$ if and only if

$$\frac{2-x^2}{x} = \frac{2-(-x)^2}{-x},$$

or $x^2 = 2$. Thus $x = \pm\sqrt{2}$ are the only solutions.

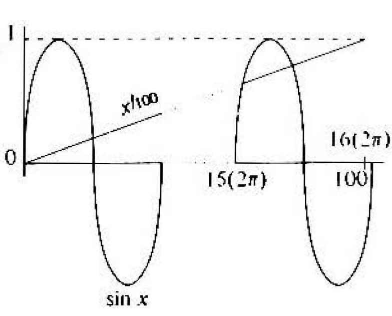
18. (C) Since $\frac{-x}{100} = \sin(-x)$, the equation has an equal number of positive and negative solutions. Also $x = 0$ is a solution. Furthermore, all positive solutions are less than or equal to 100, since

$|x| = 100\,|\sin x| \leqslant 100.$

Since $15.5 < \frac{100}{2\pi} < 16$, the graphs of $\frac{x}{100}$ and $\sin x$ are as shown in the adjoining figure. Thus there is one solution to the given equation between 0 and π and two solutions in each of the intervals from $(2k-1)\pi$, to $(2k+1)\pi$, $1 \leqslant k \leqslant 15$.

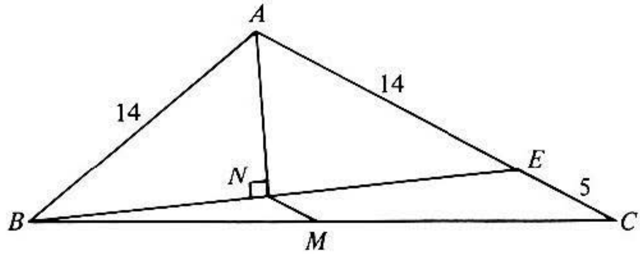
The total number of solutions is, therefore,

$$1 + 2(1 + 2\cdot 15) = 63.$$



19. (B) In the adjoining figure, BN is extended past N and meets AC at E. Triangle BNA is congruent to ΔENA, since $\measuredangle BAN = \measuredangle EAN$, $AN = AN$ and $\measuredangle ANB = \measuredangle ANE$.

Therefore N is the midpoint of BE and $AB = AE = 14$. Thus $EC = 5$. Since M is given to be the midpoint of BC, MN joins the midpoints of two sides of ΔBEC and $MN = \frac{1}{2}(EC) = \frac{5}{2}$.



20. (B) Let $\measuredangle DAR_1 = \theta$ and let θ_i be the (acute) angle the light beam and the reflecting line form at the i^{th} point of reflection. Applying the theorem on exterior angles of triangles to ΔAR_1D, then successively to the triangles $\Delta R_{i-1}R_iD$, $2 \leqslant i \leqslant n$, and finally to ΔR_nBD yields

$$\begin{aligned} \theta_1 &= \theta + 8^\circ \\ \theta_2 &= \theta_1 + 8^\circ = \theta + 16^\circ \\ \theta_3 &= \theta_2 + 8^\circ = \theta + 24^\circ \\ &\text{-------------} \\ \theta_n &= \theta_{n-1} + 8^\circ = \theta + (8n)^\circ \\ 90^\circ &= \theta_n + 8^\circ = \theta + (8n+8)^\circ. \end{aligned}$$

But θ must be positive. Therefore,

$$\begin{aligned} 0 \leqslant \theta &= 90 - (8n + 8) \\ n &\leqslant \frac{82}{8} < 11. \end{aligned}$$

If $\theta = 2^\circ$, then n takes its maximum value of 10.

21. (D) Let θ be the angle opposite the side of length c.
Now

$$\begin{aligned} (a+b+c)(a+b-c) &= 3ab \\ (a+b)^2 - c^2 &= 3ab \\ a^2 + b^2 - ab &= c^2. \end{aligned}$$

But

$$a^2 + b^2 - 2ab\cos\theta = c^2,$$

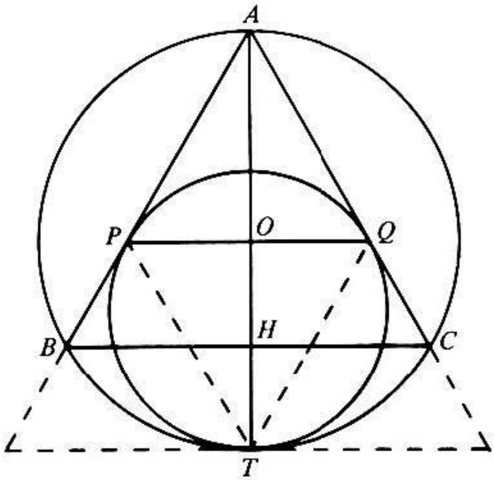
so that $ab = 2ab\cos\theta$, $\cos\theta = \frac{1}{2}$ and $\theta = 60^\circ$.

22. (D) Consider the smallest cube containing all the lattice points (i, j, k), $1 \leqslant i, j, k \leqslant 4$ in a three dimensional Cartesian coordinate system. There are 4 main diagonals. There are 24 diagonal lines parallel to a coordinate plane: 2 in each of four planes parallel to each of the three coordinate planes. There are 48 lines parallel to a coordinate axis: 16 in each of the three directions. Therefore, there are $4 + 24 + 48 = 76$ lines.

23. (C) Let O and H be the points at which PQ and BC, respectively, intersect diameter AT. Sides AB and AC form a portion of the equilateral triangle circumscribing the smaller circle and tangent to the smaller circle at T. Therefore, ΔPQT is an equilateral triangle. Since ΔAPQ is an equilateral triangle with a side in common with ΔPQT, $\Delta APQ \cong \Delta PQT$. Thus $AO = OT$ and O is the center of the larger circle.

This implies $AO = \frac{2}{3}(AH)$,

so that $PQ = \frac{2}{3}(BC) = 8$.



24. (D) Write $x + \frac{1}{x} = 2\cos\theta$ as

$$x^2 - 2x\cos\theta + 1 = 0.$$

Then $x = \cos\theta \pm \sqrt{\cos^2\theta - 1} = \cos\theta \pm i\sin\theta \quad (= e^{\pm i\theta})$. By De Moivre's theorem

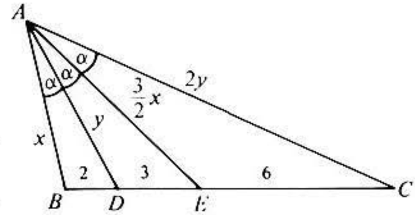
$$x^n = \cos n\theta \pm i\sin n\theta \; (= e^{\pm in\theta})$$

$$\frac{1}{x^n} = \frac{1}{\cos n\theta \pm i\sin n\theta} = \cos n\theta \mp i\sin n\theta \; (= e^{\mp i\theta}).$$

Thus

$$x^n + \frac{1}{x^n} = 2\cos n\theta.$$

25. (A) In the adjoining figure let $\measuredangle BAC = 3\alpha$, $x = AB$ and $y = AD$. Then by the angle bisector theorem



$\frac{AB}{AE} = \frac{2}{3}$ and $\frac{AD}{AC} = \frac{1}{2}$. Hence $AE = \frac{3x}{2}$ and $AC = 2y$. Using the law of cosines in ΔADB, ΔAED and ΔACE, respectively, yields

$$\frac{x^2 + y^2 - 4}{2xy} =$$

$$\frac{\frac{9}{4}x^2 + y^2 - 9}{3xy} = \frac{\frac{9}{4}x^2 + 4y^2 - 36}{6xy}.$$

The equality of the first and second expressions implies

$$3x^2 - 2y^2 = 12.$$

The equality of the first and third expressions implies

$$3x^2 - 4y^2 = -96.$$

Solving these two equations for x and y yields

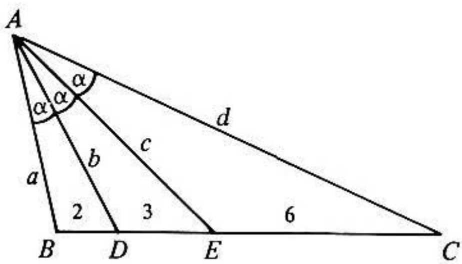
$$x = 2\sqrt{10}$$
$$y = \sqrt{54} = 3\sqrt{6}.$$

Thus the sides are $AB = 2\sqrt{10} \approx 6.3$, $AC = 6\sqrt{6} \approx 14.7$, and $BC = 11$.

OR

In the adjoining figure let $AB = a$, $AD = b$, $AE = c$ and $AC = d$. Using the angle bisector theorem

$$\frac{a}{c} = \frac{2}{3}, \qquad \frac{b}{d} = \frac{3}{6}.$$



Thus $a = \frac{2c}{3}$ and $d = 2b$. Using the formula for the length of an angle bisector*

$$b^2 + 6 = ac$$
$$c^2 + 18 = bd.$$

Using the relations above in these equations yields

$$b^2 + 6 = \frac{2c^2}{3}$$

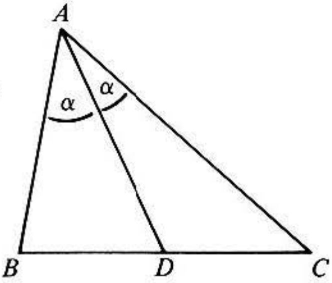
$$c^2 + 18 = 2b^2.$$

Solving these equations for c^2 and b^2 yields

$$c^2 = 90$$
$$b^2 = 54.$$

Thus, $$a = \frac{2(3\sqrt{10})}{3} = 2\sqrt{10},$$

$$d = 2(3\sqrt{6}) = 6\sqrt{6}.$$



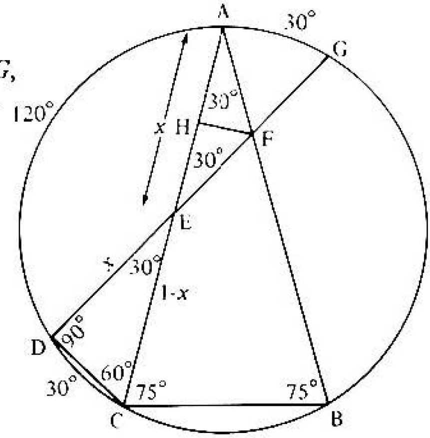
**Angle Bisector*: Let AD be the angle bisector of $\measuredangle CAB$ in $\triangle ABC$, then $(AD)^2 + (BD)(DC) = (AB)(AC)$.

26. (D) The probability that Carol tosses the first 6 on her initial throw is $\left(\frac{5}{6}\right)^2 \frac{1}{6}$. The probability she tosses the first 6 on her second throw is $\left(\frac{5}{6}\right)^5 \frac{1}{6}$. In general, the probability that the first 6 is tossed on Carol's n^{th} throw is $\left(\frac{5}{6}\right)^{3n-1} \frac{1}{6}$. Therefore, the probability that Carol will be the first one to toss a 6 is

$$\left(\frac{5}{6}\right)^2 \frac{1}{6} + \left(\frac{5}{6}\right)^5 \frac{1}{6} + \ldots + \left(\frac{5}{6}\right)^{3n-1} \frac{1}{6} + \ldots,$$

which is the sum of a geometric series with first term $a = \left(\frac{5}{6}\right)^2 \frac{1}{6}$ and common ratio $r = \left(\frac{5}{6}\right)^3$. The sum is $\frac{a}{1-r} = \frac{25}{91}$.

27. (C) In the adjoining figure line segment DC is drawn. Since $\overset{\frown}{AC} = 150^\circ$, $\overset{\frown}{AD} = \overset{\frown}{AC} - \overset{\frown}{DC} = 150^\circ - 30^\circ = 120^\circ$. Hence $\measuredangle ACD = 60^\circ$. Since $AC = DG$, $\overset{\frown}{GA} = \overset{\frown}{GD} - \overset{\frown}{AD} = \overset{\frown}{AC} - 120^\circ = 30^\circ$. Therefore, $\overset{\frown}{CG} = 180^\circ$ and $\measuredangle CDG = 90^\circ$. Thus ΔDEC is a $30^\circ - 60^\circ - 90^\circ$ right triangle.



Since we are looking for the ratio of the areas, let us assume without loss of generality that $AC = AB = DG = 1$.

Let $AE = x = DE$. Then $CE = 1 - x = \frac{x}{\sqrt{3}} \cdot 2$. Solving for x yields $AE = x = 2\sqrt{3} - 3$. Let FH be the altitude of ΔAFE on AE. Now $EH = \dfrac{AE}{2} = \dfrac{2\sqrt{3}-3}{2}$ and $FH = \left(\dfrac{2\sqrt{3}-3}{2}\right)\dfrac{\sqrt{3}}{3}$.

The area of $\Delta AFE = (EH)\,(FH) = \left(\dfrac{2\sqrt{3}-3}{2}\right)^2 \dfrac{\sqrt{3}}{3} = \dfrac{7\sqrt{3}-12}{4}$. Also,

area $\Delta ABC = \frac{1}{2}\,(AB)(AC)\sin 30^\circ = \frac{1}{2} \cdot (1) \cdot (1) \cdot \frac{1}{2} = \frac{1}{4}$. Hence

$$\frac{\text{area } \Delta AFE}{\text{area } \Delta ABC} = 7\sqrt{3} - 12.$$

28. (D) Let $g(x) = x^3 + a_2x^2 + a_1x + a_0$ be an arbitrary cubic with constants of the specified form. Because x^3 dominates the other terms for large enough x, $g(x) > 0$ for all x greater than the largest real root of g. Thus we seek a particular g in which the terms $a_2x^2 + a_1x + a_0$ "hold down" $g(x)$ as much as possible, so that the value of the largest real root is as large as possible. This suggests that the answer to the problem is the largest root of $f(x) = x^3 - 2x^2 - 2x - 2$. Call this root r_0. To verify this conjecture, note that for $x \geqslant 0$, $-2x^2 \leqslant a_2x^2$, $-2x \leqslant a_1x$, and $-2 \leqslant a_0$.
Summing these inequalities, and adding x^3 to both sides, gives $f(x) \leqslant g(x)$ for all $x \geqslant 0$. Thus for all $x > r_0$, $0 < f(x) \leqslant g(x)$. That is, no g has a root larger than r_0, so r_0 is the r of the problem.
A sketch of f shows that it is a typical S-shaped cubic, with largest root a little less than 3. In fact, $f(2) = -6$ and $f(3) = 1$. To be absolutely sure the answer is (D), not (C), compute $f\left(\dfrac{5}{2}\right)$ to see if it is negative. Indeed, $f\left(\dfrac{5}{2}\right) = -\dfrac{31}{8}$.

29. (E) Since x is the principal square root of some quantity, $x \geqslant 0$. For $x \geqslant 0$, the given equation is equivalent to

$$a - \sqrt{a + x} = x^2.$$

Since the left member of this equation is a decreasing function of x and the right member is an increasing function, one easily verifies that the equation has exactly one solution. To find this solution let $y = \sqrt{a+x}$. Then

$$\begin{aligned} a - y &= x^2 \\ a - y - y^2 &= x^2 - y^2 \\ a - y - (a + x) &= x^2 - y^2 \\ -(x + y) &= (x + y)(x - y) \\ 0 &= (x + y)(x - y + 1). \end{aligned}$$

Since $a \geqslant 1$ and $x \geqslant 0$, it follows that $y > 0$ and $x + y \neq 0$. Therefore,

$$\begin{aligned} x - y + 1 &= 0 \\ x + 1 &= y \\ x + 1 &= \sqrt{a + x} \\ (x + 1)^2 &= a + x \\ x &= \frac{-1 \pm \sqrt{4a - 3}}{2}. \end{aligned}$$

The positive solution $x = \dfrac{\sqrt{4a - 3} - 1}{2}$ is the sum.

30. (D) Since the coefficient of x^3 in the polynomial function $f(x) = x^4 - bx - 3$ is zero, the sum of the roots of $f(x)$ is zero and therefore,

$$\frac{a + b + c}{d^2} = \frac{a + b + c + d - d}{d^2} = \frac{-1}{d}.$$

Similarly,

$$\frac{a + c + d}{b^2} = \frac{-1}{b}, \frac{a + b + d}{c^2} = \frac{-1}{c}, \frac{b + c + d}{a^2} = \frac{-1}{a}.$$

Hence the equation $f\left(-\frac{1}{x}\right) = 0$ has the specified solutions:

$$\begin{aligned} \frac{1}{x^4} + \frac{b}{x} - 3 &= 0 \\ 1 + bx^3 - 3x^4 &= 0 \\ 3x^4 - bx^3 - 1 &= 0. \end{aligned}$$

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1. (E) $$\begin{array}{r|l} & x \\ \hline x^2 + 0x - 2 & x^3 + 0x^2 + 0x - 2 \\ & \underline{x^3 + 0x^2 - 2x} \\ & \qquad\qquad 2x - 2 = \text{remainder.} \end{array}$$

2. (A) The answer is $\dfrac{8x+2}{4} = 2x + \dfrac{1}{2}$.

3. (C) For $x = 2$, the expression equals
$$(2^2)^{(2^2)} = 4^4 = 4 \cdot 4 \cdot 4 \cdot 4 = 256.$$

4. (E) Let r be the radius of the semicircle. The perimeter of a semicircular region is $\pi r + 2r$. The area of the region is $\dfrac{\pi r^2}{2}$.

Therefore, $\pi r + 2r = \dfrac{\pi r^2}{2}$,
$$2\pi + 4 = \pi r,$$
$$2 + \frac{4}{\pi} = r.$$

5. (C) Since $y = \dfrac{b}{a}x$, $\dfrac{b}{a} > 1$, and $x > 0$, it follows that x is the smaller number. Also, $x + y = c$. Thus
$$x + \frac{b}{a}x = c,$$
$$ax + bx = ac,$$
$$x = \frac{ac}{a+b}.$$

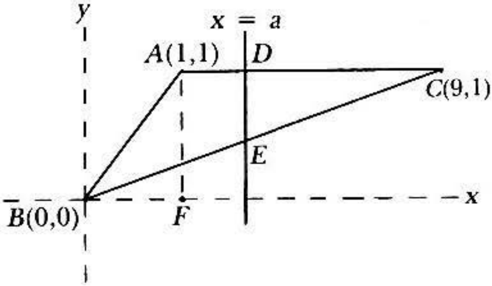
6. (D) The sum of the angles in a convex polygon of n sides is $(n-2)180°$. Therefore, if x is the unknown angle,
$$(n-2)180° = 2570° + x, \text{ with } 0° < x < 180°.$$
If $n = 17$, then $(n-2)180° = 15(180°) = 2700°$ and $x = 130°$. Smaller values of n would yield negative values of x, and larger values of n would yield values of x greater than $180°$.

7. (B)
$$x * (y + z) = (x+1)(y+z+1) - 1,$$
$$(x * y) + (x * z) = \left[(x+1)(y+1) - 1\right] + \left[(x+1)(z+1) - 1\right]$$
$$= (x+1)(y+z+2) - 2.$$
Therefore, $x * (y+z) \neq (x * y) + (x * z)$. The remaining choices can easily be shown to be true.

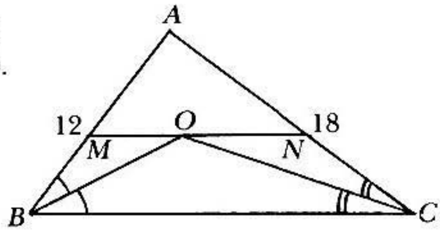
8. (B) Write
$$\binom{n}{1} = n, \ \binom{n}{2} = \frac{n(n-1)}{2} \text{ and } \binom{n}{3} = \frac{n(n-1)(n-2)}{6}.$$
Hence
$$\frac{n(n-1)}{2} - n = \frac{n(n-1)(n-2)}{6} - \frac{n(n-1)}{2}.$$
Thus
$$n^3 - 9n^2 + 14n = 0,$$
$$n(n-2)(n-7) = 0.$$
Since $n > 3$, $n = 7$ is the solution.

(The answer may also be obtained by evaluating the sequence $\binom{n}{1}$, $\binom{n}{2}$, $\binom{n}{3}$ for the values of n listed as choices.)

9. (B) In the adjoining figure, ABC is the given triangle and $x = a$ is the dividing line. Since Area $\triangle ABC = \frac{1}{2}(1)(8) = 4$, the two regions must each have area 2. Since the portion of $\triangle ABC$ to the left of the vertical line through vertex A has area less than Area $\triangle ABF = \frac{1}{2}$, the line $x = a$ is indeed right of A as shown. Since the equation of the line BC is $y = \frac{x}{9}$, E is $(a, \frac{a}{9})$.
Thus Area
$\triangle DEC = 2 = \frac{1}{2}(1 - \frac{a}{9})(9 - a)$,
or $(9 - a)^2 = 36$. Then
$9 - a = \pm 6$, and
$a = 15$ or 3. Since
the line $x = a$ must
intersect $\triangle ABC$, $x = 3$.



10. (A) Since MN is parallel to BC,
$\measuredangle MOB = \measuredangle CBO = \measuredangle OBM$,
$\measuredangle CON = \measuredangle OCB = \measuredangle NCO$.
Therefore, $MB = MO$ and $ON = NC$,
and $AM + MO + ON + AN$
$= (AM + MB) + (AN + NC)$
$= AB + AC = 12 + 18 = 30$.



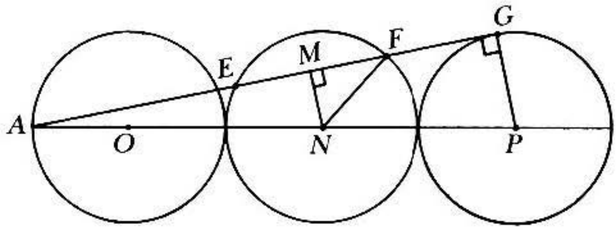
11. (C) From the set $\{0,1,\ldots,9\}$ there are sixteen pairs of numbers $\{(0, 2), (2, 0), (1, 3), (3, 1), \ldots\}$ whose difference is ± 2. All but $(0, 2)$ can be used as the first and last digit, respectively, of the required number. For each of the 15 ordered pairs there are $8 \cdot 7 = 56$ ways to fill the remaining middle two digits. Thus there are $15 \cdot 56 = 840$ numbers of the required form.

12. (A) Since $f(x) = ax^7 + bx^3 + cx - 5$,
$$f(-x) = a(-x)^7 + b(-x)^3 + c(-x) - 5.$$
Therefore, $f(x) + f(-x) = -10$ and $f(7) + f(-7) = -10$.
Hence, since $f(-7) = 7$, $f(7) = -17$.

13. (D) We have

$$\begin{aligned} p(\log_b a) &= \log_b(\log_b a), \\ \log_b(a^p) &= \log_b(\log_b a), \\ a^p &= \log_b a. \end{aligned}$$

14. (C) In the adjoining figure, MN is perpendicular to AG at M, and NF and PG are radii. Since $\triangle AMN \sim \triangle AGP$, it follows that $\frac{MN}{AN} = \frac{GP}{AP}$, or $\frac{MN}{45} = \frac{15}{75}$. Thus $MN = 9$. Applying the Pythagorean Theorem to triangle MNF yields $(MF)^2 = (15)^2 - 9^2 = 144$, so $MF = 12$. Therefore, $EF = 24$.



15. (D) We have

$$\begin{aligned} 2[x] + 3 &= 3\,[x - 2] + 5, \\ 2[x] + 3 &= 3\,([x] - 2) + 5, \\ [x] &= 4. \end{aligned}$$

Therefore, $4 < x < 5$, and $y = 2[x] + 3 = 11$. Hence, $15 < x + y < 16$. (Alternately, if one draws the graphs of $y = 2[x] + 3$ and $y = 3[x - 2] + 5$ one can see that they overlap when $4 < x < 5$).

16. (B) Each exterior unit square which is removed exposes 4 interior unit squares, so the entire surface area in square meters is

$$6 \cdot 3^2 - 6 + 24 = 72.$$

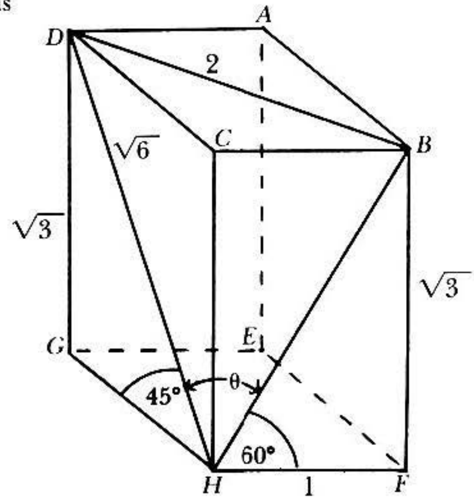
17. (C) Since

$$\begin{aligned} 0 &= 3^{2x+2} - 3^{x+3} - 3^x + 3 \\ &= 3^2(3^x)^2 - 28(3^x) + 3 \\ &= 9(3^x)^2 - 28(3^x) + 3 \\ &= (3^x - 3)\,(9(3^x) - 1), \end{aligned}$$

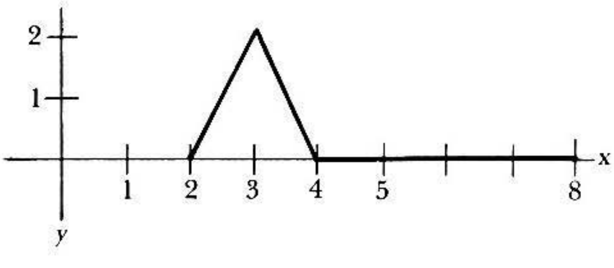
we must have $3^x = 3$ or $3^x = \frac{1}{9}$. Thus, $x = 1$ or -2 are the only solutions.

18. (D) Without loss of generality, let $HF = 1$ in the adjoining figure. Then $BH = 2$, $BF = \sqrt{3} = DG = GH$, and $DH = \sqrt{6}$. Since $DC = HC = \sqrt{3}$, $\triangle DCB \cong \triangle HCB$ and $DB = HB = 2$. Since $\triangle DBH$ is isoceles,

$$\cos\theta = \frac{\frac{1}{2}\sqrt{6}}{2} = \frac{\sqrt{6}}{4}.$$



19. (B) When $2 \leqslant x \leqslant 3$, $f(x) = (x-2) - (x-4) + (2x-6) = -4 + 2x$. Similar algebra shows that when $3 \leqslant x \leqslant 4$, $f(x) = 8 - 2x$; and when $4 \leqslant x \leqslant 8$, $f(x) = 0$. The graph of $f(x)$ given in the adjoining figure shows that the maximum and minimum of $f(x)$ are 2 and 0, respectively.



20\. (D) Since

$$x^2 + y^2 = x^3,$$
$$y^2 = x^2(x - 1).$$

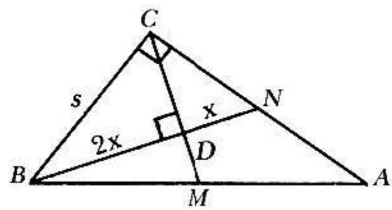
Therefore, if k is an integer satisfying $(x - 1) = k^2$, i.e., $x = 1 + k^2$, then there is a y satisfying $x^2 + y^2 = x^3$. Hence there are infinitely many solutions.

21\. (E) Since the medians of a triangle intersect at a point two thirds the distance from the vertex and one third the distance from the side to which they are drawn, we can let $x = DN$ and $2x = BD$. In right triangle BCN, using the fact that a leg of a right triangle is the geometric mean between its projection upon the hypotenuse and the hypotenuse,

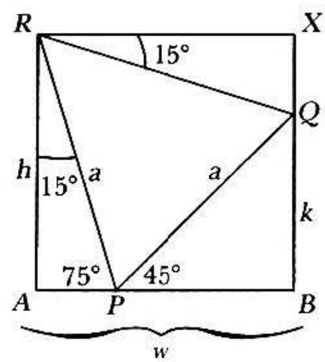
$$\frac{2x}{s} = \frac{s}{3x}.$$

Thus $s^2 = 6x^2$, or $x = \dfrac{s}{\sqrt{6}}$,

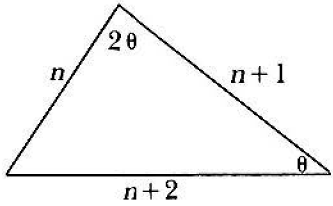
and $BN = 3x = \dfrac{3s}{\sqrt{6}} = \dfrac{s\sqrt{6}}{2}$.



22\. (E) In the adjoining figure RX is perpendicular to QB at X. Since $\measuredangle QPR = 60°$, $\triangle RPQ$ is equalateral and $RQ = a$. Also $\measuredangle ARP = \measuredangle QRX = 15°$. Therefore, $\triangle RXQ \cong \triangle RAP$. Thus $w = h$.



23. (A) In the adjoining figure, n denotes the length of the shortest side, and θ denotes the measure of the smallest angle. Using the law of sines and writing $2\sin\theta\cos\theta$ for $\sin 2\theta$ yields



$$\frac{\sin\theta}{n} = \frac{2\sin\theta\cos\theta}{n+2},$$

$$\cos\theta = \frac{n+2}{2n}.$$

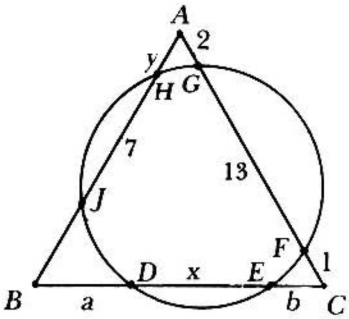
Equating $\frac{n+2}{2n}$ to the expression of $\cos\theta$ obtained from the law of cosines yields

$$\frac{n+2}{2n} = \frac{(n+1)^2+(n+2)^2-n^2}{2(n+1)(n+2)}$$

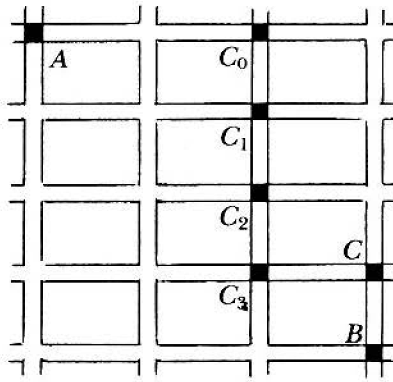
$$= \frac{(n+1)(n+5)}{2(n+1)(n+2)} = \frac{n+5}{2(n+2)}.$$

Thus $n = 4$ and $\cos\theta = \frac{4+2}{4(2)} = \frac{3}{4}$.

24. (A) In the adjoining figure, let $AH = y$, $BD = a$, $DE = x$ and $EC = b$. We are given $AG = 2$, $GF = 13$, $HJ = 7$ and $FC = 1$. Thus the length of the side of the equilateral triangle is 16. Also, using the theorem on secants drawn to a circle from an external point, we have $y(y+7) = 2(2+13)$, or $0 = y^2 + 7y - 30 = (y-3)(y+10)$. Hence $y = 3$ and $BJ = 6$. Using the same theorem we have $b(b+x) = 1(1+13) = 14$ and $a(a+x) = 6(6+7) = 78$. Also, $a+b+x = 16$. Solving these last three equations simultaneously gives $x = 2\sqrt{22}$ (also $b = 6-\sqrt{22}$, $a = 10-\sqrt{22}$).



25. (D) The probability that the student passes through C is the sum from $i = 0$ to 3 of the probabilities that he enters intersection C_i in the adjoining figure and goes east. The number of paths from A to C_i is $\binom{2+i}{i}$, because each such path has 2 eastward block segments and they can occur in any order. The probability of taking any one of these paths to C_i and then going east is $\left(\frac{1}{2}\right)^{3+i}$ because there are $3 + i$ intersections along the way (including A and C_i) where an independent choice with probability $\frac{1}{2}$ is made. So the answer is



$$\sum_{i=0}^{3} \binom{2+i}{i} \left(\frac{1}{2}\right)^{3+i}$$
$$= \frac{1}{8} + \frac{3}{16} + \frac{6}{32} + \frac{10}{64} = \frac{21}{32}.$$

Alternate Solution. Less elegantly, one may construct a tree-diagram of the respective probabilities, obtaining the values step-by-step as shown in the scheme to the right (the final 1 also serves as a check on the computations).

It is important to recognize that not all twenty of the thirty five paths leading from A to B through C are equally likely; hence answer (C) is incorrect!

$$\begin{tikzcd}
1 \arrow[r] \arrow[d] & \frac{1}{2} \arrow[r] \arrow[d] & \frac{1}{4} \arrow[r] \arrow[d] & \frac{1}{8} \arrow[d] \\
\frac{1}{2} \arrow[r] \arrow[d] & \frac{1}{2} \arrow[r] \arrow[d] & \frac{3}{8} \arrow[r] \arrow[d] & \frac{5}{16} \arrow[d] \\
\frac{1}{4} \arrow[r] \arrow[d] & \frac{3}{8} \arrow[r] \arrow[d] & \frac{3}{8} \arrow[r] \arrow[d] & \frac{1}{2} \arrow[d] \\
\frac{1}{8} \arrow[r] \arrow[d] & \frac{1}{4} \arrow[r] \arrow[d] & \frac{5}{16} \arrow[r] \arrow[d] & \frac{21}{32} \arrow[d] \\
\frac{1}{16} \arrow[r] & \frac{3}{16} \arrow[r] & \frac{11}{32} \arrow[r] & 1
\end{tikzcd}$$

26. (B) If $n^2 = (ab3c)_8$, let $n = (de)_8$. Then $n^2 = (8d + e)^2 = 64d^2 + 8(2de) + e^2$. Thus, the 3 in $ab3c$ is the first digit (in base 8) of the sum of the eights digit of e^2 (in base 8) and the units digit of $(2de)$ (in base 8). The latter is even, so the former is odd. The entire table of base 8 representations of squares of base 8 digits appears below.

e	1	2	3	4	5	6	7
e^2	1	4	11	20	31	44	61

The eights digit of e^2 is odd only if e is 3 or 5; in either case c, which is the units digit of e^2, is 1. (In fact, there are three choices for n: $(33)_8$, $(73)_8$ and $(45)_8$. The squares are $(1331)_8$, $(6631)_8$ and $(2531)_8$, respectively.)

Alternate solution sketch (using familiarity with number theory). We are given $n^2 = (ab3c)_8 = 8^3a + 8^2b + 8\cdot 3 + c$.
Since $n^2 \equiv 0$, 1 or 4 mod 8, we must have $c = 0$, 1 or 4. If $c = 0$, then $n^2 \equiv 8(8K + 3)$ mod 8, an impossibility since 8 is not a square. If $c = 4$, then $n^2 \equiv 4(8L + 7)$ mod 8, another impossibility since no odd squares have the form $8L + 7$. Thus $c = 1$.

27. (C) One cannot simply use the theorem that solutions come in conjugate pairs, because that theorem applies to polynomials with *real* coefficients only. However, one can use the technique for proving that theorem to work this problem too. Namely, conjugate both sides of the original equation

$$0 = c_4z^4 + ic_3z^3 + c_2z^2 + ic_1z + c_0,$$

obtaining

$$\begin{aligned} 0 &= c_4\bar{z}^4 - ic_3\bar{z}^3 + c_2\bar{z}^2 - ic_1\bar{z} + c_0 \\ &= c_4(-\bar{z})^4 + ic_3(-\bar{z})^3 + c_2(-\bar{z})^2 + ic_1(-\bar{z}) + c_0. \end{aligned}$$

That is, $-\bar{z} = -a + bi$ is also a solution of the original equation. (One may check by example that neither $-a-bi$ nor $a-bi$ need be a solution. For instance, consider the equation $0 = z^4 + iz$ and the solution $a + bi = \frac{1}{2}(\sqrt{3} - i)$. Neither $\frac{1}{2}(-\sqrt{3} + i)$ nor $\frac{1}{2}(\sqrt{3} + i)$ is a solution.)

28. (B) Let n be the last number on the board. Now the largest average possible is attained if 1 is erased; the average is then

$$\frac{2+3+\cdots+n}{n-1} = \frac{\frac{(n+1)n}{2}-1}{n-1} = \frac{n+2}{2}.$$ The smallest average possible is attained when n is erased; the average is then $\frac{n(n-1)}{2(n-1)} = \frac{n}{2}$. *Thus*

$$\frac{n}{2} \leqslant 35\frac{7}{17} \leqslant \frac{n+2}{2},$$

$$n \leqslant 70\,\frac{14}{17} \leqslant n+2,$$

$$68\,\frac{14}{17} \leqslant n \leqslant 70\,\frac{14}{17}.$$

Hence $n = 69$ or 70. Since $35\,\frac{7}{17}$ is the average of $(n-1)$ integers, $(35\,\frac{7}{17})\,(n-1)$ must be an integer and n is 69.

If x is the number erased, then

$$\frac{\frac{69(70)}{2} - x}{68} = 35\,\frac{7}{17}.$$

So $69{\cdot}35 - x = (35\,\frac{7}{17})\,68 = 35{\cdot}68 + 28,$

$$35 - x = 28,$$
$$x = 7.$$

29. (A) Let $m = x_0y_0z_0$ be the minimum value. By symmetry, we may assume $x_0 \leqslant y_0 \leqslant z_0$. In fact $z_0 = 2x_0$, for if $z_0 < 2x_0$, then by decreasing x_0 slightly, increasing z_0 by the same amount, and keeping y_0 fixed, we would get new values which still meet the constraints but which have a smaller product – contradiction! To show this contradiction formally, let $x_1 = x_0 - h$ and $z_1 = z_0 + h$, where $h > 0$ is so small that $z_1 \leqslant 2x_1$ also. Then x_1, y_0, z_1 also meet all the original constraints, and

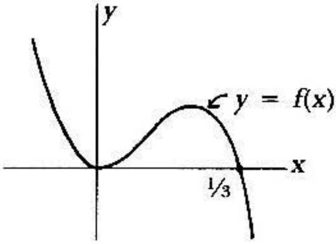
$$\begin{aligned} x_1y_0z_1 &= (x_0 - h)y_0(z_0 + h) \\ &= x_0y_0z_0 + y_0\left[h(x_0 - z_0) - h^2\right] < x_0y_0z_0. \end{aligned}$$

So $z_0 = 2x_0$, $y_0 = 1 - x_0 - z_0 = 1 - 3x_0$, and $m = 2x_0^2\,(1 - 3x_0)$. Also, $x_0 \leqslant 1 - 3x_0 \leqslant 2x_0$, or equivalently, $\frac{1}{5} \leqslant x_0 \leqslant \frac{1}{4}$. Thus m may be viewed as a value of the function $f(x) = 2x^2\,(1 - 3x)$ on the domain $D = \left\{x \mid \frac{1}{5} \leqslant x \leqslant \frac{1}{4}\right\}$. In fact, m is the *smallest* value of f on D, because minimizing f on D is just a restricted version of the original problem: for each $x \in D$, setting $y = 1 - 3x$ and $z = 2x$ gives x, y, z meeting the original constraints, and makes $f(x) = xyz$.

To minimize f on D, first sketch f for all real x. (See Figure.) Since f has a relative minimum at $x = 0$ ($f(x)$ has the same sign as x^2 for $x < \frac{1}{3}$), and cubics have at most one relative minimum, the minimum of f on D must be at one of the endpoints. In fact,

$$f(\tfrac{1}{4}) = \frac{1}{32} \leqslant f(\tfrac{1}{5}) = \frac{4}{125}.$$

(If f had another relative minimum between its two zeros, say at point $x = a$, then the equation $f(x) = f(a)$ would have at least 4 roots — draw a sketch. But a cubic equation has at most three roots!)



30. (D) Let $d_1 = a + \sqrt{b}$ and $d_2 = a - \sqrt{b}$, where $a = 15$ and $b = 220$. Then using the binomial theorem, we may obtain

$$d_1^n + d_2^n = 2\left[a^n + \binom{n}{2}a^{n-2}b + \binom{n}{4}a^{n-4}b^2 + \ldots\right],$$

where n is any positive integer. Since fractional powers of b have been eliminated in this way, and since a and b are both divisible by 5, we may conclude that $d_1^n + d_2^n$ is divisible by 10.

We now apply the above result twice, taking $n = 19$ and $n = 82$. In this way we obtain

$$d_1^{19} + d_2^{19} = 10k_1 \text{ and } d_1^{82} + d_2^{82} = 10k_2,$$

where k_1 and k_2 are positive integers. Adding and rearranging these results gives

$$d_1^{19} + d_1^{82} = 10k - (d_2^{19} + d_2^{82}),$$

where $k = k_1 + k_2$. But $d_2 = 15 - \sqrt{220} = \dfrac{5}{15 + \sqrt{220}} < \dfrac{1}{3}.$

Therefore, $d_2^{19} + d_2^{82} < 1$. It follows that the units digit of $10k - (d_2^{19} + d_2^{82})$ is 9.

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

34th ANNUAL AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION 1983

A Prize Examination Sponsored by

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This Solutions Pamphlet gives at least one solution for each problem on this year's Examination and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will find the time to share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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1. (E) Solving each equation for y^2 gives $y^2 = \frac{x}{2} = \frac{x^2}{256}$. Thus

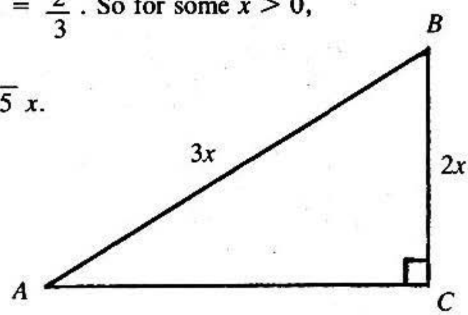
$$x = \frac{256}{2} = 128 \quad \text{since } x \neq 0.$$

2. (B) The points in question are the points of intersection of the original circle C and the circle of radius 3 cm around P. Two distinct circles intersect in at most 2 points.

3. (A) Not both p and q are odd, since then r would be an even prime greater than 2, which is impossible. Thus one of p and q is 2. Since $1<p<q$, p is 2.

4. (D) Draw in lines BF, BE and BD. There are now 4 equilateral triangles with side length 1. (For instance, $\triangle FAB$ is equilateral because $AF = AB = 1$ and $\measuredangle A = 60°$.)
Thus the total area is $4\,\frac{1^2\sqrt{3}}{4} = \sqrt{3}$.

5. (D) In the figure, $\sin A = \frac{BC}{AB} = \frac{2}{3}$. So for some $x > 0$,
$BC = 2x$, $AB = 3x$ and
$AC = \sqrt{(AB)^2-(BC)^2} = \sqrt{5}\,x$.
Thus $\tan B = \frac{AC}{BC} = \frac{\sqrt{5}}{2}$.



6. (C) By definition, a polynomial (in x) is an expression of the form $a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0$, where the a's are constants and $a_n \neq 0$. The degree is defined to be n, the highest power of x. Since

$$x^5\left(x+\frac{1}{x}\right)\left(1+\frac{2}{x}+\frac{3}{x^2}\right) = x^2(x^2+1)(x^2+2x+3),$$

one sees without actually multiplying further (if one adds the degrees of the factors on the right) that the product is a polynomial and has degree 6.

7. (B) Let x be the list price in dollars. Then

$$\begin{aligned} .1(x-10) &= .2(x-20), \\ x-10 &= 2x-40, \\ x &= 30. \end{aligned}$$

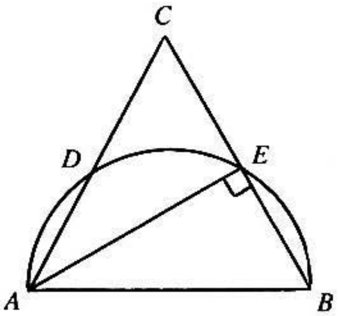
8. (A) $$f(-x) = \frac{-x+1}{-x-1} = \frac{x-1}{x+1} = \frac{1}{\left(\frac{x+1}{x-1}\right)} = \frac{1}{f(x)}.$$

9. (D) Let w be the number of women, m the number of men. From the given, $w = 11x$ and $m = 10x$ for some x; also $34w$ is the sum of the ages of the women and $32m$ is the sum for the men. Thus the average age for all is

$$\frac{34(11x) + 32(10x)}{11x + 10x} = \frac{34 \cdot 11 + 32 \cdot 10}{21} = \frac{694}{21} = 33\frac{1}{21}.$$

Note: except for the leftmost expression in the previous line, the computation of the average is the same computation one would do if there were *exactly* 11 women and 10 men.

10. (D) In the figure, $\measuredangle AEB = 90°$ since AB is a diameter. Thus AE is an altitude of equilateral $\triangle ABC$.
It follows that $\triangle ABE$ is a $30°$–$60°$–$90°$ triangle and $AE = AB\,\frac{\sqrt{3}}{2} = \sqrt{3}$.



11. (B) Let $w = x - y$. Then the given expression is $\sin w \cos y + \cos w \sin y = \sin(w+y) = \sin x$.

12. (E) $\log_7(\log_3(\log_2 x)) = 0 \Rightarrow \log_3(\log_2 x) = 1$
$\Rightarrow \log_2 x = 3 \Rightarrow x = 2^3 = 8.$

So $x^{-1/2} = \dfrac{1}{\sqrt{8}} = \dfrac{1}{2\sqrt{2}}$.

13. (E) Observe that $abc = x^2y^2z^2 = x^2c^2$, so $x^2 = \dfrac{ab}{c}$. Likewise,

$y^2 = \dfrac{ac}{b}$ and $z^2 = \dfrac{bc}{a}$. So

$$x^2 + y^2 + z^2 = \frac{ab}{c} + \frac{ac}{b} + \frac{bc}{a} = \frac{(ab)^2 + (ac)^2 + (bc)^2}{abc}.$$

14. (E) Consider the first few powers of 3, 7 and 13:

3	7	13
9	49	169
27	343	...7
81	...1	1
243	7	3

Clearly, the units digits in each case go through a cycle of length 4, with the units digit being 1 if the power is a multiple of 4. Let $u(n)$ be the units digit of n. Since 4 divides 1000,

$$\begin{aligned} u(3^{1001}) &= u(3^1) = 3, \\ u(7^{1002}) &= u(7^2) = 9, \\ u(13^{1003}) &= u(13^3) = 7. \end{aligned}$$

So $u(3^{1001}7^{1002}13^{1003}) = u(3 \cdot 9 \cdot 7) = 9$.

(This solution can be expressed much more briefly using congruences.)

Alternate solution. Any power of either $7 \cdot 13 = 91$ or $3^4 = 81$ has a units digit of 1. Thus $3^{1001}7^{1002}13^{1003} = 3 \cdot 13 \cdot 81^{250} 91^{1002}$ which clearly has a units digit of 9.

15. (C) A total of 6 can be achieved by 7 equally likely ordered triples of draws: (1,2,3), (1,3,2), (2,1,3), (2,3,1), (3,1,2), (3,2,1) and (2,2,2). Therefore the answer is $\frac{1}{7}$.

16. (D) Look at the first 1983 digits, letting z denote the 1983$^{\text{rd}}$ digit. We may break this string of digits into three segments:

$$\underbrace{.123456789}_{A} \quad \underbrace{1011\ldots9899}_{B} \quad \underbrace{100101\ldots z}_{C}.$$

There are 9 digits in A, $2 \cdot 90 = 180$ in B, hence $1983 - 189 = 1794$ in C. Dividing 1794 by 3 we get 598 with remainder 0. Thus C consists of the first 598 3-digit integers. Since the first 3-digit integer is 100 (not 101 or 001), the 598$^{\text{th}}$ 3-digit integer is 697. Thus $z = 7$.

17. (C) Write F as $a+bi$, where we see from the diagram that $a, b > 0$ and $a^2 + b^2 > 1$. Since

$$\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i,$$

we see that the reciprocal of F is in quadrant IV, since the real part on the right is positive and the imaginary coefficient is negative. Also, the magnitude of the reciprocal is

$$\frac{1}{a^2+b^2}\sqrt{a^2+(-b)^2} = \frac{1}{\sqrt{a^2+b^2}} < 1.$$

Thus the only possibility is point C.

Alternate solution. For any complex number $z \neq 0$, the argument (standard reference angle) of $\frac{1}{z}$ is the negative of the argument of z, and the modulus (magnitude) of $\frac{1}{z}$ is the reciprocal of the modulus of z. (Quick proof: if $z = re^{i\theta}$, then $\frac{1}{z} = \frac{1}{re^{i\theta}} = \frac{1}{r}e^{i(-\theta)}$.) Applying this to the point F, its reciprocal must be in quadrant IV, inside the unit circle. The only possibility is point C.

18. (B) Since $f(x)$ is a polynomial and $f(x^2+1)$ has degree 4, then $f(x)$ has degree 2. That is, $f(x) = ax^2 + bx + c$ for some constants a, b, c, and

$$\begin{aligned} f(x^2+1) = x^4 + 5x^2 + 3 &= a(x^2+1)^2 + b(x^2+1) + c \\ &= ax^4 + (2a+b)x^2 + (a+b+c). \end{aligned}$$

Since two polynomials are equal if and only if the coefficients of corresponding terms are equal, we have $a = 1$, $2a+b = 5$ and $a+b+c = 3$. Solving these gives $f(x) = x^2 + 3x - 1$. Thus $f(x^2-1) = (x^2-1)^2 + 3(x^2-1) - 1 = x^4 + x^2 - 3$.

Alternate solution. Rewrite $x^4 + 5x^2 + 3$ in terms of powers of $x^2 + 1$:

$$\begin{aligned} x^4 + 5x^2 + 3 &= (x^4 + 2x^2 + 1) + (3x^2 + 3) - 1 \\ &= (x^2+1)^2 + 3(x^2+1) - 1. \end{aligned}$$

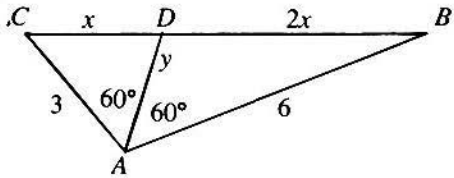
Thus setting $w = x^2 + 1$, we have $f(w) = w^2 + 3w - 1$.

(Actually, since x is real, w as defined can only take on values equal to or greater than 1. Thus so far we have shown only that $f(w) = w^2 + 3w - 1$ when $w \geq 1$. However, since $f(w)$ is given to be a polynomial, and since it is identical to the polynomial $w^2 + 3w - 1$ for infinitely many w, it follows that they are identical for all w.)

Therefore, as before, for all x

$$f(x^2-1) = (x^2-1)^2 + 3(x^2-1) - 1 = x^4 + x^2 - 3.$$

19. (A) Let $AD = y$. Since AD bisects $\measuredangle BAC$, we have $\dfrac{DB}{CD} = \dfrac{AB}{AC} = 2$; so we may set $CD = x$, $DB = 2x$ as in the figure. Applying the Law of Cosines to $\triangle CAD$ and $\triangle DAB$, we have

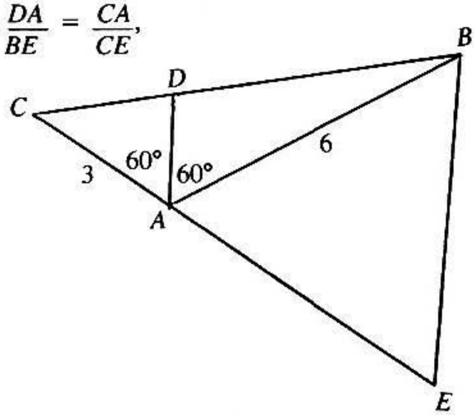


$$x^2 = 3^2 + y^2 - 3y,$$
$$(2x)^2 = 6^2 + y^2 - 6y.$$

Subtracting 4 times the first equation from the second yields $0 = -3y^2 + 6y = -3y(y-2)$. Since $y \neq 0$, $y = 2$.

Alternate solution. Extend CA to E so that $BE \| DA$ as in the new figure. Then $\triangle ABE$ is equilateral: $\measuredangle BEA = \measuredangle DAC$ by corresponding angles, $\measuredangle ABE = \measuredangle BAD$ by alternate interior angles, and $\measuredangle EAB = 180° - 120°$. Since $\triangle BEC \sim \triangle DAC$, we have $\dfrac{DA}{BE} = \dfrac{CA}{CE}$, or $\dfrac{DA}{6} = \dfrac{3}{9}$.

So $DA = 2$.



20. (C) By the relationship between the roots and the coefficients of a quadratic equation, it follows that $p = \tan\alpha + \tan\beta$, $q = \tan\alpha\tan\beta$, $r = \cot\alpha + \cot\beta$, and $s = \cot\alpha\cot\beta$. Since

$$\cot\alpha + \cot\beta = \frac{1}{\tan\alpha} + \frac{1}{\tan\beta} = \frac{\tan\alpha + \tan\beta}{\tan\alpha\tan\beta},$$

and $\cot\alpha\cot\beta = \dfrac{1}{\tan\alpha\tan\beta}$, the equalities $r = \dfrac{p}{q}$ and $s = \dfrac{1}{q}$ follow. Thus $rs = \dfrac{p}{q^2}$.

Alternate solution. The roots are reciprocals, and in general, if $x^2 - px + q = 0$ has roots m and n, then $qx^2 - px + 1 = 0$ has roots $\dfrac{1}{m}$ and $\dfrac{1}{n}$. Dividing the last equation by q shows that $x^2 - \dfrac{p}{q}x + \dfrac{1}{q} = 0$ also has roots $\dfrac{1}{m}$ and $\dfrac{1}{n}$. So $r = \dfrac{p}{q}$, $s = \dfrac{1}{q}$, and $rs = \dfrac{p}{q^2}$.

21. (D) Since $10^2 = 100 > 99 = (3\sqrt{11})^2$,
$18^2 = 324 < 325 = (5\sqrt{13})^2$,
$51^2 = 2601 > 2600 = (10\sqrt{26})^2$,

only (A) and (D) are positive numbers.

Moreover, since $a-b = \dfrac{a^2-b^2}{a+b}$,

$$10 - 3\sqrt{11} = \frac{1}{10+3\sqrt{11}} \approx \frac{1}{20} \text{ and } 51 - 10\sqrt{26} = \frac{1}{51+10\sqrt{26}} \approx \frac{1}{102}.$$

So (D) is the answer.

22. (B) We must describe geometrically those (a,b) for which the equation

$$x^2 + 2bx + 1 = 2a(x+b),$$

or equivalently, the equation

$$x^2 + 2(b-a)x + (1-2ab) = 0,$$

has no real root for x. Since a quadratic equation $Ax^2 + Bx + C = 0$ has no real root if and only if its discriminant B^2-4AC is negative, S is the set of (a,b) for which

$$[2(b-a)]^2 - 4(1-2ab) < 0,$$
$$4(a^2-2ab+b^2) - 4 + 8ab < 0,$$
$$4a^2 + 4b^2 < 4,$$
$$a^2 + b^2 < 1.$$

Thus S is the unit circle (without boundary) and the area is π.

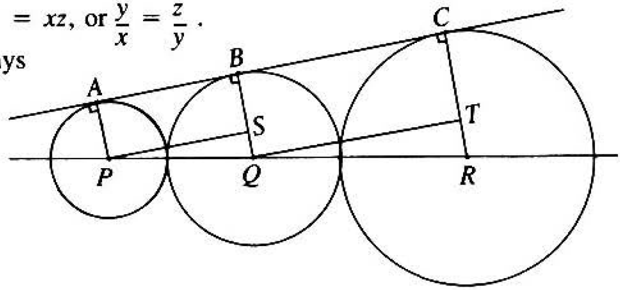
23. (A) We claim that the ratio of consecutive radii is constant, that is, the sequence of radii form a geometric progression. It follows that this ratio is $\sqrt[4]{\frac{18}{8}}$ and that the middle radius is $8\sqrt{\frac{18}{8}} = 12$.

The claim should be intuitive—the picture for any two consecutive circles looks the same except for a change of scale—but here is a proof. In the figure below, three consecutive circles are shown. Their centers P, Q and R are collinear, on the line bisecting the angle formed by the two tangents. Let A, B and C be the points of tangency, and let PS, QT be segments parallel to the upper tangent as shown. Then $\triangle PQS \sim \triangle QRT$. If we let x, y, z be the radii from smallest to largest, then $QS = y-x$, $RT = z-y$.

Thus $\dfrac{QS}{PQ} = \dfrac{RT}{QR}$ becomes $\dfrac{y-x}{x+y} = \dfrac{z-y}{y+z}$,

which simplifies to $y^2 = xz$, or $\dfrac{y}{x} = \dfrac{z}{y}$.

This last equation says that the ratio of consecutive radii is constant, as claimed.



24. (E) We will show that the ratio of perimeter to area in a triangle can be changed at will by replacing the triangle by a similar one. Thus in each class of similar right triangles, there is one with perimeter (in whatever length units, say cm) equal to area (in whatever area units), that is, with the ratio being 1. Since there are infinitely many non-similar right triangles, thus there are infinitely many non-similar (hence non-congruent) right triangles with perimeter = area.

Let a, b, c be the lengths (in cm) of the two sides and the hypotenuse respectively of an arbitrary right triangle. The ratio of perimeter (in cm) to the area (in cm^2) is

$$r = \frac{2(a+b+c)}{ab}.$$

Now consider the similar triangle with sides ka, kb, kc. The ratio is now

$$\frac{2(ka+kb+kc)}{(ka)(kb)} = \frac{1}{k}\,\frac{2(a+b+c)}{ab} = \frac{r}{k}.$$

Thus, choose $k = r$ and in the new triangle perimeter = area.

Alternate solution sketch. If a and b are the lengths of the legs of a right triangle, then the area and perimeter are numerically equal if and only if

$$\frac{1}{2}ab = a + b + \sqrt{a^2+b^2}.$$

This is one equation in two (nonnegative) real variables. Generally such an equation has infinitely many solutions. Also, one expects different solutions (a,b) to result in non-congruent triangles. Therefore, one should already believe that (E) is correct. A complete analysis (subtle, but left to the reader!) shows that this belief is correct, even though solving the above equation (by rearranging and squaring to remove the radical) introduces extraneous roots, and even though some distinct pairs of valid solutions represent congruent triangles, e.g., (6,8) and (8,6).

25. (B) $12 = \frac{60}{5} = \frac{60}{60^b} = 60^{1-b}$. So

$$\begin{aligned} 12^{[(1-a-b)/2(1-b)]} &= [60^{1-b}]^{(1-a-b)/2(1-b)} \\ &= 60^{(1-a-b)/2} = \sqrt{\frac{60}{60^a 60^b}} \\ &= \sqrt{\frac{60}{3\cdot 5}} = 2. \end{aligned}$$

26. (D) Let $P(E)$ be the probability that event E occurs. By the Inclusion-Exclusion Principle, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. So

$$p = P(A \cap B) = P(A) + P(B) - P(A \cup B) = \frac{3}{4} + \frac{2}{3} - P(A \cup B).$$

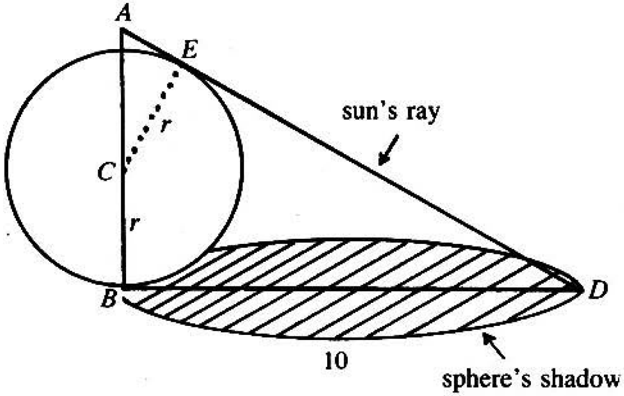
At the most, $P(A \cup B) = 1$; at the least, $P(A \cup B) =$ $\max\{P(A), P(B)\} = \frac{3}{4}$. So $\frac{3}{4} + \frac{2}{3} - 1 \le p \le \frac{3}{4} + \frac{2}{3} - \frac{3}{4}$, which is (D).

(Note: only if A and B are independent is $p = \frac{1}{2}$.)

27. (E) We first construct the figure shown, which results from bisecting the sphere-shadow configuration with a vertical plane. Since the meter stick casts a shadow of 2 m, $AB = 5$ and hence $AD = 5\sqrt{5}$. Now $\triangle CEA$ is similar to $\triangle DBA$.

Thus $\dfrac{CE}{AC} = \dfrac{BD}{AD}$ or $\dfrac{r}{5-r} = \dfrac{10}{5\sqrt{5}} = \dfrac{2}{\sqrt{5}}$.

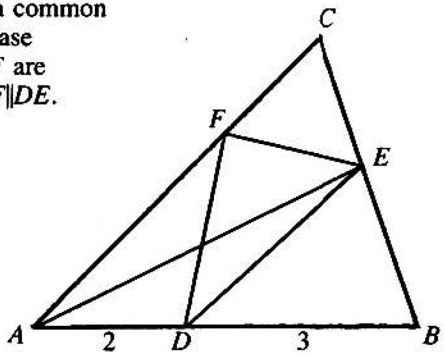
So $\sqrt{5}\, r = 10 - 2r$ or $r = \dfrac{10}{2+\sqrt{5}} = 10\sqrt{5} - 20.$



28. (C) Draw in line DE. Since Area ABE = Area ADE + Area DBE, and Area $DBEF$ = Area FDE + Area DBE, it follows that Area ADE = Area FDE, Since these two triangles have a common base DE, the altitudes on this base must be equal. That is, A and F are equidistant from line DE, so $AF \| DE$. Then, using similar triangles ABC and DBE,

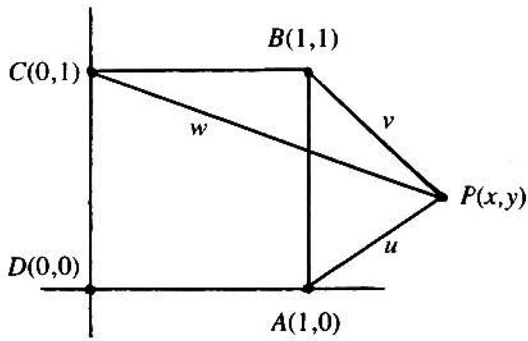
$\dfrac{EB}{CB} = \dfrac{DB}{AB} = \dfrac{3}{5}$.

Thus Area ABE = $\dfrac{3}{5}$ Area ABC = 6.

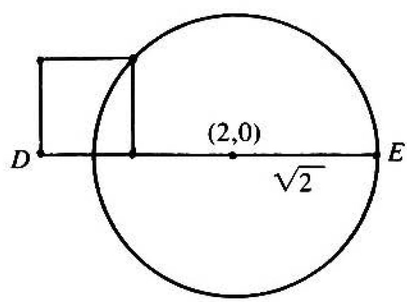


29. (C) Let the square be placed in the coordinate plane as in the first figure: D is placed at the origin so that algebraic expressions for the distance from it will be easy to interpret. Then $u^2 + v^2 = w^2$ becomes

$$[(x-1)^2+y^2] + [(x-1)^2 + (y-1)^2] = x^2+(y-1)^2,$$



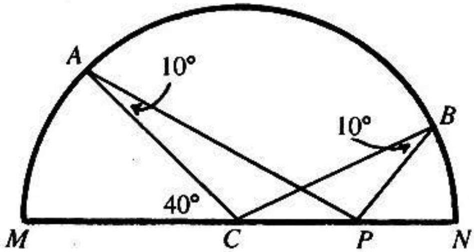
which simplifies to $x^2-4x+2+y^2 = 0$, that is, $(x-2)^2+y^2 = 2$. Thus the locus of P is a circle with center (2,0) and radius $\sqrt{2}$. From the second figure it is clear that the farthest point on this circle from D is $E = (2+\sqrt{2}, 0)$.



30. (C) In $\triangle ACP$ and $\triangle BCP$ we have (in the order given) the condition a.s.s. Since these triangles are not congruent ($\measuredangle CPA \neq \measuredangle CPB$), we must have that $\measuredangle CPA$ and $\measuredangle CPB$ are supplementary. From $\triangle ACP$ we compute

$$\measuredangle CPA = 180°-10°-(180°-40°) = 30°.$$

Thus $\measuredangle CPB = 150°$ and $\widehat{BN} = \measuredangle PCB = 180°-10°-150° = 20°$.



Alternate solution. Again $\measuredangle CPA = 30°$. Applying the Law of Sines to $\triangle ACP$ and then $\triangle BCP$, we have

$$\frac{\sin 10°}{CP} = \frac{\sin 30°}{AC} \quad \text{and} \quad \frac{\sin 10°}{CP} = \frac{\sin \measuredangle CPB}{BC}.$$

Thus $\sin \measuredangle CPB = \frac{1}{2}$ since $AC = BC$. As $\measuredangle CPB \neq \measuredangle CPA$, we must have $\measuredangle CPB = 150°$. Hence $\widehat{BN} = 20°$.

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

35th ANNUAL
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1984

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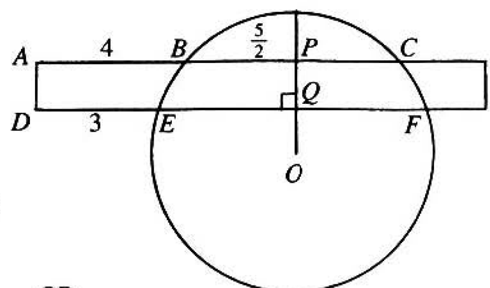
1. (C) $$\frac{1000^2}{252^2-248^2} = \frac{1000\cdot 1000}{(252+248)(252-248)} = \frac{1000}{500}\cdot\frac{1000}{4} = 500.$$

2. (B) $$\frac{x-\frac{1}{y}}{y-\frac{1}{x}} = \frac{\frac{xy-1}{y}}{\frac{xy-1}{x}} = \frac{xy-1}{y}\cdot\frac{x}{xy-1} = \frac{x}{y}.$$

3. (C) The number n must be a product of two or more primes greater than 10, not necessarily distinct. The smallest such prime is 11, so the smallest product is $11\cdot 11 = 121$.

4. (B) Draw in the perpendicular bisector of chord EF, as shown in the adjoining figure. This bisector must go through the center O of the circle. Since PO is perpendicular to chord BC as well, it bisects BC. Since $PQ \parallel AD$, we have $DQ = AP = 4 + \frac{5}{2} = \frac{13}{2}$. Thus

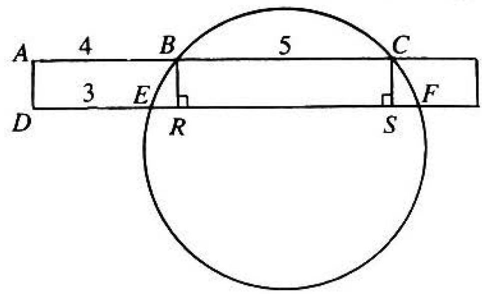
$$EF = 2\,EQ = 2\,(DQ - DE) = 7.$$



OR

Drop perpendiculars BR and CS as shown in the second figure, yielding rectangles $BCSR$ and $ABRD$. Then $ER = 1$, and by symmetry, $SF = 1$. Also, $RS = BC = 5$, so

$$EF = 1 + 5 + 1 = 7.$$

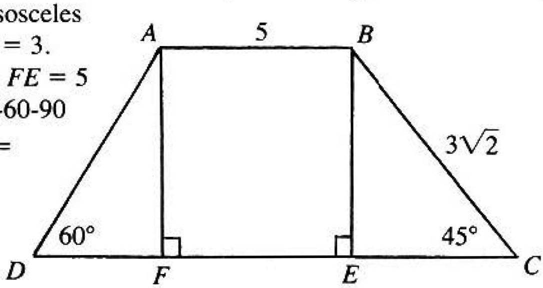


5. (D) Taking one-hundredth roots, $n^{200} < 5^{300} \Longleftrightarrow n^2 < 5^3 = 125$. The largest perfect square less than 125 is $121 = 11^2$.

6. (B) We have $b = 3g$ and $g = 9t$ (not $3b = g$ or $9g = t$!) so that $g = \frac{b}{3}$ and $t = \frac{g}{9} = \frac{b}{27}$. Thus the desired sum is (B).

7. (C) Dave travels (90)(75) cm each minute, so the entire trip is (16)(90)(75) cm long. Jack travels (100)(60) cm each minute, so it takes him (16)(90)(75)/(100)(60) or 18 minutes to get to school.

8. (D) Drop perpendiculars from A and B to DC, intersecting DC at F and E, respectively. ΔBEC is an isosceles right triangle, so $BE = EC = 3$. Since $ABEF$ is a rectangle, $FE = 5$ and $AF = 3$. ΔAFD is a 30-60-90 triangle, so $DF = \frac{1}{\sqrt{3}} AF = \sqrt{3}$. So $DC = 8 + \sqrt{3}$.



9. (D) $4^{16}5^{25} = 2^{32}5^{25} = 2^{7}10^{25} = 128 \cdot 10^{25}$. Thus the usual form is 128 followed by 25 zeros, for a total of 28 digits.

10. (B) Two given points, $1 + 2i$ and $-1 - 2i$, are symmetric with respect to the origin, and the distances from the origin to all three points are equal. Therefore the origin is the center of the square, and the fourth vertex must be symmetric to $-2 + i$ around it. Thus the fourth vertex is $2 - i$.

OR

Plotting the given three points makes it graphically clear where the fourth point must go.

11. (A) At each stage the displayed entry can be thought of as a power of x. Each reciprocation reverses the sign of the exponent of x. Each squaring doubles the exponent. Thus each pair of squaring and reciprocation multiplies the exponent by -2. Since the initial exponent is 1, the final exponent is $(-2)^n$.

12. (B) From the definition of the sequence,

$$\begin{aligned} a_2 - a_1 &= 2\cdot 1, \\ a_3 - a_2 &= 2\cdot 2, \\ &\;\;\cdots\cdot \\ a_{100} - a_{99} &= 2\cdot 99. \end{aligned}$$

Adding, one obtains

$$\begin{aligned} a_{100} - a_1 &= 2\,(1 + 2 + \cdots + 99) = 99\cdot 100 = 9900, \\ a_{100} &= 9902. \end{aligned}$$

13. (A) Note that $(\sqrt{2}+\sqrt{3}+\sqrt{5})(\sqrt{2}+\sqrt{3}-\sqrt{5}) =$ $(\sqrt{2}+\sqrt{3})^2 - 5 = (5+2\sqrt{6}) - 5 = 2\sqrt{6}$. Thus

$$\frac{2\sqrt{6}}{\sqrt{2}+\sqrt{3}+\sqrt{5}} =$$
$$\frac{2\sqrt{6}}{\sqrt{2}+\sqrt{3}+\sqrt{5}} \cdot \frac{\sqrt{2}+\sqrt{3}-\sqrt{5}}{\sqrt{2}+\sqrt{3}-\sqrt{5}} = \sqrt{2}+\sqrt{3}-\sqrt{5}.$$

Remark. In general, $\dfrac{2\sqrt{mn}}{\sqrt{m}+\sqrt{n}+\sqrt{m+n}} = \sqrt{m}+\sqrt{n}-\sqrt{m+n}.$

14. (A) Given $x^{\log_{10}x} = 10$, take logs base 10 of both sides:

$$\begin{aligned}(\log_{10}x)(\log_{10}x) &= 1,\\ \log_{10}x &= \pm 1,\\ x &= 10,\ 10^{-1}.\end{aligned}$$

Neither solution is extraneous. Thus the product of the roots is 1.

15. (A) The following statements are equivalent:

$$\begin{aligned}&\sin 2x \sin 3x = \cos 2x \cos 3x,\\ &\cos 2x \cos 3x - \sin 2x \sin 3x = 0,\\ &\cos(2x+3x) = 0,\\ &5x = 90^\circ + 180^\circ k, \quad k = 0, \pm 1, \pm 2, \dots,\\ &x = 18^\circ + 36^\circ k, \quad k = 0, \pm 1, \pm 2, \dots .\end{aligned}$$

The only correct value listed among the answers is 18°.

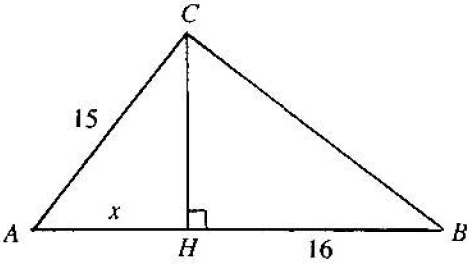
OR

By inspection of the original equation, it is sufficient that $\sin 2x = \cos 3x$ and $\sin 3x = \cos 2x$, which are both true if $2x$ and $3x$ are complementary. Thus $2x + 3x = 90^\circ$, i.e., $x = 18^\circ$, is a correct value.

16. (E) If $r_1 = 2 + a$ is a root, then so is $r_2 = 2 - a$. That is, all the distinct roots come in pairs symmetric around $x = 2$ (except for the self-symmetric value $r = 2$, if it is a root). Each pair has sum 4. Since there are exactly four distinct roots, 2 is not a root, there are exactly two pairs, and the sum of all roots is 8.

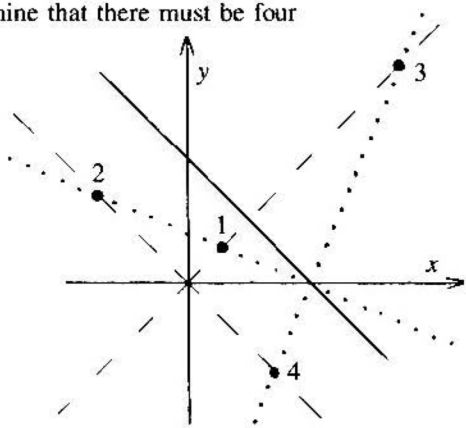
17. (C) A leg of a right triangle is the geometric mean of the hypotenuse and the projection of the leg on the hypotenuse. Setting $AH = x$, it follows that

$$225 = x(x + 16),$$
$$x^2 + 16x - 225 = 0,$$
$$(x + 25)(x - 9) = 0,$$
$$x = 9.$$



Thus $AB = 25$, $CH = \sqrt{15^2 - 9^2} = 12$, and the area of ΔABC *is* $\frac{1}{2}\cdot 25 \cdot 12 = 150$.

18. (E) From a picture we can determine that there must be four such points, with different x coordinates. To be equidistant from two nonparallel lines L and K, a point must be on *either* of the two lines which bisect a pair of opposite angles at the intersection of L and K. In particular, to be equidistant from the x and y axes, (x,y) must be on either $y = x$ or $y = -x$ (shown dashed in the adjoining figure); to be equidistant from the x-axis and $x + y = 2$, (x,y) must be on one of the lines through $(2,0)$ shown dotted in the figure. There are four points simultaneously on one of the dashed and one of the dotted lines, as indicated. One can show that the x coordinates of the four points, in the order labeled, are $2 - \sqrt{2}$, $-\sqrt{2}$, $2 + \sqrt{2}$ and $\sqrt{2}$.

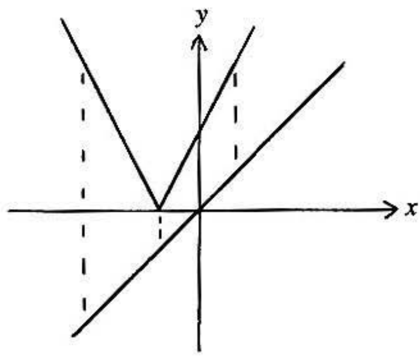


OR

The line and the axes determine a triangle. This triangle has an inscribed circle and three escribed circles. The centers of these circles, and no other points, satisfy the equal-distance condition. Hence there are four points. It is easy to see by a sketch that their x-coordinates are not all the same.

19. (D) The sum is odd if and only if an odd number of odd numbered balls is chosen. Of the $\binom{11}{6} = 462$ possible unordered sets of 6, $\binom{6}{1}\binom{5}{5} = 6$ have 1 odd numbered ball, $\binom{6}{3}\binom{5}{3} = 200$ have 3, and $\binom{6}{5}\binom{5}{1} = 30$ have 5. So the probability is $(6 + 200 + 30)/462 = 118/231$.

20. (C) The equation $|x - |2x+1|| = 3$ is satisfied by precisely those x for which the vertical distance between the graph of $y = x$ and the graph of $y = |2x+1|$ is 3. These two graphs are sketched in the adjoining figure; the latter graph is V-shaped with vertex $(-\frac{1}{2}, 0)$ and sides of slope ± 2. Thus there are exactly two x values which satisfy the original equation, because the vertical distance between the graphs increases steadily as one moves right or left from $x = -\frac{1}{2}$, and the vertical distance for that x is $\frac{1}{2}$.



OR

Either $x - |2x+1| = -3$ or $x - |2x + 1| = 3$, yielding $|2x + 1| = x + 3$ and $|2x + 1| = x - 3$, respectively. In the first case one must have $x \geq -3$, while in the second case, $x \geq 3$ must hold. In the first case, $2x + 1$ must equal $x + 3$ or $-x - 3$, yielding $x = 2$ and $x = -\frac{4}{3}$, which are solutions. In the second case, $2x + 1$ must equal $x - 3$ or $-x + 3$, and hence $x = -4$ and $x = \frac{2}{3}$; however, these values violate the condition $x \geq 3$ and therefore do not satisfy the original equation.

21. (C) From the second equation, $c(a + b) = 23$, and 23 is prime. Consequently, the two factors must be 1 and 23. Since a and b are positive integers, $a + b > 1$. Hence one must have $c = 1$ and $a + b = 23$. Upon substituting 1 for c and $23 - a$ for b into the first equation, it becomes a quadratic, $a^2 - 22a + 21 = 0$, with solutions $a = 1$ and $a = 21$. Both of these, and the corresponding values of b (22 and 2), satisfy both equations. Thus the solutions are (1,22,1) and (21,2,1).

22. (B) The vertex of any parabola $y = ax^2 + tx + c$ is on its axis of symmetry $x = \frac{-t}{2a}$. Thus $x_t = \frac{-t}{2a}$ and

$$y_t = a\left(\frac{-t}{2a}\right)^2 + t\left(\frac{-t}{2a}\right) + c = c - \frac{t^2}{4a} = c - ax_t^2.$$

Thus every point (x_t, y_t) is on the parabola $y = -ax^2 + c$. Conversely, each point (x,y) on this parabola is of the form (x_t,y_t): just set $t = -2ax$.

23. (D) Two trigonometric identities for expressing sums as products are:

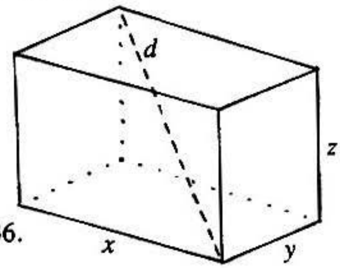
$$\sin x + \sin y = 2 \sin \frac{x+y}{2} \cos \frac{x-y}{2},$$

$$\cos x + \cos y = 2 \cos \frac{x+y}{2} \cos \frac{x-y}{2}.$$

Thus $\dfrac{\sin 10^\circ + \sin 20^\circ}{\cos 10^\circ + \cos 20^\circ} = \dfrac{\sin 15^\circ}{\cos 15^\circ} = \tan 15^\circ.$

24. (E) That the roots of $x^2 + ax + 2b = 0$ are both real implies that the discriminant $a^2 - 8b$ is nonnegative, that is, $a^2 \geq 8b$. Likewise, that $x^2 + 2bx + a = 0$ has real roots implies $4b^2 \geq 4a$ or $b^2 \geq a$. Thus $a^4 \geq 64b^2 \geq 64a$. Since $a > 0$, one obtains $a^3 \geq 64$ and so $a \geq 4$ and $b \geq 2$. Conversely, $a = 4$, $b = 2$ does result in real roots for both (identical) equations. Thus the minimum value of $a + b$ is 6.

25. (D) With x, y, z, d as in the adjoining figure, we have



(1) $2xy + 2xz + 2yz = 22,$
(2) $4x + 4y + 4z = 24,$
(3) $d^2 = x^2 + y^2 + z^2.$

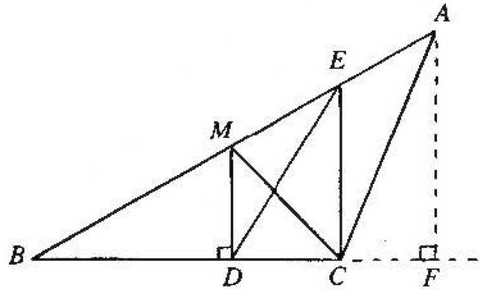
From (2) we obtain $(x+y+z)^2 = 6^2$, or

(4) $x^2 + y^2 + z^2 + 2xy + 2xz + 2yz = 36.$

Substituting (1) and (3) into (4) gives $d^2 + 22 = 36$, or $d = \sqrt{14}$. (Note: the dimensions of the solid are not uniquely determined, but d is!)

26. (B) Draw in MC as in the adjoining figure. Then ΔDMC and ΔDME have the same area, as they have the same base MD and equal altitudes on that base. Thus

$$\text{Area } \Delta BMC = \text{Area } \Delta BED.$$



Moreover, as shown below,

$$\text{Area } \Delta BMC = \tfrac{1}{2} \text{ Area } \Delta BAC.$$

Thus

$$\text{Area } \Delta BED = \tfrac{1}{2} \text{ Area } \Delta BAC = 12.$$

As for the claim in the second display, since M is the midpoint of BA, altitude MD of ΔBMC is one half altitude AF of ΔBAC. Since the triangles have the same base BC, the claim follows.

Alternately, one proves the claim by noting that

$$\begin{aligned}\text{Area } \Delta BMC &= \tfrac{1}{2}(BM)(BC)\sin B = \tfrac{1}{2}\left[\tfrac{1}{2}(BA)(BC)\sin B\right] \\ &= \tfrac{1}{2} \text{ Area } \Delta BAC.\end{aligned}$$

27. (C) Let $AC = x$ and $\measuredangle DCF = \theta$. Then $\measuredangle CBD = \theta$, and $\measuredangle ADB = 2\theta$ by the Exterior Angle Theorem. Thus $\cos\theta = \frac{1}{x}$ and $\cos 2\theta = x - 1$. Therefore,

$$\begin{aligned} 2\left(\tfrac{1}{x}\right)^2 - 1 &= x - 1, \\ 2 - x^2 &= x^3 - x^2, \\ x &= \sqrt[3]{2}. \end{aligned}$$

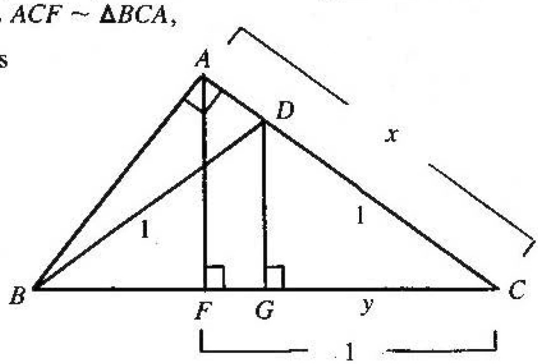
OR

Drop $DG \perp BC$. Let $AC = x$, $GC = y$. Note that $BC = 2y$, for ΔBDC is isosceles. Since $\Delta DCG \sim \Delta ACF \sim \Delta BCA$,

we obtain $\frac{1}{y} = \frac{x}{1} = \frac{2y}{x}$. Thus

$y = \frac{1}{x}$ and $y = \frac{x^2}{2}$, implying

$x^3 = 2$, or $x = \sqrt[3]{2}$.



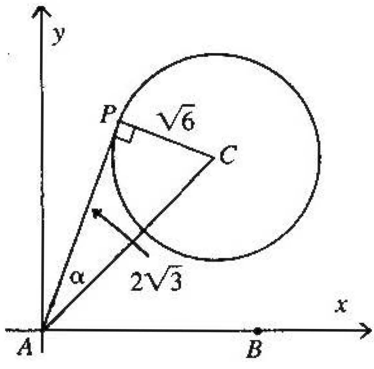
Query: How can $\sqrt[3]{2}$ be the answer to this problem when doubling the cube (i.e., constructing $\sqrt[3]{2}$) is known to be impossible?

Answer: it is true that constructing $\sqrt[3]{2}$ is impossible, if one means constructing a segment of that length, starting with a segment of length 1 and using compass and straightedge *only*. But just because such a construction is impossible, that doesn't mean a segment can't have length $\sqrt[3]{2}$. In fact, if one is allowed to use a *ruler* instead of a straightedge (a ruler has markings, a straightedge doesn't), then one *can* construct $\sqrt[3]{2}$. See Heinrich Dorrie, "100 Great Problems of Elementary Mathematics," Section 35 (Dover Pub., New York, 1965). Even if mankind knew no way to construct $\sqrt[3]{2}$, that still wouldn't mean that it doesn't exist.

Another good source for information on constructibility is Ivan Niven, "Numbers: Rational and Irrational," New Mathematical Library, Vol. 1, Mathematical Association of America, Washington, DC.

28. (C) Note that the prime factorization of 1984 is $2^6 \cdot 31$, that $x < 1984$ and that $y = (\sqrt{1984} - \sqrt{x}\,)^2 = 1984 + x - 2\sqrt{1984x}$. It follows that y is an integer if and only if $1984x$ is a perfect square, that is, if and only if x is of the form $31t^2$. Since x is less than 1984, we have $1 \le t \le 7$, yielding the pairs (31, 1519), (124, 1116) and (279, 775) for (x,y), corresponding to $t = 1,2,3$. Since $y \le x$ for $t > 3$, these are the only solutions.

29. (A) Let $P = (x,y)$, $A = (0,0)$, $C = (3,3)$ and B be any point on the positive x-axis. The locus of P is the circle with center C and radius $\sqrt{6}$, and $\frac{y}{x}$ is the slope of segment AP. Clearly this slope is greatest when AP is tangent to the circle on the left side, as in the adjoining figure (Note: $\sqrt{6} < 3$). Let $\alpha = \angle CAP$. Since $\angle BAC = 45°$, the answer is



$\tan(\alpha + 45°) = \dfrac{\tan\alpha + 1}{1 - \tan\alpha}$. Since

$\angle APC = 90°$, $\tan\alpha = \dfrac{PC}{PA}$. By the

Pythagorean Theorem, $PA = \sqrt{(AC)^2 - (PC)^2} = 2\sqrt{3}$. Thus $\tan\alpha = \dfrac{1}{\sqrt{2}}$ and the answer is $3 + 2\sqrt{2}$.

OR

The maximum value of $\frac{y}{x}$ is the maximum value of the slope of a line which contains the origin and which intersects the circle $(x-3)^2 + (y-3)^2 = 6$. Clearly the line of maximum slope, m, is tangent to the circle and is the steeper of the two lines through the origin and tangent to the circle. In short, it is the line containing AP in the figure on page 9. The following statements are equivalent:

1) $y = mx$ is tangent to the circle;
2) the system of equations

$$y = mx,$$
$$(x - 3)^2 + (y - 3)^2 = 6,$$

has only one solution (x,y);

3) the quadratic equation $(x - 3)^2 + (mx - 3)^2 = 6$, or

$$(m^2 + 1)x^2 - 6(m + 1)x + 12 = 0,$$

has a double root;

4) the discriminant is zero, i.e.,

$$36\,(m + 1)^2 - 48\,(m^2 + 1) = 0.$$

Solving the last equation for its larger root, one obtains $m = 3 + 2\sqrt{2}$.

Note: this method is more general than the previous one. The circle may be replaced by any conic and the method will still work.

30. (B) We derive a more general result: if n is an integer > 1 and $w = \cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$, then

$$\left|w + 2w^2 + \cdots + nw^n\right|^{-1} = \frac{2}{n}\sin\frac{\pi}{n}.$$

To prove this, let $S = w + 2w^2 + \cdots + nw^n$. Multiplying both sides of this equation by w and subtracting Sw from S, we get

$$S(1 - w) = w + w^2 + \cdots + w^n - nw^{n+1}.$$

Now, $w \neq 1$ since $n > 1$. Thus we may use the formula for summing a geometric series to obtain

$$S(1 - w) = \frac{w^{n+1} - w}{w - 1} - nw^{n+1}.$$

Since $w^n = 1$ (by De Moivre's formula), this reduces further to $S(1 - w) = -nw$. Thus

$$\frac{1}{S} = \frac{w-1}{nw}, \qquad \frac{1}{|S|} = \frac{|w-1|}{n}.$$

Finally, $|w - 1|$ is the length of the side of the regular n-gon inscribed in the unit circle, since 1 and w are consecutive vertices. It is well known that this side length is $2\sin\frac{\pi}{n}$ (in the isosceles triangle with vertices $0, 1, w$, drop an altitude from 0).

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

36th ANNUAL
AMERICAN HIGH SCHOOL
MATHEMATICS EXAMINATION
1985

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This Solutions Pamphlet gives at least one solution for each problem on this year's Examination and shows that all the problems can be solved using noncalculus high school mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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1. (A) $4x + 1 = 2(2x+1) - 1 = 2\cdot 8 - 1 = 15$. Or, solve $2x + 1 = 8$ to obtain $x = 7/2$. Thus $4x + 1 = 4(7/2) + 1 = 15$.

2. (E) The shaded sector has central angle $360° - 60°$, or $\frac{5}{6}$ of $360°$. Thus the perimeter consists of $\frac{5}{6}$ of the circumference of a circle of radius 1, plus 2 radii of the circle. Thus the length is

$$\frac{5}{6}(2\pi r) + 2r = \frac{5}{3}\pi + 2.$$

3. (D) Since $\triangle ABC$ is a right triangle, $AB = 13$. Also, $BN = BC = 5$ and $AM = AC = 12$. Thus

$$BM = AB - AM = 1,$$
$$MN = BN - BM = 4.$$

4. (C) If the bag contains p pennies, d dimes and q quarters, then $d = 2p$ and $q = 3d$. Thus $q = 6p$. If A is the worth of the coins in cents, then $A = p + 10(2p) + 25(6p) = 171p$. Of the answers given, only (C) is a multiple of 171.

5. (E)

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12}$$
$$= \frac{60}{120} + \frac{30}{120} + \frac{20}{120} + \frac{15}{120} + \frac{12}{120} + \frac{10}{120}.$$

Since $60 + 30 + 20 + 10 = 120$, it is clear that one *may* remove $\frac{15}{120} = \frac{1}{8}$ and $\frac{12}{120} = \frac{1}{10}$. Furthermore, one sees easily from the right hand side above that no other terms sum to $\frac{27}{120}$. Thus one *must* remove $\frac{1}{8}$ and $\frac{1}{10}$.

6. (B) Since the selection is random, the *number* of boys is 2/3 the number of girls. Thus there are 2 boys for every 3 girls, i.e., 2 boys for every 5 students. Alternately, let b and g be the number of boys and girls. Then $b = \frac{2}{3}g$ and so

$$\frac{b}{b+g} = \frac{(2/3)g}{(5/3)g} = 2/5.$$

7. (E) Going from right to left, we first add c to d, obtaining $c + d$. Then we subtract this result from b, obtaining $b - c - d$. Finally, we divide a by this last expression, obtaining (E).

8. (E) The solution of $ax+b=0$ is $-b/a$. The solution of $a'x+b'=0$ is $-b'/a'$. Thus the question becomes: which one of the five inequalities in the answers is equivalent to

$$(*) \qquad \frac{-b}{a} < \frac{-b'}{a'}\ ?$$

Multiplying by -1, one obtains

$$\frac{b}{a} > \frac{b'}{a'},$$

which is (E). Conversely, multiplying (E) by -1 gives (*), so they are equivalent.

Note. Multiplying (*) by $-aa'$ results in (B) if, but only if, $aa'>0$. Thus (*) and (B) are not equivalent.

9. (B) Observe that the first column consists of all numbers with value $16n-1$ for $n=1,2,\ldots,$ and that the first number in the next row beneath each value of $16n-1$ is $16n+1$ for the same n and appears in the second column. Since $1985 = 16\cdot 124+1$, 1985 appears in the second column.

OR

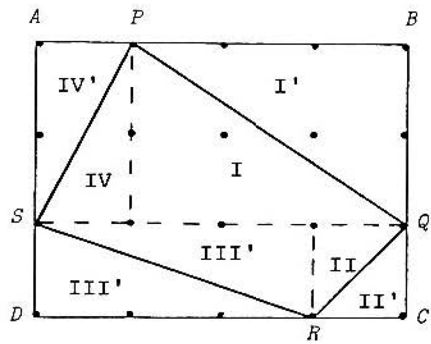
Observe that all the odd integers between 0 and 8 appear in the first row, all between 8 and 16 in the second, and in general, all between $8(n-1)$ and $8n$ in the n^{th}. Since $1985 = 248\cdot 8+1$, it follows that 1985 is the smallest odd integer in the 249^{th} row. Now observe that the entries in odd-numbered rows increase to the right and start in the second column.

10. (E) By definition, the graph of $y=f(x)$ is the set of points (x,y) in the rectilinear coordinate plane for which $y=f(x)$. Thus the graph of $y=\sin x$ is the usual periodic wave along the x-axis. Consider a circle tangent to the x-axis at the origin. If the radius is *very* large, the circle stays close to the x-axis for a long time and thus intersects the sine curve many times. By making the radius arbitrarily large, the number of intersection points may be made arbitrarily large.

11. (B) There are 2 ways to order the vowels. There would be $5! = 120$ orderings of the consonants if they were all distinct. Since there are two T's, there are $120/2 = 60$ consonant orderings. Thus there are $2\times 60=120$ reorderings of CONTEST with the vowels first.

12. (D) If $n = pq^2r^4$ is a divisor of cube c, then c must have p, q and r as primes in its factorization. Moreover, the exponents of p, q and r in the factorization must be multiples of 3 and they must be at least as great as 1, 2 and 4, respectively (1, 2, 4 being the exponents of p, q, r in n). Thus $p^3q^3r^6 = (pqr^2)^3$ is the smallest such cube.

13. (E) Method 1. Using the dotted horizontal and vertical lines shown as part of the figure, divide $PQRS$ into 4 right triangles, I, II, III, IV. Compute the area of each from the lengths of the legs and add.



Method 2. Surround $PQRS$ by rectangle $ABCD$. Note that for each right triangle in Method 1, there is a second congruent triangle in $ABCD$. Thus

$$\text{Area } PQRS =$$

$$\tfrac{1}{2}\text{Area } ABCD = \tfrac{1}{2}(3\times 4) = 6.$$

Method 3. Let P be any polygon in the plane which has all its vertices at lattice points (points with both coordinates integers). P can have any number of sides, and need not be convex, but different sides cannot cross each other. Then

$$\text{Area} = I + \tfrac{1}{2}B - 1,$$

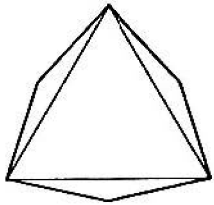
where I is the number of lattice points strictly inside P and B is the number on the boundary. Thus

$$\text{Area } PQRS = 5 + \tfrac{1}{2}(4) - 1 = 6.$$

This surprising formula is called *Pick's Theorem.* A proof may be found in Duane DeTemple & Jack Robertson, *The Equivalence of Euler's and Pick's Theorems*, The Math. Teacher, March 1974, pp. 222-226. Also see Ross Honsberger, "Ingenuity in Mathematics," New Mathematical Library, Vol. 23, pp. 27-31.

14. (C) For any convex n-gon, the sum of the interior angles is $(n-2)180°$. If an n-gon has exactly 3 obtuse interior angles, then the remaining $n-3$ angles have measure at most $90°$ each and the obtuse angles have measure less than $3\cdot180°$ together. Thus

$$(n-2)180 < (n-3)90 + 3\cdot180,$$
$$2(n-2) < n-3+6,$$
$$n < 7.$$



To show $n=6$ is possible, "distort" an equilateral triangle as shown.

15. (E) Raise both sides of $a^b = b^a$ to the power $1/a$ and substitute $b=9a$ to obtain $a^9 = 9a$. Since $a \neq 0$, $a^8 = 9$; so $a = 9^{1/8} = (3^2)^{1/8} = 3^{1/4}$.

Note. For any positive $k \neq 1$, the equations $a^b = b^a$, $b = ka$ have the unique positive simultaneous solution

$$(a,b) = \left(k^{1/(k-1)}, k^{k/(k-1)}\right).$$

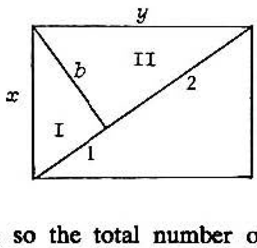
16. (B) We show that for any angles A and B for which the tangent function is defined and $A+B=45°$, $(1+\tan A)(1+\tan B) = 2$. By the addition law for tangents,

$$1 = \tan 45° = \tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B},$$
$$1 - \tan A \tan B = \tan A + \tan B,$$
$$1 = \tan A + \tan B + \tan A \tan B.$$

Thus

$$(1+\tan A)(1+\tan B) = 1 + \tan A + \tan B + \tan A \tan B$$
$$= 1 + 1 = 2.$$

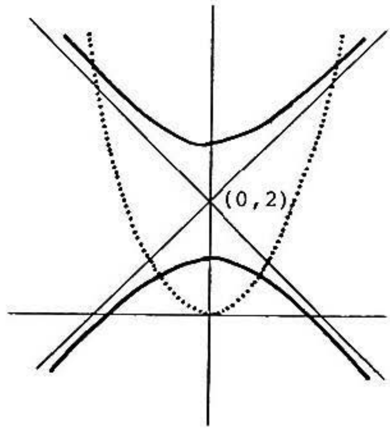
17. (B) Triangles I and II in the figure are similar. Thus $\frac{1}{b} = \frac{b}{2}$ and $b = \sqrt{2}$. The area of the rectangle is twice the sum of the areas of I and II, so the answer is $3b = 3\sqrt{2} \approx 3(1.414) \approx 4.2$.



18. (D) Jane gets twice as many marbles as George, so the total number of marbles taken must be divisible by 3; that is, the difference between 140 and the number of chipped marbles must be divisible by 3. Of the six given numbers only 23 has this property. Therefore, there are 23 chipped marbles, leaving $19+25+34 = 78$ for Jane and $18+21 = 39$ for George.

19. (A) Rearrange the terms and complete the square for the second equation to obtain

$$(y-2)^2 - x^2 = 1,$$



whose graph is a hyperbola with center (0,2), vertices at (0,1) and (0,3), and asymptotes $y = 2 \pm x$. The graph of $y = Ax^2$ is a parabola opening upward with its vertex at the origin for all positive values of A. Therefore there are two distinct points of intersection between the parabola and the lower branch of the hyperbola. The equation of the upper branch of the hyperbola is $y = 2 + \sqrt{x^2+1}$. For any given positive A, if the absolute value of x is large enough, then $Ax^2 > 2 + \sqrt{x^2+1} \approx 2 + x$. Since $Ax^2 < 2 + \sqrt{x^2+1}$ when $x = 0$, the parabola must also intersect the upper branch of the hyperbola in two distinct points for any given positive value of A.

OR

We show that the system of equations

$$(1)\ \ y = Ax^2, \qquad (2)\ \ y^2 + 3 = x^2 + 4y,$$

has exactly four distinct solutions for any $A > 0$. First, multiply (2) by A, eliminate Ax^2 using (1), and simplify, obtaining

$$(3)\ \ Ay^2 - (4A+1)y + 3A = 0.$$

Since all these steps are reversible, the system (1) and (3) has the same solution pairs as the system (1) and (2). Furthermore, (1) and (3) has two distinct solution pairs (x,y) for each positive solution y to (3) alone, namely, $(\sqrt{y/A}, y)$ and $(-\sqrt{y/A}, y)$. So it suffices to show that (3) has exactly two solutions, both positive. By the quadratic formula,

$$(4)\ \ y = \frac{4A+1 \pm \sqrt{(4A+1)^2 - 12A^2}}{2A}$$

Since the discriminant in (4) is positive for $A > 0$, there are two real roots. Since the constant term of (3) is positive and the coefficient of y is negative, both these roots must be positive.

20. (D) A smaller cube will have no face painted if it comes from the interior of the original cube, that is, if it is part of the cube of side length $n-2$ obtained by stripping away a one-unit layer from each face of the original. So there are $(n-2)^3$ unpainted smaller cubes. A smaller cube has one painted face if it comes from one of the faces of the original cube, but was not on the edge of the face. Thus there are $6(n-2)^2$ such smaller cubes. Therefore, $(n-2)^3 = 6(n-2)^2$. Since $n > 2$, we can divide out the $(n-2)^2$, leaving $n-2=6$, or $n=8$.

21. (C) There are 3 ways a^b can equal 1 when a and b are integers:

    I) $a=1$;

    II) $a=-1$, b even;

    III) $b=0$, $a \neq 0$.

In this problem $a = x^2-x-1$, $b = x+2$. We solve each case in turn.

Case I: $x^2-x-1 = 1$,
$(x-2)(x+1) = 0$,
$x=2$ or $x=-1$.

Case II: $x^2-x-1 = -1$ and $x+2$ is even,
$x^2-x = 0$,
$x = 0$ or 1.

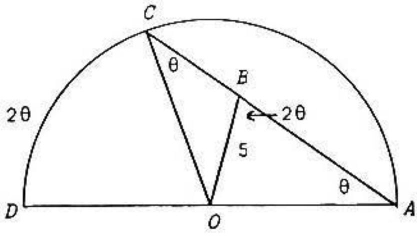
The choice $x=0$ is a solution, since then $x+2$ is even. However, $x=1$ is not a solution, since then $x+2$ would be odd.

Case III: $x+2 = 0$ and $x^2-x-1 \neq 0$,
$x = -2$.

This is a solution, since $(-2)^2-(-2)-1 = 5 \neq 0$.

Thus there are 4 solutions in all.

22. (D) Consider the half circle $ACDO$. Let $2\theta = 60° = \angle ABO = \widehat{CD}$. Then $\angle CAD = \theta$, since it is an inscribed angle. Draw in CO. Since $\triangle COA$ is isosceles, $\angle ACO = \theta$. By the Exterior Angle Theorem applied to $\triangle BOC$ at B, $\angle BOC = 2\theta - \theta = \theta$. So $\triangle BOC$ is also isosceles and $BC = 5$.



Note. The fact that $2\theta = 60°$ is irrelevant. $BC = 5$ for any θ for which the configuration exists. What is the range of values for θ?

23. (C) Note that $x^3 = y^3 = 1$, because x and y are the complex roots of $0 = x^3 - 1 = (x-1)(x^2+x+1)$. Alternatively, plot x and y in the complex plane and observe that they have modulus 1 and arguments $2\pi/3$ and $4\pi/3$. Thus $x^9 + y^9 = 1 + 1 \neq -1$. (To show that the other equations are correct is easy if one also notes that $y = x^2$ and $x + y = -1$. For instance, $x^{11} + y^{11} = x^2 + x^{22} = y + x = -1$.)

24. (C) Let $\Pr\{d_1, d_2, \ldots\}$ be the probability that the digit chosen is one of $d_1, d_2, \ldots$. Note that

$$\Pr\{d\} = \log\frac{d+1}{d} \quad \text{and} \quad \Pr\{d,d+1\} = \log\frac{d+1}{d} + \log\frac{d+2}{d+1} = \log\frac{d+2}{d}.$$

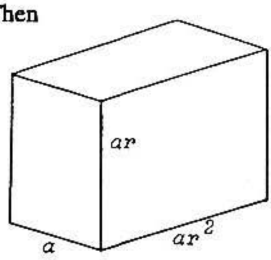
We seek a set of digits with probability $2\Pr\{2\}$. Thus

$$\begin{aligned} 2\Pr\{2\} &= 2\log\frac{3}{2} = \log\frac{9}{4} \\ &= \log\frac{5}{4} + \log\frac{6}{5} + \cdots + \log\frac{9}{8}. \\ &= \Pr\{4,5,6,7,8\}. \end{aligned}$$

Note 1. In this solution, we have not used the fact that the logs are base 10. However, this fact is necessary for the problem to make sense; otherwise the union of all possibilities (i.e, picking some digit from 1 to 9) does not have probability 1.

Note 2. If one collects a lot of measurements from nature (say, the lengths of American rivers in miles), the fraction of the time that the first significant (i.e., nonzero) digit is d is approximately $\log_{10}(d+1) - \log_{10} d$. In particular, the distribution is *not* uniform, e.g., 1 is the first digit about 30% of the time. This counterintuitive fact, sometimes called Benford's Law, can be explained if one assumes that the distribution of natural constants is independent of our units of measurement. See two articles by R. Raime: *On the Distribution of First Significant Figures*, Amer. Math. Monthly 76 (1969), pp. 342-48, and *The Peculiar Distribution of First Digits*, Scientific Amer., Dec. 1969, p. 109ff.

25. (B) Label the edges of the solid a, ar, and ar^2. Then



$$\text{Volume} = a(ar)(ar^2) = 8, \quad \text{or} \quad ar = 2.$$

$$\begin{aligned}\text{Surface Area} &= 2a^2r + 2a^2r^2 + 2a^2r^3 = 32\\ &= 2(ar)(a + ar + ar^2)\\ &= 4(a + ar + ar^2).\end{aligned}$$

But this last expression is just the sum of the edge lengths.

Note 1. One can also solve for a and r, and thus find that the edges have lengths $3-\sqrt{5}$, 2 and $3+\sqrt{5}$.

Note 2. For any rectangular solid with volume 8 and edge lengths in geometric progression, the surface area will equal the sum of the edge lengths.

26. (E) $\frac{n-13}{5n+6}$ is reducible and nonzero iff its reciprocal $\frac{5n+6}{n-13}$ exists and is reducible. By long division, $\frac{5n+6}{n-13} = 5 + \frac{71}{n-13}$. Thus it is necessary and sufficient that $\frac{71}{n-13}$ be reducible. Since 71 is a prime, $n-13$ must be a multiple of 71. So $n-13=71$, or $n=84$, is the smallest solution.

OR

We seek the smallest $n>0$ for which $n-13$ and $5n+6$ have a common factor and $n-13 \neq 0$. To make it easier to see a common factor, set $m=n-13$; then $5n+6 = 5m+71$. Clearly, m and $5m+71$ have a common factor iff m and 71 do. Since 71 is a prime, m must be one of $\ldots, -71, 0, 71, 142, \ldots$. Thus n must be one of $\ldots, -58, 13, 84, 155, \ldots$. The smallest positive value of n giving a positive fraction is $n=84$.

27. (C) We have

$$\begin{aligned}x_1 &= 3^{1/3},\\ x_2 &= (3^{1/3})^{\sqrt[3]{3}} = 3^{\sqrt[3]{3}/3},\\ x_3 &= (3^{\sqrt[3]{3}/3})^{\sqrt[3]{3}} = 3^{\sqrt[3]{9}/3},\\ x_4 &= (3^{\sqrt[3]{9}/3})^{\sqrt[3]{3}} = 3^{\sqrt[3]{27}/3} = 3^{3/3} = 3.\end{aligned}$$

One should verify that x_1, x_2 and x_3 are not integers. Here is a sketch of one way to do this. First, prove by induction that the sequence $\{x_n\}$ is increasing. Next, since $1<x_1<2$ and $x_4=3$, it suffices to show that neither x_2 nor x_3 is 2. As for x_2, note that $x_1<3/2$ (since $x_1{}^3 = 3 < 27/8$). Thus

$$x_2 = x_1^{x_1} < (3/2)^{3/2} = \sqrt{27/8} < 2.$$

To show $x_3>2$, show that $x_1>\sqrt{2}$ and do a similar manipulation.

Note. x_2 and x_3 are integers raised to radicals, e.g.,

$$x_2 = 3^{(\sqrt[3]{1/9})}.$$

There is a general theorem which says (as a special case) that such numbers cannot be integers (except when the base is 0 or 1), but the proof is very deep. For an introduction, see I. Niven, "Numbers: Rational and Irrational," New Mathematical Library, Vol. 1, Section 5.4.

28. (B) By the Law of Sines, $\dfrac{27}{\sin A} = \dfrac{48}{\sin 3A}$. Using the identity $\sin 3A = 3\sin A - 4\sin^3 A$, we have

$$\frac{48}{27} = \frac{16}{9} = \frac{\sin 3A}{\sin A} = 3 - 4\sin^2 A.$$

Solving for $\sin A$ gives $\sin A = \sqrt{11}/6$ and $\cos A = 5/6$. ($\cos A$ cannot be negative since $0 < 3A < 180°$.) Again by the Law of Sines,

$$\frac{b}{\sin(180° - 4A)} = \frac{27}{\sin A} \quad \text{or} \quad b = \frac{27\sin 4A}{\sin A}.$$

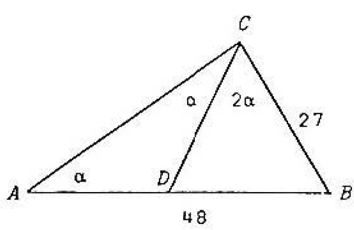
Since

$$\sin 4A = 2\sin 2A\cos 2A = 4\sin A\cos A\,(\cos^2 A - \sin^2 A),$$

we have $b = 27\cdot 4\cdot \frac{5}{6}(\frac{25-11}{36}) = 35.$

OR

Divide $\angle C$ into $\alpha = \angle A$ and 2α as shown in the figure. By the Exterior Angle Theorem, $\angle CDB = 2\alpha$. Thus $DB = CB = 27$ and $AD = 48 - 27 = 21$. Since ΔADC is isosceles also, $CD = 21$. Now apply Stewart's Theorem [see H.S.M. Coxeter & S.L. Greitzer, "Geometry Revisited," New Mathematical Library, Vol. 19], which says that for *any* point D on AB,



$$(AC)^2(BD) + (BC)^2(AD) = (AB)[(AD)(BD) + (CD)^2].$$

Thus $27b^2 + 27^2\cdot 21 = 48(21\cdot 27 + 21^2)$. Solving gives $b = 35$.

29. (C) First note that since $9ab$ ends in a zero, the sum of its digits is the same as the sum of the digits in $N = 9ab/10$. Second note that for any integer M represented as a string of k copies of the digit d, $M = \frac{d}{9}(999...99) = \frac{d}{9}(10^k - 1)$. Thus

$$\begin{aligned} N &= \tfrac{9}{10}(\tfrac{8}{9}(10^{1985}-1))(\tfrac{5}{9}(10^{1985}-1)) \\ &= \tfrac{4}{9}(10^{2\cdot 1985} - 2\cdot 10^{1985} + 1) \\ &= \tfrac{4}{9}((10^{2\cdot 1985}-1) - 2(10^{1985}-1)) \\ &= \tfrac{4}{9}(10^{2\cdot 1985}-1) - \tfrac{8}{9}(10^{1985}-1) \\ &= P - Q, \end{aligned}$$

where P is the number consisting of a sequence of $2\cdot 1985$ fours and Q consists of 1985 eights. Thus N consists of 1984 fours followed by 1 three followed by 1984 fives followed by 1 six. The sum of the digits in N is

$$1984\cdot(4+5) + (6+3) = 1985\cdot 9 = 17865.$$

Alternatively, one can replace $k = 1985$ by $k = 1, 2, 3$ and look for a pattern. When $k = 1$, $9ab = 360$ and the sum of the digits is 9. When $k = 2$, $9ab = 9\cdot 88\cdot 55 = 43560$ and the digit sum is 18. When $k = 3$, $9ab = 9\cdot 888\cdot 555 = 4435560$ and the digit sum is 27. It now seems clear (and can be proved by induction) that in general the digit sum is $9k$.

Query: we have shown that there is nothing special in this problem about 1985. Is there anything special about 8 and 5?

30. (E) Let $f(x) = 4x^2 - 40\lfloor x\rfloor + 51$ and let I_n be the interval $n \le x < n+1$ for integral n. Clearly $f(x) > 0$ for $x < 0$. For $x \ge 0$, $f(x)$ is increasing on each interval I_n since $4x^2$ is increasing and $-40\lfloor x\rfloor + 51 = -40n + 51$ is constant. Thus $f(x)$ has at most one root in each I_n and such a root will exist if and only if $f(n) \le 0$ and $f(n+1-\epsilon) > 0$ for small $\epsilon > 0$. Since $f(n+1-\epsilon)$ approaches $g(n) = 4(n+1)^2 - 40n + 51$ as ϵ approaches 0, it suffices to check if $g(n) > 0$ rather than checking $f(n+1-\epsilon)$ directly. Now,

$$\begin{aligned} f(n) \le 0 \quad &\Leftrightarrow \quad (2n-3)(2n-17) \le 0 \\ &\Leftrightarrow \quad \frac{3}{2} \le n \le \frac{17}{2}. \end{aligned}$$

So it suffices to check I_n for $n = 2, 3, 4, 5, 6, 7, 8$. Checking we find $g(n) > 0$ for $n = 2, 6, 7, 8$; so there are four roots.

OR

Since $40\lfloor x \rfloor$ is even, $40\lfloor x \rfloor - 51$ is odd, implying that $4x^2$ must also be an odd integer, say $2k+1$, and $x = \sqrt{2k+1}/2$. Substituting in the original equation, it follows that

$$\left\lfloor \frac{\sqrt{2k+1}}{2} \right\rfloor = \frac{k+26}{20},$$

hence one must have $k \equiv 14 \pmod{20}$. Furthermore,

$$\frac{k+26}{20} \le \frac{\sqrt{2k+1}}{2} < \frac{k+26}{20} + 1.$$

Treating the two inequalities separately, multiplying by 20, squaring and completing the square, one obtains $(k-74)^2 \le 70^2$ and $(k-54)^2 > 30^2$. Since x^2 must be positive, k is nonnegative, and it follows from the first inequality that $4 \le k \le 144$, and from the second one that either $k < 24$ or $k > 84$. Putting these together, one finds that either $4 \le k < 24$ or $84 < k \le 144$. In these intervals the only values of k for which $k \equiv 14 \pmod{20}$ are $k = 14, 94, 114, 134$, yielding the four solutions

$$x = \frac{\sqrt{29}}{2}, \frac{\sqrt{189}}{2}, \frac{\sqrt{229}}{2}, \frac{\sqrt{269}}{2}.$$

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

37th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
1986

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This Solutions Pamphlet gives at least one solution for each problem on this year's Examination and shows that all the problems can be solved using noncalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will find the time to share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

AMERICAN MATHEMATICS COMPETITIONS

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1. (B) $$\begin{aligned}[x-(y-z)]-[(x-y)-z] &= (x-y+z)-(x-y-z)\\ &= x-y+z-x+y+z = 2z.\end{aligned}$$

2. (A) The slope of the original line is $2/3$ and the y-intercept is 4. So L has slope $1/3$ and y-intercept 8.

3. (D) Since $\angle C=90^\circ$ and $\angle A=20^\circ$, we have $\angle ABC=70^\circ$. Thus $\angle DBC=35^\circ$. It follows that $\angle BDC=90^\circ-35^\circ=55^\circ$.

4. (B) The statement S is false if there is a whole number n such that the sum of the digits of n is divisible by 6, but n itself is *not* divisible by 6. The number 33 has these properties.

5. (A) $$\left(\sqrt[6]{27}-\sqrt{6\tfrac{3}{4}}\right)^2=\left[(3^3)^{1/6}-\left(\frac{27}{4}\right)^{1/2}\right]^2=\left[\sqrt{3}-\frac{3\sqrt{3}}{2}\right]^2$$ $$=\left[\frac{-\sqrt{3}}{2}\right]^2=\frac{3}{4}.$$

6. (C) Let h, l and w represent the height of the table and the length and width of the wood blocks, respectively, in inches. From Figure 1 on the exam, we have $l+h-w=32$. From Figure 2 we have $w+h-l=28$. Adding, $2h=60$ and $h=30$.

7. (E) Let $\lfloor x\rfloor$ be the greatest integer less than or equal to x, and let $\lceil x\rceil$ be the least integer greater than or equal to x. If $x\le 2$, then $\lfloor x\rfloor+\lceil x\rceil \le 2+2<5$, so there are no solutions with $x\le 2$. Similarly, if $x\ge 3$, then $\lfloor x\rfloor+\lceil x\rceil\ge 3+3$, so there are no solutions here. Finally, if $2<x<3$, then $\lfloor x\rfloor+\lceil x\rceil=2+3=5$, so every such x is a solution. Thus the solution set is (E).

8. (E) With about 230 million people and under 4 million square miles, there are about 60 people per square mile. Since a square mile is about $(5000\text{ ft})^2=25$ million square feet, that gives approximately $25/60$ of a million square feet per person. The closest answer is (E).

OR

Consider the approximation obtained by rounding to two significant digits, using scientific notation and simplifying. We obtain

$$\frac{(3.6\times 10^6 \text{ mi}^2)(5.3\times 10^3 \text{ ft/mi})^2}{2.3\times 10^8 \text{ people}} = \frac{(3.6)(5.3)^2}{2.3}\times 10^{6+6-8} \text{ ft}^2/\text{person}.$$

Now, $3.6/2.3$ is slightly more than 1.5, and $(5.3)^2$ is between 25 and 30. Thus the exact answer is about $(28\times 1.5)(10{,}000) = 420{,}000$. The closest answer to this among those given is (E).

9. (C) Factor each term of the given expression as the difference of two squares and group the terms according to signs to obtain

$$\left[\left(1-\tfrac{1}{2}\right)\left(1-\tfrac{1}{3}\right)\left(1-\tfrac{1}{4}\right)\cdots\left(1-\tfrac{1}{10}\right)\right]\left[\left(1+\tfrac{1}{2}\right)\left(1+\tfrac{1}{3}\right)\left(1+\tfrac{1}{4}\right)\cdots\left(1+\tfrac{1}{10}\right)\right]$$
$$= \left[\frac{1}{2}\frac{2}{3}\frac{3}{4}\cdots\frac{9}{10}\right]\left[\frac{3}{2}\frac{4}{3}\frac{5}{4}\cdots\frac{11}{10}\right] = \left[\frac{1}{10}\right]\left[\frac{11}{2}\right] = \frac{11}{20}.$$

OR

Note that

$$\left(1-\frac{1}{2^2}\right) = \frac{3}{4},$$
$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right) = \frac{3}{4}\cdot\frac{8}{9} = \frac{2}{3} = \frac{4}{6},$$
$$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right) = \frac{4}{6}\cdot\frac{15}{16} = \frac{5}{8}.$$

Clearly the pattern is

$$\left(1-\frac{1}{2^2}\right)\cdots\left(1-\frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

(This may be proved by induction.) Thus the answer is $\dfrac{11}{20}$.

10. (E) The first $24 = 4!$ words begin with A, the next 24 begin with E and the next 24 begin with H. So the 86^{th} begins with M, and it is the $86-72=14^{\text{th}}$ such word. The first six words that begin with M begin with MA and the next six with ME. So the desired word begins with MH and it is the second such word. The first word that begins with MH is MHAES, the second is MHASE. Thus E is the letter we seek.

11. (B) ΔAHB is a right triangle. In any right triangle, the median from the right angle is half as long as the hypotenuse. (In short, much of the given information is irrelevant!)

12. (B) Let c be the number John gets correct, w the number he gets wrong and u the number he leaves unanswered. That his score under the old scoring system would have been 84 means

$$30 + 4c - w = 84.$$

That his score under the new system is 93 means

$$5c + 2u = 93.$$

Since there are 30 problems,

$$c + w + u = 30.$$

Solving these three simultaneous equations gives

$$c = 15, \quad w = 6, \text{ and } \; u = 9.$$

OR

We have

$$84 = 30 + 4c - w = (c + w + u) + 4c - w = 5c + u.$$

Subtracting $5c + u = 84$ from $5c + 2u = 93$, we obtain $u = 9$.

Note: the first sentence of this alternate solution shows that the old scoring system could have been described more simply as 5 times the number correct + the number left unanswered. The second sentence shows that the difference between the new and old systems is that the new system gives you an extra point for each unanswered question.

13. (E) Since (4,2) is the vertex, $x = 4$ is the axis of symmetry. Since (2,0) is on the parabola, by symmetry so is (6,0). In other words, 2 and 6 are roots of $ax^2 + bx + c = 0$. Thus

$$y = ax^2 + bx + c = a(x-2)(x-6).$$

Substituting (4,2) we obtain

$$2 = a(2)(-2) = -4a.$$

So $a = -1/2$. Thus $y = -\frac{1}{2}x^2 + 4x - 6$ and $abc = (-\frac{1}{2})(4)(-6) = 12$.

OR

That (2,0) is on the graph means that $0 = 4a + 2b + c$. That (4,2) is on the graph means $2 = 16a + 4b + c$. That $x = 4$ is the axis of symmetry means $4 = -b/2a$. Thus we have 3 equations in the three unknowns a, b, c, which we may now solve.

14. (D) We are given

$$b \text{ hops} = c \text{ skips, or } 1 \text{ hop} = \frac{c}{b} \text{ skips;}$$

$$d \text{ jumps} = e \text{ hops, or } 1 \text{ jump} = \frac{e}{d} \text{ hops;}$$

$$f \text{ jumps} = g \text{ meters, or } 1 \text{ meter} = \frac{f}{g} \text{ jumps.}$$

Substituting repeatedly gives

$$1 \text{ meter} = \frac{f}{g} \text{ jumps} = \frac{f}{g}(\frac{e}{d} \text{ hops}) = \frac{f}{g}\Big(\frac{e}{d}(\frac{c}{b} \text{ skips})\Big) = \frac{cef}{bdg} \text{ skips.}$$

15. (C) The true average A is $(x+y+z)/3$. The student computed

$$B = \frac{\frac{x+y}{2}+z}{2} = \frac{x+y+2z}{4}.$$

Thus

$$B-A = \frac{2z-x-y}{12} = \frac{(z-x)+(z-y)}{12},$$

which is always positive since $z > x$ and $z > y$.

Note. In effect, the student averaged the four numbers x, y, z, z. Since the largest number was counted twice, the student's average is larger than the average of x, y, z.

16. (C) By the similarity of the two triangles, $\frac{PA}{PB} = \frac{PC}{PA} = \frac{CA}{AB}$; hence $\frac{PA}{PC+7} = \frac{PC}{PA} = \frac{6}{8}$, yielding the two equations

$$6(PC+7) = 8PA \quad \text{and} \quad 6PA = 8PC.$$

From these one obtains $PC = 9$.

17. (B) For any selection, at most one sock of each color will be left unpaired, and this happens if and only if an odd number of socks of that color is selected. Thus 24 socks suffice: at most 4 will be unpaired, leaving at least 20 in pairs. However, 23 will do! Since 23 is not the sum of four odd numbers, at most 3 socks out of 23 will be unpaired. On the other hand, 22 will *not* do: if the numbers of red, green, blue and black socks are 5,5,5,7, then four are unpaired, leaving 9 pairs. Thus 23 is the minimum.

OR

Proceed inductively. If we require only one pair, then it suffices to select 5 socks. Moreover, selecting 4 socks doesn't guarantee a pair since we might select one sock of each color.

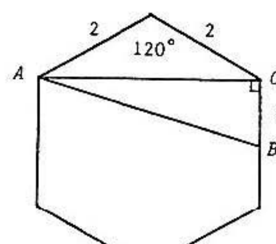
If we require two pairs, then it suffices to select 7 socks: any set of 7 socks must contain a pair; if we remove this pair, then the remaining 5 socks will contain a second pair as shown above. On the other hand, 6 socks might contain 3 greens, 1 black, 1 red and 1 blue -- hence only one pair. Thus 7 socks is the smallest number to guarantee two pairs.

Similar reasoning shows that we must draw 9 socks to guarantee 3 pairs, and in general, $2p+3$ socks to guarantee p pairs. This formula is easily proved by mathematical induction. Thus 23 socks are needed to guarantee 10 pairs.

18. (E) We claim that the minor axis has length 2, in which case the major axis has length $2+.5(2)=3$. To prove the claim, we may assume that the equation of the cylinder is $x^2+y^2=1$ and that the intersection of the plane and the axis of the cylinder is (0,0,0). By symmetry, (0,0,0) is also the center of the ellipse. Furthermore, the line of intersection of the given plane with the x-y plane contains two points on the ellipse (and the cylinder), and they have $z=0$. Finally, each "radius" of the ellipse extends from (0,0,0) to some point (x,y,z) on the cylinder and thus has length $\sqrt{x^2+y^2+z^2}=\sqrt{1+z^2}$. This has minimum value 1 when $z=0$, which is obtained for the two diametrically opposed points described earlier. Thus the minor axis does indeed have length 2.

Note. For those with good spatial intuition, it should be evident that the minor axis of the ellipse is a diameter of the cylinder.

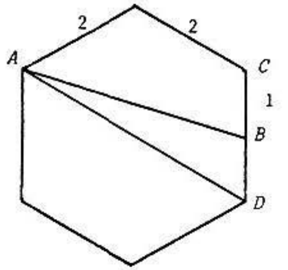
19. (A) Consider the figure. We may assume that Alice starts at A and ends at B. Observe that in ΔABC, $\angle ACB$ is right and AC has length $2\sqrt{3}$. The Pythagorean Theorem shows that $(AB)^2=13$.



OR

Consider the second figure. Observe that in $\Delta\, ABD$

1) $\angle ADB = 60^\circ$
2) $AD = 4$,
3) $BD = 1$.



By the Law of Cosines,

$$(AB)^2 = 1^2 + 4^2 - 2(1)(4)(1/2) = 13.$$

Therefore $AB = \sqrt{13}$.

20. (E) To say that x and y are inversely proportional is to say that if x is multiplied by k, then y is divided by k. Let x' and y' be the new values after x increases by $p\,\%$. Then

$$x' = \left(1+\frac{p}{100}\right)x \quad \text{and} \quad y' = \frac{y}{\left(1+\frac{p}{100}\right)} = \frac{100}{100+p}\,y\,.$$

By definition, the percentage decrease in y is

$$100\left(\frac{y-y'}{y}\right) - 100\left(1-\frac{100}{100+p}\right) = \frac{100p}{100+p}\,.$$

Note. It is a common error to think that the correct answer to this question is (A). For instance, this error appeared a few years ago in the president's annual report from a famous American university!

21. (B) The area of the shaded sector is $\frac{\theta}{2}(AC)^2$. This must equal half the area of $\Delta\, ABC$, which is $\frac{1}{2}(AC)(AB)$. Hence the shaded regions have equal area iff $\frac{\theta}{2}(AC)^2 = \frac{1}{4}(AC)(AB)$, which is equivalent to $2\theta = \frac{AB}{AC} = \tan\theta$.

22. (C) There are $\binom{10}{6} = 210$ different sets which could be picked. If the second smallest pick is 3, then one number is picked from $\{1,2\}$ and four are picked from $\{4,5,\ldots,9,10\}$. There are $\binom{2}{1}\binom{7}{4} = 70$ ways to do this. So the probability is $70 \div 210 = 1/3$.

23. (E) By the Binomial Theorem, $N = (69+1)^5 = (2\cdot5\cdot7)^5$. Thus a positive integer d is a factor of N iff $d = 2^p\, 5^q\, 7^r$, where p, q, r are each one of the 6 integers 0,1,2,3,4,5. Therefore there are $6^3 = 216$ choices for d.

24. (D) Because $p(x)$ is a factor of x^4+6x^2+25 and of $3x^4+4x^2+28x+5$, it is also a factor of $3(x^4+6x^2+25)-(3x^4+4x^2+28x+5)$, which equals $14x^2-28x+70$, or $14(x^2-2x+5)$. Therefore, $p(x)=x^2-2x+5$, and $p(1)=4$.

OR

The unique complete factorization of x^4+6x^2+25 is a difference of squares:

$$x^4+10x^2+25-4x^2 = (x^2+5+2x)(x^2+5-2x).$$

Thus $p(x)$ is either the first quadratic or the second. Long division shows that only the second quadratic is a factor of $3x^4+4x^2+28x+5$. Thus $p(x)$ is the second quadratic.

Note. It was not necessary to assume that b and c are integers. Even if they are allowed to be arbitrary complex numbers, $p(x)$ must still be x^2-2x+5. The first solution shows this. (Why?) The second does not. (Why?) Can the second solution be modified so that it does show this?

25. (B)

$$\lfloor \log_2 N \rfloor = \begin{cases} 1 & \text{for } \ 2 \le N < 2^2 \\ 2 & \text{for } \ 2^2 \le N < 2^3 \\ \ \cdot & \qquad \cdot \\ \ \cdot & \qquad \cdot \\ 9 & \text{for } \ 2^9 \le N < 2^{10} \\ 10 & \text{for } \ N = 2^{10} \end{cases}$$

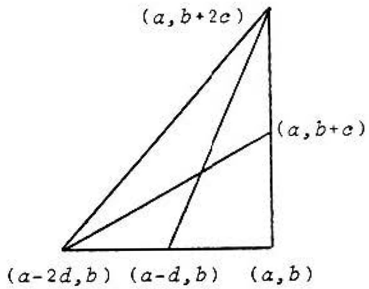
Thus the desired sum is

$$\begin{aligned} &1(2^2-2)+2(2^3-2^2)+3(2^4-2^3)+\cdots+9(2^{10}-2^9)+10 \\ &= 9\cdot 2^{10}-(2^9+2^8+2^7+\cdots+2)+10 \\ &= 9\cdot 2^{10}-(2^9+2^8+2^7+\cdots+2+1)+11 \\ &= 9\cdot 2^{10}-(2^{10}-1)+11 \\ &= 8\cdot 2^{10}+12 = 8(1024)+12 = 8204, \end{aligned}$$

where we have used the sum formula for a geometric series to obtain the next to last line.

26. (C) In *any* right triangle with legs parallel to the axes, one median to the midpoint of a leg has slope 4 times that of the other. This is easily shown by analytic geometry: any triangle of this sort may be labelled as in the figure, where a, b, c, d are arbitrary except $c \neq 0$, $d \neq 0$. (The figure has the right angle in the lower right, but the labelling allows complete generality – why?) Note that the slopes of the medians are

$$\frac{c}{2d} \text{ and } \frac{2c}{d} = 4\left(\frac{c}{2d}\right).$$

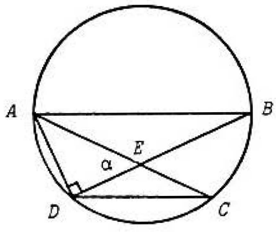


Therefore, in our problem m is either 12 or $3/4$.

In fact, both values are possible, each for infinitely many triangles. We show this for $m = 12$. Take any right triangle having legs parallel to the axes and a hypotenuse with slope 12/2, e.g., the triangle with vertices (0,0), (1,0), (1,6). Then the medians to the legs have slopes 12 and 3 (Why?). Now translate the triangle (don't rotate!) so that its medians intersect at the point where $y = 12x + 2$ and $y = 3x + 1$ intersect. This forces the medians to lie on these lines (Why?). Finally, for any central dilation of this triangle (a larger or smaller triangle with the same centroid and sides parallel to this one's sides), the medians will still lie on these lines.

27. (C) Because $AB \parallel DC$, arc AD = arc CB and CDE and ABE are similar isosceles triangles. Thus

$$\frac{\text{Area } CDE}{\text{Area } ABE} = \left(\frac{DE}{AE}\right)^2.$$



Draw in AD. Since AB is a diameter, $\angle ADB = 90^\circ$. Thus, considering right triangle ADE, $DE = AE \cos\alpha$, and

$$\left(\frac{DE}{AE}\right)^2 = \cos^2\alpha.$$

28. (C) Let s denote the length of a side of the pentagon. We compute the area of $ABCDE$ in two ways. First (Figure 1), it is the sum of the areas of the triangles OAB, OBC, OCD, ODE and OEA. Each of these has base s and altitude 1. Thus the area of the pentagon is $5s/2$. On the other hand (Figure 2), the area is the sum of the areas of the triangles ABC, ACD and ADE, which have base s and altitudes AQ, AP and AR, respectively. Thus the total area is $\frac{s}{2}(AP+AQ+AR)$. Hence $AP+AQ+AR=5$ and $AO+AQ+AR=4$.

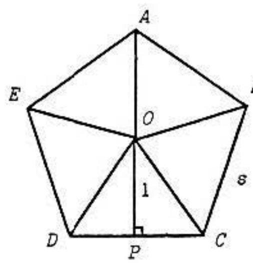


Figure 1

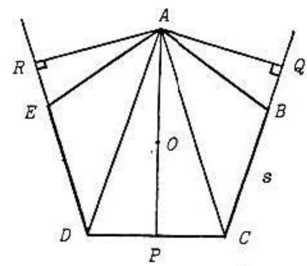


Figure 2

29. (B) Assume the altitude with length 4 falls on side a, the altitude with length 12 falls on b, and the unknown altitude with length h falls on c. Let K be the area of ΔABC. Then

$$(*) \qquad 4a = 12b = hc = 2K.$$

By the triangle inequality, $c < a+b$ and $c > a-b$. In other words,

$$\frac{2K}{h} < \frac{2K}{4}+\frac{2K}{12} \quad \text{and} \quad \frac{2K}{h} > \frac{2K}{4}-\frac{2K}{12},$$

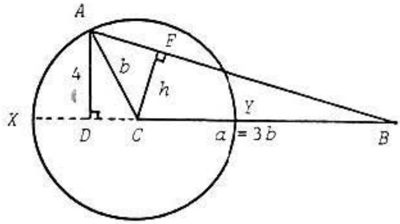
These are equivalent to

$$\frac{1}{h} < \frac{1}{4}+\frac{1}{12} = \frac{1}{3} \quad \text{and} \quad \frac{1}{h} > \frac{1}{4}-\frac{1}{12} = \frac{1}{6}.$$

Hence $3 < h < 6$. If h is an integer, it cannot exceed 5.

OR

If the altitudes of a triangle are in a given ratio, the corresponding sides are in the inverse ratio. In particular, using the same notation as above, we have $a = 3b$ from (*). Thus, thinking of side CB as fixed, A can be any point on the circle with center C and radius b as shown in the figure (except points X and Y, which don't result in a triangle). Since $\Delta FCB \sim \Delta DAB$, $\frac{h}{4} = \frac{3b}{AB}$. Furthermore, $AB < a + b = 4b$ (AB would equal $4b$ when $A = X$) and $AB > a - b = 2b$ (AB would equal $2b$ when $A = Y$). Thus



$$\frac{3b}{4b} < \frac{h}{4} < \frac{3b}{2b} \quad \text{or} \quad 3 < h < 6.$$

The largest integer value for h is 5.

This solution also suggests how to *construct* a triangle with altitudes 4, 12 and 5. From (*), $c = 4a/5$. Thus A is an intersection point of the circle shown and the circle of radius $4a/5$ centered at B. There is only one hitch: this construction requires that you know a to begin with. Do you see how to get around this problem?

30. (B) Either $x > 0$ or $x < 0$. Also, for any positive number a, $\frac{1}{2}(a + \frac{17}{a}) \geq \sqrt{17}$, with equality only if $a = \sqrt{17}$, because this inequality is equivalent to $(a - \sqrt{17})^2 \geq 0$. Thus, if $x > 0$, then considering each of the given equations in turn, one deduces that $y \geq \sqrt{17}$, $z \geq \sqrt{17}$, $w \geq \sqrt{17}$ and $x \geq \sqrt{17}$. Suppose $x > \sqrt{17}$. Then

$$y - \sqrt{17} = \frac{x^2 + 17}{2x} - \sqrt{17} = \left(\frac{x - \sqrt{17}}{2x}\right)\left(x - \sqrt{17}\right) < \frac{1}{2}(x - \sqrt{17}),$$

so that $x > y$. Similarly, $y > z$, $z > w$, and $w > x$, implying $x > x$, an obvious contradiction. Therefore $x = y = z = w = \sqrt{17}$, clearly a solution, is the only solution with $x > 0$. As for $x < 0$, note that (x, y, z, w) is a solution if and only if $(-x, -y, -z, -w)$ is a solution. Thus $x = y = z = w = -\sqrt{17}$ is the only other solution.

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

38th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
1987

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This Solutions Pamphlet gives at least one solution for each problem on this year's Examination and shows that all the problems can be solved using non-calculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

AMERICAN MATHEMATICS COMPETITIONS

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1. (D) $(1+x^2)(1-x^3) = 1(1+x^2)-x^3(1+x^2)$
$$= 1+x^2-x^3-x^5.$$

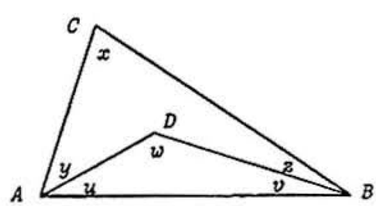
2. (E) The perimeter of ABC is 9. We delete two segments of length 1, DB and EB, and add a segment of length 1, DE. ($DE=1$ because ΔDBE is also equilateral, since $DB=BE$ and $\angle DBE=60^\circ$.) Thus the perimeter of $ADEC$ is $9-2+1=8$.

3. (C) 7, 17, 37, 47, 67 and 97 are primes; for instance, for 97 it is sufficient to check that it is not divisible by 2, 3, 5 and 7, which are the only primes less than its square root. On the other hand, 27, $57=3\times 19$, 77 and $87=3\times 29$ are not primes.

4. (B) By inspection, we see that the fraction is of the form
$$\frac{8a+8b+8c}{a+b+c} = \frac{8(a+b+c)}{a+b+c} = 8.$$
Or, compute
$$\frac{2+1+\frac{1}{2}}{\frac{1}{2^2}+\frac{1}{2^3}+\frac{1}{2^4}} = \frac{\frac{7}{2}}{\frac{1}{4}+\frac{1}{8}+\frac{1}{16}} = \frac{\frac{7}{2}}{\frac{7}{16}} = 8.$$

5. (B) Since 12.5% is 1/8, N must be at least 8. Since $50\% = 1/2 = 4/8$ and $25\% = 1/4 = 2/8$, $N=8$ is the answer.

6. (A) From the figure,
$$\begin{aligned} x &= 180-[(y+u)+(z+v)] \\ &= 180-(y+z)-(u+v) \\ &= 180-(y+z)-(180-w) \\ &= w-y-z. \end{aligned}$$



OR

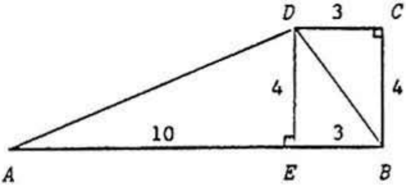
Since $ADBC$ is a quadrilateral, the sum of its interior angles is 360°. (It does not matter that the interior angle at D is a reflex angle.) Thus
$$x+y+z+(360-w) = 360,$$
$$x = w-y-z.$$

7. (C) Adding 3 to each expression in the equality of the problem, we obtain

$$c = a+2 = b+5 = d+7.$$

It is now easy to see that c is always the largest.

8. (C) Draw in DA and DB and drop perpendicular DE, forming rectangle $DCBE$. By the Pythagorean Theorem applied to triangles DCB and ADE, $DB = 5$ and $AD = \sqrt{116}$. Since $10 < \sqrt{116} < 11$, we find that $15 < AD + DB < 16$.



9. (B) The difference between the second and fourth terms is x; thus the difference between successive terms is $x/2$. Therefore $a = x/2$, $b = 3x/2$ and

$$\frac{a}{b} = \frac{x/2}{3x/2} = \frac{1}{3}.$$

10. (D) The simultaneous equations $a = bc$, $b = ca$, $c = ab$ imply

$$abc = (bc)(ca)(ab) = (abc)^2,$$

so either $abc = 0$ (ruled out) or $abc = 1$. The same simultaneous equations above also imply

$$abc = a^2 = b^2 = c^2,$$

so $|a| = |b| = |c| = 1$. It cannot be that all 3 unknowns are -1, nor can exactly one be -1, for in either case $a = bc$ is not satisfied. However, the remaining four cases,

$$(a, b, c) = (1,1,1),\ (-1,-1,1),\ (-1,1,-1) \text{ or } (1,-1,-1)$$

are all solutions. Thus there are 4 solutions in all.

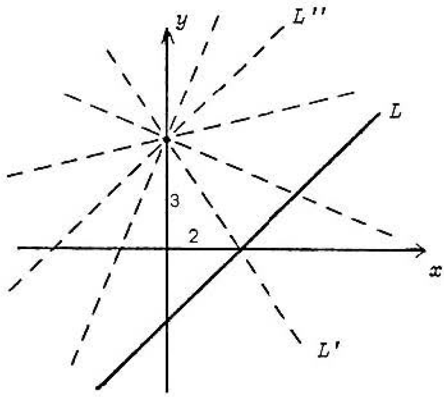
11. (E) Algebraic solution. Solving simultaneously, we obtain

$$x = \frac{5}{c+1}, \quad y = \frac{3-2c}{c+1}.$$

We wish to find all values of c for which $x, y > 0$. First, $x > 0 \Leftrightarrow c+1 > 0 \Leftrightarrow c > -1$. Next, given that $c+1 > 0$, then $y > 0 \Leftrightarrow 3-2c > 0 \Leftrightarrow 3/2 > c$. Thus $x, y > 0 \Leftrightarrow -1 < c < 3/2$.

OR

Geometric solution. We wish to find those values of c for which the lines $x - y = 2$ and $cx + y = 3$ intersect inside Quadrant I. The line $x - y = 2$ has slope 1 and x-intercept 2, and is shown as L in the figure. The line $cx + y = 3$ has y-intercept 3 and slope $-c$. Several possible choices for this line are shown dashed in the figure. For this line to intersect L in Quadrant I, it is necessary and sufficient to choose a slope between that of L', which passes through (2,0), and that of L'', which is parallel to L. Thus



$$-\frac{3}{2} < -c < 1,$$

or

$$-1 < c < \frac{3}{2}.$$

12. (D) At any given time, the letters in the box are in increasing order from the bottom, because that's the way they were put in the box and the relative order of those still there is never changed. Thus, for any j and all $i < j$, all letters i taken off the pile later than letter j must still have been in the pile when j was added, and thus are in descending order from the top. Thus they must be taken off in descending order. For $j = 4$, this is not true in (D) -- the lower numbered letters supposedly typed after 4 are 2,3,1, in that order. So (D) is not a possible typing order. All the other permutations listed meet the condition just described. For instance, for (B), the letters i with $i < 4$ which come after 4 are 3 and 1, in that order, which is descending order. All these other permutations are possible typing orders; the reader should show when each letter was added to the pile and when each was typed.

Note. This is a problem in computer science in disguise. The in-box is a "stack", an important "data structure" in computer science. The general question underlying this problem is: if items are put on a stack in order $1,2,3,\ldots,n$, then what permutations are possible orders for taking them off? We have found a necessary condition: all $i < j$ which follow j in the permutation must be in descending order. Is this condition sufficient?

13. (A) Let $d_1, d_2, \ldots, d_{600}$ be the diameters of the concentric circles in the model. The d's form an arithmetic sequence with $d_1 = 2$ cm and $d_{600} = 10$ cm. If L is the total length, then

$$\begin{aligned} L &= \pi d_1 + \pi d_2 + \cdots + \pi d_{600} \\ &= \pi(d_1 + d_2 + \cdots + d_{600}) = \pi 600\left(\frac{d_1 + d_{600}}{2}\right) \\ &= \pi 600\left(\frac{12}{2}\right) \text{ cm} = 36\pi \text{ meters.} \end{aligned}$$

OR

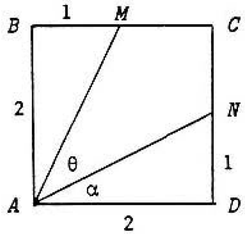
Let L be the length of the tape in cm. The thickness of paper on the roll is $(10-2)/2 = 4$ cm. Therefore, the thickness of the tape is $4/600 = 1/150$ cm. We may assume that unfolding the paper and laying it out flat has negligible effect on its cross-sectional area. Therefore, we may equate the cross-sectional area of the laid out paper, which is $L/150\,\text{cm}^2$, to the cross-sectional area while it is on the roll, which is

$$\pi 5^2 - \pi 1^2 = 24\pi \text{ cm}^2.$$

Solving for L gives $L = 3600\pi$ cm $= 36\pi$ meters.

14. (B) We may suppose that the sides of the square have length 2, so that $BM = ND = 1$. Then

$$\begin{aligned} \sin\theta &= \sin\left(\frac{\pi}{2} - 2\alpha\right) = \cos 2\alpha \\ &= 2\cos^2\alpha - 1 \\ &= 2\left(\frac{2}{\sqrt{5}}\right)^2 - 1 \\ &= \frac{3}{5}. \end{aligned}$$



Alternately, one may express Area $\triangle AMN$ in terms of $\sin\theta$, find Area $\triangle AMN$ again numerically by subtracting the areas of other (right) triangles from the area of the square, and then solve for $\sin\theta$.

15. (D) One can solve for x and y individually, and then substitute, but it's a mess: $x = \dfrac{9 \pm \sqrt{57}}{2}$ and $y = \dfrac{9 \mp \sqrt{57}}{2}$. It's sufficient, and easier, to solve for $x + y$:

$$63 = x^2y + xy^2 + x + y = xy(x+y) + (x+y) = (6+1)(x+y),$$

so $x + y = 9$ and

$$x^2 + y^2 = (x+y)^2 - 2xy = 81 - 12 = 69.$$

16. (D) The fact that VVW follows VYX establishes $W \leftrightarrow 0$ and $X \leftrightarrow 4$. Since VYX follows VYZ, X immediately follows Z, and this establishes that $Z \leftrightarrow 3$. That VVW follows VYX also establishes that V follows Y, so $Y \leftrightarrow 1$ and $V \leftrightarrow 2$. Hence $XYZ \leftrightarrow 413_5 = 108_{10}$.

17. (E) Let A, B, C and D represent the scores of Ann, Bill, Carol and Dick, respectively. Then

$$(1)\quad A + C = B + D,$$

$$(2)\quad A + B > C + D,$$

$$(3)\quad D > B + C.$$

Adding (1) and (2) shows $2A > 2D$ or $A > D$. Subtracting (1) from (2) gives $B - C > C - B$ or $B > C$. By (3), $D > B$ (since C is nonnegative), so $A > D > B > C$.

Note. All the conditions in the problem can be satisfied, for instance, with the scores $A = 7$, $B = 3$, $C = 1$ and $D = 5$.

18. (D) Let the length of the shelf be 1 unit, and let x and y denote the thicknesses of the algebra and geometry books, respectively. Then

$$Ax + Hy = 1, \quad Sx + My = 1 \quad \text{and} \quad Ex = 1.$$

Thus $E = 1/x$ and it is sufficient to solve the first two equations above for x. Multiply the first by M, the second by H, and subtract: $(AM - SH)x = M - H$. Thus

$$E = \frac{1}{x} = \frac{AM - SH}{M - H}.$$

Note. The problem asserted that A, H, S, M, E are positive integers. One possible set of integer values are $A = 6$, $H = 4$, $S = 3$ $M = 6$, $E = 12$, which yield $x = 1/12$ and $y = 1/8$.

Query. In the solution, have we used the given information that $y > x$? That the constants are distinct? If so, how?

19. (B) Since

$$\sqrt{65}-\sqrt{63} = \frac{(\sqrt{65}-\sqrt{63})(\sqrt{65}+\sqrt{63})}{\sqrt{65}+\sqrt{63}} = \frac{2}{\sqrt{65}+\sqrt{63}},$$

and since $\sqrt{65}$ and $\sqrt{63}$ are each equal to about 8, the answer must be near $1/8$. Specifically,

$$\sqrt{65}+\sqrt{63} > 7.5+7.5 = 15, \quad \text{so} \quad \sqrt{65}-\sqrt{63} < 2/15 \approx .1333.$$

Thus the answer is (A) or (B). To determine which, we must decide whether $\sqrt{65}+\sqrt{63}$ is larger or smaller than 8+8, since $2/16 = .125$ exactly. In fact, it is smaller: this is the case $n=64$ and $a=1$ of the inequality

$$\sqrt{n+a} + \sqrt{n-a} < 2\sqrt{n},$$

valid whenever $0<|a|\le n$, which one may verify by squaring. (Alternately, the fact that the graph of $y=\sqrt{x}$ is concave down shows geometrically that $\sqrt{64}$ is greater than the average of $\sqrt{63}$ and $\sqrt{65}$ — how?) Therefore, $\sqrt{65}-\sqrt{63} > 2/16$ and the answer is (B).

Note. $\sqrt{65}-\sqrt{63} \approx .125004$, as can be verified using the Binomial series or (after the test) a calculator.

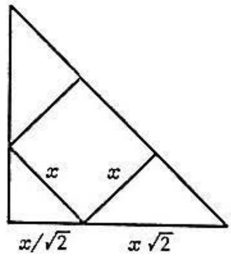
20. (A) Using $\log a + \log b = \log ab$ repeatedly, we find that the sum is

$$P = \log_{10}\left[(\tan 1^\circ)(\tan 2^\circ)\cdots(\tan 45^\circ)\cdots(\tan 89^\circ)\right]$$

Moreover, $(\tan 1^\circ)(\tan 89^\circ)=1$, $(\tan 2^\circ)(\tan 88^\circ)=1$, and so on, because $\tan\theta\tan(90-\theta)=\tan\theta\cot\theta=1$ for all θ at which both $\tan\theta$ and $\cot\theta$ are defined. Thus

$$P = \log_{10}(\tan 45^\circ) = \log_{10} 1 = 0.$$

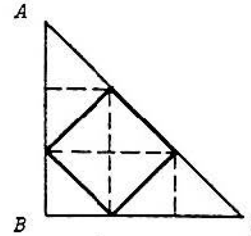
21. (B) First observe that the legs of the triangle are twice as long as the sides of the first inscribed square, so the legs have length $2\sqrt{441}=42$. Now, let x be the side of the second inscribed square, as in the figure to the right. Note that all triangles in the figure are similar isosceles right triangles. Thus



$$42 = x(\sqrt{2}+1/\sqrt{2}) = 3x/\sqrt{2},$$
$$x = 14\sqrt{2},$$
$$x^2 = 196\cdot 2 = 392.$$

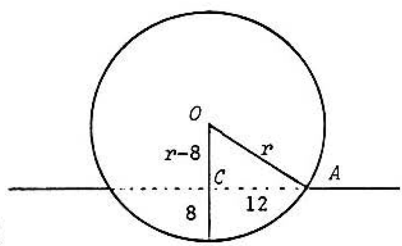
OR

The area of ΔABC is $2 \cdot 441 = 882$. Subdivide the area of ΔABC as shown on the right, and observe that 4 of the 9 congruent isosceles right triangles into which ΔABC is partitioned are inside the square inscribed the second way. It follows that the area of that square is $(4/9)(882) = 392\,\text{cm}^2$.



22. (C) In the figure, we show a cross section of the ball still in the ice. Since $\angle OCA = 90°$, $(r-8)^2+12^2 = r^2$. Solving, one finds

$$r = \frac{64+144}{16} = 13.$$



Query. We have assumed the ball floated with its center above the water, for otherwise the ball could not be removed without breaking the ice. Is there a second mathematical solution with the center below the water line?

23. (D) Applying the quadratic formula to $x^2+px-444p = 0$, we find that its discriminant, $p(p+1776)$, must be a perfect square. This implies that $p+1776$, and hence 1776, must be a multiple of p. Since $1776 = 2^4 \cdot 3 \cdot 37$, it follows that $p = 2, 3$ or 37. It is easy to see that neither $2(2+1776)$ nor $3(3+1776)$ is a perfect square; so $p = 37$, which indeed satisfies the conditions of the problem, leading to 111 and -148 as the two roots. Since $31 < 37 \leq 41$, the correct answer is (D).

OR

Rewrite the equation as $x^2 = p(444-x)$ and observe that $p \mid x^2$ (i.e., p is a factor of the integer x^2). Since p is prime, it follows that $p \mid x$ and so $x = np$ for some integer n. Substituting, the equation becomes $n^2p = 444 - np$, from which $n(n+1)p = 444$. Since the factorization of 444 into primes is $3 \cdot 4 \cdot 37$, it is evident that $p = 37$, $n = 3$ or -4, and $x = 111$ or -148.

24. (B) The only such polynomial is $f(x)=x^2$. To prove this, recall that by definition a polynomial of degree n is a function

$$f(x) = \sum_{k=0}^{n} a_k x^k = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 ,$$

where $a_n \neq 0$. For a polynomial, the equation $f(x^2) = [f(x)]^2 = f(f(x))$ is thus

$$(*) \qquad \sum_{k=0}^{n} a_k (x^2)^k = \left[\sum_{k=0}^{n} a_k x^k\right]^2 = \sum_{k=0}^{n} a_k \Big(\sum_{j=0}^{n} a_j x^j\Big)^k .$$

Polynomials are identical $\Leftrightarrow$ they are identical term by term. The highest-power terms in the three expressions in (*) are, respectively,

$$a_n x^{2n}, \quad a_n^2 x^{2n}, \quad a_n^{n+1} x^{n^2}.$$

Thus $2n = n^2$ and $a_n = a_n^2 = a_n^{n+1}$, so $n=2$ (since $n=0$ was ruled out) and $a_n=1$ (since $a_n \neq 0$). The first equation in (*) now becomes

$$(**) \qquad x^4 + bx^2 + c = (x^2 + bx + c)^2 .$$

It follows that $b=0$; otherwise, the right side of (**) has a cubic term but the left doesn't. Then it follows that $c=0$; otherwise $(x^2+c)^2$ has an x^2 term but x^4+c doesn't. So $f(x)=x^2$ is the only candidate. One easily checks that it satisfies (*).

Query. If we included the polynomials of degree 0 (the constant functions), how many more solutions would there be?

25. (C) Since $A=(0,0)$ and $B=(36,15)$, we know that base AB of $\triangle ABC$ has length $3\sqrt{12^2+5^2}=39$. We must choose C so that the height of $\triangle ABC$ is minimum. The height is the distance to $C=(x_0,y_0)$ from the line AB. This line is $5x-12y=0$. In general, the distance from (x_0,y_0) to the line $ax+by=c$ is

$$\frac{|\,ax_0+by_0-c\,|}{\sqrt{a^2+b^2}} .$$

So in this case the distance is $|5x_0-12y_0|/13$. Since x_0 and y_0 are integers, the smallest this expression could be is $1/13$. ($0/13$ is not possible, for then C would be *on* line AB and we would not have a triangle.) The value $1/13$ is achieved, for instance, with $C = (5,2)$ or $(7,3)$. Thus the minimum area is $(1/2)bh = (1/2)(39)(1/13) = 3/2$.

OR

The triangle with vertices (x_1,y_1), (x_2,y_2), (x_3,y_3) has area

$$\frac{1}{2}\ \text{abs}\begin{vmatrix} x_2-x_1 & y_2-y_1 \\ x_3-x_1 & y_3-y_1 \end{vmatrix}.$$

Setting $(x_1,y_1)=A$, $(x_2,y_2)=B$, $(x_3,y_3)=C$, we have

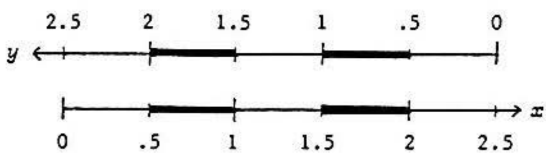
$$\text{Area } ABC = \frac{1}{2}\ \text{abs}\begin{vmatrix} 36 & 15 \\ x_3 & y_3 \end{vmatrix} = \frac{1}{2}\,|36y_3-15x_3| = \frac{3}{2}\,|12y_3-5x_3|.$$

If x_3, y_3 are integers, the least nonzero value $|12y_3-5x_3|$ could have is 1. Indeed, when $x_3=5$, $y_3=2$, it **is** 1. So 3/2 is the minimum area.

OR

This problem can be solved using Pick's Theorem. See the 1985 AHSME Solutions Pamphlet, problem 13, for a statement of the theorem and references.

26. (B) Let x be the first number, $y=2.5-x$ the second. We solve the problem graphically. By running the number line for y backwards, we can put y directly over x as follows:



We have marked in bold the segments of their number lines where x and y are rounded up. We see that either they are both rounded up (in which case the resulting integers add to 3) or they are both rounded down (in which case the resulting integers don't add to 3). Since the bold segments for x have combined length 1 out of a total length of 2.5, and the probability distribution for x is uniform, the answer is $1/(2.5) = 2/5$.

Note. We haven't worried whether 1/2 and 3/2 are rounded up or down. Why doesn't it make any difference?

27. (B) The cut $x=y$ separates the cheese into points with $x<y$ and those with $x>y$. Similarly for the other cuts. Thus, which piece a point is in depends only on the relative sizes of its coordinates, x, y, z. For instance, all points with $x<y<z$ are in the same piece. Since there are 3! ways to order x, y and z, there are 6 pieces.

OR

Since the planes $x=y$, $y=z$ and $z=x$ intersect along the line $x=y=z$, they divide all of space into just 6 pieces, not 8 the way most sets of 3 planes do. (Imagine forming the configuration by rotating a single half plane around that line; each rotation to the next position sweeps out one more piece.) Since the line $x=y=z$ is a major diagonal of the cheese, in each of the 6 pieces of space there is some point of the cheese. Thus the cheese is divided into 6 pieces also.

28. (D) If $z=x+iy$ is a complex root of a polynomial equation with real coefficients, then so is $x-iy$. (In fact, if $x+iy$ is a multiple root, so is $x-iy$, with the same multiplicity. This is important below because in any sum involving roots it is meant that each root should be counted once for each multiplicity.) Now if $x+iy$ lies on the unit circle centered at $0+0i$, then the reciprocal of $x+iy$ is $\dfrac{x-iy}{x^2+y^2}=x-iy$. Hence the reciprocal of each root is again a root. Thus the sum of the reciprocals of the roots is the same as the sum of the roots. In general, the sum of the roots of $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is $-a_{n-1}/a_n$, so in our case the sum is simply $-a$.

Note. $-c/d$ is also a correct expression for the sum, but it wasn't listed. To see that it is correct, first let

$$f(z) = z^4 + az^3 + bz^2 + cz + d \quad \text{and} \quad g(z) = dz^4 + cz^3 + bz^2 + az + 1.$$

Next observe that for $z \neq 0$

$$z^4+az^3+bz^2+cz+d = z^4\left[1+a\left(\frac{1}{z}\right)+b\left(\frac{1}{z}\right)^2+c\left(\frac{1}{z}\right)^3+d\left(\frac{1}{z}\right)^4\right].$$

Thus z is a nonzero root of $f(x)=0$ iff $1/z$ is a root of $g(x)=0$. Furthermore, all the roots of $f(x)=0$ are nonzero, and hence their reciprocals exist, iff $d \neq 0$. (Why?) Thus, whenever all the roots of $f(x)=0$ are nonzero (not just when they are on the unit circle), the sum of their reciprocals equals the sum of the roots of $g(x)=0$. By the general fact stated in the last sentence of the solution, this latter sum is $-c/d$.

When the roots are on the unit circle, we can say more: $d=\pm 1$. (Why?) Thus one of c and $-c$ is also correct, but neither alone is always correct. [Exercises: with all roots on the unit circle, find a specific f with $d=1$ and another with $d=-1$. For any such f show that the coefficients are either symmetric ($d=1$, $c=a$) or antisymmetric ($d=-1$, $c=a$, $b=-b=0$). Finally, extend everything in this Note to general n^{th} degree polynomials.]

29. (A) It is easily seen by induction that $t_k > 1$ is true for all even k, and that $0 < t_k < 1$ is true for all odd $k > 1$. Hence n is odd, and t_{n-1} is $87/19$. Because $87/19 > 4$, it follows that $n-1$ is divisible by 2 four times and that, subtracting 1 from t_{n-1} for each division by 2, term number $(n-1)/16$ is $(87/19)-4=11/19$. Continuing in this way, we find that term number $\frac{n-1}{16}-1=(n-17)/16$ is $19/11$, term number $(n-17)/32$ is $8/11$, term number $(n-49)/32$ is $11/8$, term number $(n-49)/64$ is $3/8$, term number $(n-113)/64$ is $8/3$, term number $(n-113)/256$ is $2/3$, term number $(n-369)/256$ is $3/2$, term number $(n-369)/512$ is $1/2$, term number $(n-881)/512$ is 2, and term number $(n-881)/1024$ is 1. Hence $(n-881)/1024$ equals 1, so $n=1905$. The sum of the digits of n is 15.

Note. The given recursion is just a disguised form of the standard representation of a positive rational number in continued fraction form. See Chapter 1 of C. D. Olds, "Continued Fractions," Mathematical Association of America's New Mathematical Library, Vol. 9. Each positive rational number appears exactly once in the sequence given in the problem.

30. (E) Point E is on AC, as in the original figure. To show this, we show that if $E=C$, then $\triangle ADE$ has more than half the area, hence DE is too far right. Indeed, in Figure 1 below, we may assume altitude CF is 1, in which case

$$\frac{\text{Area } EAD}{\text{Area } EAB} = \frac{AD}{AB} = \frac{1+(1/\sqrt{3})}{1+\sqrt{3}} = \frac{1}{\sqrt{3}} > \frac{1}{2}.$$

Thus we must move DE to the left, as in Figure 2, shrinking the dimensions of $\triangle EAD$ by a factor k so that

$$\begin{aligned} \text{Area } EAD &= \tfrac{1}{2}\text{ Area } CAB, \\ (1/2)k^2[1+(1/\sqrt{3})] &= (1/4)(1+\sqrt{3}), \\ k^2 &= \sqrt{3}/2, \\ k &= \sqrt[4]{3/4}. \end{aligned}$$

Thus

$$\frac{AD}{AB} = \frac{k[1+(1/\sqrt{3})]}{1+\sqrt{3}} = \frac{k}{\sqrt{3}} = \sqrt[4]{\frac{3}{4}}\,\sqrt[4]{\frac{1}{9}} = \frac{1}{\sqrt[4]{12}}.$$

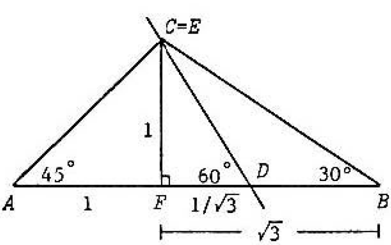


Figure 1

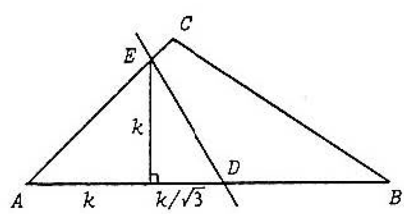


Figure 2

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

39th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
1988

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This Solutions Pamphlet gives at least one solution for each problem on this year's Examination and shows that all the problems can be solved using non-calculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

AMERICAN MATHEMATICS COMPETITIONS

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1. (D)
$$\sqrt{8}+\sqrt{18}=\sqrt{4\cdot 2}+\sqrt{9\cdot 2}$$
$$=2\sqrt{2}+3\sqrt{2}=5\sqrt{2}.$$

2. (D) The ratios of corresponding sides are equal, so
$$\frac{XY}{AB}=\frac{YZ}{BC} \quad \text{or} \quad YZ=\frac{4\cdot 5}{3}=6\frac{2}{3}.$$

3. (A) There are 4 strips, each of area 10. Also, 4 squares of side 1 are covered twice. So the total area covered is $4\cdot 10-4\cdot 1=36$.

4. (B) Rewrite the equation as $y=-\frac{2}{3}x+2$. With the equation in this form, the slope is the coefficient of x.

5. (E) We have
$$x^2+(2+b)x+2b = x^2+cx+6,$$
so
$$c = 2+b \quad \text{and} \quad 6=2b.$$
Thus $b=3$ and $c=5$.

6. (A) A parallelogram has (among other things) 4 angles. "Equiangular" means that all angles are equal, so they are all 90°. Thus all equiangular parallelograms are rectangles. (They are not all squares since adjacent sides of a parallelogram need not be equal.) Conversely, every rectangle is a 4-sided, equiangular figure with opposite sides parallel. Thus a figure is an equiangular parallelogram if and only if it is a rectangle.

7. (D)
$$60 \text{ blocks} \times \frac{512}{120}\ \frac{\text{chunks/block}}{\text{chunks/second}} = 256 \text{ seconds} \approx 4 \text{ minutes}.$$

Note. If we substitute "byte" for "chunk," we are talking standard terminology for communication between computers. A byte is a sequence of 8 "bits", each of which is a 0 or a 1. A transmission rate of 120 bytes/sec is called 1200 (not 120) "baud".

8. (B) $c=3b=3(2a)=6a$, so
$$\frac{a+b}{b+c}=\frac{a+2a}{2a+6a}=\frac{3}{8}.$$

9. (C) Consider either diagonal of the table. It has length $\sqrt{8^2+10^2} = \sqrt{164}$. At some point in turning the table, this diagonal must be perpendicular to a pair of opposite walls of the room. Thus $S \geq \sqrt{164}$. This necessary condition is also sufficient: imagine a circle of diameter $\sqrt{164}$ within the room, translate the table so that it is inscribed in this circle, rotate the table $90°$, and then translate it into the desired corner. The smallest integer satisfying $S \geq \sqrt{164}$ is 13.

10. (D) From the given,

$$2.43553 \leq C \leq 2.44177.$$

Thus, rounded to one significant digit $C=2$, to two digits $C=2.4$, and to three digits $C=2.44$ (not 2.43 even if $C=2.43553$). However, rounded to four digits C might be as low as 2.436 and as high as 2.442. Thus the most precise value the experimenter can announce in significant digits is (D).

11. (C) By examining the data, we see that the percentage increases are

$$A: \frac{10}{40} = 25\% \qquad B: \frac{20}{50} = 40\% \qquad C: \frac{30}{70} = 42\tfrac{6}{7}\%$$

$$D: \frac{30}{100} = 30\% \qquad E: \frac{40}{120} = 33\tfrac{1}{3}\%$$

Note. Greatest percentage increase is not the same as greatest absolute increase.

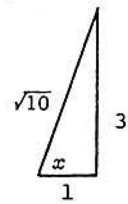
12. (A) From the units digits of the addition table of 1 through 9 (shown to the right) one sees that 0 appears 9 times and every other digit appears 8 times.

	1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9	0
2	3	4	5	6	7	8	9	0	1
3	4	5	6	7	8	9	0	1	2
4	5	6	7	8	9	0	1	2	3
5	6	7	8	9	0	1	2	3	4
6	7	8	9	0	1	2	3	4	5
7	8	9	0	1	2	3	4	5	6
8	9	0	1	2	3	4	5	6	7
9	0	1	2	3	4	5	6	7	8

Note 1. This table is called the "mod 10" addition table. Usually a 0 row and a 0 column are also included.

Note 2. Since there are 81 sums of two positive digits, the 10 units digits cannot be equally likely. This rules out (E).

13. (E) If $\sin x = 3\cos x$ then $\tan x = 3$. From the figure we conclude that



$$\sin x \cos x = \frac{3}{\sqrt{10}} \cdot \frac{1}{\sqrt{10}} = \frac{3}{10}$$

for any acute angle x. If x' is another angle with $\tan x' = 3$, $x' - x$ is a multiple of π. Thus

$$\sin x' = \pm\frac{3}{\sqrt{10}}, \qquad \cos x' = \pm\frac{1}{\sqrt{10}}.$$

So $\sin x' \cos x'$ is still $3/10$ (since $\sin x'$ and $\cos x'$ have the *same* sign).

OR

Multiplying the given equation first by $\sin x$ and then by $\cos x$ yields

$$\sin^2 x = 3\sin x \cos x ,$$
$$\cos^2 x = (1/3)\sin x \cos x .$$

Adding gives

$$1 = (10/3)\sin x \cos x ,$$

so $\sin x \cos x = 3/10$.

14. (A) The answer is

$$\frac{(-\frac{1}{2})(-\frac{3}{2}) \cdots (-\frac{1}{2}-98)(-\frac{1}{2}-99)}{100!} \div \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2}) \cdots (\frac{1}{2}-99)}{100!}$$
$$= \frac{(-\frac{1}{2}-99)}{\frac{1}{2}} = -199.$$

15. (A) By long division, one finds that $x^2 - x - 1$ divides $ax^3 + bx^2 + 1$ with quotient $ax + (a+b)$ and remainder $(2a+b)x + (a+b+1)$. But $x^2 - x - 1$ is a factor of $ax^3 + bx^2 + 1$, so the remainder is 0. In other words,

$$2a + b = 0,$$
$$a + b = -1.$$

Solving, one obtains $a = 1$, $b = -2$.

OR

Since x^2-x-1 is a factor of ax^3+bx^2+1, the quotient must be $ax-1$ (why?). Thus

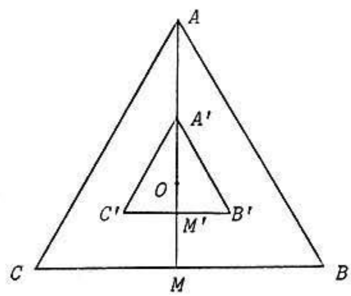
$$ax^3+bx^2+1 = (ax-1)(x^2-x-1) = ax^3+(-a-1)x^2+(1-a)x+1.$$

Equating the coefficients of x^2 on the left and right, and then the coefficients of x, we obtain

$$b = -a-1, \quad 0 = 1-a.$$

Hence $a=1$ and $b=-2$.

16. (C) Let $\triangle ABC$ and $\triangle A'B'C'$ have heights h and h'. The required ratio thus equals $(\frac{h'}{h})^2$. Let O be the common center of the triangles and let M and M' be the intersections of BC and $B'C'$ with the common altitude from A. Since altitudes are also medians in equilateral triangles, $OM=\frac{h}{3}$ and $OM'=\frac{h'}{3}$. By hypothesis, $MM'=\frac{h}{6}$ so $\frac{h}{3}=\frac{h'}{3}+\frac{h}{6}$. Thus $\frac{h'}{h}=\frac{1}{2}$ and $(\frac{h'}{h})^2=\frac{1}{4}$.



17. (C) Label the equations

$$(1) \quad |x|+x+y = 10,$$
$$(2) \quad x+|y|-y = 12.$$

If $x \le 0$, then $(1) \Rightarrow y=10$. Then $(2) \Rightarrow x=12$, a contradiction. Thus $x>0$ and (1) becomes

$$(3) \quad 2x+y=10.$$

If $y \ge 0$, $(2) \Rightarrow x=12$. Then $(3) \Rightarrow y=-14$, a contradiction. Therefore $y<0$ and equation (2) becomes

$$(4) \quad x-2y=12.$$

Solving (3) and (4) simultaneously, one finds $x=\frac{32}{5}$, $y=-\frac{14}{5}$ and $x+y=\frac{18}{5}$.

Alternatively, graph equations (1) and (2). The graph of (1) has two pieces: $y=10$ for $x \le 0$ and $2x+y=10$ for $x \ge 0$. Each piece is easy to sketch. Similarly, (2) has two pieces: $x-2y=12$ for $y \le 0$ and $x=12$ for $y \ge 0$. A quick sketch of both equations shows that the only solution is in Quadrant IV, where the problem reduces to solving the linear equations (3) and (4).

18. (B) There are 4 games in every play-off, and each game has 2 possible outcomes. For each sequence of 4 outcomes, the prizes are awarded a different way. Thus there are $2^4=16$ possible orders.

19. (B) Let N be the numerator in the expression to be simplified. The answers given suggest that N is divisible by $bx+ay$. If the differing middle terms in the two parenthesized expressions in N, namely, $2a^2y^2$ and $2b^2x^2$, were not there, divisibility would be obvious. So let us separate the middle terms off. We obtain

$$\begin{aligned} N &= bx(a^2x^2+b^2y^2)+ay(a^2x^2+b^2y^2)+bx(2a^2y^2)+ay(2b^2x^2) \\ &= (bx+ay)(a^2x^2+b^2y^2)+2a^2bxy^2+2ab^2x^2y \\ &= (bx+ay)(a^2x^2+b^2y^2)+2abxy(ay+bx). \end{aligned}$$

Thus

$$\frac{N}{bx+ay} = a^2x^2+b^2y^2+2abxy = (ax+by)^2.$$

20. (E) The height of the rectangle is $XY = DE+FB = 2\sqrt{1^2+2^2} = 2\sqrt{5}$. The area of the rectangle is the same as the square's, i.e., 4. Therefore,

$$YZ = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}} \quad \text{and} \quad \frac{XY}{YZ} = \frac{2\sqrt{5}}{2/\sqrt{5}} = 5.$$

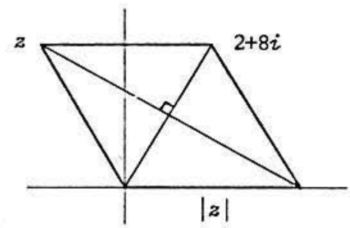
21. (E) Set $z=a+bi$. We seek $|z|^2=a^2+b^2$. So

$$\begin{aligned} z+|z| &= 2+8i \\ a+bi+\sqrt{a^2+b^2} &= 2+8i \\ a+\sqrt{a^2+b^2} = 2, &\quad b=8 \\ a+\sqrt{a^2+64} &= 2 \\ a^2+64 = (2-a)^2 &= a^2-4a+4 \\ 4a = -60, &\quad a=-15. \end{aligned}$$

Thus $a^2+b^2=225+64=289$.

OR

In the complex plane, z and $|z|$ are vectors of equal length, and $z+|z|=2+8i$ is their vector sum. So $0, z, 2+8i$, and $|z|$ form the vertices of a rhombus, as in the figure. The diagonals of a rhombus bisect each other and are perpendicular. The diagonal from 0 to $2+8i$ has slope 4 and midpoint $1+4i$. Thus the diagonal from z to $|z|$ passes through $1+4i$ with slope $-1/4$. Therefore this diagonal intersects the real axis at $x=17$. Since $|z|$ is on the real axis, we conclude that $|z|=17$ and $|z|^2=289$.



22. (A) In any triangle with sides a, b, c, the angle opposite a is acute iff $a^2 < b^2+c^2$. This follows from the Law of Cosines. Applying this fact in turn to the angle opposite x, 24 and 10, we find

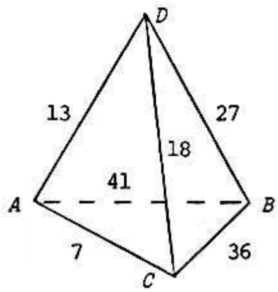
$$x^2 < 10^2+24^2 = 26^2,$$

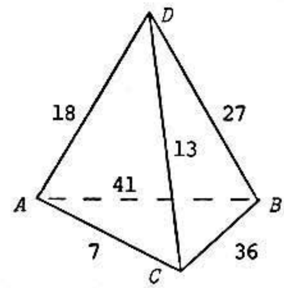
$$24^2 < x^2+10^2 \iff 476 < x^2,$$

$$10^2 < x^2+24^2.$$

The first line tells us that $x < 26$. The second tells us that $x \geq 22$ (since x is an integer). The third is satisfied for every x. Thus there are 4 integer values which meet all the conditions: 22, 23, 24, 25.

23. (B) Consider the edge of length 7 and the two triangular faces of $ABCD$ which share this edge. For both of these triangles, the other two sides must have lengths differing by less than 7, for otherwise the Triangle Inequality would be violated. Of the numbers given, only the pairs (13,18) and (36,41) satisfy this requirement, leaving the edge of length 27 as the one opposite to that of length of 7. Consequently, we must have one of the two arrangements pictured below:





The arrangement on the left is impossible because ΔABD fails to satisfy the Triangle Inequality. This leaves the arrangement on the right, in which $CD = 13$.

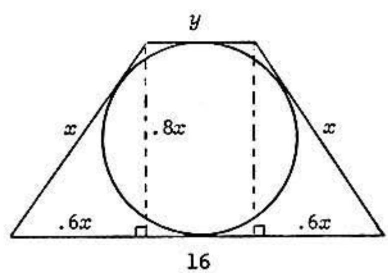
The reader may wish to verify that the second arrangement *is* possible by constructing a physical model — or by showing mathematically that it is constructible.

24. (C) By viewing the sides of the trapezoid as tangents to the circle, we find that the sums of the lengths of opposite sides are equal. (Indeed, this is true for any circumscribed quadrilateral.) Defining x and y as shown in the figure, we have

$$2y + 1.2x = 2x,$$

$$y + 1.2x = 16.$$

Then $y = 4$, $x = 10$, and the area is $\frac{1}{2}(4+16)(8) = 80$.



25. (E) Let the numbers of elements in the sets be given by $|X|=x$, $|Y|=y$ and $|Z|=z$. Then $|X \cup Y|=x+y$, $|X \cup Z|=x+z$ and $|Y \cup Z|=y+z$. The given information can be summarized in the following 3 equations in 3 unknowns:

$$\frac{37x+23y}{x+y}=29; \quad \frac{37x+41z}{x+z}=39.5; \quad \frac{23y+41z}{y+z}=33.$$

Simplifying these equations, we obtain $4x=3y$; $5x=3z$; $5y=4z$. We want the value of the fraction

$$\frac{37x+23y+41z}{x+y+z}.$$

Making the substitutions $y=4x/3$ and $z=5x/3$, we obtain

$$\frac{37x+23(\frac{4x}{3})+41(\frac{5x}{3})}{x+\frac{4x}{3}+\frac{5x}{3}}=\frac{111x+92x+205x}{3x+4x+5x}=34.$$

Queries. Where did we use the given that X, Y and Z are disjoint? Also, did we need **all** the information which was given?

26. (D) Let t be the common value of $\log_9(p)$, $\log_{12}(q)$ and $\log_{16}(p+q)$. Then

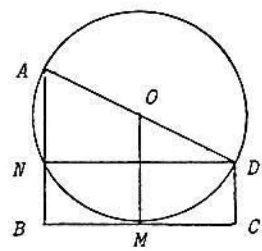
$$p=9^t, \quad q=12^t, \quad \text{and} \quad 16^t=p+q=9^t+12^t.$$

Divide the last equation by 9^t and note that

$$\frac{16^t}{9^t}=\left(\frac{4^t}{3^t}\right)^2=\left(\frac{12^t}{9^t}\right)^2=\left(\frac{q}{p}\right)^2.$$

Now let x stand for the unknown ratio q/p. From the division referred to above we obtain $x^2=1+x$, which leads easily to $x=\frac{1}{2}(1+\sqrt{5})$ since x must be the positive root.

27. (D) $ABCD$ is a trapezoid, hence its area is $BC(AB+CD)/2$. With an eye on the answers, we seek to express BC in terms of AB and CD. Let M be the point where BC is tangent to the circle, and let N be the point where AB intersects the circle. $\angle AND$ is right because AD is a diameter. Thus $BCDN$ is a rectangle and $BN=CD$. $\angle OMB$ is right because BC is a tangent. Thus OM is parallel to CD and M is the midpoint of BC. A standard theorem about tangents and secants to circles now states that $(BM)^2=BN\cdot BA$, so that $BC=2\sqrt{CD\cdot BA}$. Hence the area of $ABCD$ is $(AB+CD)\sqrt{AB\cdot CD}$. This is an integer iff $AB\cdot CD$ is a perfect square, since AB and CD are integers in all the choices given. Thus the answer is (D).



28. (D) We must solve for p when

$$\binom{5}{3}p^3(1-p)^2 = 10p^3(1-p)^2 = \frac{144}{625},$$

or $p^3(1-p)^2 = 72/5^5$. If we define

$$f(p) = p^3(1-p)^2 - \frac{72}{5^5},$$

we see that $f(0)=f(1)=-72/5^5$, and

$$f(\tfrac{1}{2}) = \frac{1}{32}-\frac{72}{5^5} \approx .03-.024 > 0.$$

Thus, since f is continuous, f has at least *two* real roots, r_1, r_2, satisfying $0<r_1<\frac{1}{2}$ and $\frac{1}{2}<r_2<1$. In fact, $r_1=2/5$ and $3/5<r_2<4/5$.

29. (A) The best fit line goes through some point (x_2,z). Claim: the correct slope for this line makes the directed vertical distances to it from (x_1,y_1) and (x_3,y_3) the same. To see this, imagine any line through (x_2,z) and rotate it until the two directed vertical distances to it are the same. Call this distance d. For any other line through (x_2,z) the directed distances to it from (x_1,y_1) and from (x_3,y_3) will be $d+e$ and $d-e$ for some $e \neq 0$. Thus the sum of the squares will increase from

$$2d^2+(z-y_2)^2$$

to

$$(d+e)^2+(d-e)^2+(z-y_2)^2 = 2d^2+2e^2+(z-y_2)^2.$$

This proves the claim. Finally, the slope from (x_1,y_1+d) to (x_3,y_3+d) is $(y_3-y_1)/(x_3-x_1)$, whatever d and z are!

Note. There are several formulas in statistics which give the slope of the best fit line for arbitrary data points. One can use any of these general formulas to solve this special case, but the algebra is quite involved.

Queries. What are d and z? Where was the hypothesis $x_3-x_2 = x_2-x_1$ used?

30. (E) Note that $x_0=0$ gives the constant sequence 0,0,..., since $f(0)= 4\cdot 0-0^2=0$. Because $f(4)=0$, $x_0=4$ gives the sequence 4,0,0,... with two different values. Similarly, $f(2)=4$ so $x_0=2$ gives the sequence 2,4,0,0,... with three values. In general, if $x_0=a_n$ gives the sequence $a_n, a_{n-1},\ldots,a_2,a_1,a_1,\ldots$ with n different values, and $f(a_{n+1})=a_n$, then $x_0=a_{n+1}$ gives a sequence with $n+1$ different values. (It could not happen that $a_{n+1}=a_i$ for some $i<n+1$; why?) Thus, it follows by induction that there is a sequence with n distinct values for every positive integer n — as soon as we verify that there *is* always a real number a_{n+1} such that $f(a_{n+1})=a_n$. This follows from the quadratic formula: First, the solutions to $f(a_{n+1})=4a_{n+1}-a_{n+1}^2=a_n$ are $a_{n+1}=2\pm\sqrt{4-a_n}$. Second, if $0\le a_n\le 4$, then a_{n+1} is real; in fact, $0\le a_{n+1}\le 4$ (why?). Third, $0=a_1\le 4$. Thus, by induction, all terms satisfy $0\le a_n\le 4$; in particular, they are all real.

AMERICAN MATHEMATICS COMPETITIONS

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

40th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)

TUESDAY, FEBRUARY 28, 1989

Sponsored by

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This Solutions Pamphlet contains at least one solution to each problem on this year's Examination and shows that all the problems can be solved using non-calculus mathematics. Routine calculations and obvious reasons for proceeding in a certain way are often omitted to give greater emphasis to the essential ideas behind each solution. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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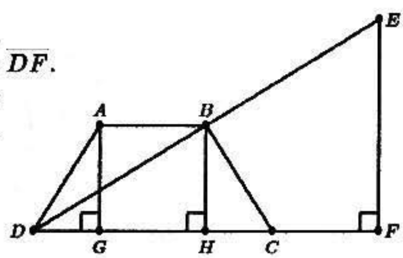
1. **(C)** $(-1)^{5^2} + 1^{2^5} = (-1)^{25} + 1^{32} = -1 + 1 = 0.$

2. **(D)** $\sqrt{\frac{1}{9} + \frac{1}{16}} = \sqrt{\frac{16+9}{144}} = \frac{5}{12}.$

3. **(D)** Let x be the length of the shorter side of one of the rectangles. Then the perimeter of this rectangle is $8x$. Since $8x = 24$, $x = 3$ and the area of the square is $(3x)^2 = 81$.

4. **(D)** Drop perpendiculars $\overline{AG}$ and $\overline{BH}$ to $\overline{DF}$. Then $GH = 4$ and $DG = HC = 3$. Since $\overline{BH} \parallel \overline{EF}$ and $DB = BE$, $DH = HF = 7$. Thus $CF = HF - HC = 7 - 3 = 4$.



5. **(E)** For an $m \times n$ grid there are $m+1$ columns of vertical toothpicks, each n toothpicks long. Thus, there are $(m+1)n$ vertical toothpicks. Likewise, there are $(n+1)m$ horizontal toothpicks. The total is $(m+1)n + (n+1)m$ toothpicks. In our case with $m = 10$ and $n = 20$, the number of toothpicks is $11 \cdot 20 + 21 \cdot 10 = 430$.

OR

Each of the 20×10 unit squares has four sides. Thus $4 \cdot (20 \cdot 10)$ counts each toothpick twice, except for the $2 \cdot (20 + 10)$ toothpicks on the perimeter of the rectangle, which are only counted once. Hence, the number of toothpicks is $\frac{1}{2}\left(4 \cdot 20 \cdot 10 + 2(20 + 10)\right) = 430$.

6. **(A)** The x-intercept of the line is $\frac{6}{a}$, and the y-intercept is $\frac{6}{b}$. Thus the area of the triangle is
$$\frac{1}{2}\cdot\frac{6}{a}\cdot\frac{6}{b}=\frac{18}{ab}.$$
If $\frac{18}{ab}=6$, then $ab=3$.

7. **(C)** Since $\angle C=30°$ and $AH\perp HC$, $\angle CAH=60°$ and, in fact, $\triangle AHC$ is a 30-60-90 triangle. Hence, $AH=\frac{1}{2}AC=AM$. Thus, $\triangle AHM$ is equilateral, $\angle AHM=60°$, and $\angle MHC=90°-60°=30°$.

OR

Since M is the midpoint of hypotenuse $\overline{AC}$ of right triangle AHC, $\overline{MH}$ and $\overline{MC}$ are radii of the circle circumscribing $\triangle AHC$. Therefore $MH=MC$ and $\angle MHC=\angle C=30°$.

8. **(D)** Let $x^2+x-n=(x-a)(x+b)$ where a and b are consecutive positive integers with $a<b$. Then by completing this table

a :	1	2	...	9	10
b :	2	3	...	10	11
$n=ab$:	2	6	...	90	110

we see that there are 9 values for n between 1 and 100.

OR

By the quadratic formula, $x^2+x-n=(x-x_1)(x-x_2)$ if and only if $x_1,x_2=\frac{-1\pm\sqrt{1+4n}}{2}$. Since every odd square is of the form $1+4n$, the answer is the number of odd squares between $1+4\cdot 1$ and $1+4\cdot 100$. There are 9 odd squares in this range, the largest of which is 19^2.

9. **(B)** Any two-element subset of the first 25 letters of the alphabet in alphabetical order together with Z will produce a suitable monogram; furthermore, all suitable monograms are of this form. The number of such two-element subsets is $\binom{25}{2} = 300$.

OR

The last initial is fixed at Z. If A is chosen for the first initial, there are 24 choices for the second. If B is chosen, there are 23 choices, etc. Therefore (using $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$) the number of monograms is

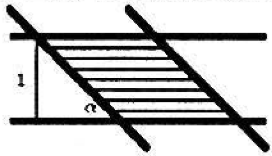
$$24 + 23 + 22 + \cdots + 1 = \frac{24 \cdot 25}{2} = 300.$$

10. **(C)** Note that $u_1 = a$, $u_2 = \dfrac{-1}{a+1}$, $u_3 = \dfrac{-1}{\frac{-1}{a+1} + 1} = \dfrac{-(a+1)}{a}$, and $u_4 = \dfrac{-1}{-\frac{(a+1)}{a} + 1} = \dfrac{-a}{-1} = a$. Hence $a = u_1 = u_4 = u_7 = \cdots = u_{16}$. Use $a = 1$ to show $u_n \neq a$ for the other given values of n.

11. **(A)** First note that a attains its largest possible value only when b, c and d are also as large as possible. The largest possible value for d is 99. Since $c < 4 \cdot 99 = 396$, the largest possible value for c is 395. Since $b < 3 \cdot 395 = 1185$, the largest possible value for b is 1184. Since $a < 2 \cdot 1184 = 2368$, the largest possible value for a is 2367.

12. **(C)** Since the vehicles moving in each direction are traveling at 60 miles per hour, the eastbound traffic is actually traveling at 120 miles per hour relative to the westbound traffic. Thus, in five minutes the eastbound driver will actually count the number of westbound vehicles in a ten-mile section of highway. Since 20 vehicles are in a ten-mile section, there will be 200 vehicles in a 100-mile section of highway.

13. **(B)** The shaded figure in the problem is a rhombus. Each side has length $(1/\sin\alpha)$ as shown by the figure to the right. Its height is 1, which is the width of each strip. Its area is *base* $\cdot$ *height* $= (1/\sin\alpha)$.



14. **(B)** Use the definitions of the tangent and cotangent functions and the identity for the cosine of the difference of two angles to obtain

$$\cot 10 + \tan 5 = \frac{\cos 10}{\sin 10} + \frac{\sin 5}{\cos 5} = \frac{\cos 10 \cos 5 + \sin 10 \sin 5}{\sin 10 \cos 5} =$$

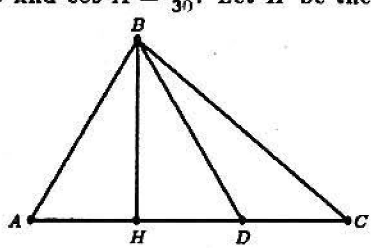
$$\frac{\cos(10-5)}{\sin 10 \cos 5} = \frac{1}{\sin 10} = \csc 10.$$

Note. This is an instance of the identity $\cot 2x + \tan x = \csc 2x$.

15. **(E)** Apply the *Law of Cosines* to $\triangle BAC$ to find $\cos A = \frac{19}{30}$. Let H be the foot of the altitude from B. Then

$$AD = 2 \cdot AH = 2 \cdot AB \cos A = \frac{19}{3}.$$

Thus $DC = \frac{8}{3}$ and $AD : DC = 19 : 8$.



OR

Let H be the foot of the altitude from B. Then, by the Pythagorean Theorem,

$$5^2 - AH^2 = BH^2 = 7^2 - (9 - AH)^2,$$

so $AH = \frac{19}{6}$, $AD = 2 \cdot AH = \frac{19}{3}$. Thus $AD : DC = AD : (9 - AD) = 19 : 8$.

16. **(B)** Since $\frac{281-17}{48-3} = \frac{88}{15}$, the lattice point (x, y) is on the line segment if and only if

$$x \text{ and } y \text{ are integers}, \quad 3 \le x \le 48, \quad \text{and} \quad y = 17 + \frac{88}{15}(x-3).$$

But y is an integer if and only if $x - 3$ is a multiple of 15. (*Why?*) The 4 lattice points are, therefore, $(3, 17)$, $(18, 105)$, $(33, 193)$ and $(48, 281)$.

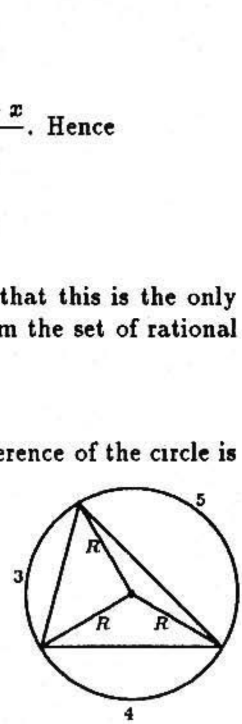
17. **(D)** Let x denote the length of each side of the triangle and y denote the length of each side of the square, so that $3x = 4y + 1989$. Then $3d = 3x - 3y = y + 1989$, so that $d = \frac{y}{3} + 663$. Because $y > 0$, $d \le 663$ is impossible. However, y may take on any positive value, so all integral values of d that exceed 663 (as well as many non-integral values) are possible. Thus, only $1, 2, \ldots, 663$ are excluded as integer values for d.

18. **(B)** Note that $\dfrac{1}{x+\sqrt{x^2+1}} \cdot \dfrac{x-\sqrt{x^2+1}}{x-\sqrt{x^2+1}} = \dfrac{\sqrt{x^2+1}-x}{1}$. Hence

$$x + \sqrt{x^2+1} - \frac{1}{x+\sqrt{x^2+1}} = 2x,$$

which is rational if and only if x is rational. To show that this is the only answer, show that each of the other sets is different from the set of rational numbers.

19. **(E)** Let R be the radius of the circle. Then the circumference of the circle is $2\pi R = 3+4+5$ so $R = \frac{6}{\pi}$. The central angle subtended by the arc of length 5 measures $\frac{5}{6/\pi} = \frac{5\pi}{6}$ radians. Likewise, the angles subtended by the arcs of length 4 and 3 have measures $\frac{2\pi}{3}$ and $\frac{\pi}{2}$ radians, respectively. The area of the given triangle is the sum of the areas of the three triangles into which it is partitioned by the radii to its vertices, so the answer is



$$\frac{1}{2}R^2\left(\sin\frac{5\pi}{6} + \sin\frac{2\pi}{3} + \sin\frac{\pi}{2}\right) =$$

$$\frac{1}{2}\left(\frac{6}{\pi}\right)^2\left(\frac{1}{2} + \frac{\sqrt{3}}{2} + 1\right) = \frac{1}{2}\left(\frac{36}{\pi^2}\right)\left(\frac{3+\sqrt{3}}{2}\right) = \frac{9}{\pi^2}(3+\sqrt{3}).$$

20. **(B)** The following three statements are equivalent:

$$\lfloor\sqrt{x}\rfloor = 12, \qquad 12 \le \sqrt{x} < 13, \qquad 144 \le x < 169;$$

and the following four statements are equivalent:

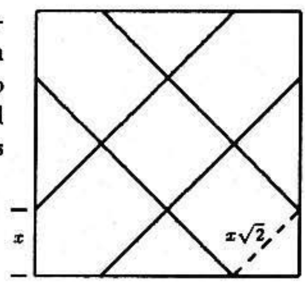
$$\lfloor\sqrt{100x}\rfloor = 120, \qquad 120 \le \sqrt{100x} < 121,$$

$$12 \le \sqrt{x} < 12.1, \qquad 144 \le x < 146.41$$

Thus, the probability, p, that a number selected at random in the interval $[144, 169)$ is also in the interval $[144, 146.41)$ is $p = \dfrac{2.41}{25} = \dfrac{241}{2500}$.

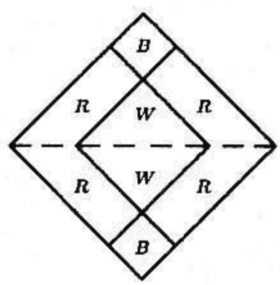
Note. Choosing x between 144 and 169 uniformly at random is not the same as choosing $\sqrt{x}$ uniformly at random between 12 and 13. This is what makes choice **(C)** wrong.

21. **(C)** Let the total area be 100 and let each red segment on the border of the flag be of length x. Then the four white triangles can be placed together to form a white square of area $100 - 36 = 64$ and side $10 - 2x$. Thus $x = 1$ and the blue area is $(x\sqrt{2})^2 = 2$, which is 2% of the total area.



OR

First note that the flag can be cut into four congruent isosceles right triangles by the two diagonals of the flag and that the percent of red, white, and blue areas in each of these triangles is the same as that in the flag. Then form a square by attaching two of these triangles along their hypotenuses as shown. For simplicity assume that this "half-flag" is a 10×10 square. The interior white square consists of 64% of the area, so it must be 8×8. Thus the two blue squares measure 1×1, so they constitute 2% of the area.



22. **(A)** There are $\binom{4}{2} = 6$ ways a block can differ from the given block in exactly two ways: **(1)** material and size, **(2)** material and color, **(3)** material and shape, **(4)** size and color, **(5)** size and shape, and **(6)** color and shape. Since there is only 1 choice for a different material, 2 choices for a different size, 3 choices for a different color, and 3 choices for a different shape, it follows that the number of blocks in each of the above 6 categories is $1\cdot 2$, $1\cdot 3$, $1\cdot 3$, $2\cdot 3$, $2\cdot 3$ and $3\cdot 3$, respectively. The answer is the sum of these six numbers.

Note. The number of blocks that differ from the given block in exactly j ways is the coefficient of x^j in $(1+x)(1+2x)(1+3x)(1+3x)$. This is an example of a *generating function*. [See almost any text on discrete mathematics or combinatorics.]

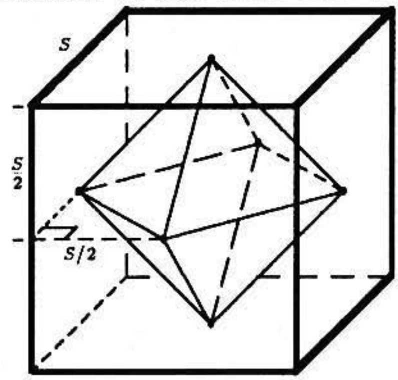
23. **(D)** If t is even, after t^2 minutes all the lattice points of the $t \times t$ square $\{(x,y)\,|\,0 \le x \le t-1,\, 0 \le y \le t-1\}$ have been visited, and the particle is at $(0,t)$. Now, $1989 = 44^2 + 53$, and $53 = 44 + 9$. After 1936 minutes the particle is at $(0,44)$, 44 minutes later its location is $(44,44)$, and 9 minutes after that it is at $(44,35)$.

24. **(B)** Let N denote the number of females in the group of 5. If $N = 0$ then $(0,5)$ is the only pair, and if $N = 1$ then $(2,5)$ is the only pair. If $N = 2$ then the possible pairs are $(4,5)$ and $(3,4)$, depending on whether or not the females are sitting next to each other. By symmetry, the pairs $(5,0)$, $(5,2)$, and $(5,4)$ and $(4,3)$ correspond to $N = 5, 4$, and 3, respectively. Thus there are 8 possible (f,m) pairs.

25. **(B)** With exactly 10 runners contributing to their teams' scores, the sum of the scores of the two teams is $1+2+3+\cdots+10 = 55$. The lowest winning score is $1+2+3+4+5 = 15$. Every winning score must be less than half of 55. Thus, there are 13 possible winning scores, provided that all scores from 15 through 27 are possible.

In fact, all integers between 15 and 40 are possible scores. Note that if a certain finishing order results in score $x < 6+7+8+9+10$ for Team A, then there is a runner from Team A in that finishing order after whom the next finisher is from Team B. If the positions for these two runners were interchanged, then the resulting finishing order would give Team A a score of $x+1$.

26. **(C)** Without loss of generality, let the length of an edge of the cube be $S = 2$. Then the volume of the cube is 8 and each edge of the octahedron is $\sqrt{2}$. Bisect the octahedron into two pyramids with square bases.



Each of the pyramids has altitude 1 and a base of area $(\sqrt{2})^2 = 2$, so the volume of each pyramid is $\frac{1}{3} \cdot 2 \cdot 1 = \frac{2}{3}$. The volume of the octahedron is thus $\frac{4}{3}$, so the required ratio is $\frac{4/3}{8} = \frac{1}{6}$.

OR

The bases of the two pyramids that make up the octahedron each have area equal to half the area of a face of the cube. The height of each pyramid is half the height of the cube. Thus

$$\frac{\text{volume of octahedron}}{\text{volume of cube}} = \frac{2\left(\frac{1}{3} \cdot \frac{S}{2} \cdot \frac{S^2}{2}\right)}{S^3} = \frac{1}{6}.$$

Note. More generally, if we form an octahedron by joining the centers of the six faces of <u>any</u> rectangular solid, then the volume of the octahedron is exactly one sixth of the volume of the rectangular solid.

Query. Is the hypothesis of perpendicularity in the preceding generalization a necessary one? Can you state and prove a further generalization?

27. **(D)** Because z is positive, solving $2(x + y) = n - z$ is equivalent to solving $x + y < \frac{n}{2}$ in positive integers x and y. The number of solutions to this inequality is the number of lattice points inside the triangle T in the first quadrant formed by the coordinate axes and the line $x + y = \frac{n}{2}$. Since $1 + 2 + \cdots + 7 = 28$, n must be chosen so that there are exactly 7 lattice points on the line $y = 1$ in T. That is, $(1, 7)$ must be inside T but $(1, 8)$ is on or outside T. Hence $1 + 7 < \frac{n}{2} \leq 1 + 8$, so that n is 17 or 18.

28. **(D)** Since

$$\tan^2 x - 9\tan x + 1 = \sec^2 x - 9\tan x = \frac{1 - 9\sin x\cos x}{\cos^2 x} = \frac{1 - \frac{9}{2}\sin 2x}{\cos^2 x},$$

we need to sum the roots of the equation $\sin 2x = \frac{2}{9}$ between $x = 0$ and $x = 2\pi$. These roots, all of which must be roots of the given equation, are

$$x = \frac{\arcsin\frac{2}{9}}{2},\ \frac{\pi - \arcsin\frac{2}{9}}{2},\ \frac{2\pi + \arcsin\frac{2}{9}}{2},\ \text{and}\ \frac{3\pi - \arcsin\frac{2}{9}}{2}$$

and their sum is 3π.

OR

For any $b > 2$ the solutions of $y^2 - by + 1 = 0$ are $y_1, y_2 = \frac{b \pm \sqrt{b^2-4}}{2}$, which are distinct and positive. These solutions are reciprocals because their product must be 1. Since the solutions $y_i = \tan x_i$ are reciprocals,

$$\tan x_2 = y_2 = \frac{1}{y_1} = \frac{1}{\tan x_1} = \cot x_1 = \tan(\frac{\pi}{2} - x_1).$$

Thus, y_1 and y_2 are tangents of two distinct, complementary, first-quadrant angles, x_1 and $x_2 = \frac{\pi}{2} - x_1$. Since $\tan(x+\pi) = \tan x$, there are four values of x between 0 and 2π: x_1, $\pi + x_1$, $\frac{\pi}{2} - x_1$ and $\frac{3\pi}{2} - x_1$. Their sum is 3π.

29. **(B)** By the *Binomial Theorem*,

$$(1+i)^{99} = \binom{99}{0} + \binom{99}{1}i + \binom{99}{2}i^2 + \binom{99}{3}i^3 + \cdots + \binom{99}{99}i^{99}.$$

Note that the real part of this series is

$$\binom{99}{0} - \binom{99}{2} + \binom{99}{4} - \binom{99}{6} + \cdots - \binom{99}{98},$$

the very sum we want to find. Since $(1+i) = \sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4})$, by *DeMoivre's Theorem* $(1+i)^{99} = 2^{(99/2)}(\cos\frac{99\pi}{4} + i\sin\frac{99\pi}{4})$. Thus the real part of $(1+i)^{99}$ is $2^{99/2}\cos\frac{99\pi}{4} = 2^{99/2}\cos\frac{3\pi}{4} = 2^{99/2}(-\frac{\sqrt{2}}{2}) = -2^{49}$.

30. **(A)** Suppose that John and Carol are two of the people. For $i = 1, 2, \ldots, 19$, let J_i and C_i be the numbers of orderings (out of all 20!) in which the i^{th} and $(i+1)^{\text{st}}$ persons are John and Carol, or Carol and John, respectively. Then $J_i = C_i = 18!$ is the number of orderings of the remaining persons.
For $i = 1, 2, \ldots, 19$, let N_i be the number of times a boy-girl or girl-boy pair occupies positions i and $i+1$. Since there are 7 boys and 13 girls, $N_i = 7 \cdot 13 \cdot (J_i + C_i)$. Thus the average value of S is

$$\frac{N_1 + N_2 + \ldots + N_{19}}{20!} = \frac{19[7 \cdot 13 \cdot (18! + 18!)]}{20!} = \frac{91}{10}.$$

OR

In general, suppose there are k boys and $n-k$ girls. For $i = 1, 2, \ldots, n-1$ let A_i be the probability that there is a boy-girl pair in positions $(i, i+1)$ in the line. Since there is either 0 or 1 pair in $(i, i+1)$, A_i is also the expected number of pairs in these positions. By symmetry, all A_i's are the same (or note that the argument below is independent of i). Thus the answer is $(n-1)A_i$.
We may consider the boys indistinguishable and likewise the girls. (*Why?*) Then an order is just a sequence of k **B**s and $n-k$ **G**s. To have a pair at $(i, i+1)$ we must have **BG** or **GB** in those positions, and the remaining $n-2$ positions must have $k-1$ boys and $n-k-1$ girls. Thus there are $2\binom{n-2}{k-1}$ sequences with a pair at $(i, i+1)$. Since there are $\binom{n}{k}$ sequences,

$$\textbf{answer} = (n-1)A_i = \frac{(n-1)2\binom{n-2}{k-1}}{\binom{n}{k}} = \frac{2k(n-k)}{n}.$$

In our case, the answer is $\dfrac{2 \cdot 7 \cdot 13}{20} = \dfrac{91}{10}$.

AMERICAN MATHEMATICS COMPETITIONS

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

41st ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)

TUESDAY, FEBRUARY 27, 1990

Sponsored by

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This Solutions Pamphlet contains at least one solution to each problem on this year's Examination and shows that all the problems can be solved using non-calculus mathematics. Routine calculations and obvious reasons for proceeding in a certain way are often omitted to give greater emphasis to the essential ideas behind each solution. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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1. **(E)** Multiply both sides of the given equation by $2 \cdot \frac{x}{2}$ to obtain $\frac{x^2}{8} = 8$. Thus $x^2 = 8^2$, or $x = \pm 8$.

2. **(E)** $\left(\frac{1}{4}\right)^{-\frac{1}{4}} = \left(\frac{1}{2^2}\right)^{-\frac{1}{4}} = (2^{-2})^{-\frac{1}{4}} = 2^{\frac{2}{4}} = 2^{\frac{1}{2}} = \sqrt{2}.$

3. **(C)** Let d be the common difference of the arithmetic sequence. Then $75 + (75+d) + (75+2d) + (75+3d) = 360$, so $d = 10$ and $75 + 3d = 105$.

OR

Since the opposite bases of a trapezoid are parallel, the smallest and largest angles must be supplementary, so the answer is $180° - 75° = 105°$.

Note. Any quadrilateral whose consecutive angles form an arithmetic sequence is a trapezoid.

4. **(B)** Since $\overline{AD} \parallel \overline{BC}$, $\triangle FDE$ is similar to $\triangle BCE$. Hence

$$\frac{FD}{BC} = \frac{DE}{CE} \quad \text{or} \quad FD = \frac{DE}{CE} \cdot BC = \frac{4}{4+16} \cdot 10 = 2.$$

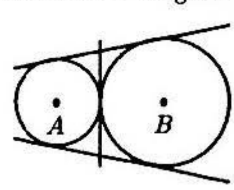
Note. The answer is independent of $\angle ABC$.

5. **(B)** If a and b are positive, then $a < b$ if and only if $a^6 < b^6$. Since all the choices are positive, raise each to the sixth power to simplify the comparison:

(A) $5 \cdot 6$ **(B)** $6 \cdot 6 \cdot 6 \cdot 5$ **(C)** $5 \cdot 5 \cdot 5 \cdot 6$ **(D)** $5 \cdot 5 \cdot 6$ **(E)** $6 \cdot 6 \cdot 5$

and note that **(B)** is largest.

6. **(D)** The set of lines that are 2 units from the point A is the set of tangents to the circle with center A and radius 2. Similarly, the set of lines that are 3 units from point B is the set of tangents to the circle with center B and radius 3. Thus the desired set of lines is the set of common tangents to the two circles. Since $AB = 5 = 2 + 3$, these two circles are tangent externally, so they have three common tangents.



7. **(A)** Let $a \geq b \geq c$ be the lengths of the three sides of the triangle. Since the longest side must be an integer that is less than half the perimeter and at least one third of the perimeter, $a = 3$. Since $3 \geq b \geq c > 0$ and $b + c = 5$, the only triangle with integral sides and perimeter 8 has sides of lengths 3, 3 and 2. The altitude to the base of length 2 of this isosceles triangle is $\sqrt{3^2 - 1^2}$, so its area is $\frac{1}{2} \cdot 2 \cdot \sqrt{8} = 2\sqrt{2}$.

   Query. If any integral perimeter larger than 8 were used, the area would not be unique. Can you prove it?

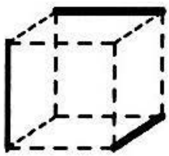
8. **(E)** For any number x in the interval $[2, 3]$, we have

$$|x - 2| + |x - 3| = (x - 2) + (3 - x) = 1.$$

OR

We can interpret $|a - b|$ as the distance between a and b. Then the given equation is the condition that the distance between x and 2 plus the distance between x and 3 equals 1. This is the case whenever $2 \leq x \leq 3$.

9. **(B)** Since there are six faces and each edge is shared by only two faces, there must be at least $\dfrac{6}{2} = 3$ black edges. The diagram shows that 3 black edges will suffice.



10. **(D)** At most three of the large cube's six faces can be seen at once. Excluding the unit cubes of the three closest edges, the three visible faces contain $3 \cdot 10^2$ unit cubes. The three edges contain $3 \cdot 10$ unit cubes plus the single, shared corner cube. Therefore the desired number is $3 \cdot 10^2 + 3 \cdot 10 + 1 = 331$.

OR

The unseen unit cubes form a $10 \times 10 \times 10$ cube. Thus the number of unit cubes that can be seen is $11^3 - 10^3 = 331$.

11. **(C)** Let N be a positive integer and d a divisor of N. Then $\frac{N}{d}$ is also a divisor of N. Thus the divisors of N occur in pairs $d, \frac{N}{d}$ and these two divisors will be distinct unless N is a perfect square and $d = \sqrt{N}$. It follows that N has an odd number of divisors if and only if N is a perfect square. There are 7 perfect squares among the numbers $1, 2, 3, \ldots, 50$.

Note. If $N > 1$ is an integer then $N = p_1^{r_1} \cdot p_2^{r_2} \cdot \ldots \cdot p_k^{r_k}$ where p_i is the i^{th} prime. The divisors of N are those $d = p_1^{s_1} \cdot p_2^{s_2} \cdot \ldots \cdot p_k^{s_k}$ with $0 \le s_i \le r_i$ for all i. Thus, N has $(r_1+1)\cdot(r_2+1)\cdot\ldots\cdot(r_k+1)$ divisors, a product which will be an odd number only when each r_i is even.

12. **(D)** Since $a \neq 0$, the only x for which $f(x) = -\sqrt{2}$ is $x = 0$. Since $f(f(\sqrt{2})) = -\sqrt{2}$, $f(\sqrt{2})$ must be 0. Thus $2a - \sqrt{2} = 0$, or $a = \frac{\sqrt{2}}{2}$.

OR

Since $f(\sqrt{2}) = 2a - \sqrt{2}$, $f\big(f(\sqrt{2})\big) = a(2a - \sqrt{2})^2 - \sqrt{2}$ which we set equal to $-\sqrt{2}$. Therefore, $a(2a - \sqrt{2})^2 = 0$. Since $a > 0$, $2a = \sqrt{2}$ and $a = \frac{\sqrt{2}}{2}$.

13. **(E)** Since $S = 5+7+9+\cdots+X$ is the sum of an arithmetic series containing $1+\frac{X-5}{2}$ terms, we have

$$S = \frac{1}{2}\left(1 + \frac{X-5}{2}\right)(5+X) = \frac{X^2+2X-15}{4} = \left(\frac{X+1}{2}\right)^2 - 4 \ge 10000.$$

Thus $\frac{X+1}{2} > 100$. Since X is odd, $X = 201$ is the printed value.

14. **(A)** Angles BAC, BCD and CBD all intercept the same circular arc. Therefore $\angle BCD = \angle CBD = x$ and $\angle D = \pi - 2x$. The given condition now becomes $\frac{\pi - x}{2} = 2(\pi - 2x)$, which has the solution $x = \frac{3}{7}\pi$.

OR

Let O be the center of the circle. Then $\angle COB = 2x$ and, from the sum of the angles of the quadrilateral $COBD$, we obtain $2x + \angle D = \pi$. The conditions of the problem yield $x + 4\angle D = \pi$ to be the sum of the angles of $\triangle ABC$. Solve these two equations in x and $\angle D$ simultaneously to find $x = 3\pi/7$.

Query. What is x if $\triangle ABC$ is an obtuse isosceles triangle?

15. **(C)** Let the four numbers be w, x, y and z with $w \le x \le y \le z$. Since each number appears three times in the four sums,

$$3(w+x+y+z) = 180 + 197 + 208 + 222 = 807.$$

Thus $w+x+y+z = 269$ and $w+x+y = 180$, so $z = 269 - 180 = 89$.

16. **(C)** If all 26 people shook hands there would be $\binom{26}{2}$ handshakes. Of these, $\binom{13}{2}$ would take place between women and 13 between spouses. Therefore there were $\binom{26}{2} - \binom{13}{2} - 13 = 13 \cdot 25 - 13 \cdot 6 - 13 = 234$ handshakes.

17. **(C)** For every 3 distinct digits selected from $\{1, 2, \ldots, 9\}$ there is exactly one way to arrange them into a number with increasing digits, and every number with increasing digits corresponds to one of these selections. Similarly, the numbers with decreasing digits correspond to the subsets with 3 elements of the set of all 10 digits. Hence our answer is

$$\binom{9}{3} + \binom{10}{3} = \frac{9 \cdot 8 \cdot 7}{1 \cdot 2 \cdot 3} + \frac{10 \cdot 9 \cdot 8}{1 \cdot 2 \cdot 3} = \frac{9 \cdot 8}{2 \cdot 3}(7+10) = 12 \cdot 17 = 204.$$

OR

By making a list of such numbers with increasing digits in decreasing order,

$$789, \underbrace{689, 679, 678}, \underbrace{589, 579, 578, 569, 568, 567}, \ldots, \underbrace{189, \ldots, 124, 123}$$

and grouping them according to their first digit, we see there are $1 + (1+2) + (1+2+3) + \cdots + (1+2+\cdots+7) = 84$ such numbers. Now make a list of such numbers with decreasing digits in increasing order:

$$210, \underbrace{310, 320, 321}, \underbrace{410, 420, 421, 430, 431, 432}, \ldots, \underbrace{910, \ldots, 986, 987}$$

and group them according to their first digit. In this case there are $1 + 3 + 6 + 10 + \cdots + 28 + 36 = 120$ such numbers, an additional 36 numbers since a number with decreasing digits can contain 0 while one with increasing digits cannot. Thus, the answer is $84 + 120 = 204$.

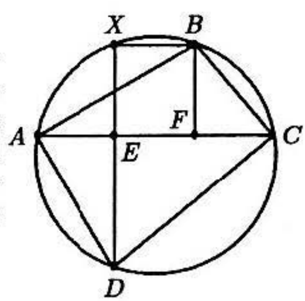
18. **(C)** By observing the repeating pattern of the units digits of consecutive integral powers of 3 we note that 25 of the given values for a yield a units digit in 3^a of 3, 25 yield 9, 25 yield 7 and 25 yield 1. Similarly, the given values for b yield 25 of each of these units digits in 7^b: 7, 9, 3, 1. Thus there are 16 possible pairs of units digits, and each pair is equally likely. Of these, three pairs, $(1,7)$, $(7,1)$ and $(9,9)$, yield a sum with units digit 8. Thus, the desired probability is $\frac{3}{16}$.

19. **(B)** Since $\dfrac{N^2+7}{N+4} = \dfrac{(N-4)(N+4)+23}{N+4}$, the numerator and denominator will have a nontrivial common factor exactly when $N+4$ and 23 have a factor in common. Because 23 is a prime, $N+4$ is a multiple of 23 when $N = -4+23k$ for some integer k. Solving $1 < -4+23k < 1990$ yields $\frac{5}{23} < k < 86\frac{16}{23}$, or $k = 1, 2, \ldots, 86$.

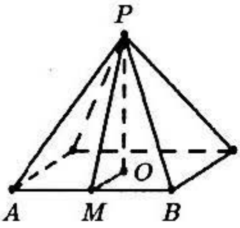
20. **(C)** Since $\angle BAF$ and $\angle ADE$ are both complementary to $\angle CAD$ they must be equal. Thus, $\triangle BAF \sim \triangle ADE$ so $\dfrac{BF}{AF} = \dfrac{AE}{DE}$, or $\dfrac{BF}{3+EF} = \dfrac{3}{5}$. By an analogous argument, $\triangle BCF \sim \triangle CDE$, $\dfrac{BF}{CF} = \dfrac{CE}{DE}$, and $\dfrac{BF}{7-EF} = \dfrac{7}{5}$. Solve these two equations simultaneously to obtain $BF = 4.2$.

OR

Note that $ABCD$ is a cyclic quadrilateral since opposite angles are supplementary. Extend $\overline{DE}$ to X on the circumcircle. Since $\angle DAB$ subtends the same arc as $\angle DXB$, $BFEX$ is a rectangle and $BF = EX$. We can consider $\overline{AC}$ and $\overline{DX}$ as intersecting chords in a circle and use $DE \cdot EX = AE \cdot EC$ to find $BF = EX = \dfrac{AE \cdot EC}{DE} = \dfrac{21}{5}$.



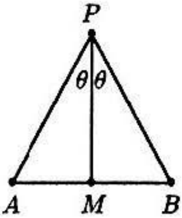
21. **(E)** Let M be the midpoint of $\overline{AB}$ and O be the center of the square. Thus $AM = OM = \frac{1}{2}$ and slant height $PM = \frac{1}{2}\cot\theta$. Hence



$$PO^2 = PM^2 - OM^2 = \frac{1}{4}\cot^2\theta - \frac{1}{4}$$
$$= \frac{\cos^2\theta - \sin^2\theta}{4\sin^2\theta} = \frac{\cos 2\theta}{4\sin^2\theta}.$$

Since $0 < \theta < 45°$, the volume is

$$\frac{1}{3}\cdot 1^2 \cdot PO = \frac{\sqrt{\cos 2\theta}}{6\sin\theta}.$$



22. **(D)** Use the polar form, $x = r(\cos\theta + i\sin\theta)$. By DeMoivre's Theorem,

$$r^6(\cos 6\theta + i\sin 6\theta) = x^6 = -64 = 2^6\big(\cos(180° + 360°k) + i\sin(180° + 360°k)\big).$$

Thus $r = 2$ and, using $k = -3, -2, -1, 0, 1, 2$, we have

$$\theta = (180° + 360°k)/6 = \pm 30°, \pm 90°, \pm 150°.$$

Since $a > 0$, $\theta = \pm 30°$, so $x = 2\big(\cos(\pm 30°) + i\sin(\pm 30°)\big) = \sqrt{3} \pm i$. The product of these two roots is $\left(\sqrt{3}+i\right)\left(\sqrt{3}-i\right) = 4$.

OR

Recall, from DeMoivre's Theorem, that the six sixth roots of -64 are equispaced around the circle of radius $\sqrt[6]{64}$. Since $\pm 2i$ are roots, exactly two of the roots are in the right half-plane and they must be conjugates. The product of any pair of conjugates is the square of their distance from the origin, so the product of these two roots is $\left(\sqrt[6]{64}\right)^2 = 4$.

23. **(B)** Let $v = \log_y x$. Then, since $\log_x y = \dfrac{1}{v}$, we solve $v + \dfrac{1}{v} = \dfrac{10}{3}$ to find $\log_y x = v = 3$ or $\frac{1}{3}$. Without loss of generality, assume $x > y$. Then $\log_y x = 3$, $x = y^3$ and $xy = y^4 = 144$ so that $y = \sqrt{12} = 2\sqrt{3}$, $x = 24\sqrt{3}$ and $\dfrac{x+y}{2} = \dfrac{24\sqrt{3} + 2\sqrt{3}}{2} = 13\sqrt{3}$.

24. **(D)** We want a weighted average, X, of 76 and 90, with weights proportional to the number of girls at Adams HS and Baker HS, respectively. We obtain these weights as follows: Let

b=number of boys at Adams,	B=number of boys at Baker,
g=number of girls at Adams,	G=number of girls at Baker.

Then, from the first column of the table we obtain $\dfrac{71b+76g}{b+g}=74$, which leads to $g=1.5b$. Similarly, the second column shows that $G=.5B$ and the first row shows that $B=4b$. Thus

$$X=\frac{76g+90G}{g+G}=\frac{76(1.5b)+90[.5(4b)]}{1.5b+[.5(4b)]}=\frac{114+180}{3.5}=84.$$

25. **(B)** Let r be the radius of each sphere. Note that the centers of the eight outer spheres form a cube of side $(1-2r)$ whose main diagonal is $4r$ units. Since the length of the diagonal of a cube is $\sqrt{3}$ times its side, $\sqrt{3}(1-2r)=4r$. Solve this equation to find $r=\dfrac{2\sqrt{3}-3}{2}$.

OR

If the radius of each sphere is r, the center of a corner sphere is $\sqrt{r^2+r^2+r^2}$ units from the closest vertex. Thus the length of the diagonal of the cube is $4r+2r\sqrt{3}$. But the length of the diagonal of a unit cube is $\sqrt{3}$. Solve $4r+2r\sqrt{3}=\sqrt{3}$ to find $r=\dfrac{2\sqrt{3}-3}{2}$.

Note. To visualize the arrangement of the spheres in the cube, begin with nine small congruent spheres with one at the center and one tangent to the three faces at each of the eight vertices of the cube. Keeping the center sphere in the center of the cube and the other eight tangent to their three faces, expand the radii of all nine spheres until the spheres are tangent.

26. **(A)** Let p_i be the person who announced "i" and let x be the number picked by p_6. Since the average of the numbers picked by p_4 and p_6 is 5, p_4 picked $10-x$. Continuing counterclockwise around the table, we find that p_2 picked $x-4$, p_{10} picked $6-x$, p_8 picked $12+x$, and p_6 picked $2-x$. Since $2-x=x$, $x=1$.

OR

Let x_i be the original number picked by the person who announced "i". We have a system of ten equations in ten unknowns which has a unique solution:

$$\frac{1}{2}(x_{10}+x_2)=1, \quad \frac{1}{2}(x_1+x_3)=2,$$
$$\vdots \qquad\qquad \vdots$$
$$\frac{1}{2}(x_8+x_{10})=9, \quad \frac{1}{2}(x_9+x_1)=10.$$

"1":**6**
"10":**5** "2":**−3**
"9":**14** "3":**−2**
"8":**13** "4":**9**
"7":**2** "5":**10**
"6":**1**

The sum of the five equations involving the variables with even subscripts yields $x_2+x_4+x_6+x_8+x_{10}=25$. Substitute $x_2+x_4=6$ and $x_8+x_{10}=18$ to obtain $x_6=1$. In the figure we show "i":$\mathbf{x}_i$ where the $\mathbf{x}_i$ yield the desired averages, "i".

Query. Suppose there had been n people instead of 10. For which n is there a unique answer?

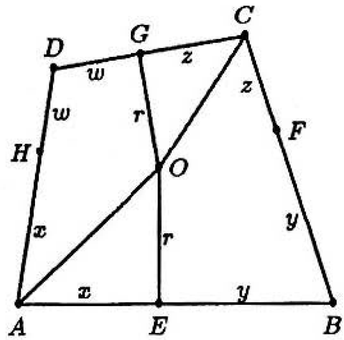
Note. This problem is an example of inverting averages. Such problems arise in many applications of mathematics, for instance, the operation of CAT scanners in medicine. To obtain information from a CAT scan, one must invert averages along continuous rays in a disk, rather than averages of discrete points on the perimeter of the disk.

27. **(C)** Let x, y and z denote the sides of a triangle, h_x, h_y and h_z the corresponding altitudes, and A the area. Since $xh_x=yh_y=zh_z=2A$, the sides are inversely proportional to the altitudes. If x, y and z form a triangle with largest side x, then $x<y+z$. Thus

$$\frac{2A}{h_x}<\frac{2A}{h_y}+\frac{2A}{h_z} \qquad \text{or} \qquad \frac{1}{h_x}<\frac{1}{h_y}+\frac{1}{h_z}. \qquad (*)$$

Only triple **(C)** fails to satisfy $(*)$. To show that the other four choices (a,b,c) *do* correspond to possible triangles, just build a triangle T with sides $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$. The altitudes of T are in the ratio a:b:c, so some triangle similar to T has altitudes a, b and c.

28. **(B)** Let the inscribed circle have center O and radius r. Label the quadrilateral $ABCD$ where $DA = 90$, $AB = 130$ and $BC = 110$. Label the points of tangency with the inscribed circle E, F, G and H, and let w, x, y and z be the distances from these points of tangency to the vertices of the quadrilateral as indicated in the figure. Since the quadrilateral is inscribed in a circle, $\angle DAB$ is supplementary to $\angle DCB$. Thus, since $\overline{OA}$ bisects $\angle DAB$ and $\overline{OC}$ bisects $\angle DCB$, $\angle OAE$ and $\angle OCG$ are complementary. Hence $\triangle OEA \sim \triangle CGO$. Thus $\frac{x}{r} = \frac{r}{z}$. Similarly, $\frac{y}{r} = \frac{r}{w}$. Hence $wy = r^2 = zx$, which leads to



$$90y = (w + x)y = wy + xy = zx + xy = (z + y)x = 110x.$$

Solve $90y = 110x$ and $x + y = 130$ simultaneously for x and y to obtain $|x - y| = 13$.

Note. This quadrilateral has an inscribed circle because $70 + 130 = 90 + 110$. The shape of this quadrilateral is unique since it is inscribed in a circle.

29. **(D)** For each positive integer b that is not divisible by 3, we must decide which of the numbers in the list $b, 3b, 9b, 27b, \cdots \leq 100$ to place in the subset. Clearly, a maximal subset can be obtained by using alternate numbers from this list starting with b. Thus, it will contain $67 = 100-33$ members that are not divisible by 3, $8 = 11-3$ members that are divisible by 9 but not by 27, and 1 member divisible by 81, for a total of $67 + 8 + 1 = 76$ elements.

Note. The maximal subset is not unique. For example, for each b between 13 and 32 that is not divisible by 3, either b or $3b$ could be used.

30. **(E)** Multiply both sides of $R_n = \dfrac{a^n + b^n}{2}$ by $a+b$ to obtain

$$(a+b)R_n = (a+b)\left(\frac{a^n + b^n}{2}\right) = \frac{a^{n+1} + b^{n+1}}{2} + ab\left(\frac{a^{n-1} + b^{n-1}}{2}\right)$$
$$= R_{n+1} + abR_{n-1}.$$

Since $a+b = 6$ and $ab = 1$, the recursion $R_{n+1} = 6R_n - R_{n-1}$ follows. Use this, together with $R_0 = 1$ and $R_1 = 3$, to calculate the units digits of $R_2, R_3, R_4, R_5, R_6, R_7, \ldots$ which are $7, 9, 7, 3, 1, 3, \ldots$, respectively. An induction argument shows that R_n and R_{n+6} have the same units digit for all nonnegative n. In particular, $R_3, R_9, \cdots, R_{12345}$ all have the same units digit, 9, since $12345 = 3 + 6 \cdot 2057$.

Note. Since $(x-a)(x-b) = (x-3-2\sqrt{2})(x-3+2\sqrt{2}) = x^2 - 6x + 1$ it follows that a and b satisfy $x^2 = 6x - 1$, so $a^{n+1} = a^{n-1}a^2 = a^{n-1}(6a-1) = 6a^n - a^{n-1}$ and similarly, $b^{n+1} = 6b^n - b^{n-1}$. This yields an alternate derivation of the recursion:

$$R_{n+1} = \frac{a^{n+1} + b^{n+1}}{2} = 6\frac{a^n + b^n}{2} - \frac{a^{n-1} + b^{n-1}}{2} = 6R_n - R_{n-1}.$$

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FOR STUDENTS AND TEACHERS

42nd ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)

TUESDAY, FEBRUARY 26, 1991

Sponsored by

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1. **(E)** If $a = 1, b = -2$ and $c = -3$, then $\boxed{a, b, c} = \frac{c+a}{c-b} = \frac{-3+1}{-3+2} = \frac{-2}{-1} = 2.$

2. **(E)** Since $3 - \pi < 0$, we have $|3 - \pi| = -(3 - \pi) = \pi - 3$.

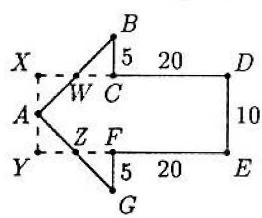
3. **(A)** $(4^{-1} - 3^{-1})^{-1} = \left(\frac{1}{4} - \frac{1}{3}\right)^{-1} = \left(-\frac{1}{12}\right)^{-1} = -12.$

4. **(C)** A triangle cannot contain one 90° angle and another angle greater than 90° since the three angles must sum to 180°. Triangles with sides having the following lengths show that each of the others is possible:
(A) $2, 3, 3$ **(B)** $1, 1, \sqrt{2}$ **(D)** $3, 4, 5$ **(E)** $3, 4, 6$

5. **(E)** Rectangle $CDEF$ has area $10 \cdot 20 = 200$. Triangle ABG is isosceles with base angles 45°. The altitude to base $\overline{BG}$ is 10, so the area of the triangle is $\frac{1}{2} \cdot 20 \cdot 10 = 100$. Thus the total area of the polygon is $200 + 100 = 300$.

OR

Extend $\overline{DC}$ to X and $\overline{EF}$ to Y where X and Y are on the line parallel to $\overline{DE}$ through A. By symmetry, $AX = AY = \frac{1}{2}DE = 5 = BC$. If $\overline{AB}$ intersects $\overline{DX}$ at W, then $\triangle AXW \cong \triangle BCW$ since $\angle WAX = \angle B = 45° = \angle BWC = \angle AWX$. Similarly, if $\overline{AG}$ intersects $\overline{EY}$ at Z, then $\triangle AYZ \cong \triangle GFZ$. Hence the area of the arrow equals the area of rectangle $DXYE$ which is $10(20 + 5 + 5) = 300$.



6. **(E)** $\sqrt{x\sqrt{x\sqrt{x}}} = \left(x\left(x \cdot x^{1/2}\right)^{1/2}\right)^{1/2} = \left(x\left(x^{3/2}\right)^{1/2}\right)^{1/2} = \left(x \cdot x^{3/4}\right)^{1/2}$

$$= \left(x^{7/4}\right)^{1/2} = x^{7/8} = \sqrt[8]{x^7}.$$

7. **(B)** Since $a = bx$, $\frac{a+b}{a-b} = \frac{bx+b}{bx-b} = \frac{b(x+1)}{b(x-1)} = \frac{x+1}{x-1}$. To show that the other four choices are incorrect, let $a = 2$ and $b = 1$.

OR

Dividing numerator and denominator by b shows that

$$\frac{a+b}{a-b} = \frac{\frac{a}{b}+1}{\frac{a}{b}-1} = \frac{x+1}{x-1}.$$

Note. More generally, if $\frac{s}{t} = \frac{u}{v} \neq 1$ then $\frac{s+t}{s-t} = \frac{u+v}{u-v}$, since both sides of the last equation equal $\frac{\frac{u}{v}t+t}{\frac{u}{v}t-t}$.

8. **(C)** The volume of liquid X in cm^3 is $3{\cdot}6{\cdot}12 = 216$. The film is a cylinder whose volume in cm^3 is $0.1\pi r^2$. Solve $216 = 0.1\pi r^2$ to find $r = \sqrt{\frac{2160}{\pi}}$.

9. **(D)** Let the population be P at time $t=0$, and suppose the population increased by $k\%$ from time $t=0$ to time $t=2$. The population at time $t=2$

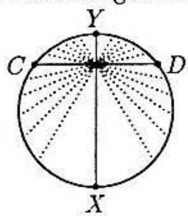
is
$$P \cdot \left(1+\frac{k}{100}\right) = \left[P \cdot \left(1+\frac{i}{100}\right)\right]\left(1+\frac{j}{100}\right).$$

Hence
$$1+\frac{k}{100} = 1+\frac{i+j}{100}+\frac{ij}{10000},$$

so
$$k = i+j+\frac{ij}{100}.$$

Let $i=j=100$ to show that the other choices are not always correct.

10. **(B)** The longest chord through P is the diameter, $\overline{XY}$, which has length 30. The shortest chord through P, $\overline{CD}$, is perpendicular to this diameter. Hence its length is $2\sqrt{15^2-9^2} = 24$. As the chords rotate through point P, their lengths will take on all real numbers between 24 and 30 twice. [See figure.] Thus, for each of the five integers k strictly between 24 and 30 there are *two* chords of length k through P. This gives a total of $2 + 5{\cdot}2 = 12$ chords with integer lengths.



11. (B) Let x denote the distance in kilometers from the top of the hill to where they meet. When they meet, Jack has been running for $\frac{5}{15} + \frac{x}{20}$ hours and Jill has been running for $\frac{5-x}{16}$ hours. Since Jack has been running $1/6$ hour longer than Jill, we solve

$$\left(\frac{5}{15} + \frac{x}{20}\right) - \frac{5-x}{16} = \frac{1}{6}$$

to find $x = 35/27$.

OR

Jack runs up the hill in 20 minutes. Therefore at the time when he starts down the hill, Jill has been running for 10 minutes and has come $16 \cdot \frac{1}{6} = \frac{8}{3}$ km up the hill. Let t be the time needed to cover the $7/3$ km that now separates them. Then

$$20t + 16t = \frac{7}{3}, \quad \text{so} \quad t = \frac{7}{108}.$$

The distance from the top of the hill is the distance that Jack travels, namely $20 \cdot \frac{7}{108} = \frac{35}{27}$ km.

12. (D) Let d be the common difference of the arithmetic sequence. The sum of the interior angles of the hexagon,

$$6m - 15d = m + (m-d) + (m-2d) + \cdots + (m-5d) = (6-2)180 = 720,$$

shows that $6m = 15d + 720 = 5(3d + 144)$, so m is divisible by 5. Because the hexagon is convex, $m \leq 175$. Because $65 + 87 + 109 + 131 + 153 + 175 = 720$, there is such a hexagon and $175°$ is the answer.

13. (D) The probability that X wins is $\frac{1}{3+1}$ and the probability that Y wins is $\frac{3}{2+3}$. The sum of the winning probabilities for all three horses must be 1, so the probability that Z wins is $1 - \frac{1}{4} - \frac{3}{5} = \frac{3}{20} = \frac{3}{17+3}$. Hence the odds against Z winning are 17-to-3.

14. **(C)** The cubes
$$x = 1, 2^3, 2^6, \ldots, 2^{3k}, \ldots, (2^{67})^3$$
have
$$d = 1, 4, 7, \ldots, 3k+1, \ldots, 202$$
divisors, respectively. In fact, for any prime p, $(p^{67})^3$ has 202 divisors.
To show that, of the choices listed, $d = 202$ is the only possible answer, we prove that for *any* perfect cube $x > 1$, d *must* be of the form $3k+1$:

If $x = p_1^{3b_1} p_2^{3b_2} \cdots p_n^{3b_n}$ where the p_i are distinct primes, then its divisors are all the numbers of the form $p_1^{a_1} p_2^{a_2} \cdots p_n^{a_n}$, with $0 \le a_i \le 3b_i$ for $i = 1, 2, \ldots, n$. Taking the product of the number of choices for each a_i yields $d = (3b_1+1)(3b_2+1)\cdots(3b_n+1) = 3k+1$ for some integer k.

15. **(B)** Divide the chairs around the table into $60/3 = 20$ sets of three consecutive seats. If fewer than 20 people are seated at the table, then at least one of these sets of three seats will be unoccupied. If the next person sits in the center of this unoccupied set, then that person will not be seated next to anyone already seated. On the other hand, if 20 people are already seated, and each occupies the center seat in one of the sets of three, then the next person to be seated must sit next to one of these 20 people.

Query. What is the answer for 59 chairs? For c chairs?

16. **(D)** If s seniors took the **AHSME** then $\frac{3}{2}s$ non-seniors took it, so $s + \frac{3}{2}s = 100$, $s = 40$ and $\frac{3}{2}s = 60$. If a is the mean score for the seniors then $\frac{2}{3}a$ is the mean score for the non-seniors, so $\dfrac{40a + 60\left(\frac{2}{3}a\right)}{100} = 100$ and $a = 125$.

17. **(D)** Since a palindrome between 1000 and 2000 begins and ends with a 1, there are 10 numbers, all of the form $1dd1$ to check. Since 3 divides 1221, 1551, and 1881, 7 divides 1001 and 1771, and $1331 = 11^3$, these six choices can be eliminated. We then note that the remaining four numbers, $1111 = 11 \cdot 101$ $1441 = 11 \cdot 131$, $1661 = 11 \cdot 151$ and $1991 = 11 \cdot 181$ all have both required properties.

OR

First note that 11 is the only two-digit prime palindrome. Since $11 \cdot 190 > 2000$, the three-digit palindrome must be less than 190. The only three-digit prime palindromes in this range are 101, 131, 151 and 181. Thus, $1111 = 11 \cdot 101$, $1441 = 11 \cdot 131$, $1661 = 11 \cdot 151$ and $1991 = 11 \cdot 181$ are the only four numbers with the two required properties.

18. **(D)** The set S consists of all complex numbers of the form

$$z = \frac{r}{3+4i} = \frac{r}{25}(3-4i)$$

for some real number r. Since S consists of all real multiples of $3-4i$, each point in S is on the line through the origin and $3-4i$, and conversely.

OR

Let $z = x+iy$. Then $(3+4i)z = 3x-4y+(3y+4x)i$, which is real if and only if $3y+4x=0$, the equation of a line.

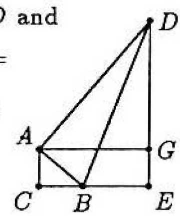
19. **(B)** Since $AB = \sqrt{3^2+4^2} = 5$ and $BD = \sqrt{5^2+12^2} = 13$, it follows that

$$\begin{aligned}\frac{m}{n} = \frac{DE}{DB} &= \sin\angle DBE = \sin\angle DBC = \sin(\angle DBA + \angle ABC)\\ &= \sin\angle DBA\cos\angle ABC + \cos\angle DBA\sin\angle ABC\\ &= \frac{12}{13}\cdot\frac{4}{5} + \frac{5}{13}\cdot\frac{3}{5} = \frac{63}{65}\end{aligned}$$

and $m+n=128$.

OR

Draw $\overline{AG}$ parallel to $\overline{CE}$ with G on $\overline{DE}$. Then $\angle GAD = \angle CAB$ since both are complementary to $\angle GAB$. Thus, triangles GAD and CAB are similar, and $GD = \frac{AD}{AB}CB = \frac{48}{5}$. Hence $DE = \frac{63}{5}$. Apply the Pythagorean theorem to triangles ABC and DAB to find $DB=13$. Therefore, $\frac{DE}{DB} = \frac{63/5}{13} = \frac{63}{65}$, so $m+n=128$.



20. **(E)** If $a = 2^x-4$ and $b = 4^x-2$ then $a+b = 4^x+2^x-6$. Since $(a+b)^3 = a^3+3a^2b+3ab^2+b^3$, a^3+b^3 will equal $(a+b)^3$ if and only if $3a^2b+3ab^2=0$. Therefore,

$$a^3+b^3=(a+b)^3 \iff 0 = 3ab(a+b) \iff a=0,\ b=0,\ \text{or } a+b=0.$$

Thus $2^x-4=0$ or $4^x-2=0$ or $4^x+2^x-6=(2^x+3)(2^x-2)=0$. Since $2^x+3=0$ has no real roots, the sum of the real roots is $2+\frac{1}{2}+1=\frac{7}{2}$.

21. **(A)** If $\frac{x}{x-1} = \sec^2\theta$ then $x = x\sec^2\theta - \sec^2\theta$, or $\sec^2\theta = x(\sec^2\theta-1) = x\tan^2\theta$. Hence $x = \frac{1}{\sin^2\theta}$ and $f(\sec^2\theta) = \sin^2\theta$.

OR

First solve $y = \dfrac{x}{x-1}$ for x to find $x = \dfrac{y}{y-1}$. Then $f(y) = \dfrac{y-1}{y}$. Hence

$$f(\sec^2\theta) = \frac{\sec^2\theta - 1}{\sec^2\theta} = 1 - \cos^2\theta = \sin^2\theta.$$

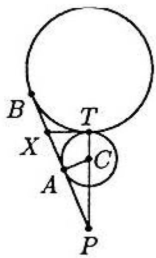
OR

Since $f\left(\dfrac{1}{1-\frac{1}{x}}\right) = f\left(\dfrac{x}{x-1}\right) = \dfrac{1}{x}$, $\qquad (*)$

$$f\left(\sec^2\theta\right) = f\left(\frac{1}{\cos^2\theta}\right) = f\left(\frac{1}{1-\sin^2\theta}\right) = \sin^2\theta$$

where for the last equality we substituted $\sin^2\theta$ for $\dfrac{1}{x}$ in $(*)$.

22. (B) Let C be the center of the smaller circle, T be the point where the two circles are tangent, and X be the intersection of the common internal tangent with $\overline{AB}$. Since tangents from a common point are equal, $BX = TX = AX = \dfrac{AB}{2} = 2$. Since $\triangle ACP \sim \triangle TXP$, it follows that

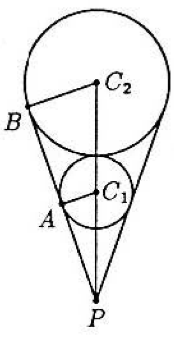


$$\frac{AC}{TX} = \frac{AP}{TP}, \quad \text{or} \quad \frac{AC}{2} = \frac{4}{\sqrt{6^2 - 2^2}}, \quad \text{so} \quad AC = \sqrt{2}.$$

Hence the area of the circle with radius AC is 2π.

OR

Let C_1, C_2, r and R be the centers and radii of the smaller and larger circle, respectively. Points P, C_1 and C_2 are collinear by symmetry. Since the right triangles PAC_1 and PBC_2 are similar,



$$\frac{r}{R} = \frac{PC_1}{PC_2} = \frac{PA}{PB} = \frac{4}{8}.$$

Thus $R = 2r$ and $PC_1 = C_1C_2 = R + r = 3r$. Apply the Pythagorean theorem to $\triangle PAC_1$ to find $4^2 + r^2 = (3r)^2$, $r^2 = 2$ and $\pi r^2 = 2\pi$.

23. **(C)** Use coordinates with $B = (0,0)$, $F = (1,0)$ and $E = (0,1)$. The equations of lines $\overline{BH}$, $\overline{IH}$ and $\overline{EI}$ are $y = x$, $y = -2x + 2$ and $y = \frac{1}{2}x + 1$, respectively. Thus $H = \left(\frac{2}{3}, \frac{2}{3}\right)$ and $I = \left(\frac{2}{5}, \frac{6}{5}\right)$. Hence the altitude of $\triangle BHF$ from vertex H is $2/3$, and thus $[BHF] = 1/3$.† Similarly, the altitude of $\triangle AIE$ from vertex I is $2/5$, so $[AIE] = 1/5$. Therefore

$$[BEIH] = [BAF] - [AIE] - [BHF] = 1 - \frac{1}{5} - \frac{1}{3} = \frac{7}{15}.$$

OR

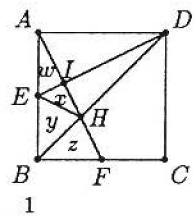
Triangles DAE and ABF have equal sides so they are congruent. Thus $\angle EAI + \angle AEI = \angle EAI + \angle BFA = 90°$, and $\triangle AIE$ is a right triangle similar to $\triangle ABF$. Since $[ABF] = 1$ and $AF = \sqrt{1^2 + 2^2} = \sqrt{5}$, we have

$$[AIE] = \frac{[AIE]}{[ABF]} = \frac{AE^2}{AF^2} = \frac{1}{5}.$$

Note that $\triangle BHF$ is similar to $\triangle DHA$ because of equal angles, and that the ratio of similarity is $\dfrac{BF}{DA} = \dfrac{1}{2}$. Hence $\dfrac{HF}{HA} = \dfrac{1}{2}$ and $\dfrac{HF}{AF} = \dfrac{1}{3}$. Thus, since $\triangle BHF$ and $\triangle BAF$ share side $\overline{BF}$ and the altitudes to that side are in the ratio 1:3, $[BHF] = \dfrac{[BHF]}{[BAF]} = \dfrac{1}{3}$.

Hence $[BEIH] = [BAF] - [AIE] - [BHF] = 1 - \dfrac{1}{5} - \dfrac{1}{3} = \dfrac{7}{15}$.

OR



Let the areas of triangles AEI, EHI, BHE and BHF be w, x, y and z, respectively. Since $BE = BF$ and $\angle EBH = \angle FBH$, triangles BHE and BHF are congruent. Hence, $y = z$.

Since $\overline{EH}$ is a median of $\triangle AHB$, we have $w + x = y$. Hence

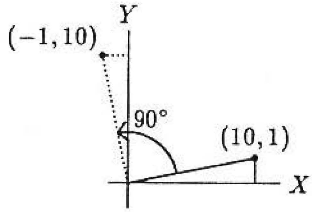
$$3y = (w + x) + y + z = \frac{1}{2} BF \cdot AB = 1, \text{ or } y = \frac{1}{3}.$$

If $\triangle ABF$ is rotated 90° clockwise about the center of the square, it coincides with $\triangle DAE$. Hence $\overline{AF} \perp \overline{DE}$, from which it follows that $\triangle AIE \sim \triangle DAE$. Hence $\dfrac{AI}{DA} = \dfrac{IE}{AE} = \dfrac{AE}{DE} = \dfrac{1}{\sqrt{5}}$, and $w = \left(\dfrac{1}{\sqrt{5}}\right)^2 [DAE] = \dfrac{1}{5}$. Since $w + x = y$, we have $[BEIH] = x + y = 2y - w = \dfrac{2}{3} - \dfrac{1}{5} = \dfrac{7}{15}$.

---

† **Notation:** $[P_1P_2 \cdots P_n]$ is the area of polygon $P_1P_2 \cdots P_n$.

24. **(D)** The point (x,y) is on the graph of G' if and only if the point $(y,-x)$ is on the graph of G, so $-x = \log_{10} y$. This last equation is equivalent to $y = 10^{-x}$, which is an equation for G'. Since $(x,y) = (10,1)$ is on G, it follows that $(x,y) = (-1,10)$ must be on G', which shows that no other choice is correct.



Note. The 90° rotation relates each (x,y) on G' to the point $(x\cos 90° + y\sin 90°, -x\sin 90° + y\cos 90°)$ on G.

25. **(D)** Since $T_n = \dfrac{(n+1)n}{2}$, $P_n = \dfrac{\frac{n(n+1)}{2}}{\frac{n(n+1)-2}{2}}P_{n-1} = \dfrac{(n+1)n}{(n+2)(n-1)}P_{n-1}$.

Therefore

$$\begin{aligned} P_{1991} &= \frac{1992\cdot 1991}{1993\cdot 1990}\cdot\left(\frac{1991\cdot 1990}{1992\cdot 1989}P_{1989}\right) = \frac{1991}{1993}\cdot\frac{1991}{1989}P_{1989} \\ &= \frac{1991}{1993}\cdot\frac{1991}{1989}\cdot\left(\frac{1990\cdot 1989}{1991\cdot 1988}P_{1988}\right) = \frac{1991}{1993}\cdot\frac{1990}{1988}P_{1988} \\ &= \cdots = \frac{1991}{1993}\cdot\frac{k+2}{k}P_k = \cdots = \frac{1991}{1993}\cdot\frac{4}{2}P_2 = \frac{1991}{1993}\cdot 3, \end{aligned}$$

so 2.9 is closest to P_{1991}.

OR

Note that

$$\begin{aligned} P_n = \prod_{k=2}^{n}\frac{k(k+1)}{(k+2)(k-1)} &= \frac{\left(\prod_{k=2}^{n}k\right)\left(\prod_{k=2}^{n}(k+1)\right)}{\left(\prod_{k=2}^{n}(k+2)\right)\left(\prod_{k=2}^{n}(k-1)\right)} \\ &= \frac{n!\left(\frac{(n+1)!}{2}\right)}{\left(\frac{(n+2)!}{2\cdot 3}\right)(n-1)!} = \frac{3n}{n+2}, \end{aligned}$$

so $P_{1991} = \dfrac{3\cdot 1991}{1993}$ which is closest to 2.9.†

---

† **Notation:** $\prod_{k=m}^{n} a_k = a_m \cdot a_{m+1}\cdot a_{m+2}\cdot\ \cdots\ \cdot a_n$.

26. (C) The fifth digit must be 5, and the second, fourth and sixth digits must be even. Since the first and third digits must be 1 and 3 in some order and $1+2+3=6$ but $1+4+3=8$ and $1+6+3=10$, the second digit must be 2. Neither 1234 nor 3214 is divisible by 4, so the fourth digit must be 6 and the sixth digit must be 4. Therefore there are two cute 6-digit integers, 123654 and 321654.

27. (C) Clear the denominator in the first equation to obtain

$$\left(x^2-(x^2-1)\right)+1=20\left(x-\sqrt{x^2-1}\right) \quad \text{or} \quad x-\sqrt{x^2-1}=\frac{1}{10}.$$

Thus $\dfrac{1}{x-\sqrt{x^2-1}}=x+\sqrt{x^2-1}=10$, so that $2x=\dfrac{1}{10}+10=10.1$. Rationalize the denominator in the problem's second expression to obtain

$$\begin{aligned} x^2+\sqrt{x^4-1}+\frac{1}{x^2+\sqrt{x^4-1}} &= x^2+\sqrt{x^4-1}+\left(x^2-\sqrt{x^4-1}\right) \\ &= 2x^2=\frac{1}{2}\cdot(10.1)^2=51.005. \end{aligned}$$

28. (B) Since the number of white marbles is either unchanged or decreases by 2 after each replacement, the number of white marbles remains even. Since every set removed that includes at least one white marble is replaced by a set containing at least one white marble, the number of white marbles can never be zero. Note that (B) is the only choice including at least two white marbles. We can attain this result in many ways. One way is to remove 3 white marbles 49 times to arrive at 149 black and 2 white marbles, and then remove 1 black and 2 white marbles 149 times.

29. (B) Since $\angle BA'P + \angle A'PB + 60^\circ = 180^\circ = \angle BA'P + 60^\circ + \angle QA'C$, it follows that $\angle A'PB = \angle QA'C$ and thus $\triangle A'PB \sim \triangle QA'C$. Let $x = AP = A'P$ and $y = QA = QA'$. Then

$$\frac{A'P}{QA'} = \frac{A'B}{QC} = \frac{PB}{A'C}, \quad \text{or} \quad \frac{x}{y} = \frac{1}{3-y} = \frac{3-x}{2}.$$

Solve to obtain $x = \frac{7}{5}$ and $y = \frac{7}{4}$. Now apply the Law of Cosines to $\triangle PAQ$,

$$PQ^2 = x^2 + y^2 - 2xy\cos 60^\circ = \frac{49}{25} + \frac{49}{16} - \frac{49}{20} = \frac{49 \cdot 21}{400},$$

which leads to $PQ = \frac{7}{20}\sqrt{21}$.

OR

Let $x = PA = PA'$ and $y = QA = QA'$. Apply the Law of Cosines to $\triangle PBA'$ to obtain $x^2 = (3-x)^2 + 1 - 2(3-x)\cos 60^\circ$ which leads to $x = 7/5$. Consider $\triangle QCA'$ in a similar fashion to find $y = 7/4$. Then complete the solution as above.

30. (B) If a set has k elements, then it has 2^k subsets. Thus we are given

$$2^{100} + 2^{100} + 2^{|C|} = 2^{|A\cup B\cup C|}, \quad \text{or} \quad 1 + 2^{|C|-101} = 2^{|A\cup B\cup C|-101}$$

Since $1 + 2^{|C|-101}$ is larger than 1 and equal to an integral power of 2, $|C| = 101$. Thus $|A\cup B\cup C| = 102$. The inclusion-exclusion formula for three sets is

$$|A\cup B\cup C| = |A| + |B| + |C| - |A\cap B| - |A\cap C| - |B\cap C| + |A\cap B\cap C|.$$

Fill in known quantities, use $|X\cap Y| = |X| + |Y| - |X\cup Y|$ (obtained from the inclusion-exclusion formula for any two sets X and Y), and then again fill in known quantities to find

$$\begin{aligned} |A\cap B\cap C| &= |A\cap B| + |A\cap C| + |B\cap C| - 199 \\ &= \big(|A| + |B| - |A\cup B|\big) + \big(|A| + |C| - |A\cup C|\big) + \\ &\qquad \big(|B| + |C| - |B\cup C|\big) - 199 \\ &= 403 - |A\cup B| - |A\cup C| - |B\cup C|. \end{aligned}$$

Since $A\cup B, A\cup C, B\cup C \subseteq A\cup B\cup C$, we have $|A\cup B|, |A\cup C|, |B\cup C| \le 102$. Thus

$$|A\cap B\cap C| = 403 - |A\cup B| - |A\cup C| - |B\cup C| \ge 403 - 3\cdot 102 = 97.$$

The example

$A = \{1,2,\ldots,100\}$, $B = \{3,4,\ldots,102\}$, $C = \{1,2,4,5,6,\ldots,101,102\}$
shows that $|A\cap B\cap C| = |\{4,5,6,\ldots,100\}| = 97$ *is* possible.

AMERICAN MATHEMATICS COMPETITIONS

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

43nd ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)

THURSDAY, FEBRUARY 27, 1992

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1. **(B)** $6^6 + 6^6 + 6^6 + 6^6 + 6^6 + 6^6 = 6 \cdot 6^6 = 6^1 \cdot 6^6 = 6^{1+6} = 6^7$.

2. **(B)** $6(8x + 10\pi) = [2{\cdot}3][2(4x + 5\pi)] = 2{\cdot}2[3(4x + 5\pi)] = 4P$.

OR

Since $P = 12x + 15\pi$, we have $6(8x + 10\pi) = 48x + 60\pi = 4(12x + 15\pi) = 4P$.

3. **(B)** Of the 80% which are coins, 60% are gold. This is 60% of 80%, or 48% of the objects in the urn.

4. **(C)** Since the slope of the line through points (x_1, y_1) and (x_2, y_2) is $\dfrac{y_2 - y_1}{x_2 - x_1}$, it follows that $m = \dfrac{m-3}{1-m}$. Therefore, $m - m^2 = m - 3$. Since $m > 0$, it follows that $m = \sqrt{3}$.

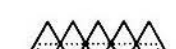
5. **(A)** The number 3^a is odd, and $b-1$ is even. Therefore $(b-1)^2$ is even and so is $(b-1)^2c$. Hence the sum $3^a + (b-1)^2c$ is odd for all choices of c. Note that $3^a + (b-1)^2c$ is odd whether a is even or odd.

6. **(D)** $\dfrac{x^y y^x}{y^y x^x} = \dfrac{x^{y-x}}{y^{y-x}} = \left(\dfrac{x}{y}\right)^{y-x}$. One may verify that none of the other choices is correct by substituting $x = 2$ and $y = 1$.

7. **(B)** $\dfrac{w}{y} = \dfrac{w}{x} \cdot \dfrac{x}{z} \cdot \dfrac{z}{y} = \dfrac{4}{3} \cdot \dfrac{6}{1} \cdot \dfrac{2}{3} = \dfrac{16}{3}$.

8. **(E)** Since the center tile is on both diagonals, it follows that there are 51 black tiles on each diagonal. Thus, there are 51 tiles on each side for a total of $51^2 = 2601$ tiles on the floor.

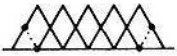
9. **(E)** Partition the figure into 16 equilateral triangles as shown. Since each side of each of these 16 triangles has length $s = \dfrac{1}{2}\left(2\sqrt{3}\right)$, the total area is $16\left(s^2\dfrac{\sqrt{3}}{4}\right) = 12\sqrt{3}$.



OR

In each of the two equilateral triangles at the ends, insert the segment connecting the midpoints of the outer side and the side on the line, as indicated. Observe that the area covered is enclosed by 5 congruent rhombi and 6 congruent equilateral triangles. The rhombi have diagonals with lengths 3 and $\sqrt{3}$, while the triangles have sides $\sqrt{3}$. Hence the area is

$$5\left(\frac{3\sqrt{3}}{2}\right)+6\left(3\cdot\frac{\sqrt{3}}{4}\right)=12\sqrt{3}.$$

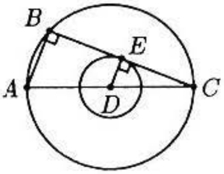


Note. There are a number of other ways to compute the area. For example: (1) compute the area of the five large triangles and subtract the area of the four small triangles of overlap; *or* (2) compute the area of the smallest trapezoid which contains the figure and subtract the area of the four small missing triangles.

10. **(D)** Since $kx = 12 + 3k$, it follows that $x = \dfrac{12}{k} + 3$. Thus the equation has an integer solution if and only if k is a factor of 12. Since $k > 0$, it follows that k is one of the six values 1, 2, 3, 4, 6 or 12.

11. **(B)** Draw the radius from the center D of the circles to the point E where $\overline{BC}$ is tangent to the smaller circle. Since $\overline{DE} \perp \overline{BC}$, DEC and ABC are similar right triangles, so

$$\frac{DE}{AB}=\frac{CD}{CA}=\frac{1}{2}.$$



Thus the radius, DE, of the smaller circle is 6 since $AB = 12$. Hence the radius of the larger circle is 18 since the ratio of the radii is $1:3$.

12. **(C)** The equation of the given line can be written as $y = \frac{1}{3}x + \frac{11}{3}$. The y-intercept, b, of the image is $-\frac{11}{3}$ and the slope, m, is $-\frac{1}{3}$. Thus $m + b = -4$.

OR

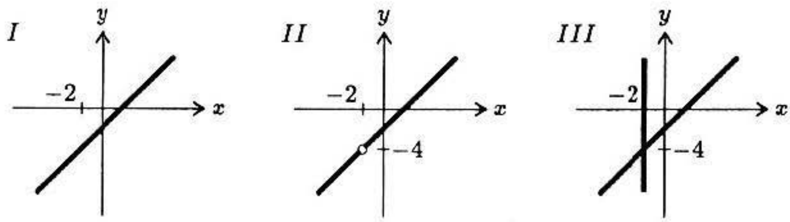
If the point (x, y) is on the reflection of the given line, then the point $(x, -y)$ is on the given line. Hence $x - 3(-y) + 11 = 0$, so $x + 3y = -11$ is an equation for the reflected line. The equation of this line can be written $y = -\frac{1}{3}x - \frac{11}{3}$, so $m + b = -4$.

13. **(C)** Multiply the numerator and denominator of the given fraction by ab to obtain

$$\frac{a^2b+a}{b+ab^2}=\frac{a(ab+1)}{b(1+ab)}=\frac{a}{b}=13.$$

Thus $a=13b$; and $a+b\le 100$ implies $14b\le 100$, so $0<b\le 7$. For each of the seven possible values of $b=1,2,3,4,5,6,7$, the pair $(13b,b)$ is a solution.

14. **(E)** The graph of $y=x-2$ is a line. The graph of $y=\dfrac{x^2-4}{x+2}$ is almost a line, but there is a point at $x=-2$ missing. The graph of $(x+2)y=x^2-4$ is a pair of intersecting lines, $x=-2$ and $y=x-2$.



15. **(B)** We compute $z_1=0$, $z_2=i$, $z_3=i-1$, $z_4=-i$, and $z_5=i-1$. Since $z_5=z_3$, it follows that $z_{111}=z_{109}=z_{107}=\cdots=z_5=z_3=i-1$, which is $\sqrt{2}$ units from the origin.

Note. The *Mandelbrot set* is defined to be the set of complex numbers c for which all the terms of the sequence defined by $z_1=0$, $z_{n+1}=z_n^2+c$ for $n\ge 1$, stay close to the origin. Thus $c=i$ is in the Mandelbrot set.

16. **(E)** A property of proportions is: *If* $\dfrac{a}{b}=\dfrac{c}{d}=\dfrac{e}{f}=k$, *then* $\dfrac{a+c+e}{b+d+f}=k$. Use this to obtain $\dfrac{x}{y}=\dfrac{y+(x+y)+x}{(x-z)+z+y}=\dfrac{2x+2y}{x+y}=2$.

Note. In general, $x=4t$, $y=2t$ and $z=3t$ for $t>0$.

OR

Simplify $\dfrac{y}{x-z}=\dfrac{x}{y}$ and $\dfrac{x+y}{z}=\dfrac{x}{y}$ to obtain $y^2=x^2-xz$ and $xy+y^2=xz$, respectively. Substitute for xz to find that $y^2=x^2-(xy+y^2)$, or $0=x^2-xy-2y^2=(x-2y)(x+y)$. Thus $\dfrac{x}{y}=2$ or $\dfrac{x}{y}=-1$. Since $x,y>0$, it follows that $\dfrac{x}{y}=2$.

17. **(B)** Since $0+1+2+\cdots+9=45$ and

$$N = 19\,\underbrace{20\,21\,\cdots\,29}_{10\cdot 2+45}\,\underbrace{30\,31\,\cdots\,39}_{10\cdot 3+45}\cdots\underbrace{80\,81\,\cdots\,89}_{10\cdot 8+45}\,90\,91\,92,$$

the sum of the digits of N is

$$\begin{aligned} S &= (1+9)+(10\cdot 2+45)+(10\cdot 3+45)+\cdots+(10\cdot 8+45)+(3\cdot 9+3) \\ &= 36\cdot 10+7\cdot 45+27+3 = 9(40+35+3)+3. \end{aligned}$$

Thus S has a factor of 3 but not 9, so the highest power of 3 which is a divisor of N is 3^1 and $k=1$.

Note. One could also compute $S=705$ and discover that it is divisible by 3 and not by 9.

OR

Note that 3 [or 9] will divide N if and only if it divides the sum of 19, 20, ..., 92. (*Why?*) Since

$$19+20+\cdots+92 = 74\cdot\frac{19+92}{2} = 37\cdot 111 = 37^2\cdot 3,$$

it follows that $k=1$.

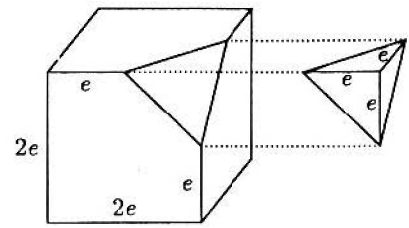
18. **(D)** If $a_1=a$ and $a_2=b$ then

$$(a_3,\ a_4,\ a_5,\ a_6,\ a_7,\ a_8) = (a+b,\ a+2b,\ 2a+3b,\ 3a+5b,\ 5a+8b,\ 8a+13b).$$

Therefore $5a+8b=a_7=120$. Since $5a=8(15-b)$ and 8 is relatively prime to 5, a must be a multiple of 8. Similarly, b must be a multiple of 5. Let $a=8j$ and $b=5k$ to obtain $40j+40k=120$, which has two solutions in positive integers, $(j,k)=(1,2)$ and $(2,1)$. Since the sequence is increasing, $(j,k)=(1,2)$. Thus $a=8\cdot 1=8$ and $b=5\cdot 2=10$, so $a_8=8a+13b=194$.

19. **(D)** Let each edge of the cube be $2e$. Then the volume of the cube is $8e^3$. Each of the 8 tetrahedra removed has an isosceles right triangle of area $e^2/2$ as a base and an altitude to this base of length e. Hence the volume of the cuboctahedron is

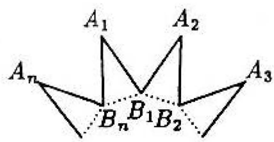
$$8e^3 - 8\left(\frac{1}{3}\cdot e\cdot\frac{e^2}{2}\right) = \frac{20e^3}{3}.$$



The ratio of the volume of the cuboctahedron to the volume of the cube is

$$\frac{20e^3/3}{8e^3} = \frac{5}{6} = 83\frac{1}{3}\%.$$

20. **(D)** Partition the n-pointed regular star into the regular n-gon $B_1B_2\cdots B_n$ and n triangles congruent to $\triangle B_1A_2B_2$, and note that the sum of the star's interior angles is



$$(n-2)180^\circ + n180^\circ = (2n-2)180^\circ.$$

Since the interior angles of the star consist of n angles congruent to A_1 and n angles congruent to $360^\circ - B_1$,

$$(2n-2)180^\circ = n\angle A_1 + n(360^\circ - \angle B_1), \quad \text{or} \quad n(\angle B_1 - \angle A_1) = 2\cdot 180^\circ.$$

Since $\angle B_1 - \angle A_1 = 10^\circ$, $n = 36$.

Note. In general, the sum of the interior angles of <u>any</u> N-sided simple closed polygon, convex or not, is $(N-2)180^\circ$.

21. **(A)** Since $\dfrac{S_1+S_2+\cdots+S_{99}}{99} = 1000$, $S_1+S_2+\cdots+S_{99} = 99000$. Thus

$$\frac{1+(1+S_1)+(1+S_2)+\cdots+(1+S_{99})}{100} = \frac{100+(S_1+S_2+\cdots+S_{99})}{100} = \frac{100+99000}{100} = 991.$$

OR

Let C be the Cesàro sum of the n-term sequence $(b_1, b_2, \ldots, b_n)$. Then

$$b_1 + (b_1 + b_2) + (b_1 + b_2 + b_3) + \cdots + (b_1 + b_2 + \cdots + b_n) = nC.$$

The Cesàro sum of the $(n+1)$-term sequence $k, b_1, b_2, \ldots, b_n$ is then

$$\begin{aligned}
&\frac{k + (k + b_1) + (k + b_1 + b_2) + \cdots + (k + b_1 + b_2 + \cdots + b_n)}{n+1} \\
&\qquad = \frac{(n+1)k + (b_1 + (b_1 + b_2) + \cdots + (b_1 + b_2 + \cdots + b_n))}{n+1} \\
&\qquad = \frac{(n+1)k + nC}{n+1} = k + \frac{n}{n+1}C.
\end{aligned}$$

In the problem, we have $n = 99$, $C = 1000$, and $k = 1$, which gives a Cesàro sum of 991.

Note. Cesàro sums are named after the nineteenth century mathematician E. Cesàro. Cesàro sums arise in many areas of mathematics such as in the study of Fourier Series.

22. **(B)** A point of intersection in the first quadrant is obtained whenever two of the segments cross to form an $\times$. An $\times$ is uniquely determined by selecting two of the points on $\mathbf{X}^+$ and two of the points on $\mathbf{Y}^+$. The maximum number of these intersections is obtained by selecting the points on $\mathbf{X}^+$ and $\mathbf{Y}^+$ so that no three of the 50 segments intersect in the same point. Therefore, the maximum number of intersections is $\binom{10}{2}\binom{5}{2} = 45 \cdot 10 = 450$.

OR

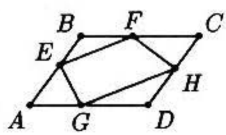
Choose ten points, $x_1 < x_2 < \ldots < x_{10}$, on $\mathbf{X}^+$. Choose y_1 on $\mathbf{Y}^+$ and draw the ten segments joining y_1 to the ten points on $\mathbf{X}^+$. Choose $y_2 > y_1$ on $\mathbf{Y}^+$, and note that, as the segments $\overline{y_2x_1}$, $\overline{y_2x_2}$, $\ldots$, $\overline{y_2x_{10}}$ are drawn, $9 + 8 + 7 + \cdots + 0 = 45$ intersections are formed. Choose $y_3 > y_2$ on $\mathbf{Y}^+$ so no segment $\overline{y_3x_i}$ goes through a previously counted intersection, and note that $2(9 + 8 + 7 + \cdots + 0) = 2 \cdot 45$ new intersections are formed. Similarly, for judiciously chosen y_4 and y_5 on $\mathbf{Y}^+$ one can generate at most $3 \cdot 45$ and $4 \cdot 45$ new intersections, respectively. Hence, the maximum number of intersections is $(1 + 2 + 3 + 4)45 = 450$ intersections.

23. **(E)** Partition $F = \{1,2,3,\ldots,50\}$ into seven subsets, F_0, F_1, ..., F_6, so that all the elements of F_i leave a remainder of i when divided by 7:

$$\begin{aligned}
F_0 &= \{7,14,21,28,35,42,49\},\\
F_1 &= \{1,8,15,22,29,36,43,50\},\\
F_2 &= \{2,9,16,23,30,37,44\},\\
F_3 &= \{3,10,17,24,31,38,45\},\\
F_4 &= \{4,11,18,25,32,39,46\},\\
F_5 &= \{5,12,19,26,33,40,47\},\\
F_6 &= \{6,13,20,27,34,41,48\}.
\end{aligned}$$

Note that S can contain at most one member of F_0, but that if S contains some member of any of the other subsets, then it can contain all of the members of that subset. Also, S cannot contain members of both F_1 and F_6, or both F_2 and F_5, or both F_3 and F_4. Since F_1 contains 8 members and each of the other subsets contains 7 members, the largest subset, S, can be constructed by selecting one member of F_0, all the members of F_1, all the members of either F_2 or F_5, and all of the members of either F_3 or F_4. Thus the largest subset, S, contains $1+8+7+7 = 23$ elements.

24. **(C)** The area of polygon $P_1P_2\cdots P_n$ will be denoted by $[P_1P_2\cdots P_n]$. Since $[ABCD] = 10$ and $AD = BC = 5$, the distance between lines $\overline{AD}$ and $\overline{BC}$ is $10/5 = 2$. Thus, if the altitude of $\triangle AEG$ from E is h, then the altitude of $\triangle BEF$ from E is $2-h$. Since $AG = BF = 2$,



$$[AEG] + [BEF] = \frac{1}{2}AG\cdot h + \frac{1}{2}BF\cdot(2-h) = h + (2-h) = 2.$$

Similarly, since the sum of the altitudes of triangles CFH and DGH from H is 2 and $CF = DG = 3$, it follows that $[CFH] + [DGH] = 3$. Hence

$$[EFHG]$$
$$= [ABCD] - ([AEG] + [BEF] + [CFH] + [DGH]) = 10 - (2+3) = 5.$$

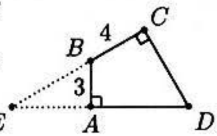
OR

Note that $ABFG$ is a parallelogram. Hence $[EFG] = \frac{1}{2}[ABFG]$, and similarly, $[HFG] = \frac{1}{2}[CDGF]$. Consequently, $[EFHG] = \frac{1}{2}[ABCD] = 5$.

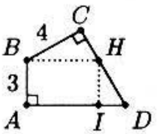
Note. Not only is the choice of E and H completely arbitrary on their respective segments, but F and G can be chosen as any two points on $\overline{BC}$ and $\overline{AD}$ as long as $BF = AG$.

Challenge. As the solution indicates, if a diagonal of a quadrilateral, Q, inscribed in a parallelogram, P, is parallel to a side of P, then the area of Q is half the area of P. Can you prove the converse?

25. **(E)** Extend $\overline{CB}$ and $\overline{DA}$ to meet at E. Since $\angle E = 30°$, $EB = 6$. Hence $EC = 10$ and $CD = \dfrac{10}{\sqrt{3}}$. In general, if $AB = x$ and $BC = y$ then $EB = 2x$ and $EC = 2x + y$, so $CD = \dfrac{2x+y}{\sqrt{3}}$.



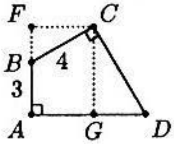
OR



Draw a line through B parallel to $\overline{AD}$ intersecting CD at H. Then drop a perpendicular from H to I on $\overline{AD}$. Note that BHC and HDI are both 30°-60°-90° triangles. Thus $CH = \dfrac{4}{\sqrt{3}}$. Since $HI = AB = 3$, it follows that $HD = \dfrac{6}{\sqrt{3}}$. Hence $CD = CH + HD = \dfrac{4}{\sqrt{3}} + \dfrac{6}{\sqrt{3}} = \dfrac{10}{\sqrt{3}}$.

OR

Draw a line through C parallel to $\overline{AD}$ and meeting $\overline{AB}$ extended at F. Then $\angle BCF = 30°$, so $BF = 2$. Drop a perpendicular from C to G on $\overline{AD}$. Since $AFCG$ is a rectangle, $CG = AF = AB + BF = 5$. Since $\angle CDG = 60°$, we have $CD = \dfrac{2}{\sqrt{3}}CG = \dfrac{10}{\sqrt{3}}$.



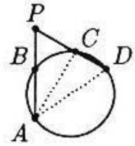
Note. One can also draw $\overline{AC}$, apply the *Law of Cosines* to both $\triangle ABC$ and $\triangle ADC$, and then equate the resulting values of AC to find CD.

26. **(B)** Since $\overline{CD} \perp \overline{AB}$, $AC = CB$ and $\angle ADB$ is inscribed in a semicircle, it follows that $\triangle ABD$ is an isosceles right triangle, $\angle BAD = \angle ABD = 45°$ and $BD = \sqrt{2}$. Note that $\angle EDF = \angle ADB = 90°$ and $DE = BE - BD = 2 - \sqrt{2}$. The area of the "smile" is

area(sector EDF) + area(sector ABE) + area(sector BAF)
− area($\triangle ABD$) − area(semicircle ADB)

$$\begin{aligned}
&= \frac{90}{360}\pi\left(2-\sqrt{2}\right)^2 + \frac{45}{360}\pi 2^2 + \frac{45}{360}\pi 2^2 - \frac{1}{2}\cdot 2\cdot 1 - \frac{1}{2}\pi 1^2 \\
&= \left(\frac{3}{2} - \sqrt{2}\right)\pi + \frac{\pi}{2} + \frac{\pi}{2} - 1 - \frac{\pi}{2} \\
&= 2\pi - \pi\sqrt{2} - 1.
\end{aligned}$$

27. **(D)** Using properties of secant segments (*power of a point*), it follows that $PD \cdot PC = PA \cdot PB = 18 \cdot 8$. But $PD \cdot PC = (PC + 7) \cdot PC$, so $PD = 16$ and $PC = 9$. Since $AP = 2PC$ and $\angle APC = 60°$, $\angle ACP = 90°$. (*Why?*) It follows that $AC = 9\sqrt{3}$ and $\angle ACD$ is also a right angle. Thus $\overline{AD}$ is a diameter of the circle. Apply the Pythagorean Theorem to triangle ACD to obtain



$$(2r)^2 = AD^2 = AC^2 + CD^2 = 3 \cdot 9^2 + 7^2 = 292,$$

from which it follows that $r^2 = 73$.

28. **(B)** The quadratic formula leads to the roots

$$z = \frac{1}{2}\left(1 \pm \sqrt{21 - 20i}\right).$$

To find $\sqrt{21 - 20i}$, let $(a + bi)^2 = 21 - 20i$ where a and b are real. Equating real and imaginary parts leads to $a^2 - b^2 = 21$ and $2ab = -20$. Solve these equations simultaneously:

$$\begin{aligned} a^2(a^2 - 21) = a^2b^2 &= 100; \\ (a^2 - 25)(a^2 + 4) &= 0; \\ a^2 - 25 &= 0; \qquad a = \pm 5,\ b = \mp 2. \end{aligned}$$

Thus $a + bi = 5 - 2i$ or $-5 + 2i$. Therefore

$$z = \frac{1}{2}\left[1 \pm (5 - 2i)\right] = 3 - i \text{ or } -2 + i.$$

The product of the real parts of these two roots is -6.

Note. One could also use the equation $a^2 - b^2 = 21$ together with $a^2 + b^2 = |a + bi|^2 = |21 - 20i| = 29$ and solve simultaneously to obtain $2a^2 = 50$, from which it follows that $a = \pm 5$.

29. **(D)** Let p be the probability that the total number of heads is even, and let q be the probability that the total number of heads is odd. Since the probability of tossing k heads and $(50-k)$ tails is $\binom{50}{k}\left(\frac{2}{3}\right)^{k}\left(\frac{1}{3}\right)^{50-k}$, we have

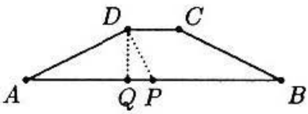
$$p=\binom{50}{0}\left(\frac{2}{3}\right)^{0}\left(\frac{1}{3}\right)^{50}+\binom{50}{2}\left(\frac{2}{3}\right)^{2}\left(\frac{1}{3}\right)^{48}+\cdots+\binom{50}{50}\left(\frac{2}{3}\right)^{50}\left(\frac{1}{3}\right)^{0}$$

and

$$q=\binom{50}{1}\left(\frac{2}{3}\right)^{1}\left(\frac{1}{3}\right)^{49}+\binom{50}{3}\left(\frac{2}{3}\right)^{3}\left(\frac{1}{3}\right)^{47}+\cdots+\binom{50}{49}\left(\frac{2}{3}\right)^{49}\left(\frac{1}{3}\right)^{1}.$$

Note that $p-q=\left(\frac{2}{3}-\frac{1}{3}\right)^{50}=\frac{1}{3^{50}}$. Since $p+q=1$, we solve for p to get $p=\frac{1}{2}\left(1+\frac{1}{3^{50}}\right)$.

30. **(B)** Since $ABCD$ is isosceles, the center of the circle, P, must be the midpoint of $\overline{AB}$. When $x=m$, the circle must be tangent to $\overline{AD}$ at D and to $\overline{BC}$ at C. (*Why?*) Let Q be the foot of the perpendicular from D to $\overline{AB}$. Then $\triangle ADP$ is a right triangle with hypotenuse $\overline{AP}$, and $\overline{DQ}$ is its altitude to the hypotenuse. Since $\triangle ADQ \sim \triangle APD$,



$$\frac{AD}{AP}=\frac{AQ}{AD},\quad \text{so}\quad m^2=AD^2=AQ\cdot AP=\frac{73}{2}\cdot\frac{92}{2}=1679.$$

AMERICAN MATHEMATICS COMPETITIONS

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

44th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)

THURSDAY, FEBRUARY 25, 1993

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1. **(D)** $\boxed{1,-1,2} = 1^{-1} - (-1)^2 + 2^1 = 1 - 1 + 2 = 2.$

2. **(D)** We have $\angle B = 180° - (55° + 75°) = 50°$. Since $\triangle BDE$ is isosceles, $\angle BED = \dfrac{180° - \angle B}{2} = \dfrac{180° - 50°}{2} = 65°.$

3. **(E)** $\dfrac{15^{30}}{45^{15}} = \dfrac{3^{30}5^{30}}{3^{15}3^{15}5^{15}} = 3^{30-15-15}5^{30-15} = 3^0 5^{15} = 5^{15}.$

OR

$$\frac{15^{30}}{45^{15}} = 15^{15}\left(\frac{15}{45}\right)^{15} = 15^{15}\left(\frac{1}{3}\right)^{15} = \left(\frac{15}{3}\right)^{15} = 5^{15}.$$

4. **(E)** Substituting 3 for x yields $3 \circ y = 12 - 3y + 3y = 12$. Thus $3 \circ y = 12$ is true for all real numbers y.

5. **(A)** Last year the bicycle and helmet cost \$160 + \$40 = \$200. This year the cost of the bicycle increased 0.05(\$160) = \$8, while the cost of the helmet increased 0.10(\$40) = \$4. Thus it costs \$12 more for the bicycle and helmet this year. This is an increase of $\dfrac{12}{200} = \dfrac{6}{100} = 6\%$.

6. **(B)** $\sqrt{\dfrac{8^{10}+4^{10}}{8^4+4^{11}}} = \sqrt{\dfrac{(2^3)^{10}+(2^2)^{10}}{(2^3)^4+(2^2)^{11}}} = \sqrt{\dfrac{2^{30}+2^{20}}{2^{12}+2^{22}}} = \sqrt{\dfrac{2^{20}(2^{10}+1)}{2^{12}(1+2^{10})}}$

$= \sqrt{2^8} = 2^4 = 16.$

7. **(E)** Since $\dfrac{R_{24}}{R_4} = \dfrac{9R_{24}}{9R_4} = \dfrac{10^{24}-1}{10^4-1} = \dfrac{(10^4)^6-1}{10^4-1}$

$= 10^{20} + 10^{16} + 10^{12} + 10^8 + 10^4 + 1 = 100010001000100010001,$

there are 15 zeros in the quotient.

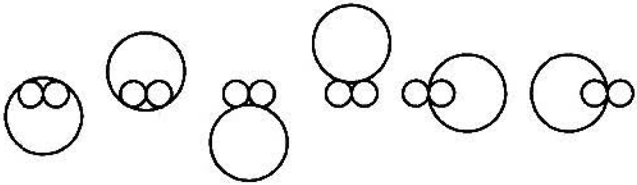
OR

Divide to compute the quotient:

```
         1 0001 0001 0001 0001 0001
1111)1111 1111 1111 1111 1111 1111
```

Note that there are $5 \times 3 = 15$ zeros in the quotient.

8. **(D)** There are six such circles. Two enclose both C_1 and C_2, two enclose neither, and two enclose exactly one of C_1 and C_2.



Query. What if the radius of the third circle were 2? $1\frac{1}{2}$? $\frac{1}{2}$?

9. **(D)** Let P be the world's population and W be its wealth. Then the $Pc/100$ citizens of $\mathcal{A}$ together own $Wd/100$ units of wealth, so each citizen of $\mathcal{A}$ owns $\dfrac{Wd/100}{Pc/100} = Wd/Pc$ units of wealth. Similarly, each citizen of $\mathcal{B}$ owns Wf/Pe units of wealth. The required ratio is therefore $\dfrac{\frac{Wd}{Pc}}{\frac{Wf}{Pe}} = \dfrac{de}{cf}$.

10. **(C)** Since $r = (3a)^{3b} = \left((3a)^3\right)^b = (27a^3)^b$, and $r = a^b x^b = (ax)^b$, we have $(27a^3)^b = (ax)^b$. Thus $27a^3 = ax$, which we solve to obtain $x = 27a^2$. To show that none of the other choices is correct, let $a = b = 1$.

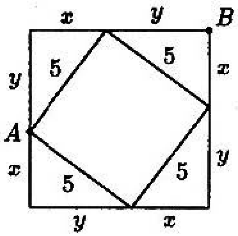
11. **(A)** Since

$$\begin{aligned} \log_2(\log_2(\log_2(x))) = 2 &\iff \log_2(\log_2(x)) = 2^2 = 4, \\ &\iff \log_2(x) = 2^4 = 16 \\ &\iff x = 2^{16} = 2^6 \cdot 2^{10} \approx 64 \cdot 1000, \end{aligned}$$

it follows that x has 5 base-ten digits.

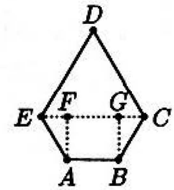
12. **(E)** $2f(x) = 2f\left(2\cdot\frac{x}{2}\right) = 2\left(\dfrac{2}{2+\frac{x}{2}}\right) = 2\left(\dfrac{4}{4+x}\right) = \dfrac{8}{4+x}$.

13. **(D)** The inscribed square will touch each edge of the larger square, dividing that edge into segments x and y units long, where $x \le y$, $x + y = 7$ and $x^2 + y^2 = 25$. Solve simultaneously to find $x = 3$ and $y = 4$. The greatest distance is therefore



$$AB = \sqrt{y^2 + (x+y)^2} = \sqrt{4^2 + 7^2} = \sqrt{65}.$$

14. **(B)** Draw $\overline{CE}$. Since $EA = BC$ and $\angle A = \angle B$, it follows that $ABCE$ is an isosceles trapezoid. Let F be the foot of the perpendicular from A to $\overline{CE}$, and G be the foot of the perpendicular from B to $\overline{CE}$. Then $EF = CG$. Since $\angle GBC = 30°$, we have $CG = \frac{1}{2}(BC) = 1$ and $BG = \frac{\sqrt{3}}{2}(BC) = \sqrt{3}$. Now $CE = CG + GF + FE = 1 + 2 + 1 = 4$, so CDE is an equilateral triangle. Thus†,



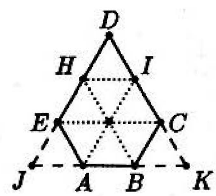
$$[ABCE] = \frac{1}{2}(BG)(AB + CE) = \frac{1}{2}\sqrt{3}(2+4) = 3\sqrt{3},$$

$$\text{and } [CDE] = \frac{\sqrt{3}}{4}(CE)^2 = \frac{\sqrt{3}}{4}(16) = 4\sqrt{3}.$$

Therefore, $[ABCDE] = [ABCE] + [CDE] = 7\sqrt{3}$.

OR

Draw $\overline{HI}$ where H is the midpoint of $\overline{ED}$ and I is the midpoint of $\overline{CD}$. Then $ABCIHE$ is a regular hexagon and $\triangle HDI$ is congruent to any of the six equilateral triangles of side 2 that make up $ABCIHE$. Thus, the area of $ABCDE$ is the sum of the areas of 7 equilateral triangles of side 2, so it is $7\left(2^2\frac{\sqrt{3}}{4}\right) = 7\sqrt{3}$.



Note. If $\overline{DE}$, $\overline{DC}$ and $\overline{AB}$ are extended to meet at J and K as in the figure, then we can compute $[ABCDE]$ as 7/9 of the area of equilateral triangle DJK.

15. **(D)** Since the degree measure of an interior angle of a regular n-sided polygon is $\dfrac{(n-2)180}{n} = 180 - \dfrac{360}{n}$, it follows that n must be a divisor of 360. Since $360 = 2^3 3^2 5^1$, its divisors are of the form $n = 2^a 3^b 5^c$ with $a = 0, 1, 2$ or 3, $b = 0, 1$ or 2 and $c = 0$ or 1. Hence, there are $(4)(3)(2) = 24$ divisors of 360. Since $n \geq 3$, we exclude the divisors 1 and 2, so there are 22 possible values of n.

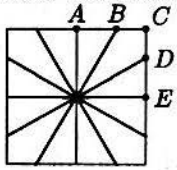
† We let $[P_1P_2 \ldots P_n]$ denote the area of polygon $P_1P_2 \ldots P_n$.

16. **(D)** The last occurrence of the n^{th} positive integer is in position number

$$1+2+\cdots+n=\frac{n(n+1)}{2}.$$

To approximate our n, consider $\frac{n^2}{2}=2000$. Since $\frac{62(63)}{2}=1953$ and $\frac{63(64)}{2}=1953+63=2016$, it follows that the 1993^{rd} term is 63, which leaves a remainder of 3 when divided by 5.

17. **(A)** Let O be the center of the clock, and label the triangle from 12 o'clock to 1 o'clock AOB, the quadrilateral from 1 o'clock to 2 o'clock $OBCD$, and the 3 o'clock position E, as indicated in the figure. Then $\triangle AOB \cong \triangle EOD$. Let $AB=1$. Since $\angle AOB=30°$, it follows that $OA=\sqrt{3}$, $[OACE]=3$ and $[AOB]=\sqrt{3}/2$. Hence $\frac{q}{t}=$



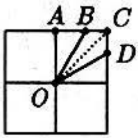
$$\frac{[OBCD]}{[AOB]}=\frac{[OACE]-2[AOB]}{[AOB]}=\frac{[OACE]}{[AOB]}-2=\frac{3}{\sqrt{3}/2}-2=2\sqrt{3}-2.$$

OR

Note that $\angle AOB=\angle BOD=30°$. Let $AO=1$. Therefore $AB=\frac{1}{\sqrt{3}}$ and $[ABO]=\frac{1}{2\sqrt{3}}$. Draw $\overline{OC}$. Compute the area of $\triangle OBC$ using $BC=1-\frac{1}{\sqrt{3}}$ as the base and $AO=1$ as the altitude. Then

$$[OBC]=\frac{1}{2}\cdot\left(1-\frac{1}{\sqrt{3}}\right)\cdot 1=\frac{3-\sqrt{3}}{6}.$$

Thus, $\frac{q}{t}=\frac{[OBCD]}{[AOB]}=\frac{2\cdot\frac{3-\sqrt{3}}{6}}{\frac{1}{2\sqrt{3}}}=2\sqrt{3}-2.$



18. **(E)** Al follows a 4-day cycle, and Barb follows a 10-day cycle. The least common multiple of 4 and 10 is 20. Together they follow a 20-day cycle, and there are 50 such cycles in 1000 days. Let us number the days in each cycle as 1, 2, ..., 20. Al rests on days 4, 8, 12, 16 and 20. Barb rests on days 8, 9, 10, 18, 19 and 20. They both rest on days 8 and 20, which is 2 days in each cycle. Thus they have $2\times 50=100$ common rest-days during their first 1000 days.

OR

Make a table: $$\begin{array}{llllllllll} \mathbf{A}: & www & r & www & \boxed{\begin{matrix} r \\ r \end{matrix}} & www & r & www & r & www & \boxed{\begin{matrix} r \\ r \end{matrix}} \cdots \\ \mathbf{B}: & www & w & www & & rrw & w & www & w & wrr & \end{array}$$

Observe that this pattern of 20 days repeats. Since there are 50 such cycles in 1000 days, it follows that the number of common rest-days is $50 \times 2 = 100$.

19. **(D)** Since m and n must both be positive, it follows that $n > 2$ and $m > 4$. Because $\dfrac{4}{m} + \dfrac{2}{n} = 1$ is equivalent to $(m-4)(n-2) = 8$, we need only find all ways of writing 8 as a product of positive integers. The four ways, $1\cdot 8$, $2\cdot 4$, $4\cdot 2$ and $8\cdot 1$, correspond to the four solutions $(m,n) = (5,10)$, $(6,6)$, $(8,4)$ and $(12,3)$.

20. **(B)** Use the quadratic formula to obtain $z = \dfrac{3i \pm \sqrt{-9+40k}}{20}$, which has discriminant $D = -9 + 40k$. If $k = 1$, then $D = 31$, so **(A)** is false. If k is a negative real number, then D is a negative real number, so **(B)** is true. If $k = i$, then $D = -9 + 40i = 16 + 40i - 25 = (4+5i)^2$, and the roots are $\frac{1}{5} + \frac{2}{5}i$ and $z = -\frac{1}{5} - \frac{1}{10}i$, so **(C)** and **(D)** are false. If $k = 0$ (which is a complex number), then the roots are 0 and $\frac{3}{10}i$, so **(E)** is false.

21. **(B)** In an arithmetic sequence with an odd number of terms, the middle term is the average of the terms. Since a_4, a_7, a_{10} form an arithmetic sequence of three terms with sum 17, $a_7 = \dfrac{17}{3}$. Since $a_4, a_5, \ldots, a_{14}$ form an arithmetic sequence of 11 terms whose sum is 77, the middle term, $a_9 = \dfrac{77}{11} = 7$. Let d be the common difference for the given arithmetic sequence. Since a_7 and a_9 differ by $2d$, $d = \dfrac{2}{3}$. Since $a_7 = a_1 + 6d$, it follows that $a_1 = \dfrac{5}{3}$. From $a_k = a_1 + (k-1)d = \dfrac{5}{3} + (k-1)\dfrac{2}{3} = 13$ we obtain $k = 18$.

22. **(C)** Intuitively, we note that the center cube in the first layer is counted most often and should be assigned the number 1 and those in the corners are used least and should be assigned 8, 9, and 10. For example, to arrive at the correct answer, assign the numbers to the bottom layer in this pattern:

$$\begin{array}{ccccccc} & & & 8 & & & \\ & & 2 & & 3 & & \\ & 7 & & 1 & & 4 & \\ 9 & & 6 & & 5 & & 10 \end{array}$$

More formally, suppose the assignment of the numbers to the bottom layer is:

$$\begin{array}{ccccccc} & & & v_1 & & & \\ & & e_1 & & e_2 & & \\ & e_3 & & c & & e_4 & \\ v_2 & & e_5 & & e_6 & & v_3 \end{array}$$

The arrangement of numbers in the second layer is:

$$\begin{array}{ccccccc} & & & (e_1+e_2+v_1) & & & \\ & (c+e_1+e_3) & & & & (c+e_2+e_4) & \\ (e_3+e_5+v_2) & & & (c+e_5+e_6) & & & (e_4+e_6+v_3) \end{array}$$

The arrangement of numbers in the third layer is

$$\begin{array}{ccc} & (2c+2e_1+2e_2+e_3+e_4+v_1) & \\ (2c+e_1+2e_3+2e_5+e_6+v_2) & & (2c+e_2+2e_4+e_5+2e_6+v_3) \end{array}$$

So $t = 6c+3(e_1+e_2+e_3+e_4+e_5+e_6)+(v_1+v_2+v_3)$ is the number assigned to the top block. Thus, the value of t is minimized when $c = 1, \{e_1, e_2, \ldots, e_6\} = \{2, 3, \ldots, 7\}$ and $\{v_1, v_2, v_3\} = \{8, 9, 10\}$. Hence, the minimum value of t is $6(1) + 3(2 + 3 + \cdots + 7) + (8 + 9 + 10) = 114$.

23. **(B)** The center of the circle is not X since $2\angle BAC \neq \angle BXC$. Thus $\overline{AD}$ bisects $\angle BXC$ and $\angle BAC$. (*Why?*) Since $\angle ABD = 90^\circ$ and $AD = 1$, it follows that $AB = AD\cos\angle BAD = \cos\frac{1}{2}\angle BAC = \cos 6^\circ$. Note that $\angle AXB = 162^\circ$ and $\angle ABX = 12^\circ$. By the *Law of Sines*,

$$\frac{AB}{\sin\angle AXB} = \frac{AX}{\sin\angle ABX} \quad \text{so} \quad \frac{\cos 6^\circ}{\sin 162^\circ} = \frac{AX}{\sin 12^\circ}.$$

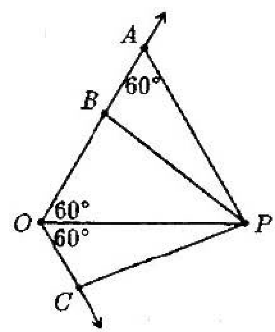
Since $\sin 162^\circ = \sin 18^\circ$, we have

$$AX = \frac{\cos 6^\circ \sin 12^\circ}{\sin 18^\circ} = \cos 6^\circ \sin 12^\circ \csc 18^\circ.$$

24. **(E)** Compute the probability that it takes 4 or fewer draws to obtain the third shiny penny and subtract this from 1. Since there are are $\binom{4}{3}$ ways to select the 3 shiny pennies in the first 4 draws and $\binom{7}{3}$ ways to select them in all, the probability that the 3 shiny pennies are drawn among the first 4 is $\frac{\binom{4}{3}}{\binom{7}{3}} = \frac{4}{35}$. Thus the requested probability is $1 - \frac{4}{35} = \frac{31}{35}$, whose numerator and denominator sum to 66.

Note. We could also have computed the first probability by considering the first 4 coins drawn. There are $\binom{7}{4}$ ways to select the first 4 coins and $\binom{3}{3}\binom{4}{1}$ ways to select the 3 shiny coins and one of the dull ones. Thus, the probability that the 3 shiny pennies are among the first 4 is $\frac{\binom{3}{3}\binom{4}{1}}{\binom{7}{4}} = \frac{4}{35}$.

25. **(E)** The figure shows the original configuration with the vertex of the 120° angle labeled O and three line segments drawn from P to S: AP, BP, CP. Point A is chosen so that $\angle APO = 60°$. Thus $\triangle APO$ is equilateral. Point B is <u>any</u> point on segment $\overline{AO}$, and C is on the other side of O with $AB = OC$. Since $PA = PO$, $\angle PAB \cong \angle POC$, and $AB = OC$, we have $\triangle PAB \cong \triangle POC$. Thus, $\angle APB \cong \angle OPC$, and hence $\angle APO \cong \angle BPC$. Since $BP = CP$ and $\angle BPC = 60°$, $\triangle BPC$ is also equilateral. Since B was an arbitrary point on segment $\overline{AO}$, it follows that there are an infinite number of the required equilateral triangles.



OR

Let O be the vertex of the $120°$ angle, choose A on the one ray so $\angle APO = 60°$ and choose <u>any</u> point B on segment $\overline{AO}$. Consider C on the other ray with $\angle BPC = 60°$. Then $PBOC$ is a cyclic quadrilateral since angles BPC and BOC are supplementary. The arc PC of the circumscribed circle is subtended by both $\angle POC$ and $\angle PBC$, so

$$\angle PBC = \angle POC = 60°. \qquad \text{Similarly,} \qquad \angle PCB = \angle POB = 60°.$$

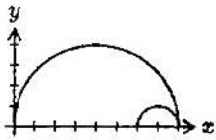
Since $\angle BPC = 60°$, $\triangle BPC$ is equiangular and therefore equilateral. Since the point B on segment $\overline{AO}$ was arbitrary, there are an infinite number of equilateral triangles PBC.

Query. If the rays form a different angle, would there still be an infinite number of equilateral triangles?

26. **(C)** Completing the squares, we have

$$f(x) = \sqrt{16-(x-4)^2} - \sqrt{1-(x-7)^2}.$$

The first term is the formula for the y-coordinate of the upper half of the circle with center at $(4,0)$ and radius 4, and the second term is the formula for the y-coordinate of the upper half-circle with center at $(7,0)$ and radius 1. By examining the graphs of these two semicircles, it is clear that $f(x)$ is real-valued only when $6 \le x \le 8$, and that the maximum value will be attained when $x = 6$. Evaluating, we get



$$f(6) = \sqrt{16-(6-4)^2} = \sqrt{12} = 2\sqrt{3}.$$

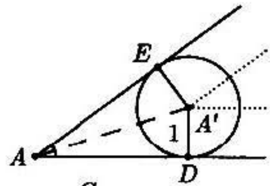
OR

Note that $f(x) = \sqrt{(8-x)x} - \sqrt{(8-x)(x-6)}$ is a real number if and only if $6 \le x \le 8$. Note that

$$f(x) = \sqrt{8-x}\left(\sqrt{x} - \sqrt{x-6}\right)\left(\frac{\sqrt{x}+\sqrt{x-6}}{\sqrt{x}+\sqrt{x-6}}\right) = \frac{6\sqrt{8-x}}{\sqrt{x}+\sqrt{x-6}}.$$

For all x such that $6 \le x \le 8$, the numerator of this last expression is maximized and its denominator is minimized when $x = 6$. Hence, its maximum is $f(6) = \dfrac{6\sqrt{2}}{\sqrt{6}} = 2\sqrt{3}$.

27. **(B)** When the circle is closest to A with its center P at A', let its points of tangency to $\overline{AB}$ and $\overline{AC}$ be D and E, respectively. The path parallel to $\overline{AB}$ is shorter than $\overline{AB}$ by AD plus the length of a similar segment at the other end. Now $AD = AE = \cot \dfrac{A}{2}$. Similar reasoning at the other vertices shows that the length of the path of P is



$$AB + BC + CA - 2\cot\frac{A}{2} - 2\cot\frac{B}{2} - 2\cot\frac{C}{2}.$$

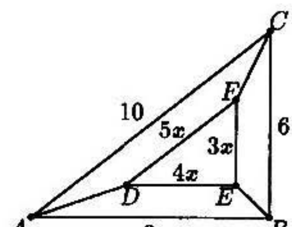
Since $\cot \dfrac{A}{2} = \dfrac{1+\cos A}{\sin A} = \dfrac{1+\frac{4}{5}}{\frac{3}{5}} = 3,$ and similarly $\cot\dfrac{B}{2} = 1$ and $\cot\dfrac{C}{2} = 2$, the length of the path is $8+6+10-6-2-4 = 12$.

OR

The locus of P is a triangle $A'B'C'$ similar to triangle ABC. Calculating as above, we have $A'B' = AB - \cot\dfrac{A}{2} - \cot\dfrac{B}{2} = 8-3-1 = 4 = \dfrac{1}{2}AB$, so the linear dimensions of $\triangle A'B'C'$ are half those of $\triangle ABC$, and its perimeter is $\dfrac{8+6+10}{2} = 12$.

OR

The locus, $\triangle DEF$, of the center of the rolling circle is similar to $\triangle ABC$, so we label its sides $3x$, $4x$ and $5x$, for some $x > 0$. The area of $\triangle ABC$ is $(AB)(BC)/2 = 24$. Partition $\triangle ABC$ into three trapezoids of altitude 1 and $\triangle DEF$, and compute the area of $\triangle ABC$ in terms of x:

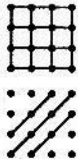


$$\begin{aligned}[ABC] &= [DABE] + [EBCF] + [FCAD] + [DEF] \\ &= \frac{1}{2}(1)(4x+8) + \frac{1}{2}(1)(3x+6) + \frac{1}{2}(1)(5x+10) + 6x^2 \\ &= 6x^2 + 6x + 12.\end{aligned}$$

Solve $6x^2 + 6x + 12 = 24$ for the positive root, $x = 1$, to find that the perimeter of $\triangle DEF$ is $3x + 4x + 5x = 12x = 12$.

Challenge. Prove that for any triangle and for any circle which rolls around inside the triangle, the perimeter of the triangle which is the locus of the center of the circle is the perimeter of the original triangle diminished by the perimeter of the similar triangle which circumscribes the circle.

28. **(D)** There are $\binom{16}{3} = 560$ sets of 3 points. We must exclude from our count those sets of three points that are collinear. There are 4 vertical and 4 horizontal lines of four points each. These 8 lines contain $8\binom{4}{3} = 32$ sets of 3 collinear points. Similarly, there are $2\binom{4}{3} + 4\binom{3}{3} = 8 + 4 = 12$ sets of 3 collinear points that determine lines of slope ± 1. Since there are no other sets of 3 collinear points, the number of triangles is $560 - 32 - 12 = 516$.



29. **(B)** Let a, b and c, with $a \le b \le c$, be the lengths of the edges of the box; and let p, q and r, with $p \le q \le r$, be the lengths of its external diagonals. The Pythagorean Theorem implies that

$$p^2 = a^2 + b^2, \qquad q^2 = a^2 + c^2 \quad \text{and} \quad r^2 = b^2 + c^2.$$

It follows that $r^2 = p^2 + q^2 - 2a^2 < p^2 + q^2$ is a necessary condition for a set $\{p, q, r\}$ to represent the lengths of the diagonals. Only choice **(B)** fails this test. The other four choices *do* correspond to actual prisms because the condition $r^2 < p^2 + q^2$ is also sufficient. To see this, just solve the equations for a, b and c:

$$a^2 = \frac{p^2 + q^2 - r^2}{2}, \qquad b^2 = \frac{p^2 - q^2 + r^2}{2} \quad \text{and} \quad c^2 = \frac{-p^2 + q^2 + r^2}{2}.$$

30. **(D)** When a number $x_0 \in [0, 1)$ is written in base-two, it has the form

$$x_0 = 0.d_1 d_2 d_3 d_4 d_5 d_6 d_7 \ldots \qquad (\text{each } d_k = 0 \text{ or } 1.)$$

The algorithm given in the problem simply moves the "binary point" one place to the right and then ignores any digits to the left of the point. That is

$$x_0 = 0.d_1 d_2 d_3 d_4 d_5 d_6 d_7 \ldots \quad \Longrightarrow \quad x_1 = 0.d_2 d_3 d_4 d_5 d_6 d_7 \ldots$$

Thus for x_0 to equal x_5 we must have

$$0.d_1 d_2 d_3 d_4 d_5 d_6 d_7 \ldots = 0.d_6 d_7 d_8 d_9 d_{10} d_{11} d_{12} \ldots$$

This happens if and only if x_0 has a repeating expansion with $d_1 d_2 d_3 d_4 d_5$ as the repeating block. There are $2^5 = 32$ such blocks. However, if $d_1 = d_2 = d_3 = d_4 = d_5 = 1$, then $x_0 = 1$. Hence there are $32 - 1 = 31$ values of $x_0 \in [0, 1)$ for which $x_0 = x_5$.

OR

We can restate the given formula as $x_n = 2x_{n-1} - \lfloor 2x_{n-1} \rfloor$, where $\lfloor t \rfloor$ is the largest integer not exceeding t. Since $\lfloor t + k \rfloor = \lfloor t \rfloor + k$ for any integer k, it follows that

$$\begin{aligned}
x_5 &= 2x_4 - \lfloor 2x_4 \rfloor \\
&= 2(2x_3 - \lfloor 2x_3 \rfloor) - \lfloor 2(2x_3 - \lfloor 2x_3 \rfloor) \rfloor = 4x_3 - 2\lfloor 2x_3 \rfloor - \lfloor 4x_3 \rfloor + 2\lfloor 2x_3 \rfloor \\
&= 4x_3 - \lfloor 4x_3 \rfloor \\
&\ \ \vdots \\
&= 32x_0 - \lfloor 32x_0 \rfloor .
\end{aligned}$$

Consequently, to have $x_5 = x_0$ it is necessary that $31x_0 = \lfloor 32x_0 \rfloor$ be an integer. But $31x_0$ is an integer for some x_0 in the prescribed domain precisely when $x_0 = n/31$ for some $n = 0, 1, \ldots, 30$. It is easy to check that for each of these 31 values of x_0 we have $x_5 = x_0$.

AMERICAN MATHEMATICS COMPETITIONS

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

45th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)

THURSDAY, FEBRUARY 24, 1994

Sponsored by

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This Solutions Pamphlet contains at least one solution to each problem on this year's Examination and shows that all the problems can be solved mathematically without the use of calculus or a calculator. Routine calculations and obvious reasons for proceeding in a certain way are often omitted to give greater emphasis to the essential ideas behind each solution. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *Remember that reproduction of these solutions is prohibited by the copyright.*

Since Prof Harold Reiter, AHSME Chair, is not in the USA this year, send correspondence about the problems and solutions to:

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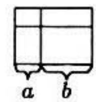
1. **(C)** $4^4 \cdot 9^4 \cdot 4^9 \cdot 9^9 = (4 \cdot 9)^4 \cdot (4 \cdot 9)^9 = (4 \cdot 9)^{4+9} = 36^{13}$.

OR

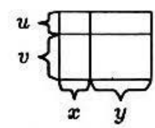
$$4^4 \cdot 9^4 \cdot 4^9 \cdot 9^9 = (4^{4+9}) \cdot (9^{4+9}) = 4^{13} 9^{13} = (4 \cdot 9)^{13} = 36^{13}.$$

2. **(B)** Rectangles of the same height have areas proportional to their bases, a and b. Hence $\dfrac{6}{14} = \dfrac{a}{b} = \dfrac{?}{35}$, so the required area is 15.



OR

We can prove that the product of the areas of the diagonally opposite rectangles is the same: If the sides are labeled x, y, u and v as shown, then the product of the areas of the lower left and upper right is $(xv)(yu)$ and the product of the other areas is $(xu)(yv)$. Hence the area of the fourth rectangle is $(35)(6)/14 = 15$.



Note. For the given data, we can find the answer by factoring the areas. Since rectangles of the same height have areas proportional to their bases, the unknown area is $r \cdot s = 3 \cdot 5 = 15$. This factoring method will not work for all given data; e.g., if 16 is substituted for 14.

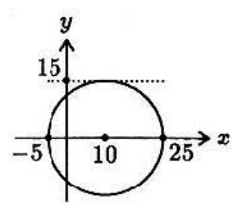
$3 \cdot 2$	$7 \cdot 2$
$r \cdot s$	$7 \cdot 5$

3. **(B)** Only one, namely $2x^x$, equals $x^x + x^x$ for all $x > 0$. Use $x = 2$ or $x = 3$ to show that the given expression is not identical to any of the other choices.

4. **(A)** The radius of the circle is $\dfrac{25-(-5)}{2} = 15$, and the midpoint of the diameter is $(10,0)$. Thus an equation of the circle is $(x-10)^2 + y^2 = 15^2$. Let $y = 15$ in this equation to find that $x = 10$.

OR

The circle has center $(10,0)$, diameter 30, and hence radius 15. Since $(x,15)$ is 15 units from the given diameter, the radius to $(x,15)$ must be perpendicular to that diameter. Thus $x = 10$.



5. **(E)** The original number n satisfies $\dfrac{n}{6} - 14 = 16$, so we must have $n = 180$. Thus, the answer Pat should have produced is $6(180) + 14 = 1094$.

6. **(A)** Calculate d, c, b and a in that order from the definition of the sequence: $1 = d+0$, so $\boxed{d=1}$; then $0 = c+d$, so $\boxed{c=-1}$; next $1 = d = b+c = b-1$, so $\boxed{b=2}$; finally, $-1 = c = a+b = a+2$, so $\boxed{a=-3}$.

OR

Use the rule for generating terms of the sequence beginning with the two terms a, b:

$$a,\ b,\ a+b,\ a+2b,\ 2a+3b,\ 3a+5b,\ \ldots$$

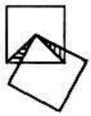
to see that $2a+3b=0$ and $3a+5b=1$. Solve simultaneously to find $a=-3$.

7. **(E)** Each square has side 10, and so has area 100. The overlap, $\triangle ABG$, has area one-fourth that of square $ABCD$. Thus, the area covered is $100+100-(100/4)=175$.

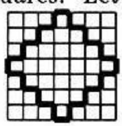
OR

Since $\triangle ABG$ is an isosceles right triangle with hypotenuse $AB=10$, it follows that the length of each of its legs is $5\sqrt{2}$ so its area is $\frac{1}{2}\left(5\sqrt{2}\right)\left(5\sqrt{2}\right)=25$. Each square has area 100. Thus, the total area covered is $2(100)-25=175$.

Note. It is not necessary that $\overline{GH}$ and $\overline{GF}$ coincide with the diagonals of square $ABCD$. The area of the overlap is constant as long as G is the center of $ABCD$.



8. **(C)** It is possible to sketch the polygon on a 7×7 lattice of squares. Let the length of a side of the polygon be s. The perimeter of the polygon is $28s$, so $s=2$. The region bounded by the polygon consists of $1+3+5+7+5+3+1=25$ squares from the lattice, so its area is $25s^2=100$.

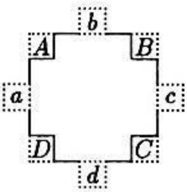


OR

Instead of counting the 25 squares as above, note that to form the figure from the 7×7 lattice, 6 $s\times s$ squares are cut from each corner. Hence the area of the region is $[(7)(7)-4(6)]s^2=100$.

OR

As shown, remove each of the four squares a, b, c, d furthest from the center of the polygon, and use them to fill in the remaining four concave sections, A, B, C, D. The area of the region bounded by the original polygon is the same as the area of the resulting 5 segment by 5 segment square. Since the length of each segment is $56/28=2$, the required area is thus $5^2 2^2=100$.



9. **(D)** Since $\angle A = 4\angle B$, and $90° - \angle B = 4(90° - \angle A)$, it follows that $90° - \angle B = 4(90° - 4\angle B)$. Thus, $\angle B = 18°$.

Note. Observe that the two conditions

(1) *$\angle A$ is k times $\angle B$,*

(2) *the complement of $\angle B$ is k times the complement of $\angle A$,*

imply that

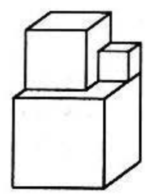
$\angle A$ is the complement of $\angle B$.

Thus, we could solve $\angle A + \angle B = 90°$ and $\angle A = 4\angle B$ simultaneously for $\angle B$.

10. **(B)** Since $m(b,c) = b$ and $m(a,e) = a$, we have

$$M(M(a, m(b,c)), m(d, m(a,e))) = M(M(a,b), m(d,a)) = M(b,a) = b.$$

11. **(D)** The sum of the surface areas of the three cubes is $6 + 24 + 54 = 84$. We can attach each cube to the other two so that each attachment subtracts twice the area of a face of the smaller cube from the total. The remaining surface area is $84 - 2(1) - 2(1) - 2(4) = 72$.



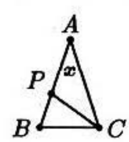
12. **(D)** $\left(i - i^{-1}\right)^{-1} = \left(i - \frac{1}{i}\right)^{-1} = \left(\frac{i^2 - 1}{i}\right)^{-1} = \left(\frac{-2}{i}\right)^{-1} = -\frac{i}{2}.$

OR

Since $i^{-1} = -i$, it follows that

$$\left(i - i^{-1}\right)^{-1} = (i - (-i))^{-1} = (2i)^{-1} = \frac{1}{2i} = -\frac{i}{2}.$$

13. **(B)** Let $\angle A = x°$. Then $\angle PCA = x°$ since $AP = PC$. By the exterior angle theorem, $\angle BPC = \angle A + \angle PCA = 2x°$. Since $PC = CB$, it follows that $\angle B = 2x°$. Thus $\angle ACB = 2x°$ since $AB = AC$. Therefore, $x + 2x + 2x = 180$, or $\angle A = x° = 36°$.



14. **(B)** There are 101 terms. In an arithmetic series, the sum is the number of terms times the average of the first and last terms. Thus the desired sum is $101 \cdot \left(\frac{20 + 40}{2}\right) = 101 \cdot 30 = 3030.$

OR

There are 101 terms. Write the sum of the last fifty terms in reverse order under the first fifty:

$$\begin{array}{l}
20+20\frac{1}{5}+20\frac{2}{5}+\cdots+29\frac{3}{5}+29\frac{4}{5}+ \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 30+ \\
40+39\frac{4}{5}+39\frac{3}{5}+\cdots+30\frac{2}{5}+30\frac{1}{5}. \\
\hline
\underbrace{60+60\ \ +60\ \ +\cdots+60\ \ +60}_{\text{50 terms}}\ \ +30
\end{array}$$

Thus the sum is $50(60)+30=3030.$

OR

Since $\frac{1}{5}+\frac{2}{5}+\frac{3}{5}+\frac{4}{5}=2$, regroup the given expression as indicated to find the sum:

$$\begin{aligned}
&(5(20)+2)+(5(21)+2)+\cdots+(5(39)+2)+40 \\
&\qquad\qquad = 5(20+21+\cdots+39)+20(2)+40 \\
&\qquad\qquad = 5\left(20\cdot\frac{20+39}{2}\right)+80 = 2950+80 = 3030.
\end{aligned}$$

15. **(B)** The squares of only two integers in $\{1,2,3,\ldots,10\}$ have odd tens digits, $4^2=16$ and $6^2=36$. Since $(n+10)^2=n^2+(20n+100)$ and the tens digit in $20n+100$ must be even, it follows that the tens digit in $(n+10)^2$ will be odd if and only if the tens digit in n^2 is odd. Inductively, we conclude that only numbers in $\{1,2,3,\ldots,100\}$ that end in 4 or 6 will have squares with an odd tens digit. There are exactly $10\times 2=20$ such numbers.

OR

Since neither 0^2 nor 100^2 have odd tens digits, we replace the given set with $\{0,1,2,\ldots,99\}$, and count the n of the form

$$n=10m+d, \qquad d=0,1,\ldots,9, \qquad m=0,1,\ldots,9$$

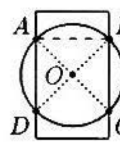
for which n^2 has an odd tens digit. Since $n^2=10(10m^2+2md)+d^2$ and $10m^2+2md$ is even, it follows that the tens digit of n^2 will be odd if and only if the tens digit of d^2 is odd. There are two digits, $d=4$ and $d=6$, for which the tens digit of d^2 is odd. Since there are 10 choices for m to pair with these two choices for d, there are $2\times 10=20$ integers n in the set whose squares have odd tens digits.

16. **(B)** Let r be the number of red marbles and n the total number of marbles originally in the bag.

Then $$\frac{r-1}{n-1}=\frac{1}{7} \quad \text{and} \quad \frac{r}{n-2}=\frac{1}{5}.$$

Solve $7r-7=n-1$ and $5r=n-2$ simultaneously to find $n=22$.

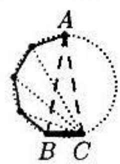
17. **(D)** Let O be the center of the circle and rectangle, and let the circle and rectangle intersect at A, B, C and D as shown. Since $AO = OB = 2$ and $AB = 2\sqrt{2}$, the width of the rectangle, it follows that $\angle AOB = 90°$. Hence $\angle AOD = \angle DOC = \angle COB = 90°$. The sum of the areas of sectors AOB and DOC is $2\left(\frac{1}{4}\left(\pi 2^2\right)\right) = 2\pi$. The sum of the areas of isosceles right triangles AOD and COB is $2\left(\frac{1}{2}\cdot 2^2\right) = 4$. Thus, the area of the region common to both the rectangle and the circle is $2\pi + 4$.



18. **(C)** Since $180° = \angle A + \angle B + \angle C = \angle A + 4\angle A + 4\angle A$, it follows that $\angle A = 20°$. Therefore, $\widehat{BC} = 2\angle A = 40°$, which is $1/9$ of $360°$. Thus the polygon has 9 sides.

OR

If $\angle C$ is partitioned into four angles congruent to $\angle A$, the four chords associated with the arcs subtended by these angles will be congruent to $\overline{BC}$. These four chords plus four obtained analogously from $\angle B$, together with $\overline{BC}$, form the $n = 9$ sides of the inscribed regular polygon.



Note. In general, if $\angle B = \angle C = k\angle A$, then $n = 2k+1$.

19. **(C)** The maximum number of disks that can be drawn without having ten with the same label is 414: all 45 disks labeled "**1**" through "**9**", and nine of each of the other 41 types. The 415^{th} draw must result in ten disks with the same label, so 415 is the minimum number of draws that guarantees at least ten disks with the same label.

20. **(B)** Since $y = xr$, $z = xr^2$ and $3z - 2y = 2y - x$, by substitution

$$3xr^2 - 2xr = 2xr - x \qquad \text{or} \qquad 3r^2 - 4r + 1 = 0. \qquad (x \neq 0)$$

Thus $(3r-1)(r-1) = 0$. Since $x \neq y$, it follows that $r \neq 1$. Thus $r = 1/3$.

21. **(C)** We check all odd positive integers with non-zero digits, the sum of whose digits is 4:

$$13,\ 31,\ 121,\ 211,\ 1111$$

and find that $121 = 11^2$ and $1111 = 11 \cdot 101$ are the two counterexamples.

22. **(C)** The two end chairs must be occupied by students, so the professors have seven middle chairs from which to choose, with no two adjacent. If these chairs are numbered from 2 to 8, the three chairs can be:

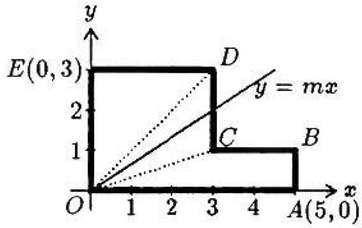
$$(2,4,6),\ (2,4,7),\ (2,4,8),\ (2,5,7),\ (2,5,8)$$
$$(2,6,8),\ (3,5,7),\ (3,5,8),\ (3,6,8),\ (4,6,8).$$

Within each triple, the professors can arrange themselves in $3!$ ways, so the total number is $10 \times 6 = 60$.

OR

Imagine the six students standing in a row before they are seated. There are 5 spaces between them, each of which may be occupied by at most one of the 3 professors. Therefore, there are $P(5,3) = 5 \times 4 \times 3 = 60$ ways the three professors can select their places.

23. **(E)** The area of the region is $3^2 + (2)(1) = 11$. Label the vertices as indicated in the figure. Since the area of trapezoid $OABC$ is $\frac{2+5}{2} < \frac{11}{2}$ and the area of triangle ODE is $\frac{3^2}{2} < \frac{11}{2}$, it follows that the desired line, $y = mx$, intersects the line $x = 3$ at some point $(3, 3m)$, where $1 < 3m < 3$. The area of the trapezoid above the line $y = mx$ is

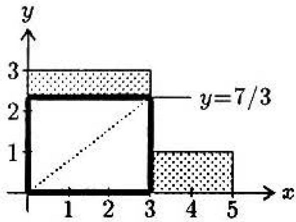


$$\frac{[3 + (3 - 3m)]}{2}(3) = \frac{18 - 9m}{2},$$

which equals $\frac{11}{2}$ when $m = \frac{7}{9}$.

OR

The area to the right of the line $x = 3$ in the L-shaped region is 2. Since $\frac{2}{3} \times 3 = 2$, the area above the line $y = 7/3$ is also 2. The diagonal of the rectangle which remains when these two rectangles of area 2 are discarded is the line which bisects the area of the L-shaped region. This diagonal connects the origin with $(3, 7/3)$ and has slope $7/9$.



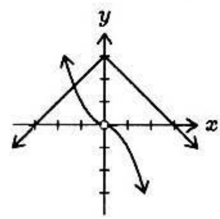
24. **(C)** Since the mean is 10, the sum of the observations must be 50. The median of 12 forces one observation to be 12, two more to be no more than 12, and the remaining two to be at least 12. If a maximal observation is increased by x, the sum of those no larger than 12 must be reduced by x in order to keep the mean at 10. However, this expands the range. Thus the minimum range will occur when three observations are 12 and the remaining two observations are equal and sum to $50 - 3(12) = 14$. Hence the sample $7, 7, 12, 12, 12$ minimizes the range, and the smallest value that the range can assume is $12 - 7 = 5$.

25. **(A)** If $x > 0$, then $x + y = 3$ and $y + x^2 = 0$. Eliminate y from these simultaneous equations to obtain $x^2 - x + 3 = 0$, which has no real roots. If $x < 0$, then we have $-x + y = 3$ and $-y + x^2 = 0$, which have a simultaneous real solution, so $x - y = -3$.

Note. One need not obtain the solution, $(x, y) = \left(\frac{1-\sqrt{13}}{2}, \frac{7-\sqrt{13}}{2}\right)$, to find the answer.

OR

Sketch $y = 3 - |x|$ and $y = \dfrac{-x^3}{|x|} = \begin{cases} x^2 & \text{if } x < 0 \\ -x^2 & \text{if } x > 0. \end{cases}$



Note that the graphs cross only on the half-line $y = x + 3$, $x < 0$. Therefore $x - y = -3$.

26. **(A)** The measure of each interior angle of a regular k-gon is $180° - \dfrac{360°}{k}$. In this problem, each vertex of the m-gon is surrounded by one angle of the m-gon and two angles of the n-gons. Therefore,

$$\left(180° - \frac{360°}{m}\right) + 2\left(180° - \frac{360°}{n}\right) = 360°,$$

and $m = 10$ gives $n = 5$.

Note. The equation may be written in the form $(m - 2)(n - 4) = 8$. Its only solutions in positive integers are $(m, n) = (3, 12)$, $(4, 8)$, $(6, 6)$, and $(10, 5)$.

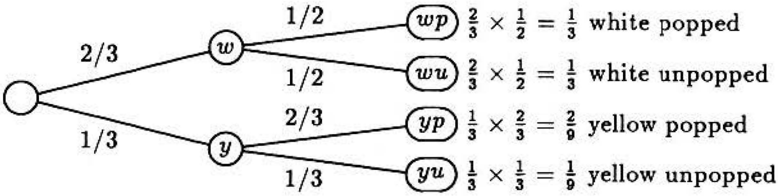
OR

Each interior angle of a regular decagon measures $(180° - 36°)$. The interior angles of the two n-gons at one of its vertices must fill $360° - (180° - 36°) = 216°$. The regular polygon each of whose interior angles measures $216°/2 = 108°$ is the pentagon, so $n = 5$.

27. **(D)** Let the total number of kernels be $3n$, so that there are $2n$ white kernels and n yellow kernels. Then $5n/3$ of the kernels will pop (n white and $2n/3$ yellow). Hence the probability that the popped kernel was white is $\dfrac{n}{5n/3} = \dfrac{3}{5}$.

OR

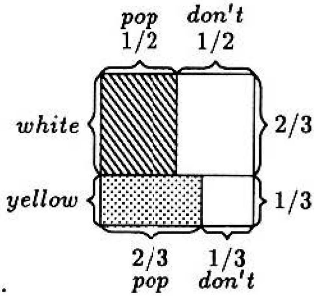
Make a probability tree diagram:



Since $\dfrac{1}{3} + \dfrac{2}{9} = \dfrac{5}{9}$ of the kernels popped and $\dfrac{1}{3}$ of the kernels are white and popped, the probability that the popped kernel was white is $\dfrac{1/3}{5/9} = \dfrac{3}{5}$.

OR

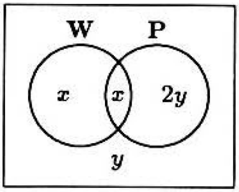
Make a diagram as indicated, letting areas represent the probabilities. The ratio of the area shaded with line segments to the area shaded with segments or dots is $\dfrac{\left(\frac{1}{2}\right)\left(\frac{2}{3}\right)}{\left(\frac{1}{2}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)} = \dfrac{3}{5}$.



Note. This is an application of Bayes' Theorem.

OR

Use a Venn diagram where the universal set is the set of kernels, **W** is the set of white kernels, and **P** is the set of kernels which will pop. Let x be the number of kernels in $\mathbf{W} \cap \mathbf{P}$ and $\mathbf{W} - \mathbf{P}$, and let y be the number of kernels not in $\mathbf{W} \cup \mathbf{P}$. Then there are $2y$ kernels in $\mathbf{P} - \mathbf{W}$. Thus, we are given $\dfrac{x+x}{x+x+2y+y} = \dfrac{2}{3}$, so $x = 3y$. The probability that a kernel in **P** is also in **W** is $\dfrac{x}{x+2y} = \dfrac{3y}{3y+2y} = \dfrac{3}{5}$.



28. **(C)** Write the equation of the line in the *two-intercept form*:

$$\frac{x}{p} + \frac{y}{b} = 1, \quad \text{where } p \text{ is prime and integer } b > 0.$$

Substitute $x = 4$ and $y = 3$ to obtain

$$\frac{4}{p} + \frac{3}{b} = 1 \qquad \text{or} \qquad b = \frac{3p}{p-4} = 3 + \frac{12}{p-4}.$$

There are only two primes, $p = 5$ and $p = 7$, which yield a positive integer b. Therefore, there are two lines with the requested properties,

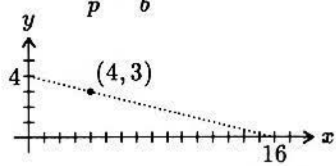
$$\frac{x}{5} + \frac{y}{15} = 1 \quad \text{and} \quad \frac{x}{7} + \frac{y}{7} = 1.$$

OR

Let p and b be the x- and y-intercepts of such a line. Then $p > 4$. Since the points $(p, 0)$, $(4, 3)$ and $(0, b)$ are collinear, by computing the slope of the line in two different ways, we find $\dfrac{b-3}{-4} = \dfrac{-3}{p-4}$, or $(p-4)(b-3) = 12$. Thus, $(p-4)$ must be one of 1, 2, 3, 4, 6 or 12 and p must be an odd prime. There are only two such primes, $p = 4 + 1 = 5$ and $p = 4 + 3 = 7$.

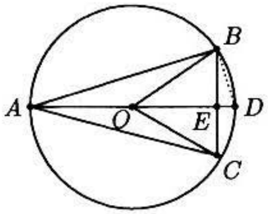
OR

Since both intercepts must be positive, the lines $\dfrac{x}{p} + \dfrac{y}{b} = 1$ with the desired properties must have negative slope. Thus, the integer b is larger than 3, so $b \geq 4$. Similarly, $p \geq 5$. Draw the line from $(0, 4)$ through $(4, 3)$ to see that $p \leq 16$.



There are four primes between 4 and 16: $p = 5, 7, 11$ and 13. From $\dfrac{4}{p} + \dfrac{3}{b} = 1$ it follows that $b = \dfrac{3p}{p-4}$, which is an integer only when $p = 5$ or $p = 7$. Thus there are two such lines.

29. **(A)** Draw and label the figure as shown, where O is the center of the circle. In radians, $\angle BOC = r/r = 1$, so $\angle BAC = 1/2$ and $\angle BAE = 1/4$. Since $\overline{BE} \perp \overline{EA}$, it follows that



$$\frac{AB}{BC} = \frac{1}{2}\frac{AB}{BE} = \frac{1}{2}\csc\angle BAE = \frac{1}{2}\csc\frac{1}{4}.$$

OR

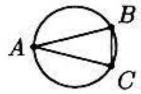
Since $\angle BAC = \frac{1}{2}$, it follows that $\angle ACB = \frac{1}{2}\left(\pi - \frac{1}{2}\right) = \frac{\pi}{2} - \frac{1}{4}$. The length of a chord subtended by the inscribed angle β in a circle of radius r is $2r\sin\beta$, so

$$AB = 2r\sin\angle ACB = 2r\sin\left(\frac{\pi}{2} - \frac{1}{4}\right) = 2r\cos\frac{1}{4}.$$
$$BC = 2r\sin\angle BAC = 2r\sin\frac{1}{2} = 4r\sin\frac{1}{4}\cos\frac{1}{4},$$

Therefore, $\displaystyle\frac{AB}{BC} = \frac{2r\cos\frac{1}{4}}{4r\sin\frac{1}{4}\cos\frac{1}{4}} = \frac{1}{2\sin\frac{1}{4}} = \frac{1}{2}\csc\frac{1}{4}.$

OR

By the *Law of Sines*, $\displaystyle\frac{AB}{BC} = \frac{\sin C}{\sin A}$. But $2\angle C + \angle A = \pi$, so $\displaystyle\angle C = \frac{\pi}{2} - \frac{\angle A}{2}$ and $\displaystyle\sin C = \cos\frac{A}{2}$. Since $\displaystyle\angle A = \frac{1}{2}$,



$$\frac{AB}{BC} = \frac{\sin C}{\sin A} = \frac{\cos\frac{A}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}} = \frac{1}{2\sin\frac{A}{2}} = \frac{1}{2}\csc\frac{A}{2} = \frac{1}{2}\csc\frac{1}{4}.$$

30. **(C)** When n dice are rolled, the sum can be any integer from n to $6n$. The sum $n+k$ can be obtained in the same number of ways as the sum $6n-k$, and this number of ways increases as k increases from 0 to $\lfloor 5n/2 \rfloor$. Minimize $S = n+k$ by choosing n and k as small as possible with $6n - k = 1994$. Since the least multiple of 6 that is greater than or equal to 1994 is $1998 = 6(333)$, S is smallest when $n = 333$ and $k = 4$. Consequently, $S = n + k = 337$.

OR

On a standard die, 6 and 1, 5 and 2, and 4 and 3 are on opposite sides. To obtain a sum of 1994 with the most sixes on the top faces of the dice requires that 332 sixes and 1 two face up. Then 332 ones and 1 five will face down, and $332 + 5 = 337$.

AMERICAN MATHEMATICS COMPETITIONS

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

46th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)

THURSDAY, FEBRUARY 16, 1995

Sponsored by

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1. **(B)** The average changes from $\dfrac{87+83+88}{3} = 86$ to $\dfrac{87+83+88+90}{4} = 87$, an increase of 1.

2. **(D)** Square both sides of the given equation to obtain $2+\sqrt{x} = 9$. Thus $\sqrt{x} = 7$, and $x = 49$, which satisfies the given equation.

3. **(B)** The total price advertised on television is

$$\$29.98 + \$29.98 + \$29.98 + \$9.98 = \$99.92,$$

so this is $\$99.99 - \$99.92 = \$0.07$ less than the in-store price.

OR

The three payments are each 2 cents less than \$30, and the shipping & handling charge is 2 cents less than \$10, so the total price advertised on television is 8 cents less than \$100. The total in-store price is 1 cent less than \$100, so the amount saved by buying the appliance from the television advertiser is 7 cents.

4. **(B)** Since $M = 0.3Q = 0.3(0.2P) = 0.06P$ and $N = 0.5P$, we have

$$\frac{M}{N} = \frac{0.06P}{0.5P} = \frac{6}{50} = \frac{3}{25}\ .$$

5. **(C)** The number of ants is approximately the product

$$(300 \text{ ft}) \times (400 \text{ ft}) \times (12 \text{ in/ft})^2 \times (3 \text{ ants/in}^2) = 300 \times 400 \times 144 \times 3 \text{ ants},$$

which is $3 \times 4 \times 1.44 \times 3 \times 10^{2+2+2} \approx 50 \times 10^6$.

6. **(C)** Think of A as the bottom. Fold B up to be the back. Then x folds upward to become the left side and C folds forward to become the right side, so C is opposite x.

7. **(C)** The length of the flight path is approximately the circumference of Earth at the equator, which is

$$C = 2\pi \cdot 4000 = 8000\pi \text{ miles.}$$

The time required is

$$\frac{8000\pi}{500} = 16\pi \begin{cases} > 16(3.1) = 49.6 \text{ hours} \\ < 16(3.2) = 51.2 \text{ hours,} \end{cases}$$

so the best choice is 50 hours.

Query. What is a negligible height; i.e., for which heights above the equator would the flight-time be closer to choice **(C)** than to **(D)**?

8. **(C)** Because $\triangle ABC$ is a right triangle, the *Pythagorean Theorem* implies that $BA = 10$. Since $\triangle DBE \sim \triangle ABC$,

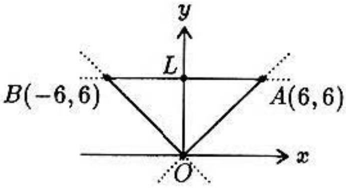
$$\frac{BD}{BA} = \frac{DE}{AC}. \qquad \text{So} \qquad BD = \frac{DE}{AC}(BA) = \frac{4}{6}(10) = \frac{20}{3}.$$

OR

Since $\sin B = \dfrac{DE}{BD}$, we have $BD = \dfrac{DE}{\sin B}$. Moreover, $BA = 10$ by the *Pythagorean Theorem*, so $\sin B = \dfrac{AC}{BA} = \dfrac{3}{5}$. Hence $BD = \dfrac{4}{3/5} = \dfrac{20}{3}$.

9. **(D)** Since all the acute angles in the figure measure 45°, all the triangles must be isosceles right triangles. It follows that all the triangles must enclose one, two or four of the eight small triangular regions. Besides the eight small triangles, there are four triangles that enclose two of the small triangular regions and four triangles that enclose four, making a total of 16.

10. **(E)** Let O be the origin, and let A and B denote the points where $y = 6$ intersects $y = x$ and $y = -x$ respectively. Let $\overline{OL}$ denote the altitude to side $\overline{AB}$ of $\triangle OAB$. Then $OL = 6$. Also, $AL = BL = 6$. Thus, the area of $\triangle OAB$ is



$$\frac{1}{2}(AB)(OL) = \frac{1}{2} \cdot 12 \cdot 6 = 36.$$

OR

Let $A' = (6,0)$. Then $\triangle A'OA \cong \triangle LOB$, so the area of triangle AOB equals the area of square $A'OLA$, which is $6^2 = 36$.

OR

Use the determinant formula for the area of the triangle: $\dfrac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ 6 & 6 & 1 \\ -6 & 6 & 1 \end{vmatrix} = 36.$

11. **(C)** Condition (i) requires that a be one of the two digits, 4 or 5. Condition (ii) requires that d be one of the two digits, 0 or 5. Condition (iii) requires that the ordered pair (b, c) be one of these six ordered pairs:

$$(3,4),\ (3,5),\ (3,6),\ (4,5),\ (4,6),\ (5,6).$$

Therefore, there are $2 \times 2 \times 6 = 24$ numbers N satisfying the conditions.

12. **(D)** Since f is a linear function, it has the form $f(x) = mx + b$. Because $f(1) \le f(2)$, we have $m \ge 0$. Similarly, $f(3) \ge f(4)$ implies $m \le 0$. Hence, $m = 0$, and f is a constant function. Thus, $f(0) = f(5) = 5$.

13. **(C)** The addition in the columns containing the ten-thousands and hundred-thousands digits is incorrect. The only digit common to both these columns is 2. Changing these 2's to 6's makes the arithmetic correct. Changing the other two 2's to 6's has no effect on the correctness of the remainder of the addition, and no digit other than 2 could be changed to make the addition correct. Thus, $d = 2$, $e = 6$, and $d + e = 8$.

14. **(E)** Since
$$\begin{aligned} f(3) &= a(\ \ 3)^4 - b(\ \ 3)^2 + 3 + 5 \\ \text{and} \quad f(-3) &= a(-3)^4 - b(-3)^2 - 3 + 5, \\ \hline \text{we have} \quad f(3) - f(-3) &= \qquad\qquad\qquad\quad 6. \end{aligned}$$

Thus, $f(3) = f(-3) + 6 = 2 + 6 = 8$.
Note. For any x, $f(x) - f(-x) = 2x$, so $f(x) = f(-x) + 2x$.

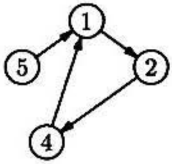
OR

Since $\quad 2 = f(-3) = 81a - 9b - 3 + 5$

we have $\quad b = 9a.$

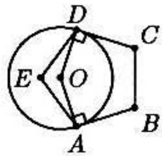
Thus $\quad f(3) = 81a - 9b + 3 + 5 = 81a - 9(9a) + 8 = 8.$

15. **(D)** With the first jump, the bug moves to point 1, with the second to 2, with the third to 4 and with the fourth it returns to 1. Thereafter, every third jump it returns to 1. Thus, after $n > 0$ jumps, the bug will be on 1, 2 or 4, depending on whether n is of the form $3k+1$, $3k+2$ or $3k$, respectively. Since $1995 = 3(665)$, the bug will be on point 4 after 1995 jumps.



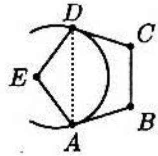
16. **(E)** Let A denote the number in attendance in Atlanta, and let B denote the number in attendance in Boston. We are given $45,000 \le A \le 55,000$ and $0.9B \le 60,000 \le 1.1B$, so $54,546 \le B \le 66,666$. Hence the largest possible difference between A and B is $66,666 - 45,000 = 21,666$, so the correct choice is **(E)**.

17. **(E)** Let O be the center of the circle. Since the sum of the interior angles in any n-gon is $(n-2)180°$, the sum of the angles in $ABCDO$ is $540°$. Since $\angle ABC = \angle BCD = 108°$ and $\angle OAB = \angle ODC = 90°$, it follows that the measure of $\angle AOD$, and thus the measure of minor arc AD, equals $144°$.



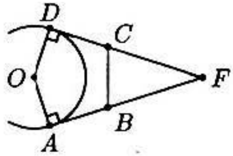
OR

Draw $\overline{AD}$. Since $\triangle AED$ is isosceles with $\angle AED = 108°$, it follows that $\angle EDA = \angle EAD = 36°$. Consequently, $\angle ADC = 108° - 36° = 72°$. Since $\angle ADC$ is a tangent-chord angle for the arc in question, the measure of the arc is $2(72°) = 144°$.



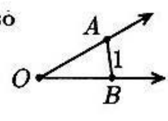
OR

Let O be the center of the circle, and extend $\overline{DC}$ and $\overline{AB}$ to meet at F. Since $\angle DCB = 108°$ and $\triangle BCF$ is isosceles, it follows that $\angle AFD = [180° - 2(180° - 108°)] = 36°$. Since $\angle ODF = \angle OAF = 90°$, in quadrilateral $OAFD$ we have angles AOD and AFD supplementary, so the measures of angle AOD and the minor arc AD are $180° - 36° = 144°$.



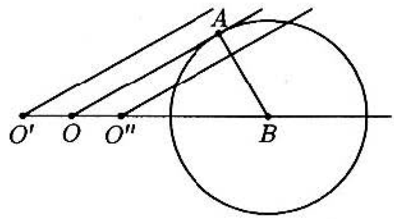
Note. A circle can be drawn tangent to two intersecting lines at given points on those lines if and only if those points are equidistant from the point of intersection of the lines.

18. **(D)** By the *Law of Sines*, $\dfrac{OB}{\sin \angle OAB} = \dfrac{AB}{\sin \angle AOB} = \dfrac{1}{1/2}$, so $OB = 2\sin \angle OAB \le 2 \sin 90° = 2$, with equality if and only if $\angle OAB = 90°$.



OR

Consider B to be fixed on a ray originating at a variable point O, and draw another ray so the angle at O is $30°$. A possible position for A is any intersection of this ray with the circle of radius 1 centered at B. The largest value for OB for which there *is* an intersection point A occurs when $\overline{OA}$ is tangent to the circle. Since $\triangle OBA$ is a $30°$-$60°$-$90°$ triangle with $AB = 1$, it follows that $OB = 2$ is largest.

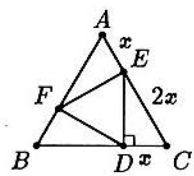


19. **(C)** Since CDE is a right triangle with $\angle C = 60°$, we have $CE = 2DC$. Also, $\angle BFD = 90° = \angle FEA$. To see that $\angle BFD = 90°$, note that

$$\angle BDF + \angle FDE + 90° = \angle BDF + 60° + 90° = 180°.$$

Thus $\angle BDF = 30°$ and since $\angle DBF = 60°$, $\angle BFD = 90°$. That $\angle FEA = 90°$ follows similarly. Since $\triangle DEF$ is equilateral, the three small triangles are congruent and $AE = DC$. Let $AC = 3x$. Then $EC = 2x$ and $DE = \sqrt{3}x$. The desired ratio is

$$\left(\frac{DE}{AC}\right)^2 = \left(\frac{\sqrt{3}x}{3x}\right)^2 = \frac{1}{3}.$$



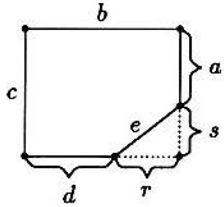
20. **(B)** The quantity $ab+c$ will be even if ab and c are both even or both odd. Furthermore, ab will be odd only when both a and b are odd, so the probability of ab being odd is $\frac{3}{5}\cdot\frac{3}{5} = \frac{9}{25}$. Thus the probability of ab being even is $1 - \frac{9}{25} = \frac{16}{25}$. Hence, the required probability is $\frac{16}{25}\cdot\frac{2}{5} + \frac{9}{25}\cdot\frac{3}{5} = \frac{59}{125}$.

21. **(E)** The diagonals of a rectangle are of the same length and bisect each other. The given diagonal has length $\sqrt{(-4-4)^2 + (-3-3)^2} = 10$ and midpoint $(0,0)$. The other diagonal must have end points on the circle of radius 5 centered at the origin and must have integer coordinates for each end point. We must find integer solutions to $x^2 + y^2 = 5^2$. The only possible diagonals, other than the given diagonal, are the segments:

$\overline{(0,5)(0,-5)}$, $\overline{(5,0)(-5,0)}$, $\overline{(3,4)(-3,-4)}$, $\overline{(-3,4)(3,-4)}$, $\overline{(4,-3)(-4,3)}$.

Each of these five, with the original diagonal, determines a rectangle.

22. **(E)** Let the sides of the pentagon be a, b, c, d and e, and let r and s be the legs of the triangular region cut off as shown. The equation $r^2 + s^2 = e^2$ has no solution in positive integers when $e = 19$ or $e = 31$. Therefore, e equals 13, 20 or 25, and the possibilities for $\{r, s, e\}$ are the well-known Pythagorean triples



$$\{5,12,13\}, \quad \{12,16,20\}, \quad \{15,20,25\}, \quad \{7,24,25\}.$$

Since 16, 15 and 24 do not appear among any of the pairwise differences of $\{13,19,20,25,31\}$, the only possibility is $\{5,12,13\}$. Then $a = 19$, $b = 25$, $c = 31$, $d = 20$ and $e = 13$.
Hence, the area of the pentagon is $31 \times 25 - \frac{1}{2}(12 \times 5) = 775 - 30 = 745$.

23. **(D)** Since the longest side of a triangle must be less than the sum of the other two sides, it follows that $4 < k < 26$. For the triangle to be obtuse, either $11^2 + 15^2 < k^2$, or $11^2 + k^2 < 15^2$. Therefore the 13 suitable values of k are 5, 6, 7, 8, 9, 10, 19, 20, 21, 22, 23, 24 and 25.

24. **(A)** Note that

$$C = A \log_{200} 5 + B \log_{200} 2 = \log_{200} 5^A + \log_{200} 2^B = \log_{200}\left(5^A \cdot 2^B\right),$$

so $200^C = 5^A \cdot 2^B$. Therefore, $5^A \cdot 2^B = 200^C = \left(5^2 \cdot 2^3\right)^C = 5^{2C}2^{3C}$. By uniqueness of prime factorization,† $A = 2C$ and $B = 3C$. Letting $C = 1$ we get $A = 2$, $B = 3$ and $A + B + C = 6$. The triplet $(A, B, C) = (2, 3, 1)$ is the only solution with no common factor greater than 1.

25. **(B)** Since the median and mode are both 8 and the range is 18, the list must take on one of these two forms:

	(I) :	$a, b, 8, 8, a+18$	where $a \le b \le 8 \le a + 18$
or	(II) :	$c, 8, 8, d, c+18$	where $c \le 8 \le d \le c + 18$.

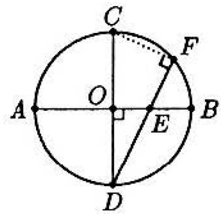
The sum of the five integers must be 60, since their mean is 12. In case (I), the requirement that $2a + b + 34 = 60$ contradicts $a, b \le 8$. In case (II), $2c + d + 34 = 60$ and $c \le 8 \le d \le c + 18$ lead to these six pairs, (c, d):

$$(8,10), \ (7,12), \ (6,14), \ (5,16), \ (4,18), \ (3,20).$$

Thus, the second largest entry in the list can be any of the six numbers $d = 10, 12, 14, 16, 18, 20$.

† An application of the *Fundamental Theorem of Arithmetic*.

26. **(C)** Draw segment $\overline{FC}$. Angle CFD is a right angle since arc CFD is a semicircle. Then right triangles DOE and DFC are similar, so



$$\frac{DO}{DF} = \frac{DE}{DC}.$$

Let $DO = r$ and $DC = 2r$. Substituting, we have

$$\frac{r}{8} = \frac{6}{2r}, \quad 2r^2 = 48, \quad r^2 = 24.$$

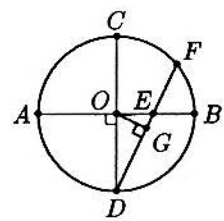
Then the area of the circle is $\pi r^2 = 24\pi$.

OR

Let $OA = OB = r$ and $OE = x$. Substituting into $AE \cdot EB = DE \cdot EF$ gives $(r+x)(r-x) = 6 \cdot 2$ so $r^2 - x^2 = 12$.
In right triangle EOD, $r^2 + x^2 = 36$.
Add to find $2r^2 = 48$. Thus, the area of the circle is $\pi r^2 = 24\pi$.

OR

Construct $\overline{OG} \perp \overline{DF}$ with G on $\overline{DF}$. Then $DG = \frac{1}{2}DF = 4$. Since $\overline{OG}$ is an altitude to the hypotenuse of right triangle EOD, we have $\dfrac{DE}{DO} = \dfrac{DO}{DG}$. Let $DO = r$. Then $\dfrac{6}{r} = \dfrac{r}{4}$, so $r^2 = 24$, and the area of the circle is $\pi r^2 = 24\pi$.



27. **(E)** Calculating the first five values of f,

$$f(1) = 0, \quad f(2) = 2, \quad f(3) = 6, \quad f(4) = 14, \quad f(5) = 30,$$

we are led to the conjecture that $f(n) = 2^n - 2$. We prove this by induction:

Observe that each of the interior numbers in row n is used twice and each of the end numbers is used once as a term in computing the interior terms of row $n+1$; i.e.,

$$f(n+1) = [2f(n) - 2(n-1)] + 2n = 2f(n) + 2,$$

so if $f(n) = 2^n - 2$, then $f(n+1) = 2f(n) + 2 = 2(2^n - 2) + 2 = 2^{n+1} - 2$.

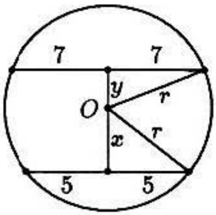
Therefore, we seek the remainder when $f(100) = 2^{100} - 2$ is divided by 100. Use the fact that 76^2 has remainder 76 when divided by 100.† We find

† **Query:** What other positive integers N have the property that N^2 has remainder N when divided by 100?

$$\begin{aligned} 2^{10} &= 100K + 24, \\ 2^{20} &= 100L + 76, \\ 2^{40} &= 100M + 76, \\ 2^{80} &= 100N + 76, \\ 2^{100} &= 100Q + 76, \end{aligned}$$

for positive integers K, L, M, N, Q, so $f(100) = 2^{100} - 2$ has remainder 74 when divided by 100.

28. **(E)** Let x be the distance from the center O of the circle to the chord of length 10, and let y be the distance from O to the chord of length 14. Let r be the radius. Then,

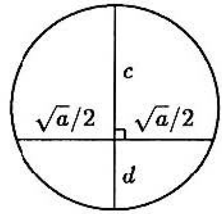


$$\begin{aligned} & x^2 + 25 = r^2, \\ & y^2 + 49 = r^2, \\ \text{so} \quad & x^2 + 25 = y^2 + 49. \\ \text{Therefore,} \quad & x^2 - y^2 = (x-y)(x+y) = 24. \end{aligned}$$

If the chords are on the same side of the center of the circle, $x - y = 6$. If they are on opposite sides, $x + y = 6$. But $x - y = 6$ implies that $x + y = 4$, which is impossible. Hence $x + y = 6$ and $x - y = 4$. Solve these equations simultaneously to get $x = 5$ and $y = 1$. Thus, $r^2 = 50$, and the chord parallel to the given chords and midway between them is 2 units from the center. If the chord is of length $2d$, then $d^2 + 4 = 50$, $d^2 = 46$, and $a = (2d)^2 = 184$.

OR

The diameter perpendicular to the chords is divided by the chord of length $\sqrt{a}$ into segments with lengths c and d as shown.



Then $$cd = \left(\frac{\sqrt{a}}{2}\right)^2 = \frac{a}{4}.$$

Treat the chords 3 units above and 3 units below similarly:

$$(c-3)(d+3) = \left(\frac{14}{2}\right)^2$$

$$(c+3)(d-3) = \left(\frac{10}{2}\right)^2.$$

Adding the last two equations, we get $2cd - 18 = 49 + 25 = 74$. Thus, $2cd = 92$ so $a = 4cd = 184$.

29. **(C)** Since the three factors, a, b and c, must be distinct, we seek the number of positive integer solutions to

$$abc = 2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11, \qquad \text{with} \qquad a < b < c.$$

The prime factors of a, b and c must be disjoint subsets of $S = \{2, 3, 5, 7, 11\}$, no more than one subset can be empty, and the union of the subsets must be S. The numbers of elements in the subsets can be: $0, 1, 4$; $0, 2, 3$; $1, 1, 3$; or $1, 2, 2$.

In the $0, 1, 4$ case, there are 5 ways to choose three subsets with these sizes. In the $0, 2, 3$ case, there are $\binom{5}{2} = 10$ ways to choose the three subsets.

In the $1, 1, 3$ case, there are $\binom{5}{3} = 10$ ways to choose the three subsets.

In the $1, 2, 2$ case, there are 5 ways of choosing the one-element subset and $\frac{1}{2} \cdot \binom{4}{2} = 3$ ways of dividing the remaining four elements into two subsets of two elements each, yielding 15 ways of choosing the three subsets in this case.

Thus there are a total of $5 + 10 + 10 + 15 = 40$ ways of choosing our three subsets and, therefore, 40 ways of expressing 2310 in the required manner. Since factorization into primes is unique, these 40 triplets of sets give distinct solutions.

OR

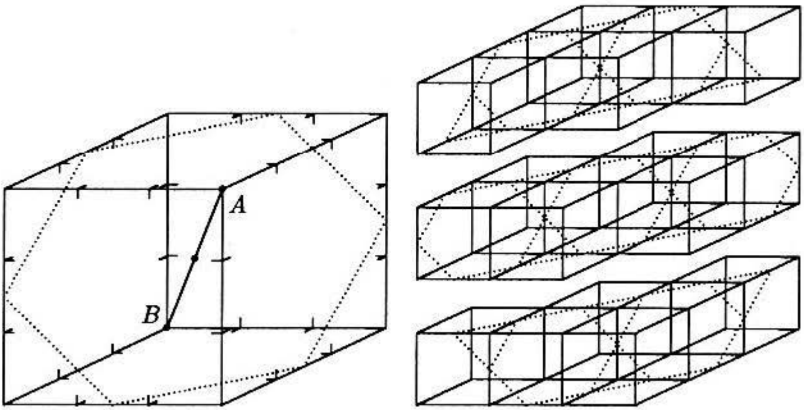
There are $3^5 = 243$ ordered triples, (a, b, c), of integers such that $abc = 2310$, since each of the five prime factors of $2310 = 2 \cdot 3 \cdot 5 \cdot 7 \cdot 11$ divides exactly one of a, b or c. In three of these 243 ordered triples, two of a, b, c equal 1. In the remaining 240 ordered triples, a, b and c are distinct, since 2310 is square-free. Each unordered triple whose product is 2310 is represented by $3! = 6$ of the 240 ordered triples (a, b, c), so the answer is $240/6 = 40$.

30. **(D)** Suppose the coordinates of the vertices of the unit cubes occur at (i, j, k) for all $i, j, k \in \{0, 1, 2, 3\}$. The equation of the plane that bisects the large cube's diagonal from $(0,0,0)$ to $(3,3,3)$ is $x + y + z = 9/2$. That plane meets a unit cube if and only if the ends of the unit cube's diagonal from (i, j, k) to $(i+1, j+1, k+1)$ lie on opposite sides of the plane. Therefore, this problem is equivalent to counting the number of the 27 triples (i, j, k) with $i, j, k \in \{0, 1, 2\}$ for which $i + j + k < 4.5 < i + j + k + 3$. Only 8 of these 27 triples do not satisfy these inequalities:

$$(0,0,0),\ (1,0,0),\ (0,1,0),\ (0,0,1),\ (1,2,2),\ (2,1,2),\ (2,2,1),\ (2,2,2).$$

Therefore, $27 - 8 = 19$ of the unit cubes are intersected by the plane.

A sketch can help you visualize the 19 unit cubes intersected by the plane. Suppose the plane is perpendicular to the interior diagonal $\overline{AB}$ at its midpoint. That plane intersects the surface of the large cube in a regular hexagon.



The sketch shows that nineteen of the twenty-seven unit cubes are intersected by this plane, with six each in the bottom and top layers and seven in the middle layer. The corner unit cube at vertex A and the three unit cubes adjacent to it are missed by this plane, as are the four symmetric to these at vertex B.

AMERICAN MATHEMATICS COMPETITIONS

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

47th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)

THURSDAY, FEBRUARY 15, 1996

Sponsored by

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1. **(D)** The mistake occurs in the tens column where any of the digits of the addends can be decreased by 1 or the 5 in the sum changed to 6 to make the addition correct. The largest of these digits is 7.

2. **(A)** Walter gets an extra \$2 per day for doing chores exceptionally well. If he never did them exceptionally well, he would get \$30 for 10 days of chores. The extra \$6 must be for 3 days of exceptional work.

3. **(E)** $\dfrac{(3!)!}{3!} = \dfrac{6!}{3!} = 6 \cdot 5 \cdot 4 = 120.$

OR

$\dfrac{(3!)!}{3!} = \dfrac{6!}{6} = 5! = 5 \cdot 4 \cdot 3 \cdot 2 = 120.$

4. **(D)** The largest possible median will occur when the three numbers not given are larger than those given. Let a, b, and c denote the three missing numbers, where $9 \le a \le b \le c$. Ranked from smallest to largest, the list is

$$3, 5, 5, 7, 8, 9, a, b, c,$$

so the median value is 8.

5. **(E)** The largest fraction is the one with largest numerator and smallest denominator. Choice **(E)** has both.

6. **(E)** Since $0^z = 0$ for any $z > 0$, $f(0) = f(-2) = 0$. Since $(-1)^0 = 1$,

$$f(0) + f(-1) + f(-2) + f(-3) = (-1)^0(1)^2 + (-3)^{-2}(-1)^0 = 1 + \frac{1}{(-3)^2} = \frac{10}{9}.$$

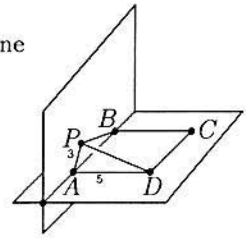
7. **(B)** The sum of the children's ages is 10 because $\$9.45 - \$4.95 = \$4.50 = 10 \times \$0.45$. If the twins were 3 years old or younger, then the third child would not be the youngest. If the twins are 4, the youngest child must be 2.

8. **(D)** Since $3 = k \cdot 2^r$ and $15 = k \cdot 4^r$, we have

$$5 = \frac{15}{3} = \frac{k \cdot 4^r}{k \cdot 2^r} = \frac{2^{2r}}{2^r} = 2^r.$$

Thus, by definition, $r = \log_2 5$.

9. **(B)** Since line segment AD is perpendicular to the plane of PAB, angle PAD is a right angle. In right triangle PAD, $PA = 3$ and $AD = AB = 5$. By the Pythagorean Theorem, $PD = \sqrt{3^2 + 5^2} = \sqrt{34}$. The fact that $PB = 4$ was not needed.

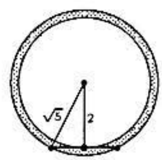


10. **(D)** There are 12 edges, 12 face diagonals, and 4 space diagonals for a total of $12 + 12 + 4 = 28$.

OR

Each pair of vertices of the cube determines a line segment. There are $\binom{8}{2} = \frac{8!}{(8-2)!2!} = \frac{8\cdot 7}{2} = 28$ such pairs.

11. **(D)** The endpoints of each of these line segments are at distance $\sqrt{2^2 + 1^2} = \sqrt{5}$ from the center of the circle. The region is therefore an annulus with inner radius 2 and outer radius $\sqrt{5}$. The area covered is $\pi(\sqrt{5})^2 - \pi(2)^2 = \pi$.



Note. The area of the annular region covered by the segments of length 2 does not depend on the radius of the circle.

12. **(B)** Since k is odd, $f(k) = k + 3$. Since $k + 3$ is even,

$$f(f(k)) = f(k+3) = (k + 3)/2.$$

If $(k + 3)/2$ is odd, then

$$27 = f(f(f(k))) = f((k+3)/2) = (k + 3)/2 + 3,$$

which implies that $k = 45$. This is not possible because $f(f(f(45))) = f(f(48)) = f(24) = 12$. Hence $(k + 3)/2$ must be even, and

$$27 = f(f(f(k))) = f((k+3)/2) = (k + 3)/4,$$

which implies that $k = 105$. Checking, we find that

$$f(f(f(105))) = f(f(108)) = f(54) = 27.$$

Hence the sum of the digits of k is $1 + 0 + 5 = 6$.

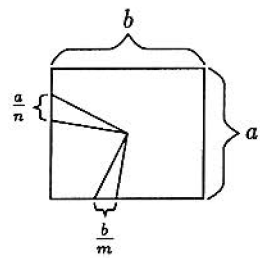
13. **(D)** Let x be the number of meters that Moonbeam runs to overtake Sunny, and let r and mr be the rates of Sunny and Moonbeam, respectively. Because Sunny runs $x-h$ meters in the same time that Moonbeam runs x meters, it follows that $\dfrac{x-h}{r}=\dfrac{x}{mr}$. Solving for x, we get $x=\dfrac{hm}{m-1}$.

14. **(C)** Since $E(100)=E(00)$, the result is the same as $E(00)+E(01)+E(02)+E(03)+\cdots+E(99)$, which is the same as

$$E(00010203\ldots 99).$$

There are 200 digits, and each digit occurs 20 times, so the sum of the even digits is $20(0+2+4+6+8)=20(20)=400$.

15. **(B)** Let the base of the rectangle be b and the height a. Triangle A has an altitude of length $b/2$ to a base of length a/n, and triangle B has an altitude of length $a/2$ to a base of length b/m. Thus the required ratio of areas is



$$\frac{\frac{1}{2}\cdot\frac{a}{n}\cdot\frac{b}{2}}{\frac{1}{2}\cdot\frac{b}{m}\cdot\frac{a}{2}}=\frac{m}{n}.$$

16. **(D)** There are 15 ways in which the third outcome is the sum of the first two outcomes.

$\boxed{(1,1,2)}$	$\boxed{(2,1,3)}$	$(3,1,4)$	$(4,1,5)$	$(5,1,6)$
$\boxed{(1,2,3)}$	$\boxed{(2,2,4)}$	$\boxed{(3,2,5)}$	$\boxed{(4,2,6)}$	
$(1,3,4)$	$\boxed{(2,3,5)}$	$(3,3,6)$		
$(1,4,5)$	$\boxed{(2,4,6)}$			
$(1,5,6)$				

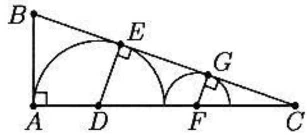
Since the three tosses are independent, all of the 15 possible outcomes are equally likely. At least one "2" appears in exactly eight of these outcomes, so the required probability is $8/15$.

17. **(E)** In the 30°-60°-90° triangle CEB, $BC=6\sqrt{3}$. Therefore, $FD=AD-AF=6\sqrt{3}-2$. In the 30°-60°-90° triangle CFD, $CD=FD\sqrt{3}=18-2\sqrt{3}$. The area of rectangle $ABCD$ is

$$(BC)(CD)=\left(6\sqrt{3}\right)\left(18-2\sqrt{3}\right)=108\sqrt{3}-36\approx 151.$$

18. **(D)** Let D and F denote the centers of the circles. Let C and B be the points where the x-axis and y-axis intersect the tangent line, respectively. Let E and G denote the points of tangency as shown. We know that $AD = DE = 2$, $DF = 3$, and $FG = 1$. Let $FC = u$ and $AB = y$. Triangles FGC and DEC are similar, so

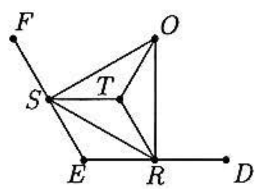
$$\frac{u}{1} = \frac{u+3}{2},$$



which yields $u = 3$. Hence, $GC = \sqrt{8}$. Also, triangles BAC and FGC are similar, which yields

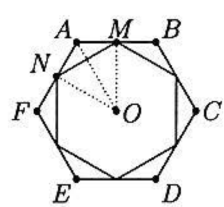
$$\frac{y}{1} = \frac{BA}{FG} = \frac{AC}{GC} = \frac{8}{\sqrt{8}} = \sqrt{8} = 2\sqrt{2}.$$

19. **(D)** Let R and S be the vertices of the smaller hexagon adjacent to vertex E of the larger hexagon, and let O be the center of the hexagons. Then, since $\angle ROS = 60°$, quadrilateral $ORES$ encloses $1/6$ of the area of $ABCDEF$, $\triangle ORS$ encloses $1/6$ of the area of the smaller hexagon, and $\triangle ORS$ is equilateral. Let T be the center of $\triangle ORS$. Then triangles TOR, TRS, and TSO are congruent isosceles triangles with largest angle $120°$. Triangle ERS is an isosceles triangle with largest angle $120°$ and a side in common with $\triangle TRS$, so $ORES$ is partitioned into four congruent triangles, exactly three of which form $\triangle ORS$. Since the ratio of the area enclosed by the small regular hexagon to the area of $ABCDEF$ is the same as the ratio of the area enclosed by $\triangle ORS$ to the area enclosed by $ORES$, the ratio is $3/4$.

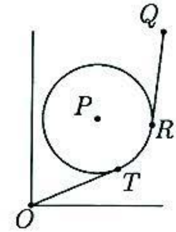


OR

Let M and N denote the midpoints of $\overline{AB}$ and $\overline{AF}$, respectively. Then $MN = AM\sqrt{3}$ since $\triangle AMO$ is a $30°$-$60°$-$90°$ triangle and $MN = MO$. It follows that the hexagons are similar, with similarity ratio $\frac{1}{2}\sqrt{3}$. Thus the desired quotient is $(\frac{1}{2}\sqrt{3})^2 = \frac{3}{4}$.

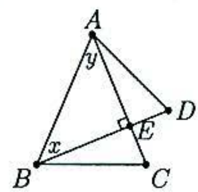


20. **(C)** Let $O = (0,0), P = (6,8)$, and $Q = (12,16)$. As shown in the figure, the shortest route consists of tangent $\overline{OT}$, minor arc TR, and tangent $\overline{RQ}$. Since $OP = 10$, $PT = 5$, and $\angle OTP$ is a right angle, it follows that $\angle OPT = 60°$ and $OT = 5\sqrt{3}$. By similar reasoning, $\angle QPR = 60°$ and $QR = 5\sqrt{3}$. Because O, P, and Q are collinear (why?), $\angle RPT = 60°$, so arc TR is of length $5\pi/3$. Hence the length of the shortest route is $2\left(5\sqrt{3}\right) + \dfrac{5\pi}{3}$.



21. **(D)** Let $\angle ABD = x$ and $\angle BAC = y$. Since the triangles ABC and ABD are isosceles, $\angle C = (180° - y)/2$ and $\angle D = (180° - x)/2$. Then, noting that $x + y = 90°$, we have

$$\angle C + \angle D = (360° - (x + y))/2 = 135°.$$

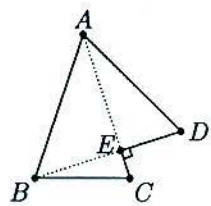


OR

Consider the interior angles of pentagon $ADECB$. Since triangles ABC and ABD are isosceles, $\angle C = \angle B$ and $\angle D = \angle A$. Since $\overline{BD} \perp \overline{AC}$, the interior angle at E measures $270°$. Since $540°$ is the sum of the interior angles of any pentagon,

$$\begin{aligned}&\angle A + \angle B + \angle C + \angle D + \angle E\\ &= 2\angle C + 2\angle D + 270° = 540°,\end{aligned}$$

from which it follows that $\angle C + \angle D = 135°$.



22. **(B)** Because all quadruples are equally likely, we need only examine the six clockwise orderings of the points: $ACBD, ADBC, ABCD, ADCB, ABDC$, and $ACDB$. Only the first two of these equally likely orderings satisfy the intersection condition, so the probability is $2/6 = 1/3$.

23. **(B)** Let a, b, and c be the dimensions of the box. It is given that

$$140 = 4a + 4b + 4c \qquad \text{and} \qquad 21 = \sqrt{a^2 + b^2 + c^2},$$

hence

$$35 = a + b + c \qquad (1) \qquad \text{and} \qquad 441 = a^2 + b^2 + c^2 \qquad (2).$$

Square both sides of (1) and combine with (2) to obtain

$$\begin{aligned} 1225 &= (a+b+c)^2 \\ &= a^2 + b^2 + c^2 + 2ab + 2bc + 2ca \\ &= 441 + 2ab + 2bc + 2ca. \end{aligned}$$

Thus the surface area of the box is $2ab + 2bc + 2ca = 1225 - 441 = 784$.

24. **(B)** The k^{th} 1 is at position

$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}$$

and $\dfrac{49(50)}{2} < 1234 < \dfrac{50(51)}{2}$, so there are 49 1's among the first 1234 terms. All the other terms are 2's, so the sum is $1234(2) - 49 = 2419$.

OR

The sum of all the terms through the occurrence of the k^{th} 1 is

$$\begin{aligned} & 1 + (2+1) + (2+2+1) + \cdots + (\underbrace{2+2+\cdots+2}_{k-1}+1) \\ = \;& 1 + 3 + 5 + \cdots + (2k-1) \\ = \;& k^2. \end{aligned}$$

The k^{th} 1 is at position

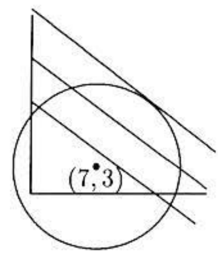
$$1 + 2 + 3 + \cdots + k = \frac{k(k+1)}{2}.$$

It follows that the last 1 among the first 1234 terms of the sequence occurs at position 1225 for $k = 49$. Thus, the sum of the first 1225 terms is $49^2 = 2401$, and the sum of the next nine terms, all of which are 2's, is 18, for a total of $2401 + 18 = 2419$.

25. **(B)** The equation $x^2 + y^2 = 14x + 6y + 6$ can be written

$$(x-7)^2 + (y-3)^2 = 8^2,$$

which defines a circle of radius 8 centered at $(7,3)$. If k is a possible value of $3x + 4y$ for (x, y) on the circle, then the line $3x + 4y = k$ must intersect the circle in at least one point. The largest value of k occurs when the line is tangent to the circle, and is therefore perpendicular to the radius at the point of tangency. Because the slope of the tangent line is $-3/4$, the slope of the radius is $4/3$. It follows that the point on the circle that yields the maximum value of $3x + 4y$ is one of the two points of tangency,



$$x = 7 + \frac{3 \cdot 8}{5} = \frac{59}{5}, \qquad y = 3 + \frac{4 \cdot 8}{5} = \frac{47}{5},$$

or

$$x = 7 - \frac{3 \cdot 8}{5} = \frac{11}{5}, \qquad y = 3 - \frac{4 \cdot 8}{5} = -\frac{17}{5}.$$

The first point of tangency gives

$$3x + 4y = 3 \cdot \frac{59}{5} + 4 \cdot \frac{47}{5} = \frac{177}{5} + \frac{188}{5} = 73,$$

and the second one gives $3x + 4y = \frac{33}{5} - \frac{68}{5} = -7$. Thus 73 is the desired maximum, while -7 is the minimum.

OR

Suppose that $k = 3x + 4y$ is a possible value. Substituting $y = (k - 3x)/4$ into $x^2 + y^2 = 14x + 6y + 6$, we get $16x^2 + (k - 3x)^2 = 224x + 24(k - 3x) + 96$, which simplifies to

$$25x^2 - 2(3k + 76)x + (k^2 - 24k - 96) = 0. \qquad (1)$$

If the line $3x + 4y = k$ intersects the given circle, the discriminant of (1) must be nonnegative. Thus we get $(3k + 76)^2 - 25(k^2 - 24k - 96) \geq 0$, which simplifies to

$$(k - 73)(k + 7) \leq 0.$$

Hence $-7 \leq k \leq 73$.

26. **(B)** The hypothesis of equally likely events can be expressed as

$$\frac{\binom{r}{4}}{\binom{n}{4}} = \frac{\binom{r}{3}\binom{w}{1}}{\binom{n}{4}} = \frac{\binom{r}{2}\binom{w}{1}\binom{b}{1}}{\binom{n}{4}} = \frac{\binom{r}{1}\binom{w}{1}\binom{b}{1}\binom{g}{1}}{\binom{n}{4}}$$

where r, w, b, and g denote the number of red, white, blue, and green marbles, respectively, and $n = r + w + b + g$. Eliminating common terms and solving for r in terms of w, b, and g, we get

$$r - 3 = 4w, \qquad r - 2 = 3b, \quad \text{and} \quad r - 1 = 2g.$$

The smallest r for which w, b, and g are all positive integers is $r = 11$, with corresponding values $w = 2$, $b = 3$, and $g = 5$. So the smallest total number of marbles is $11 + 2 + 3 + 5 = 21$.

27. **(D)** From the description of the first ball we find that $z \geq 9/2$, and from that of the second, $z \leq 11/2$. Because z must be an integer, the only possible lattice points in the intersection are of the form $(x, y, 5)$. Substitute $z = 5$ into the inequalities defining the balls:

$$x^2 + y^2 + \left(z - \frac{21}{2}\right)^2 \leq 6^2 \qquad \text{and} \qquad x^2 + y^2 + (z-1)^2 \leq \left(\frac{9}{2}\right)^2.$$

These yield

$$x^2 + y^2 + \left(-\frac{11}{2}\right)^2 \leq 6^2 \quad \text{and} \quad x^2 + y^2 + (4)^2 \leq \left(\frac{9}{2}\right)^2,$$

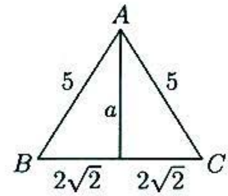
which reduce to

$$x^2 + y^2 \leq \frac{23}{4} \quad \text{and} \quad x^2 + y^2 \leq \frac{17}{4}.$$

If $(x, y, 5)$ satisfies the second inequality, then it must satisfy the first one. The only remaining task is to count the lattice points that satisfy the second inequality. There are 13:

$$\begin{array}{ccccc} (-2,0,5), & (2,0,5), & (0,-2,5), & (0,2,5), & (-1,-1,5), \\ (1,-1,5), & (-1,1,5), & (1,1,5), & (-1,0,5), & (1,0,5), \\ (0,-1,5), & (0,1,5), & \text{and} & (0,0,5). & \end{array}$$

28. **(C)** Let h be the required distance. Find the volume of pyramid $ABCD$ as a third of the area of a triangular base times the altitude to that base in two different ways, and equate these volumes. Use the altitude $\overline{AD}$ to $\triangle BCD$ to find that the volume is 8. Next, note that h is the length of the altitude of the pyramid from D to $\triangle ABC$. Since the sides of $\triangle ABC$ are 5, 5, and $4\sqrt{2}$, by the Pythagorean Theorem the altitude to the side of length $4\sqrt{2}$ is $a = \sqrt{17}$. Thus, the area of $\triangle ABC$ is $2\sqrt{34}$, and the volume of the pyramid is $2\sqrt{34}h/3$. Equating the volumes yields



$$2\sqrt{34}h/3 = 8, \quad \text{and thus} \quad h = 12/\sqrt{34} \approx 2.1.$$

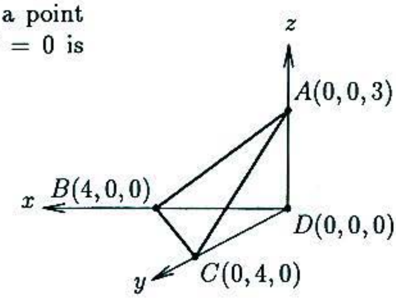
OR

Imagine the parallelepiped embedded in a coordinate system as shown in the diagram. The equation for the plane (in intercept form) is $\frac{x}{4} + \frac{y}{4} + \frac{z}{3} = 1$. Thus, it can be expressed as $3x + 3y + 4z - 12 = 0$. The formula for the distance d from a point (a, b, c) to the plane $Rx + Sy + Tz + U = 0$ is given by

$$d = \frac{|Ra + Sb + Tc + U|}{\sqrt{R^2 + S^2 + T^2}},$$

which in this case is

$$\frac{|-12|}{\sqrt{3^2 + 3^2 + 4^2}} = \frac{12}{\sqrt{34}} \approx 2.1.$$



29. **(C)** Let $2^{e_1}3^{e_2}5^{e_3}\cdots$ be the prime factorization of n. Then the number of positive divisors of n is $(e_1+1)(e_2+1)(e_3+1)\cdots$. In view of the given information, we have

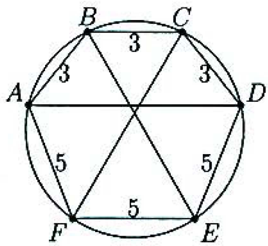
$$28 = (e_1+2)(e_2+1)P$$

and

$$30 = (e_1+1)(e_2+2)P,$$

where $P = (e_3+1)(e_4+1)\cdots$. Subtracting the first equation from the second, we obtain $2 = (e_1 - e_2)P$, so either $e_1 - e_2 = 1$ and $P = 2$, or $e_1 - e_2 = 2$ and $P = 1$. The first case yields $14 = (e_1+2)e_1$ and $(e_1+1)^2 = 15$; since e_1 is a nonnegative integer, this is impossible. In the second case, $e_2 = e_1 - 2$ and $30 = (e_1+1)e_1$, from which we find $e_1 = 5$ and $e_2 = 3$. Thus $n = 2^5 3^3$, so $6n = 2^6 3^4$ has $(6+1)(4+1) = 35$ positive divisors.

30. **(E)** In hexagon $ABCDEF$, let $AB = BC = CD = 3$ and let $DE = EF = FA = 5$. Since arc BAF is one third of the circumference of the circle, it follows that $\angle BCF = \angle BEF = 60°$. Similarly, $\angle CBE = \angle CFE = 60°$. Let P be the intersection of $\overline{BE}$ and $\overline{CF}$, Q that of $\overline{BE}$ and $\overline{AD}$, and R that of $\overline{CF}$ and $\overline{AD}$. Triangles EFP and BCP are equilateral, and by symmetry, triangle PQR is isosceles and thus also equilateral.
Furthermore, $\angle BAD$ and $\angle BED$ subtend the same arc, as do $\angle ABE$ and $\angle ADE$. Hence triangles ABQ and EDQ are similar. Therefore,



$$\frac{AQ}{EQ} = \frac{BQ}{DQ} = \frac{AB}{ED} = \frac{3}{5}.$$

It follows that

$$\frac{\frac{AD-PQ}{2}}{PQ+5} = \frac{3}{5} \quad \text{and} \quad \frac{3-PQ}{\frac{AD+PQ}{2}} = \frac{3}{5}.$$

Solving the two equations simultaneously yields $AD = 360/49$, so $m+n = 409$.

OR

In hexagon $ABCDEF$, let $AB = BC = CD = a$ and let $DE = EF = FA = b$. Let O denote the center of the circle, and let r denote the radius. Since the arc BAF is one-third of the circle, it follows that $\angle BAF = \angle FOB = 120°$. By using the Law of Cosines to compute BF two ways, we have $a^2+ab+b^2 = 3r^2$. Let $\angle AOB = 2\theta$. Then $a = 2r\sin\theta$, and

$$\begin{aligned} AD &= 2r\sin(3\theta) \\ &= 2r\sin\theta\cdot(3-4\sin^2\theta) \\ &= a\left(3-\frac{a^2}{r^2}\right) \\ &= 3a\left(1-\frac{a^2}{a^2+ab+b^2}\right) \\ &= \frac{3ab(a+b)}{a^2+ab+b^2}. \end{aligned}$$

Substituting $a = 3$ and $b = 5$, we get $AD = 360/49$, so $m+n = 409$.

OR

In hexagon $ABCDEF$, let $AB = BC = CD = 3$ and minor arcs AB, BC, and CD each be x°, let $DE = EF = FA = 5$ and minor arcs DE, EF, and FA each be y°. Then

$$3x^\circ + 3y^\circ = 360^\circ, \quad \text{so} \quad x^\circ + y^\circ = 120^\circ.$$

Therefore, $\angle BAF = 120^\circ$, so $BF^2 = 3^2 + 5^2 - 2 \cdot 3 \cdot 5 \cos 120^\circ = 49$ by the Law of Cosines, so $BF = 7$. Similarly, $CE = 7$. Using Ptolemy's Theorem in quadrilateral $BCEF$, we have

$$BE \cdot CF = CF^2 = 15 + 49 = 64 \quad \text{so} \quad CF = 8.$$

Using Ptolemy's Theorem in quadrilateral $ABCF$, we find $AC = 39/7$. Finally, using Ptolemy's Theorem in quadrilateral $ABCD$, we have $AC^2 = 3(AD) + 9$ and, since $AC = 39/7$, we have $AD = 360/49$ and $m + n = 409$.

Note. *Ptolemy's Theorem:* If a quadrilateral is inscribed in a circle, the product of the diagonals equals the sum of the products of the opposite sides.

AMERICAN MATHEMATICS COMPETITIONS

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

48th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)

THURSDAY, FEBRUARY 13, 1997

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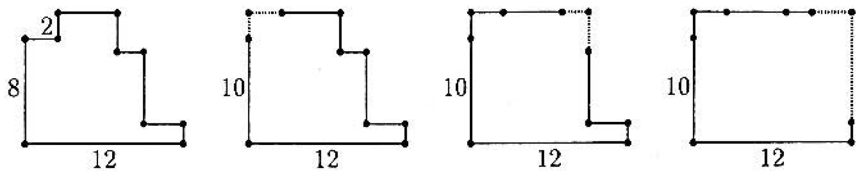
Note: Throughout this pamphlet, $[P_1P_2\dots P_n]$ denotes the area enclosed by polygon $P_1P_2\dots P_n$.

1. **(C)** Since $\mathtt{a}\times 3$ has units digit 9, $\mathtt{a}$ must be 3. Hence $\mathtt{b}\times 3$ has units digit 2, so $\mathtt{b}$ must be 4. Thus, $\mathtt{a}+\mathtt{b}=7$.

2. **(D)** Each polygon in the sequence below has the same perimeter, which is 44.



3. **(D)** Since each summand is nonnegative, the sum is zero only when each term is zero. Hence the only solution is $x=3$, $y=4$, and $z=5$, so the desired sum is 12.

4. **(A)** If a is 50% larger than c, then $a=1.5c$. If b is 25% larger than c, then $b=1.25c$. So $\frac{a}{b}=\frac{1.5c}{1.25c}=\frac{6}{5}=1.20$, and $a=1.20b$. Therefore, a is 20% larger than b.

5. **(C)** Let x and y denote the width and height of one of the five rectangles, with $x<y$. Then $5x+4y=176$ and $3x=2y$. Solve simultaneously to get $x=16$ and $y=24$. The perimeter in question is $2\cdot 16+2\cdot 24=80$.

6. **(B)** The 200 terms can be grouped into 100 odd-even pairs, each with a sum of -1. Thus the sum of the first 200 terms is $-1\cdot 100=-100$, and the average of the first 200 terms is $-100/200=-0.5$.

7. **(D)** Not all seven integers can be larger than 13. If six of them were each 14, then the seventh could be $-(6\times 14)-1$, so that the sum would be -1.

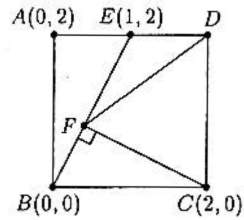
8. **(D)** The cost of 25 books is $C(25)=25\times \$11=\$275$. The cost of 24 books is $C(24)=24\times \$12=\$288$, while 23 and 22 books cost $C(23)=23\times \$12=\$276$ and $C(22)=22\times \$12=\$264$, respectively. Thus it is cheaper to buy 25 books than 23 or 24 books. Similarly, 49 books cost less than 45, 46, 47, or 48 books. In these six cases the total cost is reduced by ordering more books. There are no other cases.

OR

A discount of \$1 per book is given on orders of at least 25 books. This discount is larger than $2 \times \$12$, the cost of two books at the regular price. Thus, $n = 23$ and $n = 24$ are two values of n for which it is cheaper to order more books. Similarly, we receive an additional \$1 discount per book when we buy at least 49 books. This discount would enable us to buy 4 more books at \$10 per book, so there are 4 more values of n: $45, 46, 47,$ and 48, for a total of 6 values.

9. **(C)** In right triangle BAE, $BE = \sqrt{2^2 + 1^2} = \sqrt{5}$. Since $\Delta CFB \sim \Delta BAE$, it follows that $[CFB] = \left(\frac{CB}{BE}\right)^2 \cdot [BAE] = \left(\frac{2}{\sqrt{5}}\right)^2 \cdot \frac{1}{2}(2 \cdot 1) = \frac{4}{5}$. Then $[CDEF] = [ABCD] - [BAE] - [CFB] = 4 - 1 - \frac{4}{5} = \frac{11}{5}$.

OR



Draw the figure in the plane as shown with B at the origin. An equation of the line BE is $y = 2x$, and, since the lines are perpendicular, an equation of the line CF is $y = -\frac{1}{2}(x - 2)$. Solve these two equations simultaneously to get $F = (2/5, 4/5)$ and

$$[CDEF] = [DEF] + [CDF] = \frac{1}{2}(1)\left(2 - \frac{4}{5}\right) + \frac{1}{2}(2)\left(2 - \frac{2}{5}\right) = \frac{11}{5}.$$

10. **(D)** There are 36 equally likely outcomes as shown in the following table.

(1,1)	$\boxed{(1,2)}$	$\boxed{(1,4)}$	$\boxed{(1,4)}$	(1,5)	$\boxed{(1,6)}$
$\boxed{(2,1)}$	(2,2)	(2,4)	(2,4)	$\boxed{(2,5)}$	(2,6)
(3,1)	$\boxed{(3,2)}$	$\boxed{(3,4)}$	$\boxed{(3,4)}$	(3,5)	$\boxed{(3,6)}$
(3,1)	$\boxed{(3,2)}$	$\boxed{(3,4)}$	$\boxed{(3,4)}$	(3,5)	$\boxed{(3,6)}$
(5,1)	$\boxed{(5,2)}$	$\boxed{(5,4)}$	$\boxed{(5,4)}$	(5,5)	$\boxed{(5,6)}$
$\boxed{(6,1)}$	(6,2)	(6,4)	(6,4)	$\boxed{(6,5)}$	(6,6)

Exactly 20 of the outcomes have an odd sum. Therefore, the probability is $\frac{20}{36} = \frac{5}{9}$.

OR

The sum is odd if and only if one number is even and the other is odd. The probability that the first number is even and the second is odd is $\frac{1}{3} \cdot \frac{1}{3}$, and the probability that the first is odd and the second is even is $\frac{2}{3} \cdot \frac{2}{3}$. Therefore, the required probability is $(\frac{1}{3})^2 + (\frac{2}{3})^2 = \frac{5}{9}$.

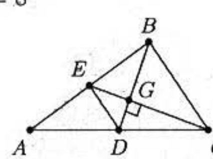
11. **(D)** The average for games six through nine was $(23 + 14 + 11 + 20)/4 = 17$, which exceeded her average for the first five games. Therefore, she scored at most $5 \cdot 17 - 1 = 84$ points in the first five games. Because her average after ten games was more than 18, she scored at least 181 points in the ten games, implying that she scored at least $181 - 84 - 68 = 29$ points in the tenth game.

12. **(E)** Since $mb > 0$, the slope and the y-intercept of the line are either both positive or both negative. In either case, the line slopes away from the positive x-axis and does not intersect it. The answer is therefore $(1997, 0)$. Note that the other four points lie on lines for which $mb > 0$. For example, $(0, 1997)$ lies on $y = x + 1997$; $(0, -1997)$ lies on $y = -x - 1997$; $(19, 97)$ lies on $y = 5x + 2$; and $(19, -97)$ lies on $y = -5x - 2$.

13. **(E)** Let $N = 10x + y$. Then $10x + y + 10y + x = 11(x + y)$ must be a perfect square. Since $1 \le x + y \le 18$, it follows that $x + y = 11$. There are eight such numbers: 29, 38, 47, 56, 65, 74, 83, and 92.

14. **(B)** Let x be the number of geese in 1996, and let k be the constant of proportionality. Then $x - 39 = 60k$ and $123 - 60 = kx$. Solve the second equation for k, and use that value to solve for x in the first equation, obtaining $x - 39 = 60 \cdot \frac{63}{x}$. Thus $x^2 - 39x - 3780 = 0$. Factoring yields $(x - 84)(x + 45) = 0$. Since x is positive, it follows that $x = 84$.

15. **(D)** Let the medians meet at G. Then $CG = (2/3)CE = 8$ and the area of triangle BCD is $(1/2)BD \cdot CG = (1/2) \cdot 8 \cdot 8 = 32$. Since BD is a median, triangles ABD and DBC have the same area. Hence the area of the triangle is 64.



OR

Since the medians are perpendicular, the area of the quadrilateral $BCDE$ is half the product of the diagonals $\frac{1}{2}(12)(8) = 48$. (Why?) However, D and E are midpoints, which makes the area of triangle AED one fourth of the area of triangle ABC. Thus the area of $BCDE$ is three fourths of the area of triangle ABC. It follows that the area of triangle ABC is 64.

16. **(D)** If only three entries are altered, then either two lines are not changed at all, or some entry is the only entry in its row and the only entry in its column that is changed. In either case, at least two of the six sums remain the same. However, four alterations are enough. For example, replacing 4 by 5, 1 by 3, 2 by 7, and 6 by 9 results in the array

$$\begin{bmatrix} 5 & 9 & 7 \\ 8 & 3 & 9 \\ 3 & 5 & 7 \end{bmatrix}$$

for which the six sums are all different.

17. **(A)** The line $x = k$ intersects $y = \log_5(x+4)$ and $y = \log_5 x$ at $(k, \log_5(k+4))$ and $(k, \log_5 k)$, respectively. Since the length of the vertical segment is 0.5,

$$0.5 = \log_5(k+4) - \log_5 k = \log_5 \frac{k+4}{k},$$

so $\frac{k+4}{k} = \sqrt{5}$. Solving for k yields $k = \frac{4}{\sqrt{5}-1} = 1 + \sqrt{5}$, so $a + b = 6$.

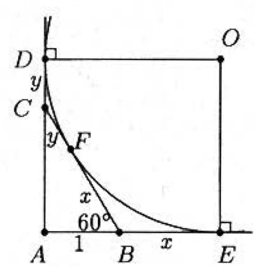
18. **(E)** When 10 is added to a number in the list, the mean increases by 2, so there must be five numbers in the original list whose sum is $5 \cdot 22 = 110$. Since 10 is the smallest number in the list and m is the median, we may assume

$$10 \le a \le m \le b \le c,$$

denoting the other members of the list by $a, b,$ and c. Since the mode is 32, we must have $b = c = 32$; otherwise, $10 + m + a + b + c$ would be larger than 110. So $a + m = 36$. Since decreasing m by 8 decreases the median by 4, a must be 4 less than m. Solving $a + m = 36$ and $m - a = 4$ for m gives $m = 20$.

19. **(D)** Let D and E denote the points of tangency on the y- and x-axes, respectively, and let $\overline{BC}$ be tangent to the circle at F. Tangents to a circle from a point are equal, so $BE = BF$ and $CD = CF$. Let $x = BF$ and $y = CF$. Because $x + y = BC = 2$, the radius of the circle is

$$\frac{(1+x)+(\sqrt{3}+y)}{2} = \frac{3+\sqrt{3}}{2} \approx 2.37.$$



OR

Let r be the radius of the circle. The area of square $AEOD$, r^2, may also be expressed as the sum of the areas of quadrilaterals $OFBE$ and $ODCF$ and triangle ABC. This is given by $rx + ry + \frac{\sqrt{3}}{2}$, where $x + y = 2$. Thus

$$r^2 = 2r + \frac{\sqrt{3}}{2}.$$

Solving for r using the quadratic formula yields the positive solution

$$r = 1 + \sqrt{1 + \frac{\sqrt{3}}{2}} \approx 2.37.$$

Note. The circle in question is called an *escribed* circle of the triangle ABC.

20. **(A)** Since

$$1 + 2 + 3 + \cdots + 100 = (100)(101)/2 = 5050,$$

it follows that the sum of any sequence of 100 consecutive positive integers starting with $a + 1$ is of the form

$$\begin{aligned}(a{+}1) + (a{+}2) + (a{+}3) + \cdots + (a{+}100) &= 100a + (1 + 2 + 3 + \cdots + 100) \\ &= 100a + 5050.\end{aligned}$$

Consequently, such a sum has 50 as its rightmost two digits. Choice **A** is the sum of the 100 integers beginning with 16,273,800.

21. **(C)** Since $\log_8 n = \frac{1}{3}(\log_2 n)$, it follows that $\log_8 n$ is rational if and only if $\log_2 n$ is rational. The nonzero numbers in the sum will therefore be all numbers of the form $\log_8 n$, where n is an integral power of 2. The highest power of 2 that does not exceed 1997 is 2^{10}, so the sum is:

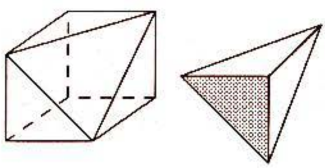
$$\log_8 1 + \log_8 2 + \log_8 2^2 + \log_8 2^3 + \cdots + \log_8 2^{10} =$$

$$0 + \frac{1}{3} + \frac{2}{3} + \frac{3}{3} + \cdots + \frac{10}{3} = \frac{55}{3}.$$

Challenge. Prove that $\log_2 3$ is irrational. Prove that, for every integer n, $\log_2 n$ is rational if and only if n is an integral power of 2.

22. **(E)** Let A, B, C, D, and E denote the amounts Ashley, Betty, Carlos, Dick, and Elgin had for shopping, respectively. Then $A - B = \pm 19$, $B - C = \pm 7$, $C - D = \pm 5$, $D - E = \pm 4$, and $E - A = \pm 11$. The sum of the left sides is zero, so the sum of the right sides must also be zero. In other words, we must choose some subset S of $\{4, 5, 7, 11, 19\}$ which has the same element-sum as its complement. Since $4 + 5 + 7 + 11 + 19 = 46$, the sum of the members of S is 23. Hence S is either the set $\{4, 19\}$ or its complement $\{5, 7, 11\}$. Thus either $A - B$ and $D - E$ are the only positive differences or $B - C$, $C - D$, and $E - A$ are. In the former case, expressing A, B, C, and D in terms of E, we get $5E + 6 = 56$, which yields $E = 10$. In the latter case, the same strategy yields $5E - 6 = 56$, which leads to non-integer values. Hence $E = 10$.

23. **(D)** The polyhedron is a unit cube with a corner cut off. The missing corner may be viewed as a pyramid whose altitude is 1 and whose base is an isosceles right triangle (shaded in the figure). The area of the base is $1/2$. The pyramid's volume is therefore $(1/3)(1/2)(1) = 1/6$, so the polyhedron's volume is $1 - 1/6 = 5/6$.



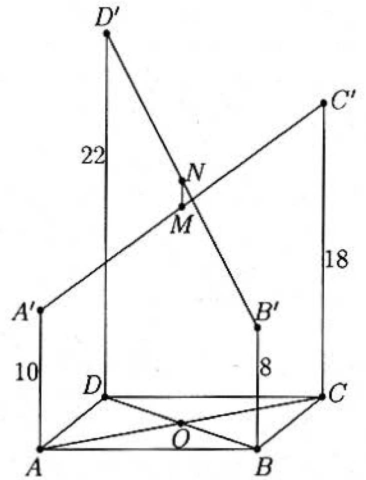
24. **(B)** The number of five-digit rising numbers that begin with 1 is $\binom{8}{4} = 70$, since the rightmost four digits must be chosen from the eight-member set $\{2, 3, 4, 5, 6, 7, 8, 9\}$, and, once they are chosen, they can be arranged in increasing order in just one way. Similarly, the next $\binom{7}{4} = 35$ integers in the list begin with 2. So the 97^{th} integer in the list is the 27^{th} among those that begin with 2. Among those that begin with 2, there are $\binom{6}{3} = 20$ that begin with 23 and $\binom{5}{3} = 10$ that begin with 24. Therefore, the 97^{th} is the 7^{th} of those that begin with 24. The first six of those beginning with 24 are 24567, 24568, 24569, 24578, 24579, 24589, and the seventh is 24678. The digit 5 is not used in the representation.

OR

As above, note that there are 105 integers in the list starting with either 1 or 2, so the 97^{th} one is ninth from the end. Count backwards: 26789, 25789, 25689, 25679, 25678, 24789, 24689, 24679, $\boxed{24678}$. Thus 5 is a missing digit.

25. **(B)** Let O be the intersection of $\overline{AC}$ and $\overline{BD}$. Then O is the midpoint of $\overline{AC}$ and $\overline{BD}$, so $\overline{OM}$ and $\overline{ON}$ are the midlines in trapezoids $ACC'A'$ and $BDD'B'$, respectively. Hence $OM = (10+18)/2 = 14$ and $ON = (8+22)/2 = 15$. Since $OM \| AA'$, $ON \| BB'$, and $AA' \| BB'$, it follows that O, M, and N are collinear. Therefore,

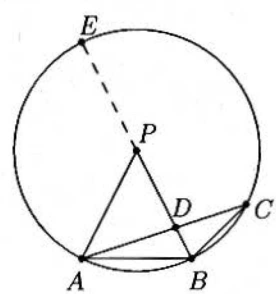
$$MN = |OM - ON| = |14 - 15| = 1.$$



Note. In general, if $AA' = a$, $BB' = b$, $CC' = c$, and $DD' = d$, then $MN = |a - b + c - d|/2$.

26. **(A)** Construct a circle with center P and radius PA. Then C lies on the circle, since the angle ACB is half angle APB. Extend $\overline{BP}$ through P to get a diameter $\overline{BE}$. Since A, B, C, and E are concyclic,

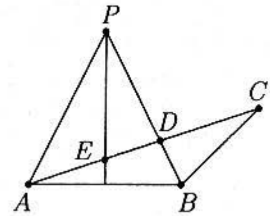
$$\begin{aligned} AD \cdot CD &= ED \cdot BD \\ &= (PE + PD)(PB - PD) \\ &= (3+2)(3-2) \\ &= 5. \end{aligned}$$



OR

Let E denote the point where $\overline{AC}$ intersects the angle bisector of angle APB. Note that $\triangle PED \sim \triangle CBD$. Hence $DE/2 = 1/DC$ so $DE \cdot DC = 2$. Apply the *Angle Bisector Theorem* to $\triangle APD$ to obtain

$$\frac{EA}{DE} = \frac{PA}{PD} = \frac{3}{2}.$$



Thus $DA \cdot DC = (DE + EA) \cdot DC = (DE + 1.5DE) \cdot DC = 2.5DE \cdot DC = 5.$

27. **(D)** We may replace x with $x+4$ in

$$f(x+4) + f(x-4) \;=\; f(x) \tag{1}$$

to get

$$f(x+8) + f(x) \;=\; f(x+4). \tag{2}$$

From (1) and (2), we deduce that $f(x+8) = -f(x-4)$. Replacing x with $x+4$, the latter equation yields $f(x+12) = -f(x)$. Now replacing x in this last equation with $x+12$ yields $f(x+24) = -f(x+12)$. Consequently, $f(x+24) = f(x)$ for all x, so that a least period p exists and is at most 24. On the other hand, the function $f(x) = \sin\left(\frac{\pi x}{12}\right)$ has fundamental period 24, and satisfies (1), so $p \geq 24$. Hence $p = 24$.

OR

Let x_0 be arbitrary, and let $y_k = f(x_0 + 4k)$ for $k = 0, 1, 2, \ldots$. Then $f(x+4) = f(x) - f(x-4)$ for all x implies $y_{k+1} = y_k - y_{k-1}$, so if $y_0 = a$ and $y_1 = b$, then $y_2 = b - a$, $y_3 = -a$, $y_4 = -b$, $y_5 = a - b$, $y_6 = a$, and $y_7 = b$. It follows that the sequence (y_k) is periodic with period 6 and, since x_0 was arbitrary, f is periodic with period 24. Since $f(x) = \sin(\frac{\pi x}{12})$ has fundamental period 24 and satisfies $f(x+4) + f(x-4) = f(x)$, it follows that $p \geq 24$. Hence $p = 24$.

28. **(E)** If $c \geq 0$, then $ab - |a+b| = 78$, so $(a-1)(b-1) = 79$ or $(a+1)(b+1) = 79$. Since 79 is prime, $\{a, b\}$ is $\{2, 80\}$, $\{-78, 0\}$, $\{0, 78\}$, or $\{-80, -2\}$. Hence $|a + b| = 78$ or $|a + b| = 82$, and, from the first equation in the problem statement, it follows that $c < 0$, a contradiction.

On the other hand, if $c < 0$, then $ab + |a + b| = 116$, so $(a + 1)(b + 1) = 117$ in the case that $a + b > 0$ and $(a - 1)(b - 1) = 117$ in the case that $a + b < 0$. Since $117 = 3^2 \cdot 13$, we distinguish the following cases:

$$\begin{aligned}
\{a, b\} = \{0, 116\} &\quad \text{yields} \quad c = -97; \\
\{a, b\} = \{2, 38\} &\quad \text{yields} \quad c = -21; \\
\{a, b\} = \{8, 12\} &\quad \text{yields} \quad c = -1; \\
\{a, b\} = \{-116, 0\} &\quad \text{yields} \quad c = -97; \\
\{a, b\} = \{-38, -2\} &\quad \text{yields} \quad c = -21; \\
\{a, b\} = \{-12, -8\} &\quad \text{yields} \quad c = -1.
\end{aligned}$$

Since a and b are interchangeable, each of these cases leads to two solutions, for a total of 12.

29. **(B)** Suppose $1 = x_1 + x_2 + \cdots + x_n$ where $x_1, x_2, \ldots, x_n$ are special and $n \leq 9$. For $k = 1, 2, 3, \ldots$, let a_k be the number of elements of $\{x_1, x_2, \ldots, x_n\}$ whose k^{th} decimal digit is 7. Then

$$1 = \frac{7a_1}{10} + \frac{7a_2}{10^2} + \frac{7a_3}{10^3} + \cdots,$$

which yields

$$\frac{1}{7} = 0.\overline{142857} = \frac{a_1}{10} + \frac{a_2}{10^2} + \frac{a_3}{10^3} + \cdots.$$

Hence $a_1 = 1, a_2 = 4, a_3 = 2, a_4 = 8$, etc. In particular, this implies that $n \geq 8$. On the other hand,

$x_1 = 0.\overline{700}$, $x_2 = x_3 = 0.\overline{07}$, $x_4 = x_5 = 0.\overline{077777}$, and $x_6 = x_7 = x_8 = 0.\overline{000777}$

are 8 special numbers whose sum is

$$\frac{700700 + 2(70707) + 2(77777) + 3(777)}{999999} = 1.$$

Thus the smallest n is 8.

30. **(C)** In order that $D(n) = 2$, the binary representation of n must consist of a block of 1's followed by a block of 0's followed by a block of 1's. Among the integers n with d-digit binary representations, how many are there for which $D(n) = 2$? If the 0's block consists of just one 0, there are $d - 2$ possible locations for the 0. If the block consists of multiple 0's, then there are $\binom{d-2}{2}$ such blocks, since only the first and last places for the 0's need to be identified. Thus there are $(d-2) + \frac{1}{2}(d-2)(d-3) = \frac{1}{2}(d-2)(d-1)$ values of n with d binary digits such that $D(n) = 2$. The binary representation of 97 has seven digits, so all the 3-, 4-, 5-, and 6-digit binary integers are less than 97. (We need not consider the 1- and 2-digit binary integers.) The sum of the values of $\frac{1}{2}(d-2)(d-1)$ for $d = 3, 4, 5$, and 6 is 20. We must also consider the 7-digit binary integers less than or equal to $1100001_2 = 97$. If the initial block of 1's contains three or more 1's, then the number would be greater than 97; by inspection, if there are one or two 1's in the initial 1's block, there are respectively five or one acceptable configurations of the 0's block. It follows that the number of solutions of $D(n) = 2$ within the required range is $20 + 5 + 1 = 26$.

OR

Note that $D(n) = 2$ holds exactly when the binary representation of n consists of an initial block of 1's, followed by a block of 0's, and then a final block of 1's. The number of nonnegative integers $n \le 2^7 - 1 = 127$ for which $D(n) = 2$ is thus $\binom{7}{3} = 35$, since for each n, the corresponding binary representation is given by selecting the position of the leftmost bit in each of the three blocks. If $98 \le n \le 127$, the binary representation of n is either (a) $110XXXX_2$ or (b) $111XXXX_2$. Consider those n's for which $D(n) = 2$. By the same argument as above, there are three of type (a), namely $1101111_2 = 111$, $1100111_2 = 103$, and $1100011_2 = 99$. There are $\binom{4}{2} = 6$ of type (b). It follows that the number of solutions of $D(n) = 2$ for which $1 \le n \le 97$ is $35 - (3 + 6) = 26$.

AMERICAN MATHEMATICS COMPETITIONS

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

49th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)

TUESDAY, FEBRUARY 10, 1998

Sponsored by

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1. (E) Only the rectangle that goes in position II must match on both vertical sides. Since rectangle D is the only one for which these matches exist, it must be the one that goes in position II. Hence the rectangle that goes in position I must be E.

2 9 E 7 0	5 7 D 4 8	1 4 A 6 9
0 1 B 3 6	8 3 C 5 2	

2. (E) We need to make the numerator large while making the denominator small. The smallest the denominator can be is $0+1=1$. The largest the numerator can be is $9+8=17$. The fraction $17/1$ is an integer, so $A+B=17$.

3. (D) The subtraction problem posed is equivalent to the addition problem

$$\begin{array}{r} 4\ 8\ \mathtt{b} \\ \underline{+\ \mathtt{c}\ 7\ 3} \\ 7\ \mathtt{a}\ 2 \end{array}$$

which is easier to solve. Since $\mathtt{b}+3=12$, $\mathtt{b}$ must be 9. Since $1+8+7$ has units digit $\mathtt{a}$, $\mathtt{a}$ must be 6. Because $1+4+\mathtt{c}=7$, $\mathtt{c}=2$. Hence $\mathtt{a}+\mathtt{b}+\mathtt{c}$ $=6+9+2=17$.

4. (E) Notice that the operation has the property that, for any r, a, b, and c,

$$[ra, rb, rc] = \frac{ra+rb}{rc} = [a, b, c].$$

Thus all three of the expressions $[60,30,90], [2,1,3]$, and $[10,5,15]$ have the same value, which is 1. So $[[60,30,90],[2,1,3],[10,5,15]] = [1,1,1] = 2$.

5. (C) Factor the left side of the given equation:

$$2^{1998}-2^{1997}-2^{1996}+2^{1995} = (2^3-2^2-2+1)2^{1995} = 3\cdot 2^{1995} = k\cdot 2^{1995},$$

so $k=3$.

6. (C) The number 1998 has prime factorization $2\cdot 3^3\cdot 37$. It has eight factor-pairs: $1\times 1998 = 2\times 999 = 3\times 666 = 6\times 333 = 9\times 222 = 18\times 111 = 27\times 74 =$ $37\times 54 = 1998$. Among these, the smallest difference is $54-37=17$.

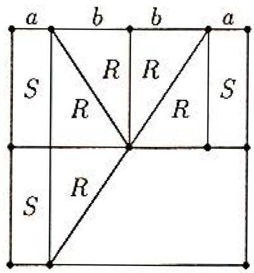
7. (D) $\sqrt[3]{N\sqrt[3]{N\sqrt[3]{N}}} = \sqrt[3]{N\sqrt[3]{N \cdot N^{\frac{1}{3}}}} = \sqrt[3]{N\sqrt[3]{N^{\frac{4}{3}}}} = \sqrt[3]{N \cdot N^{\frac{4}{9}}} = \sqrt[3]{N^{\frac{13}{9}}} = N^{\frac{13}{27}}$.

OR

$$\sqrt[3]{N\sqrt[3]{N\sqrt[3]{N}}} = \left(N\left(N(N)^{\frac{1}{3}}\right)^{\frac{1}{3}}\right)^{\frac{1}{3}} = \left(N\left(N^{\frac{1}{3}} \cdot N^{\frac{1}{9}}\right)\right)^{\frac{1}{3}} = N^{\frac{1}{3}} \cdot N^{\frac{1}{9}} \cdot N^{\frac{1}{27}} = N^{\frac{13}{27}}.$$

8. (D) The area of each trapezoid is 1/3, so $\frac{1}{2} \cdot \frac{1}{2}(x + \frac{1}{2}) = \frac{1}{3}$. Simplifying yields $x + \frac{1}{2} = \frac{4}{3}$, and it follows that $x = 5/6$.

OR



$2S + 2R = S + 3R$
$\therefore S = R$
$\therefore b = 2a$
$a + b + b + a = 1$
$\therefore 2b + a = 5/6$

9. (D) Let N be the number of people in the audience. Then $0.2N$ people heard 60 minutes, $0.1N$ heard 0 minutes, $0.35N$ heard 20 minutes, and $0.35N$ heard 40 minutes. In total, the N people heard

$$60(0.2N) + 0(0.1N) + 20(0.35N) + 40(0.35N) = 12N + 0 + 7N + 14N = 33N$$

minutes, so they heard an average of 33 minutes each.

10. (A) Let x and y denote the dimensions of the four congruent rectangles. Then $2x + 2y = 14$, so $x + y = 7$. The area of the large square is $(x + y)^2 = 7^2 = 49$.

11. (D) The four vertices determine six possible diameters, namely, the four sides and two diagonals. However, the two diagonals are diameters of the same circle. Thus there are five circles.

12. (A) Note that $N = 7^{5^{3^{2^{11}}}}$, which has only 7 as a prime factor.

13. (E) Factor 144 into primes, $144 = 2^4 \cdot 3^2$, and notice that there are at most two 6's and no 5's among the numbers rolled. If there are no 6's, then there must be two 3's since these are the only values that can contribute 3 to the prime factorization. In this case the four 2's in the factorization must be the result of two 4's in the roll. Hence the sum $3 + 3 + 4 + 4 = 14$ is a possible value for the sum. Next consider the case with just one 6. Then there must be one 3, and the three remaining 2's must be the result of a 4 and a 2. Thus, the sum $6 + 3 + 4 + 2 = 15$ is also possible. Finally, if there are two 6's, then there must also be two 2's or a 4 and a 1, with sums of $6+6+2+2 = 16$ and $6 + 6 + 4 + 1 = 17$. Hence 18 is the only sum not possible.

OR

Since 5 does not divide 144 and $6^3 > 144$, there can be no 5's and at most two 6's. Thus the only ways the four dice can have a sum of 18 are: $4, 4, 4, 6$; $2, 4, 6, 6$; and $3, 3, 6, 6$. Since none of these products is 144, the answer is (E).

14. (A) Because the parabola has x-intercepts of opposite sign and the y-coordinate of the vertex is negative, a must be positive, and c, which is the y-intercept, must be negative. The vertex has x-coordinate $-b/2a = 4 > 0$, so b must be negative.

15. (C) The regular hexagon can be partitioned into six equilateral triangles, each with area one-sixth of the original triangle. Since the original equilateral triangle is similar to each of these, and the ratio of the areas is 6, it follows that the ratio of the sides is $\sqrt{6}$.

16. (B) The area of the shaded region is

$$\frac{\pi}{2}\left(\left(\frac{a+b}{2}\right)^2 + \left(\frac{a}{2}\right)^2 - \left(\frac{b}{2}\right)^2\right) = \frac{\pi}{2}\frac{a+b}{2}\left(\frac{a+b}{2} + \frac{a-b}{2}\right) = \frac{\pi(a+b)a}{4}$$

and the area of the unshaded region is

$$\frac{\pi}{2}\left(\left(\frac{a+b}{2}\right)^2 - \left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2\right) = \frac{\pi}{2}\frac{a+b}{2}\left(\frac{a+b}{2} + \frac{b-a}{2}\right) = \frac{\pi(a+b)b}{4}.$$

Their ratio is a/b.

17. (E) Note that $f(x) = f(x+0) = x+f(0) = x+2$ for any real number x. Hence $f(1998) = 2000$. The function defined by $f(x) = x + 2$ has both properties: $f(0) = 2$ and $f(x + y) = x + y + 2 = x + (y + 2) = x + f(y)$.

OR

Note that

$$2 = f(0) = f(-1998 + 1998) = -1998 + f(1998).$$

Hence $f(1998) = 2000$.

18. (A) Suppose the sphere has radius r. We can write the volumes of the three solids as functions of r as follows:

$$\text{Volume of cone} = A = \frac{1}{3}\pi r^2(2r) = \frac{2}{3}\pi r^3,$$

$$\text{Volume of cylinder} = M = \pi r^2(2r) = 2\pi r^3, \text{ and}$$

$$\text{Volume of sphere} = C = \frac{4}{3}\pi r^3.$$

Thus, $A - M + C = 0$.

Note: The AMC logo is designed to show this classical result of Archimedes.

19. (C) The area of the triangle is $\frac{1}{2}$(base)(height) $= \frac{1}{2} \cdot (5 - (-5)) \cdot |5\sin\theta| = 25|\sin\theta|$. There are four values of θ between 0 and 2π for which $|\sin\theta| = 0.4$, and each value corresponds to a distinct triangle with area 10.

OR

The vertex $(5\cos\theta, 5\sin\theta)$ lies on a circle of diameter 10 centered at the origin. In order that the triangle have area 10, the altitude from that vertex must be 2. There are four points on the circle that are 2 units from the x-axis.

20. (C) There are eight ordered triples of numbers satisfying the conditions: $(1,2,10), (1,3,9), (1,4,8), (1,5,7), (2,3,8), (2,4,7), (2,5,6)$, and $(3,4,6)$. Because Casey's card gives Casey insufficient information, Casey must have seen a 1 or a 2. Next, Tracy must not have seen a $6, 9$, or 10, since each of these would enable Tracy to determine the other two cards. Finally, if Stacy had seen a 3 or a 5 on the middle card, Stacy would have been able to determine the other two cards. The only number left is 4, which leaves open the two possible triples $(1,4,8)$ and $(2,4,7)$.

21. (C) Let r be Sunny's rate. Thus $\frac{h}{r}$ and $\frac{h+d}{r}$ are the times it takes Sunny to cover h meters and $h+d$ meters, respectively. Because Windy covers only $h-d$ meters while Sunny is covering h meters, it follows that Windy's rate is $\frac{(h-d)r}{h}$. While Sunny runs $h+d$ meters, the number of meters Windy runs is $\frac{(h-d)r}{h} \cdot \frac{h+d}{r} = h - \frac{d^2}{h}$. Sunny's victory margin over Windy is $\frac{d^2}{h}$.

22. (C) Express each term using a base-10 logarithm, and note that the sum equals

$$\log 2/\log 100! + \log 3/\log 100! + \cdots + \log 100/\log 100! = \log 100!/\log 100! = 1.$$

OR

Since $1/\log_k 100!$ equals $\log_{100!} k$ for all positive integers k, the expression equals $\log_{100!}(2 \cdot 3 \cdot \cdots \cdot 100) = \log_{100!} 100! = 1$.

23. (D) Complete the squares in the two equations to bring them to the form

$$(x-6)^2 + (y-3)^2 = 7^2 \qquad \text{and} \qquad (x-2)^2 + (y-6)^2 = k + 40.$$

The graphs of these equations are circles. The first circle has radius 7, and the distance between the centers of the circles is 5. In order for the circles to have a point in common, therefore, the radius of the second circle must be at least 2 and at most 12. It follows that $2^2 \le k + 40 \le 12^2$, or $-36 \le k \le 104$. Thus $b - a = 140$.

24. (C) There are 10,000 ways to write the last four digits $d_4d_5d_6d_7$, and among these there are $10000 - 10 = 9990$ for which not all the digits are the same. For each of these, there are exactly two ways to adjoin the three digits $d_1d_2d_3$ to obtain a memorable number. There are ten memorable numbers for which the last four digits are the same, for a total of $2 \cdot 9990 + 10 = 19990$.

OR

Let A denote the set of telephone numbers for which $d_1d_2d_3$ and $d_4d_5d_6$ are identical and B the set for which $d_1d_2d_3$ is the same as $d_5d_6d_7$. A number $d_1d_2d_3$-$d_4d_5d_6d_7$ belongs to $A \cap B$ if and only if $d_1 = d_4 = d_5 = d_2 = d_6 = d_3 = d_7$. Hence, $n(A \cap B) = 10$. Thus, by the *Inclusion-Exclusion Principle*,

$$n(A \cup B) = n(A) + n(B) - n(A \cap B) = 10^3 \cdot 1 \cdot 10 + 10^3 \cdot 10 \cdot 1 - 10 = 19990.$$

25. (B) The crease in the paper is the perpendicular bisector of the segment that joins (0,2) to (4,0). Thus the crease contains the midpoint $(2,1)$ and has slope 2, so the equation $y = 2x - 3$ describes it. The segment joining (7,3) and (m,n) must have slope $-\frac{1}{2}$, and its midpoint $\left(\frac{7+m}{2}, \frac{3+n}{2}\right)$ must also satisfy the equation $y = 2x - 3$. It follows that

$$-\frac{1}{2} = \frac{n-3}{m-7} \quad \text{and} \quad \frac{3+n}{2} = 2 \cdot \frac{7+m}{2} - 3, \text{ so}$$

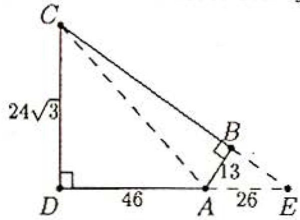
$$2n + m = 13 \quad \text{and} \quad n - 2m = 5.$$

Solve these equations simultaneously to find that $m = 3/5$ and $n = 31/5$, so that $m + n = 34/5 = 6.8$.

OR

As shown above, the crease is described by the equation $y = 2x - 3$. Therefore, the slope of the line through (m,n) and $(7,3)$ is $-1/2$, so the points on the line can be described parametrically by $(x,y) = (7 - 2t, 3 + t)$. The intersection of this line with the crease $y = 2x - 3$ is found by solving $3 + t = 2(7 - 2t) - 3$. This yields the parameter value $t = 8/5$. Since $t = 8/5$ determines the point on the crease, use $t = 2(8/5)$ to find the coordinates $m = 7 - 2(16/5) = 3/5$ and $n = 3 + (16/5) = 31/5$.

26. (B) Extend $\overline{DA}$ through A and $\overline{CB}$ through B and denote the intersection by E. Triangle ABE is a 30°-60°-90° triangle with $AB = 13$, so $AE = 26$. Triangle CDE is also a 30°-60°-90° triangle, from which it follows that $CD = (46 + 26)/\sqrt{3} = 24\sqrt{3}$. Now apply the *Pythagorean Theorem* to triangle CDA to find that $AC = \sqrt{46^2 + (24\sqrt{3})^2} = 62$.



OR

Since the opposite angles sum to a straight angle, the quadrilateral is cyclic, and AC is the diameter of the circumscribed circle. Thus AC is the diameter of the circumcircle of triangle ABD. By the *Extended Law of Sines*,

$$AC = \frac{BD}{\sin 120^\circ} = \frac{BD}{\sqrt{3}/2}.$$

We determine BD by the *Law of Cosines*:

$$BD^2 = 13^2 + 46^2 + 2 \cdot 13 \cdot 46 \cdot \frac{1}{2} = 2883 = 3 \cdot 31^2, \text{ so } BD = 31\sqrt{3}.$$

Hence $AC = 62$.

27. (E) After step one, twenty $3 \times 3 \times 3$ cubes remain, eight of which are corner cubes and twelve of which are edge cubes. At this stage each $3 \times 3 \times 3$ corner cube contributes 27 units of area and each $3 \times 3 \times 3$ edge cube contributes 36 units of area. The second stage of the tunneling process takes away 3 units of area from each of the eight $3 \times 3 \times 3$ corner cubes (1 for each exposed surface), but adds 24 units to the area (4 units for each of the six 1×1 center facial cubes removed). The twelve $3 \times 3 \times 3$ edge cubes each lose 4 units but gain 24 units. Therefore, the total surface area of the figure is

$$8 \cdot (27 - 3 + 24) + 12 \cdot (36 - 4 + 24) = 384 + 672 = 1056.$$

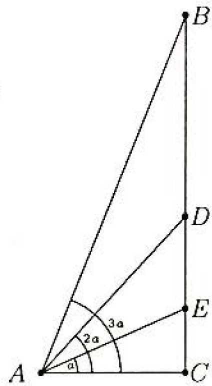
28. (B) Let E denote the point on $\overline{BC}$ for which $\overline{AE}$ bisects $\angle CAD$. Because the answer is not changed by a similarity transformation, we may assume that $AC = 2\sqrt{5}$ and $AD = 3\sqrt{5}$. Apply the *Pythagorean Theorem* to triangle ACD to obtain $CD = 5$, then apply the *Angle Bisector Theorem* to triangle CAD to obtain $CE = 2$ and $ED = 3$. Let $x = DB$. Apply the Pythagorean Theorem to triangle ACE to obtain $AE = \sqrt{24}$, then apply the Angle Bisector Theorem to triangle EAB to obtain $AB = (x/3)\sqrt{24}$. Now apply the Pythagorean Theorem to triangle ABC to get

$$(2\sqrt{5})^2 + (x+5)^2 = \left(\frac{x}{3}\sqrt{24}\right)^2,$$

from which it follows that $x = 9$. Hence $BD/DC = 9/5$, and $m + n = 14$.

OR

Denote by a the measure of angle CAE. Let $AC = 2u$, and $AD = 3u$. It follows that $CD = \sqrt{5}u$. We may assume $BD = \sqrt{5}$. (Otherwise, we could simply modify the triangle with a similarity transformation.) Hence, the ratio CD/BD we seek is just u. Since $\cos 2a = 2/3$, we have $\sin a = 1/\sqrt{6}$. Applying the *Law of Sines* in triangle ABD yields



$$\frac{\sin D}{AB} = \frac{\sin a}{\sqrt{5}} = \frac{2/3}{\sqrt{(2u)^2 + (\sqrt{5}(1+u))^2}} = \frac{1/\sqrt{6}}{\sqrt{5}}.$$

Solve this for u to get

$$\begin{aligned}
2\sqrt{5}\sqrt{6} &= 3\sqrt{4u^2 + 5(1 + 2u + u^2)} \\
120 &= 9(9u^2 + 10u + 5) \\
0 &= 27u^2 + 30u - 25 \\
0 &= (9u - 5)(3u + 5)
\end{aligned}$$

so $u = 5/9$ and $m + n = 14$.

OR

Again, let $a = \angle CAE$. We are given that $\cos 2a = 2/3$ and we wish to compute

$$\frac{CD}{BD} = \frac{AC \tan 2a}{AC(\tan 3a - \tan 2a)} = \left(\frac{\tan 3a}{\tan 2a} - 1\right)^{-1}.$$

Let $y = \tan a$. Trigonometric identities yield (upon simplification)

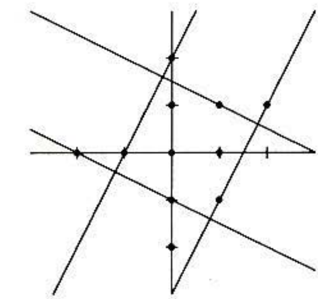
$$\left(\frac{\tan 3a}{\tan 2a} - 1\right)^{-1} = \frac{2(1-3y^2)}{(1+y^2)^2} \qquad \text{and} \qquad \frac{2}{3} = \cos 2a = \frac{1-y^2}{1+y^2}.$$

Thus $y^2 = 1/5$ and

$$\frac{CD}{BD} = \frac{2(1-3/5)}{(6/5)^2} = \frac{5}{9}.$$

Alternatively, starting with $a = \cos^{-1}(2/3)/2$, electronic calculation yields $\tan(3a)/\tan(2a) = 2.8 = 14/5$, so $CD/BD = 5/9$.

29. (D) If a square encloses three collinear lattice points, then it is not hard to see that the square must also enclose at least one additional lattice point. It therefore suffices to consider squares that enclose only the lattice points (0,0), (0,1), and (1,0). If a square had two adjacent sides, neither of which contained a lattice point, then the square could be enlarged slightly by moving those sides parallel to themselves. To be largest, therefore, a square must contain a lattice point on at least two non-adjacent sides. The desired square will thus have parallel sides that contain (1,1) and at least one of $(-1,0)$ and $(0,-1)$. The size of the square is determined by the separation between two parallel sides. Because the distance between parallel lines through (1,1) and $(0,-1)$ can be no larger than $\sqrt{5}$, the largest conceivable area for the square is 5. To see that this is in fact possible, draw the lines of slope 2 through $(-1,0)$ and $(1,-1)$, and the lines of slope $-1/2$ through $(1,1)$ and $(0,-1)$. These four lines can be described by the equations $y = 2x + 2$, $y = 2x - 3$, $2y + x = 3$, and $2y + x = -2$, respectively. They intersect to form a square whose area is **5**, and whose vertices are $(-1/5, 8/5)$, $(9/5, 3/5)$, $(4/5, -7/5)$, and $(-6/5, -2/5)$. There are only three lattice points inside this square.



30. (E) Factor a_n as a product of prime powers:

$$a_n = n(n+1)(n+2)\cdots(n+9) = 2^{e_1}3^{e_2}5^{e_3}\cdots.$$

Among the ten factors $n, n+1, \ldots, n+9$, five are even and their product can be written $2^5m(m+1)(m+2)(m+3)(m+4)$. If m is even then $m(m+2)(m+4)$ is divisible by 16 and thus $e_1 \geq 9$. If m is odd, then $e_1 \geq 8$. If $e_1 > e_3$, then the rightmost nonzero digit of a_n is even. If $e_1 \leq e_3$, then the rightmost nonzero digit of a_n is odd. Hence we seek the smallest n for which $e_3 \geq e_1$. Among the ten numbers $n, n+1, \ldots, n+9$, two are divisible by 5 and at most one of these is divisible by 25. Hence $e_3 \geq 8$ if and only if one of $n, n+1, \ldots, n+9$ is divisible by 5^7. The smallest n for which a_n satisfies $e_3 \geq 8$ is thus $n = 5^7 - 9$, but in this case the product of the five even numbers among $n, n+1, \ldots, n+9$ is $2^5m(m+1)(m+2)(m+3)(m+4)$ where m is even, namely $(5^7-9)/2 = 39058$. As noted earlier, this gives $e_1 \geq 9$. For $n = 5^7 - 8 = 78117$, the product of the five even numbers among $n, n+1, \ldots, n+9$ is $2^5m(m+1)(m+2)(m+3)(m+4)$ with $m = 39059$. Note that in this case $e_1 = 8$. Indeed, $39059 + 1$ is divisible by 4 but not by 8, and $39059 + 3$ is divisible by 2 but not by 4. Compute the rightmost nonzero digit as follows. The odd numbers among $n, n+1, \ldots, n+9$ are $7811\underline{7}, 7811\underline{9}, 7812\underline{1}, 7812\underline{3}, 78125 = 5^7$ and the product of the even numbers $78118, 78120, 78122, 78124, 78126$ is $2^5 \cdot 39059 \cdot 39060 \cdot 39061 \cdot 39062 \cdot 39063 = 2^5 \cdot 3905\underline{9} \cdot (2^2 \cdot 5 \cdot 195\underline{3}) \cdot 3906\underline{1} \cdot (2 \cdot 1953\underline{1}) \cdot 3906\underline{3}$. (For convenience, we have underlined the needed unit digits.) Having written $n(n+1)\cdots(n+9)$ as $2^8 5^8$ times a product of odd factors not divisible by 5, we determine the rightmost nonzero digit by multiplying the units digits of these factors. It follows that, for $n = 5^7 - 8$, the rightmost nonzero digit of a_n is the units digit of $7 \cdot 9 \cdot 1 \cdot 3 \cdot 9 \cdot 3 \cdot 1 \cdot 1 \cdot 3 = (9 \cdot 9) \cdot (7 \cdot 3) \cdot (3 \cdot 3)$, namely 9.

AMERICAN MATHEMATICS COMPETITIONS

AHSME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

50th ANNUAL
AMERICAN HIGH SCHOOL MATHEMATICS EXAMINATION
(AHSME)
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Sponsored by

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This Solutions Pamphlet contains at least one solution to each problem on this year's Examination and shows that all the problems can be solved mathematically without the use of calculus or a calculator. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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This Solutions Pamphlet was prepared during the tenure of the American Mathematics Competitions Executive Director, Dr. Walter E. Mientka.

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1. **(E)** Pairing the first two terms, the next two terms, etc. yields

$$\begin{aligned}&1-2+3-4+\cdots-98+99=\\&(1-2)+(3-4)+\cdots+(97-98)+99=\\&-1-1-1-\cdots-1+99=50,\end{aligned}$$

since there are 49 of the -1's.

OR

$$\begin{aligned}&1-2+3-4+\cdots-98+99=\\&1+[(-2+3)+(-4+5)+\cdots+(-98+99)]=\\&1+[1+1+\cdots+1]=1+49=50.\end{aligned}$$

2. **(A)** Triangles with side lengths of $1,1,1$ and $2,2,2$ are equilateral and not congruent, so (A) is false. Statement (B) is true since all triangles are convex. Statements (C) and (E) are true since each interior angle of an equilateral triangle measures 60°. Furthermore, all three sides of an equilateral triangle have the same length, so (D) is also true.

3. **(E)** The desired number is the arithmetic average or mean:

$$\frac{1}{2}\left(\frac{1}{8}+\frac{1}{10}\right)=\frac{1}{2}\cdot\frac{18}{80}=\frac{9}{80}.$$

4. **(A)** A number one less than a multiple of 5 is has a units digit of 4 or 9. A number whose units digit is 4 cannot be one greater than a multiple of 4. Thus, it is sufficient to examine the numbers of the form $10d+9$ where d is one of the ten digits. Of these, only $9, 29, 49, 69$ and 89 are one greater than a multiple of 4. Among these, only 29 and 89 are prime and their sum is 118.

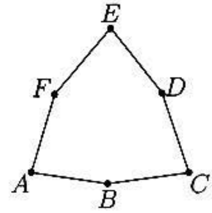
5. **(C)** If the suggested retail price was P, then the marked price was $0.7P$. Half of this is $0.35P$, so Alice paid 35% of the suggested retail price.

6. **(D)** Note that

$$2^{1999}\cdot 5^{2001}=2^{1999}\cdot 5^{1999}\cdot 5^2=10^{1999}\cdot 25=25\overbrace{0\ldots0}^{1999\text{ zeros}}.$$

Hence the sum of the digits is 7.

7. **(B)** The sum of the angles in a convex hexagon is 720° and each angle must be less than 180°. If four of the angles are acute, then their sum would be less than 360°, and therefore at least one of the two remaining angles would be greater than 180°, a contradiction. Thus there can be at most three acute angles. The hexagon shown has three acute angles, A, C, and E.



OR

The result holds for *any* convex n-gon. The sum of the exterior angles of a convex n-gon is 360°. Hence at most three of these angles can be obtuse, for otherwise the sum would exceed 360°. Thus the largest number of acute angles in any convex n-gon is three.

8. **(D)** Let w and $2w$ denote the ages of Walter and his grandmother, respectively, at the end of 1994. Then their respective years of birth are $1994 - w$ and $1994 - 2w$. Hence $(1994 - w) + (1994 - 2w) = 3838$, and it follows that $w = 50$ and Walter's age at the end of 1999 will be 55.

9. **(D)** The next palindromes after 29792 are 29892, 29992, 30003, and 30103. The difference $30103 - 29792 = 311$ is too far to drive in three hours without exceeding the speed limit of 75 miles per hour. Ashley could have driven $30003 - 29792 = 211$ miles during the three hours for an average speed of $70\frac{1}{3}$ miles per hour.

10. **(C)** Since both I and III cannot be false, the digit must be 1 or 3. So either I or III is the false statement. Thus II and IV must be true and (C) is necessarily correct. For the same reason, (E) must be incorrect. If the digit is 1, (B) and (D) are incorrect, and if the digit is 3, (A) is incorrect.

11. **(A)** The locker labeling requires $137.94/0.02 = 6897$ digits. Lockers 1 through 9 require 9 digits, lockers 10 through 99 require $2 \cdot 90 = 180$ digits, and lockers 100 through 999 require $3 \cdot 900 = 2700$ digits. Hence the remaining lockers require $6897 - 2700 - 180 - 9 = 4008$ digits, so there must be $4008/4 = 1002$ more lockers, each using four digits. In all, there are $1002 + 999 = 2001$ student lockers.

12. **(C)** The x-coordinates of the intersection points are precisely the zeros of the polynomial $p(x) - q(x)$. This polynomial has degree at most three, so it has at most three zeros. Hence, the graphs of the fourth degree polynomial functions intersect at most three times. Finding an example to show that three intersection points can be achieved is left to the reader.

13. **(C)** Since $a_{n+1} = \sqrt[3]{99} \cdot a_n$ for all $n \geq 1$, it follows that $a_1, a_2, a_3, \ldots$ is a geometric sequence whose first term is 1 and whose common ratio is $r = \sqrt[3]{99}$. Thus

$$a_{100} = a_1 \cdot r^{100-1} = \left(\sqrt[3]{99}\right)^{99} = 99^{33}.$$

14. **(A)** Tina and Alina each sang either 5 or 6 times. If N denotes the number of songs sung by trios, then $3N = 4 + 5 + 5 + 7 = 21$ or $3N = 4 + 5 + 6 + 7 = 22$ or $3N = 4 + 6 + 6 + 7 = 23$. Since the girls sang as trios, the total must be a multiple of 3. Only 21 qualifies. Therefore, $N = 21/3 = 7$ is the number of songs the trios sang.

Challenge. Devise a schedule for the four girls so that each one sings the required number of songs.

15. **(E)** From the identity $1 + \tan^2 x = \sec^2 x$ it follows that $1 = \sec^2 x - \tan^2 x = (\sec x - \tan x)(\sec x + \tan x) = 2(\sec x + \tan x)$, so $\sec x + \tan x = 0.5$.

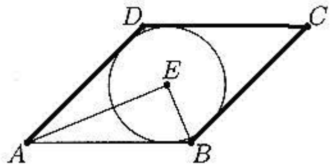
OR

The given relation can be written as $\dfrac{1 - \sin x}{\cos x} = 2$. Squaring both sides yields $\dfrac{(1 - \sin x)^2}{1 - \sin^2 x} = 4$, hence $\dfrac{1 - \sin x}{1 + \sin x} = 4$. It follows that $\sin x = -\dfrac{3}{5}$ and that

$$\cos x = \frac{1 - \sin x}{2} = \frac{1 - (-3/5)}{2} = \frac{4}{5}.$$

Thus $\sec x + \tan x = \frac{5}{4} - \frac{3}{4} = 0.5$.

16. **(C)** Let E be the intersection of the diagonals of a rhombus $ABCD$ satisfying the conditions of the problem. Because these diagonals are perpendicular and bisect each other, $\triangle ABE$ is a right triangle with sides 5, 12, and 13 and area 30. Therefore the altitude drawn to side AB is $60/13$, which is the radius of the inscribed circle centered at E.



17. **(C)** From the hypothesis, $P(19) = 99$ and $P(99) = 19$. Let

$$P(x) = (x - 19)(x - 99)Q(x) + ax + b,$$

where a and b are constants and $Q(x)$ is a polynomial. Then

$$99 = P(19) = 19a + b \text{ and } 19 = P(99) = 99a + b.$$

It follows that $99a - 19a = 19 - 99$, hence $a = -1$ and $b = 99 + 19 = 118$. Thus the remainder is $-x + 118$.

18. **(E)** Note that the range of $\log x$ on the interval $(0, 1)$ is the set of all negative numbers, infinitely many of which are zeros of the cosine function. In fact, since $\cos(x) = 0$ for all x of the form $\frac{\pi}{2} \pm n\pi$,

$$\begin{aligned} f(10^{\frac{\pi}{2}-n\pi}) &= \cos(\log(10^{\frac{\pi}{2}-n\pi})) \\ &= \cos\left(\frac{\pi}{2} - n\pi\right) \\ &= 0 \end{aligned}$$

for all positive integers n.

19. **(C)** Let $DC = m$ and $AD = n$. By the *Pythagorean Theorem*, $AB^2 = AD^2 + DB^2$. Hence $(m + n)^2 = n^2 + 57$, which yields $m(m + 2n) = 57$. Since m and n are positive integers, the only possibilities are $m = 1$, $n = 28$ and $m = 3$, $n = 8$. The second of these gives the least possible value of $AC = m + n$, namely 11.

20. **(E)** For $n \geq 3$,

$$a_n = \frac{a_1 + a_2 + \cdots + a_{n-1}}{n-1}.$$

Thus $(n-1)a_n = a_1 + a_2 + \cdots + a_{n-1}$. It follows that

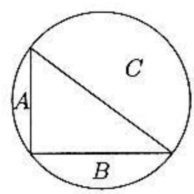
$$a_{n+1} = \frac{a_1 + a_2 + \cdots + a_{n-1} + a_n}{n} = \frac{(n-1) \cdot a_n + a_n}{n} = a_n,$$

for $n \geq 3$. Since $a_9 = 99$ and $a_1 = 19$, it follows that

$$99 = a_3 = \frac{19 + a_2}{2},$$

and hence that $a_2 = 179$. (The sequ is $19, 179, 99, 99, \ldots$.)

21. **(B)** Since $20^2 + 21^2 = 29^2$, the converse of the *Pythagorean Theorem* applies, so the triangle has a right angle. Thus its hypotenuse is a diameter of the circle, so the region with area C is a semicircle and is congruent to the semicircle formed by the other three regions. The area of the triangle is 210, hence $A + B + 210 = C$. To see that the other options are incorrect, note that



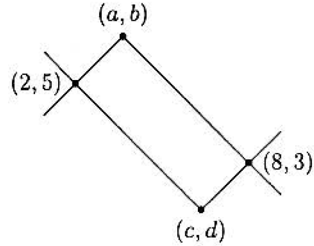
(A) $A + B < A + B + 210 = C$;

(C) $A^2 + B^2 < (A+B)^2 < (A + B + 210)^2 = C^2$;

(D) $20A + 21B < 29A + 29B < 29(A + B + 210) = 29C$; and

(E) $\dfrac{1}{A^2} + \dfrac{1}{B^2} > \dfrac{1}{A^2} > \dfrac{1}{C^2}$.

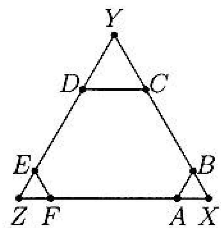
22. **(C)** The first graph is an inverted 'V-shaped' right angle with vertex at (a, b) and the second is a V-shaped right angle with vertex at (c, d). Thus $(a, b), (2, 5), (c, d)$, and $(8, 3)$ are consecutive vertices of a rectangle. The diagonals of this rectangle meet at their common midpoint, so the x-coordinate of this midpoint is $(2+8)/2 = (a+c)/2$. Thus $a+c = 10$.



OR

Use the given information to obtain the equations $5 = -|2-a| + b$, $5 = |2-c| + d$, $3 = -|8-a| + b$, and $3 = |8-c| + d$. Subtract the third from the first to eliminate b and subtract the fourth from the second to eliminate d. The two resulting equations $|8-a| - |2-a| = 2$ and $|2-c| - |8-c| = 2$ can be solved for a and c. To solve the former, first consider all $a \le 2$, for which the equation reduces to $8 - a - (2 - a) = 2$, which has no solutions. Then consider all a in the interval $2 \le a \le 8$, for which the equation reduces to $8 - a - (a - 2) = 2$, which yields $a = 4$. Finally, consider all $a \ge 8$, for which the equation reduces to $a - 8 - (a - 2) = 2$, which has no solutions. The other equation can be solved similarly to show that $c = 6$. Thus $a + c = 10$.

23. **(E)** Extend $\overline{FA}$ and $\overline{CB}$ to meet at X, $\overline{BC}$ and $\overline{ED}$ to meet at Y, and $\overline{DE}$ and $\overline{AF}$ to meet at Z. The interior angles of the hexagon are $120°$. Thus the triangles XYZ, ABX, CDY, and EFZ are equilateral. Since $AB = 1$, $BX = 1$. Since $CD = 2$, $CY = 2$. Thus $XY = 7$ and $YZ = 7$. Since $YD = 2$ and $DE = 4, EZ = 1$. The area of the hexagon can be found by subtracting the areas of the three small triangles from the area of the large triangle:



$$7^2\left(\frac{\sqrt{3}}{4}\right) - 1^2\left(\frac{\sqrt{3}}{4}\right) - 2^2\left(\frac{\sqrt{3}}{4}\right) - 1^2\left(\frac{\sqrt{3}}{4}\right) = \frac{43\sqrt{3}}{4}.$$

24. **(B)** Any four of the six given points determine a unique convex quadrilateral, so there are exactly $\binom{6}{4} = 15$ favorable outcomes when the chords are selected randomly. Since there are $\binom{6}{2} = 15$ chords, there are $\binom{15}{4} = 1365$ ways to pick the four chords. So the desired probability is $15/1365 = 1/91$.

25. **(B)** Multiply both sides of the equation by 7! to obtain

$$3600 = 2520a_2 + 840a_3 + 210a_4 + 42a_5 + 7a_6 + a_7.$$

It follows that $3600 - a_7$ is a multiple of 7, which implies that $a_7 = 2$. Thus,

$$\frac{3598}{7} = 514 = 360a_2 + 120a_3 + 30a_4 + 6a_5 + a_6.$$

Reason as above to show that $514 - a_6$ is a multiple of 6, which implies that $a_6 = 4$. Thus, $510/6 = 85 = 60a_2 + 20a_3 + 5a_4 + a_5$. Then it follows that $85 - a_5$ is a multiple of 5, whence $a_5 = 0$. Continue in this fashion to obtain $a_4 = 1, a_3 = 1$, and $a_2 = 1$. Thus the desired sum is $1 + 1 + 1 + 0 + 4 + 2 = 9$.

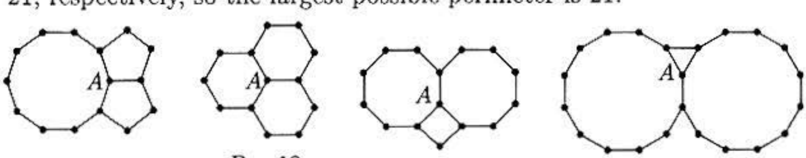
26. **(D)** The interior angle of a regular n-gon is $180(1 - 2/n)$. Let a be the number of sides of the congruent polygons and let b be the number of sides of the third polygon (which could be congruent to the first two polygons). Then

$$2 \cdot 180\left(1 - \frac{2}{a}\right) + 180\left(1 - \frac{2}{b}\right) = 360.$$

Clearing denominators and factoring yields the equation

$$(a - 4)(b - 2) = 8,$$

whose four positive integral solutions are $(a, b) = (5, 10), (6, 6), (8, 4)$, and $(12, 3)$. These four solutions give rise to polygons with perimeters of 14, 12, 14 and 21, respectively, so the largest possible perimeter is 21.



$P = 14$ $P = 12$ $P = 14$ $P = 21$

27. **(A)** Square both sides of the equations and add the results to obtain

$$9(\sin^2 A + \cos^2 A) + 16(\sin^2 B + \cos^2 B) + 24(\sin A \cos B + \sin B \cos A) = 37.$$

Hence, $24\sin(A+B) = 12$. Thus $\sin C = \sin(180^\circ - A - B) = \sin(A+B) = \frac{1}{2}$, so $\angle C = 30^\circ$ or $\angle C = 150^\circ$. The latter is impossible because it would imply that $A < 30^\circ$ and consequently that $3\sin A + 4\cos B < 3 \cdot \frac{1}{2} + 4 < 6$, a contradiction. Therefore $\angle C = 30^\circ$.

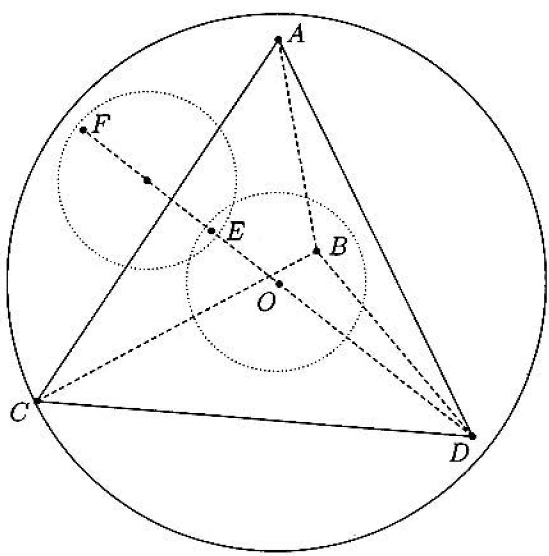
Challenge. Prove that there is a unique such triangle (up to similarity), the one for which $\cos A = \frac{5-12\sqrt{3}}{37}$ and $\cos B = \frac{66-3\sqrt{3}}{74}$.

28. **(E)** Let a, b, and c denote the number of -1's, 1's, and 2's in the sequence, respectively. We need not consider the zeros. Then a, b, c are nonnegative integers satisfying $-a + b + 2c = 19$ and $a + b + 4c = 99$. It follows that $a = 40 - c$ and $b = 59 - 3c$, where $0 \le c \le 19$ (since $b \ge 0$), so

$$x_1^3 + x_2^3 + \cdots + x_n^3 = -a + b + 8c = 19 + 6c.$$

The lower bound is achieved when $c = 0$ $(a = 40, b = 59)$. The upper bound is achieved when $c = 19$ $(a = 21, b = 2)$. Thus $m = 19$ and $M = 133$, so $M/m = 7$.

29. **(C)** Let A, B, C, and D be the vertices of the tetrahedron. Let O be the center of both the inscribed and circumscribed spheres. Let the inscribed sphere be tangent to the face ABC at the point E, and let its volume be V. Note that the radius of the inscribed sphere is OE and the radius of the circumscribed sphere is OD. Draw $\overline{OA}$, $\overline{OB}$, $\overline{OC}$, and $\overline{OD}$ to obtain four congruent tetrahedra $ABCO, ABDO, ACDO$, and $BCDO$, each with volume $1/4$ that of the original tetrahedron. Because the two tetrahedra $ABCD$ and $ABCO$ share the same base, $\triangle ABC$, the ratio of the distance from O to face ABC to the distance from D to face ABC is $1/4$; that is, $OD = 3 \cdot OE$. Thus the volume of the circumscribed sphere is $27V$. Extend $\overline{DE}$ to meet the circumscribed sphere at F. Then $DF = 2 \cdot DO = 6 \cdot OE$. Thus $EF = 2 \cdot OE$, so the sphere with diameter $\overline{EF}$ is congruent to the inscribed sphere, and thus has volume V. Similarly each of the other three spheres between the tetrahedron and the circumscribed sphere have volume V. The five congruent small spheres have no volume in common and lie entirely inside the circumscribed sphere, so the ratio $5V/27V$ is the probability that a point in the circumscribed sphere also lies in one of the small spheres. The fraction $5/27$ is closer to 0.2 than it is to any of the other choices.



30. **(D)** Let $m+n=s$. Then $m^3+n^3+3mn(m+n)=s^3$. Subtracting the given equation from the latter yields

$$s^3-33^3=3mns-99mn.$$

It follows that $(s-33)(s^2+33s+33^2-3mn)=0$, hence either $s=33$ or $(m+n)^2+33(m+n)+33^2-3mn=0$. The second equation is equivalent to $(m-n)^2+(m+33)^2+(n+33)^2=0$, whose only solution, $(-33,-33)$, qualifies. On the other hand, the solutions to $m+n=33$ satisfying the required conditions are $(0,33),(1,32),(2,31),\ldots,(33,0)$, of which there are 34. Thus there are 35 solutions altogether.