SOLUTION KEY

MATHEMATICAL CONTEST

1950

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In the original American Mathematics Competitions archives, there exist no answers page or solutions booklet for this year's contest. In the interest of providing a complete reference for those using these contests for practice or learning purposes we are providing a set of answers and solutions for this year.

The solutions were taken from:

> C**ontest Problem Book No 1: Annual High School Mathematics Examinations 1950-1960** (New Mathematical Library) by Charles T. Salkind (Washington, D.C.: Mathematical Association of America, 1975)

Part 1

1. Numbers proportional to 2, 4, 6 are $2x$, $4x$, $6x$, respectively.
$2x + 4x + 6x = 64; \quad \therefore x = 5\frac{1}{3}, \ 2x = 10\frac{2}{3}.$

2. $16 = 8g - 4; \quad \therefore g = 2\frac{1}{2}; \quad \therefore R = (2\frac{1}{2})\,10 - 4 = 21.$

3. $x^2 - \frac{8x}{4} + \frac{5}{4} = 0; \quad \therefore r_1 + r_2 = -\frac{-8}{4} = 2.$

4. $\frac{a^2 - b^2}{ab} - \frac{b(a - b)}{a(b - a)} = \frac{a^2 - b^2}{ab} + \frac{b^2}{ab} = \frac{a}{b}.$

5. Denote the terms in the geometric progression by
$$a_1 = 8, \ a_2 = 8r, \cdots, a_7 = 8r^6 = 5832.$$
$\therefore r^6 = 729; \quad \therefore r = 3$ and $a_5 = 8r^4 = 648.$

6. Solve the second equation for x, substitute into the first equation and simplify:
$$x = -\frac{y+3}{2}, \ 2\left(-\frac{y+3}{2}\right)^2 + 6\left(-\frac{y+3}{2}\right) + 5y + 1 = 0;$$
$\therefore y^2 + 10y - 7 = 0.$

7. Placing 1 as indicated shifts the given digits to the left, so that t is now the hundreds' digit and u is now the tens' digit.
$\therefore 100t + 10u + 1.$

8. Let r be the original radius; then
$$2r = \text{new radius}, \ \pi r^2 = \text{original area}, \ 4\pi r^2 = \text{new area}.$$
$$\text{Percent increase in area} = \frac{4\pi r^2 - \pi r^2}{\pi r^2} \cdot 100 = 300.$$

9. Of all triangles inscribable in a semi-circle, with the diameter as base, the one with the greatest area is the one with the largest altitude (the radius); that is, the isosceles triangle. Thus the area is $\frac{1}{2} \cdot 2r \cdot r = r^2$.

10. $\frac{\sqrt{3} - \sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{3} + \sqrt{2}}{\sqrt{3} + \sqrt{2}} = \frac{1}{3 + \sqrt{6}}.$

11. $C = \frac{en}{R + nr} = \frac{e}{(R/n) + r}$. Therefore an increase in n makes the denominator smaller and, consequently, the fraction C larger.

12. It is a theorem in elementary geometry that the sum of the exterior angles of any (convex) polygon is a constant, namely two straight angles.

13. Set each of the factors equal to zero:
$$x = 0, \ x - 4 = 0, \ x^2 - 3x + 2 = (x - 2) \cdot (x - 1) = 0;$$
$$\therefore x = 0, 4, 2, 1.$$

14. Geometrically, the problem represents a pair of parallel lines, with slope equal to $\frac{2}{3}$, and hence there is no intersection point. Algebraically, if

$$a_1x + b_1y = c_1 \quad \text{and} \quad a_2x + b_2y = c_2\,,$$

then, multiplying the first equation by b_2, the second by b_1, and subtracting, we have

$$(a_1\,b_2 - a_2\,b_1)x = c_1\,b_2 - c_2\,b_1\,; \qquad \therefore\ x = \frac{c_1\,b_2 - c_2\,b_1}{a_1\,b_2 - a_2\,b_1}.$$

Thus if $a_1b_2 - a_2b_1 = 0$ and $c_1 \neq 0$, $c_2 \neq 0$, x is undefined and no solution exists.

15. It is understood that we restrict ourselves to rational coefficients and integral powers of x. With this restriction $x^2 + 4$ is not factorable.

Part 2

16. The binomial expansion of $(x + y)^n$ has $n + 1$ terms. Thus

$$[(a + 3b)^2(a - 3b)^2]^2 = (a^2 - 9b^2)^4$$

has $4 + 1 = 5$ terms.

17. To obtain a formula directly from the table involves work beyond the indicated scope. Consequently, the answer must be found by testing the choices offered. (A) is immediately eliminated since, for equally spaced values of x, the values of y are not equally spaced; $\therefore y$ cannot be a linear function of x. (B) fails beyond the second set of values. (C) satisfies the entire table. (D) fails with the first set of values.

18. Of the four distributive laws given in statements (1), (2), (3) and (5), only (1) holds in the real number field. Statement (4) is not to be confused with $\log\,(x/y) = \log x - \log y$. $\therefore$ (E) is the correct choice.

19. The job requires md man-days, so that one man can do the job in md days. Therefore, $(m + r)$ men can do the job in $md/(m + r)$ days;

or

$$\frac{x}{d} = \frac{m}{m + r}; \quad \therefore\ x = \frac{md}{m + r}.$$

20. By ordinary division—in this instance a long and tedious operation—the answer (D) is found to be correct. By the Remainder Theorem, we have $R = 1^{13} + 1 = 2$. Note: The use of synthetic division shortens the work considerably, but synthetic division itself is generally regarded as beyond the indicated scope of this examination. The Remainder Theorem is likewise beyond the scope of this examination.

21. $lh = 12$, $hw = 8$ and $lw = 6$. Eliminating h we obtain $l = 3w/2$. Eliminating l, $3w^2/2 = 6$. $\therefore w = 2$, $l = 3$, $h = 4$, and

$$V = lwh = 24;$$

or

$$V^2 = (lwh)^2 = lh \cdot hw \cdot lw = 12 \cdot 8 \cdot 6 = 4^2 \cdot 6^2. \qquad \therefore V = 4 \cdot 6 = 24.$$

22. Let P be the original price. Then a discount of 10% gives a new price $P - .1P = .9P$. Following this by a discount of 20%, we have

$$.9P - .2(.9P) = .72P.$$

Thus the net discount is $P - .72P = .28P$ or 28%;

or

$$\text{net discount} = d_1 + d_2 - d_1 d_2 = \frac{10}{100} + \frac{20}{100} - \frac{2}{100} = 28\%$$

23. Let R be the yearly rent. Then $\frac{5\frac{1}{2}}{100} \cdot 10{,}000 = R - \frac{12\frac{1}{2}}{100}R - 325$;

$\therefore R = \$1000$ and $\frac{R}{12} \sim \$83.33$.

24. $\sqrt{x-2} = 4 - x$, $\quad x - 2 = 16 - 8x + x^2$,

$$0 = x^2 - 9x + 18 = (x-6)(x-3); \qquad \therefore x = 6,\ x = 3.$$

A check is obtained only with $x = 3$.

25. $\log_5\left(\frac{5^3 \cdot 5^4}{5^2}\right) = \log_5 5^5 = 5\log_5 5 = 5.$

26. $b = \log_{10}m + \log_{10}n = \log_{10}mn; \qquad \therefore mn = 10^b; \qquad \therefore m = 10^b/n;$

or

$\log_{10}m = \log_{10}10^b - \log_{10}n = \log_{10}(10^b/n); \qquad \therefore m = 10^b/n.$

27. If a car travels a distance d at rate r_1 and returns the same distance at rate r_2, then

$$\text{average speed} = \frac{\text{total distance}}{\text{total time}} = \frac{2d}{d/r_1 + d/r_2} = \frac{2r_1r_2}{r_1 + r_2};$$

$$\therefore x = \frac{2 \cdot 30 \cdot 40}{70} = \frac{240}{7} \sim 34 \text{ mph.}$$

28. Notice that A and B both travel for the same amount of time. Since A travels 48 miles while B travels 72 miles, we have

$$t = \frac{48}{r} = \frac{72}{r+4}; \ \therefore r = 8.$$

29. Since the first machine addresses 500 envelopes in 8 minutes, it addresses $500/8$ envelopes in 1 minute and $2 \cdot 500/8$ in 2 minutes. If the second machine addresses 500 envelopes in x minutes, then it addresses $2 \cdot 500/x$ envelopes in 2 minutes. To determine x so that both machines together address 500 envelopes in 2 minutes we use the condition

$$\frac{2}{8} \cdot 500 + \frac{2}{x} \cdot 500 = 500 \quad \text{or} \quad \frac{500}{8} + \frac{500}{x} = 250 \quad \text{or} \quad \frac{1}{8} + \frac{1}{x} = \frac{1}{2}.$$

30. If B is the original number of boys and G the original number of girls, then

$$\frac{B}{G-15} = 2, \frac{B-45}{G-15} = \frac{1}{5}; \qquad \therefore G = 40.$$

31. Let x be the number of pairs of blue socks, and let y be the price of a pair of blue socks. Then $2y$ is the price of a pair of black socks and $x \cdot 2y + 4 \cdot y = \tfrac{3}{2}(4 \cdot 2y + x \cdot y)$. Divide by y and solve for x.

$$\therefore x = 16; \qquad \therefore 4:x = 4:16 = 1:4.$$

32. Before sliding, the situation is represented by a right triangle with hypotenuse 25, and horizontal arm 7, so that the vertical arm is 24. After sliding, the situation is represented by a right triangle with hypotenuse 25, and vertical arm 20, so that the horizontal arm is 15. $15 - 7 = 8$ so that (D) is the correct choice.

33. The cross section of the pipe is πR^2.

$$\therefore \text{large pipe/small pipe} = 6^2/1^2 = 36.$$

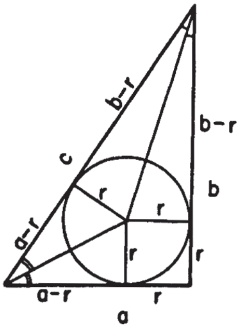
The large pipe has the carrying capacity of 36 small pipes.

34. Since $C = 2\pi r$, letting r_2 be the original radius, r_1 the final radius, we have $25 - 20 = 2\pi r_1 - 2\pi r_2 = 2\pi(r_1 - r_2)$; $\quad \therefore r_1 - r_2 = 5/2\pi$.

35. For any right triangle we have $a - r + b - r = c$ where r is the radius of the inscribed circle.

$$\begin{aligned} \therefore 2r &= a + b - c \\ &= 24 + 10 - 26 \\ &= 8; \end{aligned}$$

$\therefore r = 4.$



Part 3

36. Let C = merchant's cost, L = list price, M = marked price, S = selling price, and P = profit. Then

$$C = L - \frac{1}{4}L = \frac{3}{4}L, \quad S = M - \frac{1}{5}M = \frac{4}{5}M, \quad S = C + P,$$

$$\frac{4}{5}M = \frac{3}{4}L + \frac{1}{4} \cdot \frac{4}{5}M, \quad M = \frac{5}{3} \cdot \frac{3}{4}L = \frac{5}{4}L.$$

37. Reference to the graph of $y = \log_a x$, $a > 1$, or to the equality $x = a^y$, shows that (A), (B), (C), and (D) are all correct.

38. $\begin{vmatrix} 2x & 1 \\ x & x \end{vmatrix} = 2x \cdot x - 1 \cdot x = 3; \qquad \therefore x = -1, \tfrac{3}{2}.$

39. This sequence is an infinite geometric progression with $r = \frac{1}{2} < 1$. Thus if a is its first term and s its sum, we have

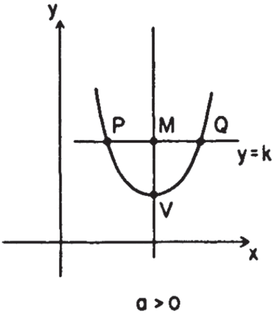
$$s = \frac{a}{1 - r} = \frac{2}{1 - \frac{1}{2}} = 4.$$

Hence (4) and (5) are correct statements.

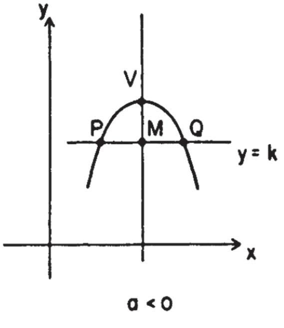
40. $\dfrac{x^2 - 1}{x - 1} = \dfrac{(x + 1)(x - 1)}{x - 1} = x + 1$, for all $x \neq 1$;

the limit of $x + 1$ as $x \to 1$ is 2, so that (D) is the correct choice. Note: This problem is beyond the scope of the contest.

41. The graph of the function $y = ax^2 + bx + c$, $a \neq 0$, is a parabola with its axis of symmetry (the line MV extended) parallel to the y-axis.



$a > 0$



$a < 0$

We wish to find the coordinates of V. Let $y = k$ be any line parallel to the x-axis which intersects the parabola at two points, say P and Q. Then the abscissa (the x-coordinate) of the midpoint of the line segment PQ is the abscissa of V. Since P and Q are the intersections of the line $y = k$ with the parabola, the abscissas of P and Q must satisfy the equation

$$ax^2 + bx + c = k \quad \text{or} \quad ax^2 + bx + c - k = 0.$$

Its roots are:

$$x = -\frac{b}{2a} + \frac{1}{2a}\sqrt{D} \quad \text{and} \quad x' = -\frac{b}{2a} - \frac{1}{2a}\sqrt{D},$$

where $D = b^2 - 4ac$ is the discriminant.

$\therefore$ the abscissa of $V = \dfrac{x + x'}{2} = -\dfrac{b}{2a}$, and the

$$\text{ordinate of } V = a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c = -\frac{b^2}{4a} + c = \frac{4ac - b^2}{4a};$$

if $a > 0$, the ordinate of V is the minimum value of the function $ax^2 + bx + c$ and if $a < 0$, it is the maximum value.

42. If $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}} = 2$, then the exponent, which is again $x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$ is also 2, and we have $x^2 = 2$. Therefore $x = \sqrt{2}$. Note: This problem is beyond the scope of the contest.

43. Rearrange the terms:

$$s_1 = \frac{1}{7} + \frac{1}{7^3} + \cdots = \frac{1/7}{1 - (1/7^2)} = \frac{7}{48};$$

$$s_2 = \frac{2}{7^2} + \frac{2}{7^4} + \cdots = \frac{2/7^2}{1 - (1/7^2)} = \frac{2}{48};$$

$$s = s_1 + s_2 = \frac{3}{16}.$$

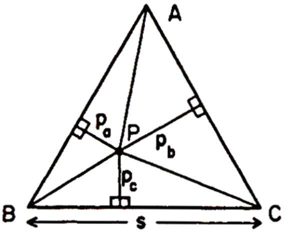
44. Looking at $y = \log_b x$ or equivalently $x = b^y$, we see that $y = 0$ for $x = 1$. That is, the graph cuts the x-axis at $x = 1$.

45. $d = n(n - 3)/2 = 100 \cdot 97/2 = 4850$.

46. In the new triangle $\overline{AB} = \overline{AC} + \overline{BC}$, that is, C lies on the line AB. Consequently, the altitude from C is zero. Therefore, the area of the triangle is zero.

47. By similar triangles, $(h - x)/h = 2x/b$. $\therefore x = bh/(2h + b)$.

48. Let P be an arbitrary point in the equilateral triangle ABC with sides of length s, and denote the perpendicular segments by p_a, p_b, p_c. Then



$$\text{Area } ABC = \text{Area } APB + \text{Area } BPC + \text{Area } CPA$$

$$= \tfrac{1}{2}\,(sp_a + sp_b + sp_c) = \tfrac{1}{2}s(p_a + p_b + p_c).$$

Also, Area $ABC = \frac{1}{2}s \cdot h$, where h is the length of the altitude of ABC. Therefore, $h = p_a + p_b + p_c$ and this sum does not depend on the location of P.

49. Perhaps the easiest approach to this problem is through coordinate geometry.

Let the triangle be $A(0, 0)$, $B(2, 0)$ and $C(x, y)$. Then A_1, the midpoint of BC, has the coordinates $[(x + 2)/2,\ y/2]$. Therefore, using the distance formula, we have

$$\sqrt{\left(\frac{x+2}{2}\right)^2 + \left(\frac{y}{2}\right)^2} = \frac{3}{2} \quad \text{or} \quad \left(\frac{x+2}{2}\right)^2 + \left(\frac{y}{2}\right)^2 = \frac{9}{4}$$

or

$$(x+2)^2 + y^2 = 9.$$

This equation represents a circle with radius 3 inches and center 4 inches from B along BA;

or

as extreme values we may have $a - b = 2$ and $a - b = 0$, using the notation a, b, c with its usual meaning. From the median formula, with $c = 2$ and $m_a = 3/2$, we have $2b^2 + 8 = 9 + a^2$. $\therefore 2b^2 - a^2 = 1$. Using the given extreme values in succession, we have $b = 5$ and $a = 7$, and $b = 1$ and $a = 1$. Therefore, the maximum distance between the extreme positions of C is 6 inches. These facts, properly collated, lead to the same result as that shown above.

50. After the two-hour chase, at 1:45 P.M. the distance between the two ships is 4 miles. Let t be the number of hours needed for the privateer to overtake the merchantman. Let D and $D + 4$ be the distances covered in t hours by the merchantman and the privateer, respectively. Then Rate of merchantman$\cdot t = D$, and Rate of privateer$\cdot t = D + 4$ or

$$\frac{8\cdot 17}{15}t = D + 4, \quad \text{and} \quad 8t = D; \qquad \therefore \left(\frac{8\cdot 17}{15} - 8\right) t = 4.$$

$\therefore t = 3\frac{3}{4}$ hours, so that the ships meet at 5:30.

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Part 1

1. The pedagogic value of this extremely simple problem can be enhanced by a discussion of necessary restrictions on the number-nature of M and N, the permissibility of negative numbers, and so forth.

2. Let the dimensions of the field be w and $2w$. Then $6w = x$, $w = x/6$, and $2w = x/3$. $\therefore A = w \cdot 2w = x^2/18$.

3. $A = \frac{1}{2}d^2 = \frac{1}{2}(a + b)^2$.

4. $S = 1(13 \cdot 10) + 2[2(13 \cdot 5) + 2(10 \cdot 5)] = 590$.

5. $S_1 = 10{,}000 + 1000 = 11{,}000$, $S_2 = 11{,}000 - 1100 = 9900$, $S_1 - S_2 = 1100$. $\therefore$ (B) is the correct choice.

6. If the dimensions are designated by l, w, h, then the bottom, side, and front areas are lw, wh, hl. The product of these areas is

$$l^2w^2h^2 = (lwh)^2 = \text{the square of the volume.}$$

7. Relative error = error/measurement. Since $.02/10 = .2/100$, the relative errors are the same.

8. Let M be the marked price of the article. $M - .1M = .9M$ = new price. To restore $.9M$ to M requires the addition of $.1M$. $.1M = \frac{1}{9}(.9M)$, and $\frac{1}{9}$, expressed as a percent, is $11\frac{1}{9}\%$.

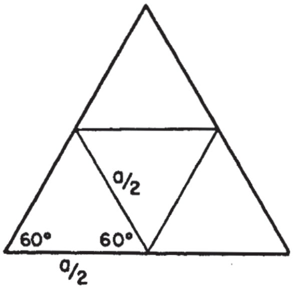
9. Let P_k be the perimeter of the kth triangle. Then $P_{k+1} = \frac{1}{2}P_k$.

$$S = P_1 + P_2 + P_3 + \cdots$$

$$= 3a + \frac{1}{2} \cdot 3a + \frac{1}{2} \cdot \frac{3}{2}a + \cdots$$

$$= 3a\left(1 + \frac{1}{2} + \frac{1}{4} + \cdots\right)$$

$$= 3a\,\frac{1}{1-(1/2)} = 6a.$$



10. (C) is incorrect since doubling the radius of a given circle quadruples the area.

11. The new series is $a^2 + a^2r^2 + a^2r^4 + \cdots$; $\therefore S = a^2/(1 - r^2)$.

12. In 15 minutes the hour hand moves through an angle of $\frac{1}{4}$ of $30° = 7\frac{1}{2}°$. Therefore, the angle between the hour hand and the minute hand is $22\frac{1}{2}°$.

13. In one day A can do $1/9$ of the job. Since B is 50 % more efficient than A, B can do $(3/2)(1/9) = 1/6$ of the job in one day. Therefore, B needs 6 days for the complete job.

14. (C) is incorrect since some terms (primitives) must necessarily remain undefined.

15. $n^3 - n = (n-1)(n)(n+1)$, for integral values of n, represents the product of three consecutive integers. Since in a pair of consecutive integers, there is a multiple of two, and in a triplet of consecutive integers, there is a multiple of three, then $n^3 - n$ is divisible by 6.

Part 2

16. The condition $c = b^2/4a$ implies equal real roots of $f(x) = 0$ for real coefficients, i.e., the curve touches the x-axis at exactly one point and, therefore, the graph is tangent to the x-axis.

17. (A), (B), (C), and (E) are of the form $y = kx$ or $xy = k$, but (D) is not.

18. Let the factors be $Ax + B$ and $Cx + D$. Then

$$\begin{aligned}(Ax+B)(Cx+D) &= ACx^2 + (AD+BC)x + BD \\ &= 21x^2 + ax + 21.\end{aligned}$$

$\therefore AC = 21$, $BD = 21$. Since 21 is odd, all its factors are odd. Therefore each of the numbers A and C is odd. The same is true for B and D. Since the product of two odd numbers is odd, AD and BC are odd numbers; $a = AD + BC$, the sum of two odd numbers, is even.

19. Such a number can be written as $P \cdot 10^3 + P = P(10^3 + 1) = P(1001)$, where P is the block of three digits that repeats. $\therefore$ (E) is the correct choice.

20. $$(x+y)^{-1}(x^{-1}+y^{-1}) = \frac{1}{x+y}\left(\frac{1}{x}+\frac{1}{y}\right) = \frac{1}{x+y}\cdot\frac{x+y}{xy} = \frac{1}{xy} = x^{-1}y^{-1}.$$

21. (C) is not always correct since, if z is negative, $xz < yz$.

22. Since $\log_{10}(a^2 - 15a) = 2$, then $a^2 - 15a = 10^2 = 100$. The solution set for this quadratic equation is $\{20, -5\}$.

23. Since $V = \pi r^2 h$, we must have $\pi(r+x)^2 h = \pi r^2(h+x)$. $\therefore x = (r^2 - 2rh)/h$. For $r = 8$ and $h = 3$, we have $x = 5\frac{1}{3}$.

24. $$\frac{2^{n+4} - 2(2^n)}{2(2^{n+3})} = \frac{2^n \cdot 2^4 - 2\cdot 2^n}{2\cdot 2^n \cdot 2^3} = \frac{2\cdot 2^n(2^3-1)}{2\cdot 2^n\cdot 2^3} = \frac{7}{8}.$$

25. Let s_1 be the side of the square, a_1 its apothem; then $s_1^2 = 4s_1$ and since $2a_1 = s_1$, we have $4a_1^2 = 8a_1$ or $a_1 = 2$. Let s_2 be the side of the equilateral triangle, h its altitude and a_2 its apothem; then $s_2^2\sqrt{3}/4 = 3s_2$. Since $h = 3a_2$, and $s_2 = 2h/\sqrt{3} = 6a_2/\sqrt{3}$, we have $(36a_2^2/3)\sqrt{3}/4 = 3\cdot 6a_2/\sqrt{3}$, and $a_2 = 2 = a_1$.

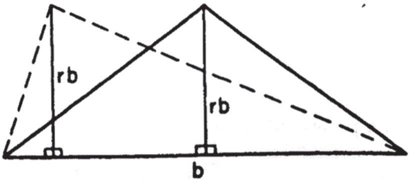
26. After simplification, we have $x^2 - x - m(m + 1) = 0$. For equal roots the discriminant $D = 1 + 4m(m + 1) = 0 = (2m + 1)^2$.

$$\therefore m = -\tfrac{1}{2}.$$

27. We may eliminate (A) and (D) by proving that corresponding angles are not equal. If (A) is false, (B) is certainly false. We may eliminate (C) by placing the point very close to one side of the triangle. Thus none of the four statements is true.

28. $P = kAV^2$, $1 = k \cdot 1 \cdot 16^2$; $\quad \therefore k = 1/16^2$;
$36/9 = (1/16^2) \cdot (1) \cdot (V^2)$; $\quad \therefore V = 32$ (mph).

29. Let the ratio of altitude to base be r. The triangles in the figure satisfy condition (D) but have different shapes. Hence (D) does not determine the shape of a triangle.



30. $$\frac{1}{x} = \frac{1}{20} + \frac{1}{80};$$

$\therefore x = 16$ (inches);

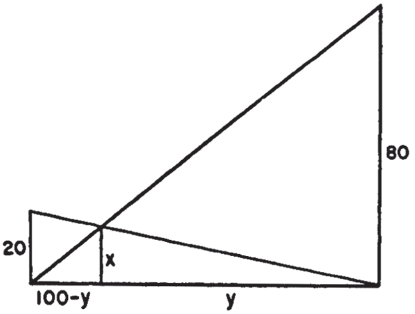
or

$$\frac{20}{100} = \frac{x}{y}, \qquad y = 5x$$

and $$\frac{80}{100} = \frac{x}{100 - y}$$

$$= \frac{x}{100 - 5x};$$

$x = 16$ (inches).



31. Let n be the number of people present. Single out a particular individual P. P shakes hands with $(n - 1)$ persons, and this is true for each of the n persons. Since each handshake is between two persons, the total number of handshakes is $n(n - 1)/2 = 28$. $\quad \therefore n = 8$;

or

$$C(n, 2) = n(n - 1)/2 = 28.$$

32. The inscribed triangle with maximum perimeter is the isosceles right triangle; in this case $\overline{AC} + \overline{BC} = \overline{AB}\sqrt{2}$. Therefore, in general, $\overline{AC} + \overline{BC} \leq \overline{AB}\sqrt{2}$.

33. $$(x^2 - 2x = 0) \leftrightarrow (x^2 = 2x) \leftrightarrow (x^2 - 2x + 1 = 1)$$
$$\leftrightarrow (x^2 - 1 = 2x - 1).$$

$\therefore$ (C) is the correct choice.

34. In general, $a^{\log_a N} = N$, with suitable restrictions on a and N;

or

let $10^{\log_{10} 7} = N$. Taking logarithms of both sides of the equation to the base 10, we have $\log_{10} 7 \cdot \log_{10} 10 = \log_{10} N$. $\therefore N = 7$.

35. Since $c^y = a^z$, $c = a^{z/y}$. $\therefore c^q = a^{(z/y)q} = a^x$; $\therefore x = zq/y$;
$$\therefore xy = qz.$$

Part 3

36. To prove that a geometric figure is a locus it is essential to (1) include all proper points, and (2) exclude all improper points. By this criterion, (B) is insufficient to guarantee condition (1); i.e., (B) does not say that every point satisfying the conditions lies on the locus.

37. Let N denote the number to be found; then we may write the given information as
$$N = 10a_9 + 9 = 9a_8 + 8 = \cdots = 2a_1 + 1.$$

Hence
$$N + 1 = 10(a_9 + 1) = 9(a_8 + 1) = \cdots = 2(a_1 + 1),$$

i.e., $N + 1$ has factors $2, 3, 4, \cdots, 10$ whose least common multiple is $2^3 \cdot 3^2 \cdot 5 \cdot 7 = 2520$. $\therefore N = 2519$.

38. $.03 = 600/x_1$ and $.02 = 600/x_2$; $\therefore x_1 = 20{,}000$ and $x_2 = 30{,}000$; $\therefore x_2 - x_1 = 10{,}000$ (feet).

39. Since the distance d traveled by the stone and the sound is the same, $d = r_1 t_1 = r_2 t_2$. $\therefore$ The times of travel are inversely proportional to the rates of fall: $r_1/r_2 = t_2/t_1$.

$$\therefore \frac{16t^2/t}{1120} = \frac{7.7 - t}{t}; \qquad \therefore t = 7 \text{ (seconds)}; \qquad \therefore 16t^2 = 16 \cdot 7^2 = 784 \text{ (feet)}.$$

40. Since $x^3 + 1 = (x + 1)(x^2 - x + 1)$ and
$$x^3 - 1 = (x - 1)(x^2 + x + 1),$$

the given expression, when simplified, is 1 (with suitable restrictions on the value of x).

41. Since the y-differences are 2, 4, 6, 8, a quadratic function is suggested, thus eliminating (C). A quick check of the other choices shows that (B) is correct;

or

Make the check with each of the choices given.

42. $x = \sqrt{1 + x}$, $x^2 = 1 + x$, $x^2 - x - 1 = 0$, and $x \sim 1.62$.

$$\therefore 1 < x < 2.$$

43. (E) is incorrect because, when the product of two positive quantities is given, their sum is least when they are equal. To prove this, let one of the numbers be x; then the other number can be written $x + h$ (h may be negative). Then if $x(x + h) = p$,

$$x = (-h + \sqrt{h^2 + 4p})/2. \qquad \therefore x + x + h = \sqrt{h^2 + 4p},$$

which has its least value when $h = 0$.

44. Inverting each of the expressions, we have the set of equations:

$$\frac{1}{y} + \frac{1}{x} = \frac{1}{a}, \qquad \frac{1}{z} + \frac{1}{x} = \frac{1}{b}, \qquad \frac{1}{y} + \frac{1}{z} = \frac{1}{c}.$$

$$\therefore \frac{1}{x} - \frac{1}{z} = \frac{1}{a} - \frac{1}{c}, \qquad \frac{1}{x} + \frac{1}{z} = \frac{1}{b}; \qquad \therefore \frac{2}{x} = \frac{1}{a} + \frac{1}{b} - \frac{1}{c}.$$

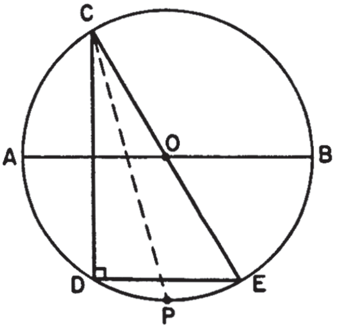
$$\therefore x = \frac{2abc}{ac + bc - ab}.$$

45. Log 8 = log 2^3 = 3 log 2; log 9 = log 3^2 = 2 log 3; log 10 = 1 and log 5 = log (10/2) = log 10 − log 2 = 1 − log 2. From these and from the information supplied in the problem,

$$\log 2 = \tfrac{1}{3} \log 8, \qquad \log 3 = \tfrac{1}{2} \log 9, \qquad \text{and} \qquad \log 5 = 1 - \log 2$$

can be found; consequently, so can the logarithm of any number representable as a product of powers of 2, 3 and 5. Since 17 is the only number in the list not representable in this way, (A) is the correct choice.

46. Extend CO so that it meets the circle at E. Since CE is a diameter, $CD \perp DE$. The bisector of angle OCD bisects the subtended arc DE, that is, arc DP = arc PE. Regardless of the position of C, the corresponding chord DE is always parallel to AB, and hence P always bisects the arc AB.



47. Since $\frac{1}{r^2}+\frac{1}{s^2}=\frac{r^2+s^2}{r^2s^2}=\frac{(r+s)^2-2rs}{(rs)^2}$, and since

$$r+s=-\frac{b}{a} \quad \text{and} \quad rs=\frac{c}{a}, \quad \text{we have} \quad \frac{1}{r^2}+\frac{1}{s^2}=\frac{b^2-2ac}{c^2}.$$

48. The area of the large square is $2a^2$, where a is the radius of the circle. The area of the small square is $4a^2/5$. The ratio is 2:5.

49. From the given information it follows that for the right triangle ABC, $(a/2)^2+b^2=25$ and $a^2+(b/2)^2=40$. $\therefore a^2=36$ and $b^2=16$.

$$\therefore c^2=a^2+b^2=52; \qquad \therefore c=2\sqrt{13}.$$

50. Let t_1, t_2, t_3 be the number of hours, respectively, that the car travels forward, back to pick up Dick, then forward to the destination. Then we may write

$$25\cdot t_1-25\cdot t_2+25\cdot t_3=100 \qquad \text{for car}$$
$$5\cdot t_1+5\cdot t_2+25\cdot t_3=100 \qquad \text{for Dick}$$
$$25\cdot t_1+5\cdot t_2+5\cdot t_3=100 \qquad \text{for Harry.}$$

This system of simultaneous equations is equivalent to the system

$$t_1-t_2+t_3=4$$
$$t_1+t_2+5t_3=20$$
$$5t_1+t_2+t_3=20$$

whose solution is $t_1=3$, $t_2=2$, $t_3=3$.
Hence $t_1+t_2+t_3=8$ = total number of hours.

COMMITTEE ON CONTESTS AND AWARDS
OF THE
METROPOLITAN NEW YORK SECTION
OF THE
MATHEMATICAL ASSOCIATION OF AMERICA

ANSWER SHEET FOR THE TEST GIVEN MAY 10, 1951

PLEASE ENTER THE STUDENT'S SCORE ON THE COVER OF THE EXAMINATION BOOKLET. THE THREE HIGHEST RANKING PAPERS ARE TO BE SENT TO THE COMMITTEE ON AWARDS.

1.	B	18.	D	35.	A
2.	D	19.	E	36.	B
3.	B	20.	C	37.	D
4.	D	21.	C	38.	A
5.	B	22.	B	39.	A
6.	D	23.	B	40.	C
7.	B	24.	D	41.	B
8.	C	25	A	42.	C
9.	D	26.	E	43.	E
10.	C	27.	E	44.	E
11.	C	28.	C	45.	A
12.	C	29.	D	46.	A
13.	C	30.	C	47.	D
14.	C or D, C and D	31.	D	48.	C
15.	E	32	D	49.	D
16.	C	33.	C	50.	D
17.	D	34.	A		

THE COMMITTEE IS ANXIOUS TO IMPROVE THE TEST AND IT IS SEEKING YOUR AID IN THIS MATTER. FOR FILING PURPOSES PLEASE SEND IN CRITICISMS OF THIS TEST ON $8\frac{1}{2}$ x 11 PAPER TO W.H. FAGERSTROM, CHAIRMAN OF COMMITTEE ON AWARDS, CITY COLLEGE, NEW YORK 31, N.Y.

SOLUTION KEY

MATHEMATICAL CONTEST

1952

Sponsored by

THE MATHEMATICAL ASSOCIATION OF AMERICA

In the original American Mathematics Competitions archives, there exist no answers page or solutions booklet for this year's contest. In the interest of providing a complete reference for those using these contests for practice or learning purposes we are providing a set of answers and solutions for this year.

The solutions were taken from:

> **Contest Problem Book No 1: Annual High School Mathematics Examinations 1950-1960** (New Mathematical Library) by Charles T. Salkind (Washington, D.C.: Mathematical Association of America, 1975)

Part 1

1. Since $A = \pi r^2$ and π is irrational and r is rational, A is irrational. (The product of a rational number and an irrational number is irrational.)

2. Average $= \dfrac{20\cdot 80 + 30\cdot 70}{20 + 30} = 74.$

3. $a^3 - a^{-3} = a^3 - \dfrac{1}{a^3} = \left(a - \dfrac{1}{a}\right)\left(a^2 + 1 + \dfrac{1}{a^2}\right).$

4. For $P - 1$ pounds the charge is 3 cents per pound. For the first pound the charge is 10 cents. $\therefore C = 10 + 3(P - 1)$;

or

The charge for each pound is 3 cents with an additional charge of 7 cents for the first pound. $\therefore C = 3P + 7 = 3P - 3 + 10 = 10 + 3(P - 1)$.

5. $y = mx + b$, $m = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{12 + 6}{6 - 0} = 3$; $\therefore y = 3x + b$. Substituting, $-6 = 3\cdot 0 + b$; $\therefore b = -6$; $\therefore y = 3x - 6$. This equation is satisfied by (3, 3).

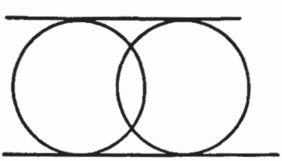
6. The roots are $(7 \pm \sqrt{49 + 36})/2$ and their difference is

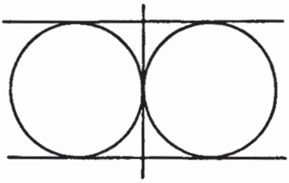
$$\frac{7+\sqrt{85}}{2} - \frac{7-\sqrt{85}}{2} = \sqrt{85} \quad \text{or} \quad \frac{7-\sqrt{85}}{2} - \frac{7+\sqrt{85}}{2} = -\sqrt{85}.$$

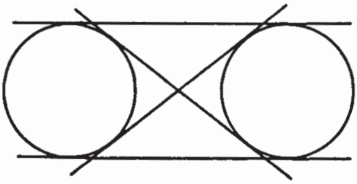
Of the given choices, (E) is correct.

7. $(x^{-1} + y^{-1})^{-1} = \dfrac{1}{(1/x) + (1/y)} = \dfrac{xy}{x + y}.$

8. Two equal circles in a plane cannot have only one common tangent







9. $ma - mb = cab, \quad ma = b(m + ca); \qquad \therefore b = ma/(m + ca).$

10. $t_1 = \dfrac{d}{10}, \quad t_2 = \dfrac{d}{20}, \quad t_1 + t_2 = d\left(\dfrac{1}{10} + \dfrac{1}{20}\right);$

$$\therefore A = \frac{2d}{d[(1/10) + (1/20)]} = \frac{40}{3} = 13\tfrac{1}{3}\ (\text{mph});$$

or

$$\frac{1}{A} = \frac{1}{2}\left(\frac{1}{10} + \frac{1}{20}\right); \qquad \therefore A = \frac{40}{3}.$$

11. The value $x = 1$ makes the denominator zero. Since division by zero is not permitted, $f(1)$ is undefined, so that (C) is the correct choice.

12. Let the first two terms be a and ar where $-1 < r < 1$.

$$\therefore a(1 + r) = 4\tfrac{1}{2}.$$

Since $s = a/(1 - r) = 6, \quad a = 6(1 - r).$

$$\therefore 6(1 - r)(1 + r) = 4\tfrac{1}{2}; \qquad \therefore r = \pm\tfrac{1}{2}; \qquad \therefore a = 3 \text{ or } 9.$$

13. The minimum value of the function is at the turning point of the graph where $x = -p/2$. (See 1950, Problem 41.)

14. $12{,}000 = H - \tfrac{1}{5}H; \qquad \therefore H = 15{,}000, \quad 12{,}000 = S + \tfrac{1}{5}S;$

$$\therefore S = 10{,}000, \quad H + S = 25{,}000.$$

Therefore, the sale resulted in a loss of \$1000.

15. Since $6^2 + 8^2 = 100 > 9^2$, the triangle is acute, so that (C) is a correct choice. A check of (D) by the Law of Sines or the Law of Cosines shows that it is incorrect. (B) is obviously incorrect by the Law of Sines. Note: The elimination of (B) and (D) involves trigonometry, and so, in part, this problem is beyond the indicated scope.

Part 2

16. $bh = \left(\dfrac{11}{10}b\right)(xh) \qquad \therefore x = \dfrac{10}{11}$ and h is reduced by $\dfrac{1}{11}$ or $9\tfrac{1}{11}\%$;

or

$$bh = \left(b + \frac{1}{10}b\right)\left(h - \frac{r}{100}h\right) = bh + \frac{1}{10}bh - \frac{r}{100}bh - \frac{r}{1000}bh.$$

$$\therefore \frac{1}{10} - \frac{r}{100} - \frac{r}{1000} = 0 \text{ and } r = 9\tfrac{1}{11}\ (\text{per cent}).$$

17. Let C be the cost, L the list price, S the selling price and M the marked price. $C = \frac{4}{5}L$, $S = C + \frac{1}{5}S$, $S = \frac{4}{5}M$.

$$\therefore \frac{4}{5}\cdot\frac{4}{5}M = \frac{4}{5}L; \qquad \therefore M = \frac{5}{4}L; \qquad \therefore \text{(C) is the correct choice.}$$

18. $\log p + \log q = \log (pq)$. Thus we must have $pq = p + q$.

$$\therefore p = q/(q-1).$$

19. Since BD bisects angle ABE,

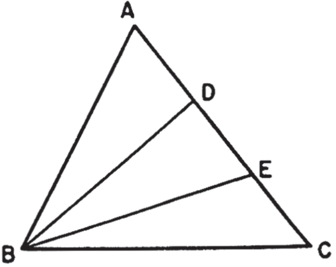
we have $\dfrac{\overline{AD}}{\overline{DE}} = \dfrac{\overline{AB}}{\overline{BE}}$;

since BE bisects angle DBC,

we have $\dfrac{\overline{DE}}{\overline{EC}} = \dfrac{\overline{BD}}{\overline{BC}}$.

Hence

$$\frac{\overline{AD}}{\overline{EC}} = \frac{\overline{DE}(\overline{AB}/\overline{BE})}{\overline{DE}(\overline{BC}/\overline{BD})} = \frac{(\overline{AB})(\overline{BD})}{(\overline{BE})(\overline{BC})}.$$



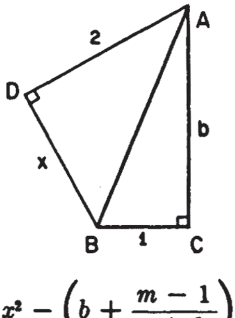
20. $x = \frac{3}{4}y \qquad \therefore \frac{x-y}{y} = -\frac{1}{4} \qquad \therefore$ (E) is incorrect.

21. Each such angle is the vertex angle of an isosceles triangle whose base angles are each, in degrees, $180 - (n-2)180/n = 360/n$.

$$\therefore \text{angle} = 180 - (720/n) = (180n - 720)/n = 180(n-4)/n.$$

22. $x^2 + 4 = b^2 + 1$;

$$\therefore x = \sqrt{b^2 - 3}.$$



23. The equation is equivalent to $x^2 - \left(b + \frac{m-1}{m+1}a\right)x + c\frac{m-1}{m+1} = 0$. Since the coefficient of x is equal to the sum of the roots, it must be zero. We have $b + \frac{m-1}{m+1}a = 0$. $\therefore bm + b + ma - a = 0$;

$$\therefore m = \frac{a-b}{a+b}.$$

24. Since $\overline{AB} = 20$ and $\overline{AC} = 12$, $\overline{BC} = 16$. Since $\triangle BDE \sim \triangle BCA$, $\dfrac{\text{Area } \triangle BDE}{\text{Area} \triangle BCA} = \dfrac{10^2}{16^2}$. But Area $\triangle BCA = \frac{1}{2}\cdot 12\cdot 16 = 96$.

$$\therefore \text{Area } \triangle BDE = 37\tfrac{1}{2},$$

and the required area is $96 - 37\frac{1}{2} = 58\frac{1}{2}$.
Note: the method here shown is based on subtracting the area of $\triangle BDE$ from that of $\triangle BCA$. It is a worthwhile exercise to find the area of the quadrilateral by decomposing it into two right triangles.

25. Since the times are inversely proportional to the rates (see 1951, Problem 39),

$$\frac{t}{30 - t} = \frac{1080/3}{8} = 45. \quad \therefore t = 30\cdot\frac{45}{46} \text{ and}$$

$$d = 8t = 8\cdot 30\cdot\frac{45}{46} \sim 245.$$

26. $r^3 + \dfrac{1}{r^3} = \left(r + \dfrac{1}{r}\right)\left(r^2 - 1 + \dfrac{1}{r^2}\right)$. But $\left(r + \dfrac{1}{r}\right)^2 = r^2 + 2 + \dfrac{1}{r^2} = 3$.

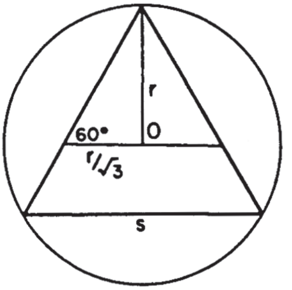
$$\therefore r^2 + \frac{1}{r^2} = 1; \qquad \therefore r^2 - 1 + \frac{1}{r^2} = 0; \qquad \therefore r^3 + \frac{1}{r^3} = 0.$$

27. Let P_1 and P_2 be the perimeters of the smaller and larger triangles respectively. We have

$$r = s\sqrt{3}/2 \text{ and } s = 2r/\sqrt{3}.$$

$$\therefore P_1 = 2r\sqrt{3} \text{ and } P_2 = 3r\sqrt{3};$$

$$\therefore P_1 : P_2 = 2:3.$$



28. (C) is shown to be the correct choice by direct check.
Note: The fact that the y-differences are 4, 6, 8, and 10 can be used t
eliminate (A) and (B) immediately.

29. Let $\overline{KB} = x$ and $\overline{CK} = y$. Then $y(8 - y) = x(10 - x)$ and $y^2 = (5 - x)^2 + 5^2$. $\therefore 8y - y^2 = 50 - y^2$; $\therefore y = 25/4$.
$\therefore x = \dfrac{10 - (15/2)}{2} = \dfrac{5}{4}$ and $10 - x = \dfrac{35}{4}$,
so that (A) is the correct choice.

30. Using $S = \dfrac{n}{2}[2a + (n - 1)d]$, we have $\dfrac{10}{2}(2a + 9d) = 4\cdot\dfrac{5}{2}(2a + 4d)$
$\therefore d = 2a$; $\quad \therefore a:d = 1:2$.

31. Single out any one point A. Joined to the remaining 11 points, A yield 11 lines. Since a line is determined by two points, we have for the 12 points, $\frac{1}{2}\cdot 12\cdot 11 = 66$ (lines);

or

$$C\ (12, 2) = \tfrac{1}{2}\cdot 12\cdot 11 = 66.$$

32. Time = distance/rate; $\therefore t = 30/x.$

33. For a given perimeter the circle has the largest area;

or

let P be the common perimeter, A_1 and A_2 the areas of the circle and the square, respectively. Then

$$P = 2\pi r, \qquad r = \frac{P}{2\pi}, \qquad A_1 = \frac{P^2}{4\pi};$$

$$P = 4s, \qquad s = \frac{P}{4}, \qquad A_2 = \frac{P^2}{16};$$

Since $4\pi < 16$, $A_1 > A_2$.

34. Let the original price be P. Then $P_1 = \left(1 + \frac{p}{100}\right) P.$

$$P_2 = \left(1 - \frac{p}{100}\right) P_1 = \left(1 - \frac{p}{100}\right)\left(1 + \frac{p}{100}\right)P = 1;$$

$$\therefore P = \frac{1}{1 - (p^2/100^2)} = \frac{10{,}000}{10{,}000 - p^2}.$$

35. $$\frac{\sqrt{2}}{(\sqrt{2}+\sqrt{3})-\sqrt{5}}\cdot\frac{(\sqrt{2}+\sqrt{3})+\sqrt{5}}{(\sqrt{2}+\sqrt{3})+\sqrt{5}} = \frac{2+\sqrt{6}+\sqrt{10}}{2\sqrt{6}};$$

$$\frac{2+\sqrt{6}+\sqrt{10}}{2\sqrt{6}}\cdot\frac{\sqrt{6}}{\sqrt{6}} = \frac{2\sqrt{6}+6+2\sqrt{15}}{2\cdot 6} = \frac{3+\sqrt{6}+\sqrt{15}}{6}.$$

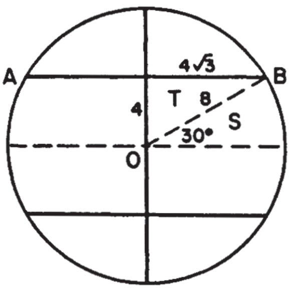
Part 3

36. $$\frac{x^3+1}{x^2-1} = \frac{(x+1)(x^2-x+1)}{(x+1)(x-1)} = \frac{x^2-x+1}{x-1} \text{ for } x \neq -1.$$

$$\lim_{x\to -1}\frac{x^3+1}{x^2-1} = \lim_{x\to -1}\frac{x^2-x+1}{x-1} = \frac{3}{-2} = -\frac{3}{2}.$$

For continuity at $x = -1$, we must define $\frac{x^3+1}{x^2-1} = -\frac{3}{2}.$

Note: This problem is beyond the indicated scope.



37. By symmetry, the required area is $4(T + S)$.

$T = \frac{1}{2}4 \cdot 4\sqrt{3} = 8\sqrt{3}$,

$S = (30/360) \cdot \pi 8^2 = 5\frac{1}{3}\pi$;

$\therefore A = 32\sqrt{3} + 21\frac{1}{3}\pi$.

38. $1400 = \frac{1}{2} \cdot 50(8a + 8b)$. $\therefore a + b = 7$. This indeterminate equation is satisfied by the following pairs of integral values: 1, 6; 2, 5; 3, 4. $\therefore$ (D) is the correct choice.

39. $l^2 + w^2 = d^2$, $l + w = p/2$, $(l + w)^2 = l^2 + 2lw + w^2 = p^2/4$, $2lw = p^2/4 - d^2$, $(l - w)^2 = l^2 - 2lw + w^2 = 2d^2 - p^2/4$, and $l - w = \sqrt{8d^2 - p^2}/2$.

40. We are told that the values of $f(x)$ listed correspond to

$$f(x),\ f(x + h),\ f(x + 2h),\ \cdots,\ f(x + 7h).$$

Observe that the difference between successive values is given by

$$\begin{aligned} f(x + h) - f(x) &= a(x + h)^2 + b(x + h) + c - (ax^2 + bx + c) \\ &= 2ahx + ah^2 + bh. \end{aligned}$$

Since this difference is a linear function of x, it must change by the same amount whenever x is increased by h. But the successive differences of the listed values

3844 3969 4096 4227 4356 4489 4624 4761

are 125 127 131 129 133 135 137

so that, if only one value is incorrect, 4227 is that value.

41. $V + y = \pi(r + 6)^2 h = \pi r^2(h + 6)$; $\therefore (r + 6)^2 \cdot 2 = r^2 \cdot 8$

$\therefore 3r^2 - 12r - 36 = 0$; $\therefore r = 6$.

42. $D = .PQQQ \cdots = .a_1 \cdots a_r b_1 \cdots b_s b_1 \cdots b_s \cdots$. So (A), (B), (C) and (E) are all correct choices. To check that (D) is incorrect, we have

$10^{r+s}D - 10^r D = PQ - P$. $\therefore 10^r(10^s - 1)D = P(Q - 1)$.

43. For each semi-circle the diameter is $2r/n$, and the length of its arc is $\pi r/n$. The sum of n such arcs is $n \cdot \pi r/n = \pi r =$ semi-circumference.

44. Given $10u + t = k(u + t)$. Let $10t + u = m(u + t)$.

$\therefore 11(t + u) = (k + m)(u + t)$; $\therefore k + m = 11$; $\therefore m = 11 - k$.

45. The Arithmetic Mean is $(a + b)/2$, the Geometric Mean is $\sqrt{ab}$, and the Harmonic Mean is $2ab/(a + b)$. The proper order for decreasing magnitude is (E);

or

Since $(a - b)^2 > 0$, we have $a^2 + b^2 > 2ab$; $\quad \therefore\ a^2 + 2ab + b^2 > 4ab$,
$a + b > 2\sqrt{ab}$, and $(a + b)/2 > \sqrt{ab}$.
Since $a^2 + 2ab + b^2 > 4ab$, we have
$1 > 4ab/(a + b)^2$; $\quad \therefore\ ab > 4a^2b^2/(a + b)^2$, and $\sqrt{ab} > 2ab/(a + b)$.

46. Area (new) $= (d + l)(d - l) = d^2 - l^2 = w^2$; $\quad \therefore$ (C) is the correct choice.

47. From equation (1) $z = y^2$.

From equation (2) $2^z = 2^{2x+1}$, $z = 2x + 1$; $\quad \therefore\ x = \dfrac{z - 1}{2} = \dfrac{y^2 - 1}{2}$.

From equation (3) $\dfrac{y^2 - 1}{2} + y + y^2 = 16$. This equation has one integral root $y = 3$. $\quad \therefore\ x = 4,\ y = 3,\ z = 9$

48. Let f and s denote the speeds of the faster and slower cyclists, respectively. The distances (measured from the starting point of the faster cyclist) at which they meet are

(1) $f \cdot r = k + s \cdot r$ when they travel in the same direction, and
(2) $f \cdot t = k - s \cdot t$ when the slower one comes to meet the other.

From (2), $f + s = k/t$; from (1), $f - s = k/r$.

$$\therefore\ 2f = \frac{k}{r} + \frac{k}{t} = \frac{k(r + t)}{rt}, \qquad 2s = \frac{k}{t} - \frac{k}{r} = \frac{k(r - t)}{rt},$$

and $\dfrac{f}{s} = \dfrac{r + t}{r - t}$.

49. By subtracting from ΔABC the sum of ΔCBF, ΔBAE, and ΔACD and restoring $\Delta CDN_1 + \Delta BFN_3 + \Delta AEN_2$, we have $\Delta N_1N_2N_3$.

$$\Delta CBF = \Delta BAE = \Delta ACD = \tfrac{1}{3}\Delta ABC.$$

From the assertion made in the statement of the problem, it follows that
$\Delta CDN_1 = \Delta BFN_3 = \Delta AEN_2 = \frac{1}{7}\cdot\frac{1}{3}\Delta ABC = \frac{1}{21}\Delta ABC$.
$\therefore\ \Delta N_1N_2N_3 = \Delta ABC - 3\cdot\frac{1}{3}\Delta ABC + 3\cdot\frac{1}{21}\Delta ABC = \frac{1}{7}\Delta ABC$.

50. Rearrange the terms:

$$S_1 = 1 + \frac{1}{4} + \frac{1}{16} + \cdots = \frac{4}{3} \text{ and } S_2 = \frac{\sqrt{2}}{4} + \frac{\sqrt{2}}{16} + \frac{\sqrt{2}}{64} + \cdots = \frac{\sqrt{2}}{3}$$

$\therefore\ S = S_1 + S_2 = \frac{1}{3}(4 + \sqrt{2})$.

SOLUTION KEY

Fourth Annual

MATHEMATICAL CONTEST

1953

Sponsored by

THE MATHEMATICAL ASSOCIATION OF AMERICA

In the original American Mathematics Competitions archives, there exist no answers page or solutions booklet for this year's contest. In the interest of providing a complete reference for those using these contests for practice or learning purposes we are providing a set of answers and solutions for this year.

The solutions were taken from:

Contest Problem Book No 1: Annual High School Mathematics Examinations 1950-1960 (New Mathematical Library) by Charles T. Salkind (Washington, D.C.: Mathematical Association of America, 1975)

Part 1

1. The profit on each orange sold is $4 - 3\frac{1}{3} = \frac{2}{3}$ cent. $\therefore \frac{2}{3}n = 100,\ n = 150.$

2. $D = D_1 + D_2 - D_1D_2,\ D = 20 + 15 - 3 = 32.$

$$\therefore S = 250 - (.32)(250) = (.68)(250),$$

that is, 68% of 250. (See 1950, Problem 22.)

3. $x^2 + y^2$ has no linear factors in the real field. $x^2 + y^2 = (x + iy)(x - iy)$ in the complex field.
Note: This problem is probably beyond the indicated scope.

4. Set each factor equal to zero, and solve the three resulting equations. The roots are $0, -4, -4,$ and 4. (The factor $x^2 + 8x + 16 = (x + 4)^2$ has the double root -4.)

5. $x = 6^{2.5} = 6^2 \cdot 6^{1/2} = 36\sqrt{6}.$

6. $\frac{5}{2}[(5q + 1) - (q + 5)] = \frac{5}{2}(4q - 4) = 10(q - 1).$

7. Multiply numerator and denominator by $\sqrt{a^2 + x^2}$ and simplify. (D) is the correct choice.

8. For intersection $8/(x^2 + 4) = 2 - x.$
$\therefore x^3 - 2x^2 + 4x = x(x^2 - 2x + 4) = 0; \qquad \therefore x = 0.$

9. $\dfrac{4\frac{1}{2} + x}{9 + x} = \dfrac{7}{10}; \qquad \therefore x = 6.$

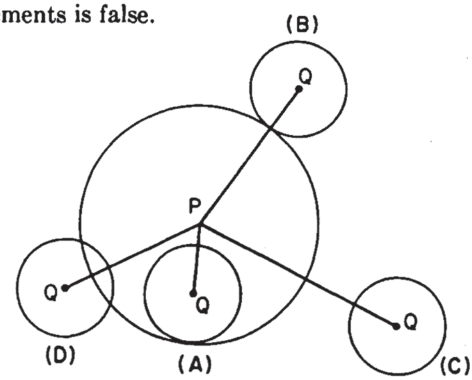
10. The circumference of the wheel is 6π. $\qquad \therefore N = 5280/6\pi = 880/\pi.$

11. $C_1 = 2\pi r,\ C_2 = 2\pi(r + 10); \qquad \therefore C_2 - C_1 = 20\pi \sim 60$ (feet).

12. $A_1/A_2 = \pi 4^2/\pi 6^2 = 4/9.$

13. $A = \frac{1}{2}bh = \frac{1}{2}h(b_1 + b_2); \qquad \therefore b_1 + b_2 = b = 18.$
$\therefore m = \frac{1}{2}(b_1 + b_2) = 9.$

14. Each assertion is realized in the accompanying figure. Hence, none of these statements is false.



15. Designate a side of the original square by s. Then the radius r of the inscribed circle is $s/2$. A side of the second square (inscribed in the circle) is $r\sqrt{2} = s/\sqrt{2}$. Therefore, the area of the second square is $s^2/2$, while that of the original square is s^2. $\therefore$ (B) is the correct choice.

Part 2

16. S = Selling price = cost + profit + expense = $C + .10S + .15S$; $\therefore S = 4/3C$; $\therefore$ (D) is the correct choice.

17. $\frac{1}{x} = \frac{1}{2}\left(\frac{1}{6} + \frac{1}{4}\right)$ $\therefore x = 4.8$;

or

Let y be the amount invested at 4%. Then $.04y = .06(4500 - y)$ and $y = 2700$. Therefore the total interest is

$.04(2700) + .06(1800) = 216$, and $(216/4500)\cdot 100 = 4.8\%$.

18. $x^4 + 4 = (x^4 + 4x^2 + 4) - 4x^2$

$$= (x^2 + 2)^2 - (2x)^2 = (x^2 + 2x + 2)(x^2 - 2x + 2).$$

19. $\frac{3}{4}x \cdot \left(\frac{3}{4}y\right)^2 = \frac{27}{64}xy^2$, and the decrease is $xy^2 - \frac{27}{64}xy^2 = \frac{37}{64}xy^2$.

20. $x^4 + x^3 - 4x^2 + x + 1 = x^2\left[\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 4\right] = 0.$

Since $x^2 + \frac{1}{x^2} = \left(x + \frac{1}{x}\right)^2 - 2$, we have

$x^2(y^2 - 2 + y - 4) = x^2(y^2 + y - 6) = 0.$

21. $x^2 - 3x + 6 = 10^1 = 10$ or $x^2 - 3x - 4 = 0$; $\therefore x = 4$ or -1

22. $27 \cdot \sqrt[4]{9} \cdot \sqrt[3]{9} = 3^3 \cdot 3^{1/2} \cdot 3^{2/3} = 3^{25/6}$, $\log_3 3^{25/6} = 4\frac{1}{6}$.

23. Multiply both sides of the equation by $\sqrt{x+10}$.
$x + 10 - 6 = 5\sqrt{x+10}$, $x^2 - 17x - 234 = 0$, $x = 26$ or -9.
-9 fails to check, i.e., it is an extraneous root.

24. $100a^2 + 10a(b + c) + bc = 100a^2 + 100a + bc$. For equality we must have $b + c = 10$.

25. $ar^n = ar^{n+1} + ar^{n+2}$; $\therefore r^2 + r = 1$; $\therefore r = (\sqrt{5} - 1)/2$.

26. Let x be the required length. Then $x^2/15^2 = 2/3$.
$\therefore x^2 = 150$; $\therefore x = 5\sqrt{6}$.

27. $S = \pi + \dfrac{\pi}{4} + \dfrac{\pi}{16} + \cdots$, $S = \pi/[1 - (1/4)] = 4\pi/3$.

28. The bisector of an angle of the triangle divides the opposite sides into segments proportional to the other two sides, i.e., $y/c = x/b$. It follows that $x/b = (x + y)/(b + c) = a/(b + c)$.

29. (D) is the correct choice. Consult sections on approximate numbers for the explanation.
Note: For a full understanding of the topic, a knowledge of calculus is needed.

30. B pays A $9000 - \frac{1}{10} \cdot 9000 = \$8100$.
A pays B $8100 + \frac{1}{10} \cdot 8100 = \$8910$.
Thus A has lost $8910 - 8100 = \$810$.

31. 30 ft. per second corresponds to 1 click per second. In miles per hour, $30 \cdot 60^2/5280 = 225/11$ mph corresponds to 1 click per second. 1 mph corresponds to $11/225$ clicks per second, which is approximately 1 click in 20 seconds. So x miles per hour corresponds approximately to x clicks in 20 seconds.

32. The diagonals of the quadrilateral formed lie along the two perpendicular lines joining the midpoints of the opposite sides of the rectangle. These diagonals are of different lengths, and they are perpendicular bisectors of each other. Therefore, the figure is a rhombus;

or

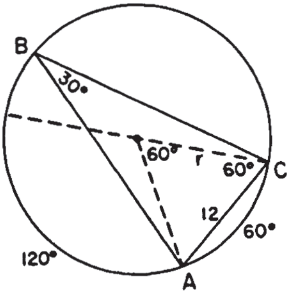
the sides of the quadrilateral can be proved equal by congruent triangles. Also it can be proved that no interior angle is a right angle.

33. Let x be one of the equal sides of the triangle. $2p = 2x + x\sqrt{2}$;

$$\therefore x = \frac{2p}{2+\sqrt{2}}, \; A = \frac{1}{2}x^2 = \frac{1}{2} \cdot \frac{4p^2}{6+4\sqrt{2}} \cdot \frac{6-4\sqrt{2}}{6-4\sqrt{2}} = p^2(3 - 2\sqrt{2}).$$

34. Let the 12-inch side be AC.

Since $\measuredangle B = 30°$, $\widehat{AC} = 60°$.
$\therefore \overline{AC} = r$;
$\therefore 2r = d = 24$ (inches).



35.
$$f(x + 2) = (x + 2)(x + 2 - 1)/2 \text{ and}$$
$$f(x + 1) = (x + 1)(x + 1 - 1)/2.$$

So $(x + 1)/2 = f(x + 1)/x$; $\quad \therefore f(x + 2) = (x + 2)f(x + 1)/x$.
Note: The symbolism here is beyond the indicated scope.

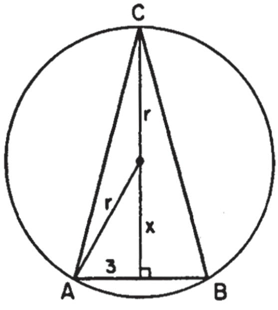
Part 3

36. By actual division, the remainder is found to be $18 + m$. Since $x - 3$ is a factor, $18 + m = 0$, and, therefore, $m = -18$. $\therefore$ (C) is the correct choice;

or

by the Factor Theorem Converse, $36 - 18 + m = 0$. $\therefore m = -18$.

37. Let x be the distance from the center
of the circle to the base and let
Then
$$h = r + x \text{ be the altitude of } \triangle ABC.$$
$$h^2 + 9 = 144. \therefore h = \sqrt{135}.$$
$$\begin{aligned} r^2 &= 3^2 + x^2 \\ &= 3^2 + (h - r)^2 \\ &= 3^2 + h^2 - 2hr + r^2. \\ \therefore r &= (9 + h^2)/2h \\ &= (9 + 135)/2\sqrt{135} \\ &= 144\sqrt{135}/270 = 8\sqrt{15}/5. \end{aligned}$$



38. $F[3, f(4)]$ means the value of $b^2 + a$ when $a = 3$ and $b = 2$, namely, 7.
Note: This problem is beyond the indicated scope.

39. For permissible values of a and b, $\log_a b \cdot \log_b a \equiv 1$;

or

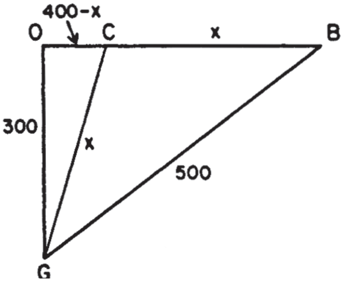
Let $x = \log_a b$ and $y = \log_b a$. $\quad \therefore a^x = b$ and $b^y = a$.
$\therefore a^{xy} = b^y = a$; $\quad \therefore xy = 1$; $\quad \therefore \log_a b \cdot \log_b a = 1$.

40. To contradict the given statement, preface it with "It is not true that $\cdots$". Therefore, the negation is (C);

or

in symbols, given $\forall_x[x \in M \mid x \text{ is honest}]$.
$\therefore \sim \forall_x = \exists_x[x \in M \mid x \text{ is dishonest}]$.

41. $\overline{OB} = 400$ rods.
$x^2 = 300^2 + (400 - x)^2$,
$800x = 250{,}000$.
$\therefore x = 312\frac{1}{2}$ (rods);
$\therefore$ (E) is the correct choice.



42. $t^2 = 41^2 - 9^2 = 40^2$.

43. Let s be the price of the article, n the number of sales. Then

$$[(1 + p)s][(1 - d)n] = sn, \quad p - d - pd = 0, \quad d = p/(1 + p).$$

44. Let the true equation be $x^2 + bx + c = 0$, the equation obtained by the first student be $x^2 + bx + c' = 0$, and the equation obtained by the second student be $x^2 + b'x + c = 0$.

$$x^2 + bx + c' = x^2 - 10x + 16 = 0 \quad \text{and}$$
$$x^2 + b'x + c = x^2 + 10x + 9 = 0.$$
$$\therefore x^2 + bx + c = x^2 - 10x + 9 = 0.$$

45. For $a \neq b$, Arithmetic Mean > Geometric Mean. For $a = b$, Arithmetic Mean = Geometric Mean. (See 1952, Problem 45.)
$\therefore (a + b)/2 \geq \sqrt{ab}$.

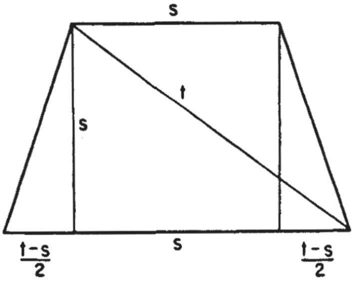
46. $\sqrt{s^2 + l^2} = s + l - \dfrac{l}{2} = s + \dfrac{l}{2}, \qquad \dfrac{3}{4}l^2 = sl; \qquad \therefore \dfrac{s}{l} = \dfrac{3}{4}.$

47. For all $x > 0$, $1 + x < 10^x$. $\quad\therefore \log(1 + x) < x$.
Note: This problem may be beyond the indicated scope.

48. $t^2 = s^2 + \left(\dfrac{t + s}{2}\right)^2$,

$5s^2 + 2ts - 3t^2 = 0 = (5s - 3t)(s + t)$;

$$\therefore \frac{s}{t} = \frac{3}{5}.$$



49. The smallest possible value of $\overline{AC} + \overline{BC}$ is obtained when C is the intersection of the y-axis, with the line that leads from A to the mirror image (the mirror being the y-axis) $B':(-2, 1)$ of B. This is true because $\overline{CB'} = \overline{CB}$ and a straight line is the shortest path between two points. The line through A and B' is given by

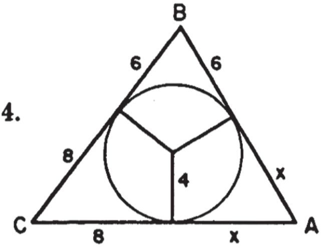
$$y = \frac{5-1}{5+2}x + k = \frac{4}{7}x + k.$$

To find k, we use the fact that the line goes through A:

$$5 = \frac{4}{7}\cdot 5 + k, \quad k = 5 - \frac{20}{7} = \frac{15}{7} = 2\tfrac{1}{7}:$$

$\therefore$ C has coordinates $(0, 2\tfrac{1}{7})$.

50. Denoting the sides of the triangle by a, b, c we observe that $a = 8 + 6 = 14$, $b = 8 + x$, $c = x + 6$. $\therefore$ $2s = a + b + c = 2x + 28$, $s = x + 14$. On the one hand, the area of the triangle is half the product of the perimeter and the radius of the inscribed circle; on the other hand, it is given in terms of s so that



$$\text{Area} = r\cdot s = 4(x+14) = \sqrt{s(s-a)(s-b)(s-c)} = \sqrt{48x(x+14)}$$

or $(x + 14)^2 = 3x(x + 14)$, $x + 14 = 3x$. $\therefore$ $x = 7$, and the shortest side is $c = 6 + 7 = 13$;

or

$$\sin\frac{B}{2} = \frac{4}{\sqrt{4^2+6^2}} = \frac{2}{\sqrt{13}}, \quad \cos\frac{B}{2} = \frac{3}{\sqrt{13}},$$

$$\sin B = 2\sin\frac{B}{2}\cos\frac{B}{2} = \frac{12}{13}, \quad \text{similarly } \sin C = \frac{4}{5}.$$

Using the Law of Sines,

$$\frac{\sin B}{b} = \frac{\sin C}{c}, \qquad \frac{12/13}{8+x} = \frac{4/5}{6+x}.$$

Thus $x = 7$.

SOLUTION KEY

Fifth Annual

MATHEMATICAL CONTEST

1954

Sponsored by

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C**ontest Problem Book No 1: Annual High School Mathematics Examinations 1950-1960** (New Mathematical Library) by Charles T. Salkind (Washington, D.C.: Mathematical Association of America, 1975)

Part 1

1. $(5 - \sqrt{y^2 - 25})^2 = 25 - 10\sqrt{y^2 - 25} + y^2 - 25 = y^2 - 10\sqrt{y^2 - 25}.$

2. Since $x = 1$ is not a solution of the original equation, the solution set of the first equation is the number 4 only;

or

the graph of $y = \dfrac{2x^2}{x-1} - \dfrac{2x+7}{3} + \dfrac{4-6x}{x-1} + 1$ is the straight line $y = \dfrac{4}{3} \cdot \dfrac{x^2 - 5x + 4}{x - 1} = \dfrac{4}{3}(x - 4)$ with the point $(1, -4)$ deleted. Since the graph crosses the x-axis at $(4, 0)$, its only root is 4.

3. $x = k_1 y^3$, $y = k_2 z^{1/5}$; $\quad \therefore x = k_1 k_2^3 z^{3/5} = k_3 z^{3/5}$ and (C) is the correct choice.

4. $132 = 2^2 \cdot 3 \cdot 11$ and $6432 = 2^5 \cdot 3 \cdot 67$. Their H.C.D. $= 2^2 \cdot 3 = 12$ and $12 - 8 = 4$.

5. $A = \frac{1}{2}ap = \frac{1}{2} \cdot 5\sqrt{3} \cdot 60 = 150\sqrt{3}$ (square inches).

6. $x^0 = 1$ for any number x $(x \neq 0)$ and $-32 = -2^5$.

$$\therefore \frac{1}{16} + 1 - \frac{1}{\sqrt{64}} - \frac{1}{(-2^5)^{4/5}} = \frac{1}{16} + 1 - \frac{1}{8} - \frac{1}{16} = \frac{7}{8}.$$

7. Let C be the original price of the dress. Then $25 = C - 2.50$ or $C = 27.50$. $250/2750 = 1/11 \sim 9\%$.

8. Let s be the side of the square and h the altitude of the triangle. Then $\frac{1}{2}h \cdot 2s = s^2$. $\quad \therefore h = s$ or $h/s = 1$.

9. $(13 + r)(13 - r) = 16 \cdot 9 \quad \therefore r^2 = 25,\ r = 5.$

10. Let $a = b = 1$. Then $(1 + 1)^6 = 2^6 = 64$;

or

$1 + 6 + 15 + 20 + 15 + 6 + 1 = 64.$

11. Let S be the selling price, M the marked price and C the cost. Then

$$S = M - \frac{1}{3}M = \frac{2}{3}M, \quad C = \frac{3}{4}S = \frac{3}{4} \cdot \frac{2}{3}M = \frac{1}{2}M$$

and

$$\frac{C}{M} = \frac{1}{2}.$$

12. The system is inconsistent (see 1950, Problem 14); hence, no solution;

or

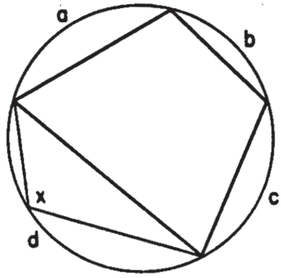
graphically, we have two parallel lines;

or

$\begin{vmatrix} 2 & -3 \\ 4 & -6 \end{vmatrix} = 0$; hence no solution.

13. $\measuredangle x = \frac{1}{2}(a + b + c)$. Thus the sum of the four angles

$$S = \tfrac{1}{2}(a + b + c + b + c + d + c + d + a + d + a + b)$$
$$= \tfrac{3}{2}(a + b + c + d) = \tfrac{3}{2}\cdot 360° = 540°.$$



14. $$\sqrt{1 + \left(\frac{x^4 - 1}{2x^2}\right)^2} = \sqrt{\frac{4x^4 + x^8 - 2x^4 + 1}{4x^4}} = \sqrt{\left(\frac{x^4 + 1}{2x^2}\right)^2}$$
$$= \frac{x^4 + 1}{2x^2} = \frac{x^2}{2} + \frac{1}{2x^2}.$$

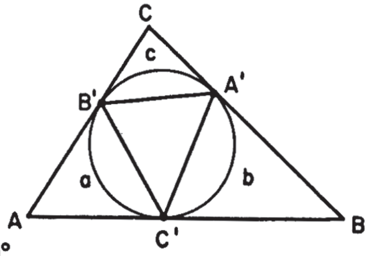
15. $\log 125 = \log (1000/8) = \log 1000 - \log 8 = 3 - \log 2^3 = 3 - 3 \log 2.$

Part 2

16. $f(x + h) - f(x) = 5(x + h)^2 - 2(x + h) - 1 - (5x^2 - 2x - 1)$
$= 10xh + 5h^2 - 2h = h(10x + 5h - 2).$
Note: The symbolism in this problem may be beyond the indicated scope, but the problem itself is quite simple.

17. There are a number of ways of determining that (A) is the correct answer. One (obvious) way is to plot a sufficient number of points.
Note: Graphs of cubics, generally, are beyond the indicated scope.

18. $2x - 3 > 7 - x$. Adding $x + 3$ to both sides of the inequality, we have $3x > 10$ or $x > \frac{10}{3}$.



19. $\measuredangle A' = \frac{a}{2}, \quad \measuredangle A = 180° - a;$

$\therefore a = 180° - \measuredangle A.$

$\measuredangle A' = \frac{a}{2} = 90° - \frac{\measuredangle A}{2};$

$\therefore \measuredangle A' < 90°.$

Similarly, $\measuredangle B' < 90°, \quad \measuredangle C' < 90°.$

20. By Descartes' Rule of Signs (B) is shown to be correct;

or

solve to find the roots $-1, -2, -3;$

or

graph $y = x^3 + 6x^2 + 11x + 6.$

21. $2\sqrt{x} + \frac{2}{\sqrt{x}} = 5.$ Squaring, we obtain $4x + 8 + \frac{4}{x} = 25.$ Multiply both sides of the equation by x and obtain $4x^2 - 17x + 4 = 0.$

22. Adding, we have

$$\frac{2x^2 - x - 4 - x}{(x+1)(x-2)} = \frac{2(x-2)(x+1)}{(x+1)(x-2)} = 2$$

for values of x other than 2 or -1.

23. $S = C + \frac{1}{n}C; \quad \therefore C = \frac{n}{n+1}S; \quad \therefore M = \frac{1}{n}C = \frac{1}{n}\cdot\frac{n}{n+1}S = \frac{S}{n+1}$

24. $2x^2 - x(k-1) + 8 = 0.$ For real, equal roots, the discriminant $(k-1)^2 - 64 = 0$ and $k - 1 = \pm 8. \quad \therefore k = 9$ or $-7.$

25. The product of the roots is $c(a-b)/a(b-c).$ Since one root is 1, the other is the fraction shown.

26. Let $x = \overline{BD}$ and let r be the radius of the small circle. Draw the line from the center of each of the circles to the point of contact of the tangent and the circle. By similar triangles,

$$\frac{x+r}{r} = \frac{x+5r}{3r}. \qquad \therefore x = r.$$

27. $\frac{1}{3}\pi r^2 h = \frac{1}{2}\cdot\frac{4}{3}\pi r^3;\qquad \therefore \frac{h}{r} = \frac{2}{1}.$

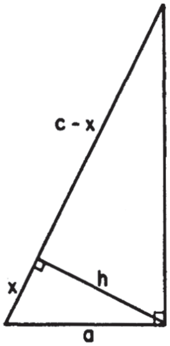
28. $\frac{mr}{nt} = \frac{4}{3}\cdot\frac{9}{14} = \frac{6}{7};\qquad \therefore mr = 6k$ and $nt = 7k.$

$$\therefore \frac{3mr - nt}{4nt - 7mr} = \frac{18k - 7k}{28k - 42k} = \frac{11k}{-14k} = -\frac{11}{14}.$$

29. In any right triangle (see accompanying figure), we have $x\cdot c = a^2$ and $(c - x)c = b^2.$

Since $\frac{a}{b} = \frac{1}{2}$, it follows that

$$\frac{x}{c - x} = \frac{x\cdot c}{(c - x)\cdot c} = \frac{a^2}{b^2} = \frac{1}{4}.$$



30. Denote by a, b, c the respective number of days it would take A, B, C to complete the job if he were working alone; then $1/a, 1/b, 1/c$ denote the fractions of the job done by each in one day. Since it takes A and B together two days to do the whole job, they do half the job in one day, i.e.,

$\frac{1}{a} + \frac{1}{b} = \frac{1}{2}$, and similarly, $\frac{1}{b} + \frac{1}{c} = \frac{1}{4}$, $\quad \frac{1}{a} + \frac{1}{c} = \frac{5}{12}.$

$\therefore \frac{1}{a} - \frac{1}{b} = \frac{1}{6}, \quad \frac{2}{a} = \frac{2}{3}, \quad a = 3.$

31. $\measuredangle BOC = 180 - \frac{1}{2}(\measuredangle B + \measuredangle C)$. But $\frac{1}{2}(\measuredangle B + \measuredangle C) = \frac{1}{2}\cdot 140 = 70.$
$\therefore \measuredangle BOC = 110$ (degrees).

32. $x^4 + 64 = (x^4 + 16x^2 + 64) - 16x^2$
$= (x^2 + 8)^2 - (4x)^2 = (x^2 + 4x + 8)(x^2 - 4x + 8).$
(See also 1953, Problem 18.)

33. The exact interest formula is complex. Approximately, he has the use of \$59 for 1 year. $6\cdot 100/59 = 10+$. ∴ (D) is the correct choice.

34. 1/3 is the infinite repeating decimal $0.333\cdots$. Thus 1/3 is greater than 0.33333333 by the amount $0.00000000333\cdots = 1/3\cdot 10^8$;

or

$$0.33333333 = 3\cdot 10^{-1} + 3\cdot 10^{-2} + \cdots + 3\cdot 10^{-8} = \frac{3\cdot 10^{-1}(1 - 10^{-8})}{1 - 10^{-1}}$$

$$= \frac{3(1 - 10^{-8})}{9} = \frac{1}{3} - \frac{10^{-8}}{3} = \frac{1}{3} - \frac{1}{3\cdot 10^8}$$

35. $x + \sqrt{d^2 + (h + x)^2} = h + d,\ \sqrt{d^2 + (h + x)^2} = h + d - x$
$d^2 + h^2 + 2hx + x^2 = h^2 + d^2 + x^2 + 2hd - 2hx - 2dx,\ \ 2hx + dx = hd;$
$\therefore x = hd/(2h + d).$

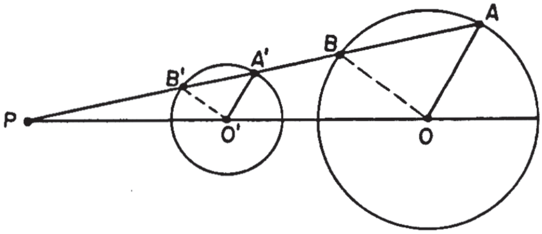
Part 3

36. $\frac{2}{x} = \frac{1}{20} + \frac{1}{10},\ x = \frac{40}{3},\ \frac{x}{15} = \frac{40/3}{15} = \frac{8}{9}.$ (See also 1951, Problem 39.)

37. $\measuredangle m = \measuredangle p + \measuredangle d,\ \measuredangle d = \measuredangle q - \measuredangle m$ (there are two vertical angles each $\measuredangle m$). $\therefore \measuredangle m = \measuredangle p + \measuredangle q - \measuredangle m;\quad \therefore \measuredangle m = \frac{1}{2}(\measuredangle p + \measuredangle q).$

38. $3^x\cdot 3^3 = 135;\quad \therefore 3^x = 5;$
$\therefore x\log 3 = \log 5 = \log(10/2) = \log 10 - \log 2.$
$\therefore x = (1 - 0.3010)/0.4771 \sim 1.47.$

39. Let PA be any line segment through P such that A lies on the given circle with center O and radius r. Let A' be the midpoint of PA and let O' be the midpoint of PO. To see that the required locus is a circle with center O' and radius $\frac{1}{2}r$, consider the similar triangles POA and $PO'A'$. For any point A on the given circle, $O'A' = \frac{1}{2}OA = \frac{1}{2}r$. $\therefore$ (E) is the correct choice.



40. $a^3 + \frac{1}{a^3} = \left(a + \frac{1}{a}\right)^3 - 3\left(a + \frac{1}{a}\right) = 3\sqrt{3} - 3\sqrt{3} = 0.$

41. $S = -(-8/4) = 2$;

or

find one root, -3, by trial. The reduced equation is $4x^2 - 20x - 3 = 0$, where the sum of the roots is 5. $\therefore\ S = 5 - 3 = 2$.

42. From the graphs, or from a consideration of the positions of the lowest points, it is found that (D) is the correct answer. (See 1950, Problem 41.)

43. $c = a - r + b - r$ (see 1950, Problem 35). $\therefore\ 10 = a + b - 2$; $\therefore\ P = a + b + c = 10 + 2 + 10 = 22$.

44. We seek an integer whose square is between 1800 and 1850. There is one, and only one, such integer, namely $1849 = 43^2$. $\therefore\ 1849 - 43 = 1806$, the year of birth;

or

let y be an integer between 0 and 50. $\therefore\ 1800 + y + x = x^2$. For x to be integral, the discriminant $1 + 7200 + 4y$ must be an odd square. The value $y = 6$ makes the discriminant $7225 = 85^2$. The year of birth is, therefore, $1800 + 6 = 1806$.

45. Let d denote the distance from A measured along AC and let $l(d)$ denote the length of the segment parallel to BD and d units from A. Then by similar triangles, we have:

For $d \leq \dfrac{\overline{AC}}{2}$,

$$\frac{(1/2)l}{d} = \frac{(1/2)\,\overline{BD}}{(1/2)\overline{AC}}, \quad \text{or } l = 2kd \text{ where } k = \frac{\overline{BD}}{\overline{AC}} = \text{constant}.$$

For $d \geq \dfrac{\overline{AC}}{2}$,

$$\frac{(1/2)l}{\overline{AC} - d} = \frac{(1/2)\,\overline{BD}}{(1/2)\,\overline{AC}}, \quad \text{or } l = 2k(\overline{AC} - d) = -2kd + 2k\overline{AC}.$$

The graph of l as a function of d evidently is linear in d. Its slope $2k$ is positive for $d < \overline{AC}/2$; its slope $-2k$ is negative for $d > \overline{AC}/2$. Hence (D) is the correct choice.

46. The distance from the center of the circle to the intersection point of the tangents is 3/8.

$$\therefore\ \overline{CD} = \frac{3}{8} + \frac{3}{16} = \frac{9}{16} = x + \frac{1}{2}; \qquad \therefore\ x = \frac{1}{16}\ (\text{inch}).$$

47. Since $\overline{MT} = \dfrac{\sqrt{p^2 - 4q^2}}{2}$, $\overline{AT} = \overline{AM} + \overline{MT} = \dfrac{p + \sqrt{p^2 - 4q^2}}{2}$ and $\overline{TB} = \overline{AB} - \overline{AT} = \dfrac{p - \sqrt{p^2 - 4q^2}}{2}$. The required equation is, therefore, $x^2 - px + q^2 = 0$.

48. Let x be the distance from the point of the accident to the end of the trip and R the former rate of the train. Then the normal time for the trip, in hours, is

$$\frac{x}{R} + 1 = \frac{x + R}{R}.$$

Considering the time for each trip, we have

$$1 + \frac{1}{2} + \frac{x}{(3/4)R} = \frac{x + R}{R} + 3\frac{1}{2}$$

and

$$1 + \frac{90}{R} + \frac{1}{2} + \frac{x - 90}{(3/4)R} = \frac{x + R}{R} + 3.$$

$\therefore x = 540$ and $R = 60$; $\therefore$ the trip is $540 + 60 = 600$ (miles).

49. $(2a + 1)^2 - (2b + 1)^2 = 4a^2 + 4a + 1 - 4b^2 - 4b - 1$
$= 4[a(a + 1) - b(b + 1)].$

Since the product of two consecutive integers is divisible by 2, the last expression is divisible by 8.

50. $210 + x - 12x = 84$; $\therefore 11x = 126$;
$\therefore x = 126/11$, $126 \cdot 60/11 \cdot 30 \sim 23$ and $210 + x + 84 = 12x$;
$\therefore 11x = 294$; $\therefore x = 294/11$, $294 \cdot 60/11 \cdot 30 \sim 53$.
$\therefore$ the times are 7:23 and 7:53.

SOLUTION KEY

Sixth Annual

MATHEMATICAL CONTEST

1955

Sponsored by

THE MATHEMATICAL ASSOCIATION OF AMERICA

In the original American Mathematics Competitions archives, there exist no answers page or solutions booklet for this year's contest. In the interest of providing a complete reference for those using these contests for practice or learning purposes we are providing a set of answers and solutions for this year.

The solutions were taken from:

> C**Contest Problem Book No 1: Annual High School Mathematics Examinations 1950-1960** (New Mathematical Library) by Charles T. Salkind (Washington, D.C.: Mathematical Association of America, 1975)

Part 1

1. $\frac{3}{8} = .375$; $\therefore \frac{3}{8}\cdot 10^{-6} = 0.000000375$. $\therefore$ (D) is the correct choice.

2. $150° - \frac{25}{60}\cdot 30° = 137\frac{1}{2}° = 137° \, 30'$.

3. Let A be the arithmetic mean of the original numbers. Then their sum is $10A$ and $A(\text{new}) = (10A + 200)/10 = A + 20$. $\therefore$ (B) is the correct choice.

4. Multiplying both sides of the equation by $(x - 1)(x - 2)$, we have $2x - 2 = x - 2$. $\therefore x = 0$.

5. $y = k/x^2$, $16 = k/1$, $k = 16$; $\therefore y = 16/x^2$, $y = 16/8\cdot 8 = 1/4$;

or

$y_1/y_2 = x_2^2/x_1^2$, $16/y_2 = 8^2/1^2$; $\therefore y_2 = 1/4$.

6. Let n be the number bought at each price, and x be the selling price of each. $2nx = (10n/3) + 4n$; $\therefore x = 11/3$, that is, 3 for 11 cents.

7. $W_2 = W_1 - W_1/5 = 4W_1/5$. $\therefore W_1 = 5W_2/4$. The increase needed is $W_2/4$ or 25% of W_2.

8. $x^2 - 4y^2 = (x + 2y)(x - 2y) = 0$; $\therefore x + 2y = 0$ and $x - 2y = 0$. Each of these equations represents a straight line.

9. This is a right triangle. For any right triangle it can be shown (see 1950, Problem 35) that

$$a - r + b - r = c. \qquad \therefore 2r = a + b - c = 8 + 15 - 17 = 6$$

and $r = 3$.

10. The train is moving for $\frac{a}{40}$ hours and is at rest for nm minutes or $\frac{n\cdot m}{60}$ hours. $\therefore \frac{a}{40} + \frac{nm}{60} = \frac{3a + 2mn}{120}$

11. The negation is: It is false that no slow learners attend this school. Therefore, some slow learners attend this school.

12. $\sqrt{5x - 1} = 2 - \sqrt{x - 1}$, and $5x - 1 = 4 - 4\sqrt{x - 1} + x - 1$. $\therefore 4x - 4 = -4\sqrt{x - 1}$, $x - 1 = -\sqrt{x - 1}$. Squaring again, we get

$$x^2 - 2x + 1 = x - 1, \quad x^2 - 3x + 2 = 0, \quad x = 1 \text{ or } 2.$$

Since 2 does not satisfy the original equation, we have one root, $x = 1$.

13. $\dfrac{(a^{-2} + b^{-2})(a^{-2} - b^{-2})}{a^{-2} - b^{-2}} = a^{-2} + b^{-2}.$

14. The dimensions of R are $1.1s$ and $0.9s$; its area is $0.99s^2$.
$\therefore\ R/S = .99s^2/s^2 = 99/100.$

15. Let x be the radius of the larger circle. Then $\pi r^2/\pi x^2 = 1/3$, $x = r\sqrt{3}$, $r\sqrt{3} - r = r(\sqrt{3} - 1) \sim 0.73r$.

Part 2

16. When $a = 4$ and $b = -4$, $a + b = 0$. Therefore, the expression becomes meaningless for these values.

17. $\log x - 5 \log 3 = \log (x/3^5)$; $\quad\therefore\ 10^{-2} = x/3^5$; $\quad\therefore\ x = 2.43.$

18. For real coefficients a zero discriminant implies that the roots are real and equal.

19. The numbers are 7 and -1; the required equation is, therefore,

$$x^2 - 6x + 7 = 0.$$

20. The expression $\sqrt{25 - t^2} + 5$ can never equal zero, since it is the sum of 5 and a non-negative number. (By $\sqrt{25 - t^2}$ we mean the positive square root.) $\quad\therefore$ (A) is the correct choice.

21. Since $\frac{1}{2}hc = A$, $h = 2A/c$.

22. Since a single discount D, equal to three successive discounts D_1, D_2, and D_3, is $D = D_1 + D_2 + D_3 - D_1D_2 - D_2D_3 - D_3D_1 + D_1D_2D_3$ (see also 1950, Problem 22), then the choices are:

$$.20 + .20 + .10 - .04 - .02 - .02 + .004 = .424 \text{ and}$$
$$.40 + .05 + .05 - .02 - .02 - .0025 + .001 = .4585.$$

The saving is $.0345 \cdot 10{,}000 = 345$ (dollars).

23. The counted amount in cents is $25q + 10d + 5n + c$. The correct amount is $25(q - x) + 10(d + x) + 5(n + x) + (c - x)$. The difference is $-25x + 10x + 5x - x = -11x$. $\quad\therefore\ 11x$ cents should be subtracted.

24. The graph of $y = 4x^2 - 12x - 1$ is a vertical parabola opening upward with the turning point (a minimum) at $(3/2, -10)$. (See 1950, Problem 41.);

or

algebraically, the turning point occurs at

$$x = -b/2a = -(-12/8) = 3/2.$$

For this value of x, $y = 4x^2 - 12x - 1 = -10$.

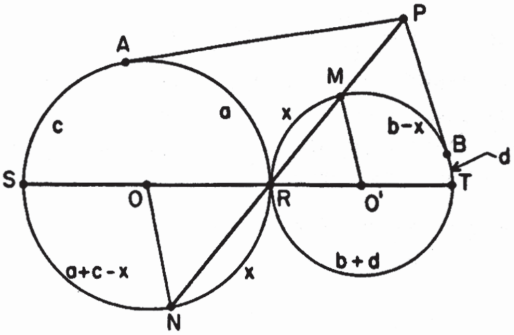
25. $x^4 + 2x^2 + 9 = (x^4 + 6x^2 + 9) - (4x^2)$
$= (x^2 + 3)^2 - (2x)^2 = (x^2 - 2x + 3)(x^2 + 2x + 3)$.
$\therefore$ (E) is the correct choice.

26. See 1951, Problem 5.

27. $r^2 + s^2 = (r + s)^2 - 2rs = p^2 - 2q$.

28. For $x \neq 0$, $ax^2 + bx + c \neq ax^2 - bx + c$;
for $x = 0$, $ax^2 + bx + c = ax^2 - bx + c$.
$\therefore$ There is one intersection point $(0, c)$; $\therefore$ (E) is the correct choice.

29. First, draw the line connecting P and R and denote its other intersections with the circles by M and N; see accompanying figure. The arcs MR and NR contain the same number of degrees; so we may denote each arc by x. To verify this, note that we have two isosceles triangles with a base angle of one equal to a base angle of the other. $\therefore \measuredangle NOR = \measuredangle MO'R$.



$$\measuredangle APR = \frac{1}{2}\{(c + a + c - x) - a\} = \frac{1}{2}\{2c - x\}$$

$$\measuredangle BPR = \frac{1}{2}\{b + d + d - (b - x)\} = \frac{1}{2}\{2d + x\}$$

and the sum of angles APR and BPR is

$$\measuredangle BPA = c + d.$$

The desired angle is

$$\begin{aligned} 360^\circ - \measuredangle BPA &= 360^\circ - (c + d) \\ &= (180^\circ - c) + (180^\circ - d) \\ &= a + b. \end{aligned}$$

30. The real roots are, respectively: ± 3; 0 and 2/3; and 3. $\therefore$ (B) is the correct choice.

31. Let A_1 and A_2 denote the areas of the small and the original triangle, respectively. The median m of the trapezoid is the arithmetic mean of the parallel sides, that is, $m = \frac{1}{2}(b + 2)$, where b denotes the shorter parallel side of the trapezoid. To find b, observe that

$$\frac{A_1}{A_2} = \frac{b^2}{2^2} = \frac{1}{2}; \qquad \therefore b = \sqrt{2} \qquad \text{and} \qquad m = \frac{1}{2}(\sqrt{2} + 2).$$

32. $D = 4b^2 - 4ac = 0; \qquad \therefore b^2 - ac = 0; \qquad \therefore b^2 = ac, \quad a/b = b/c.$

33. Let x be the number of degrees the hour hand moves in the time interval between 8 a.m. and the beginning of the trip, and let y be the number of degrees it moves between 2 p.m. and the end of the trip. The number of degrees traversed by the minute hand during these intervals is $240 + x$ and $60 + y + 180$, respectively. Since the minute hand traverses 12 times as many degrees as the hour hand in any given interval, we have

$$12x = 240 + x, \quad \therefore x = \frac{240}{11} \text{ and } 12y = 60 + y + 180, \quad \therefore y = \frac{240}{11};$$

that is, the hour hand is just as many degrees past the 8 at the beginning as it is past the 2 at the end of the trip. Since the 8 and the 2 are 180° apart, the initial and final positions of the hour hand differ also by 180°, so that (A) is the correct choice.

34. The shortest length consists of the two external tangents t and the two circular arcs l_1 and l_2.

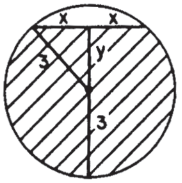
$$t = 6\sqrt{3}, \; 2t = 12\sqrt{3}, \; l_1 = \frac{120}{360}\cdot 2\pi\cdot 3 = 2\pi, \; l_2 = \frac{240}{360}\cdot 2\pi\cdot 9 = 12\pi.$$

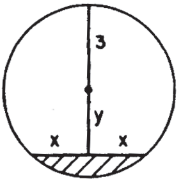
$$\therefore \text{length} = 12\sqrt{3} + 14\pi$$

35. The first boy takes $\frac{n}{2} + 1$ marbles, leaving $\frac{n}{2} - 1$ marbles. The second boy takes $\frac{1}{3}\left(\frac{n}{2} - 1\right)$. The third boy, with twice as many, must necessarily have $\frac{2}{3}\left(\frac{n}{2} - 1\right)$, so that n is indeterminate; i.e. n may be any even integer of the form $2 + 6a$, with $a = 0, 1, 2, \cdots$.

Part 3

36. The area of the rectangular surface is $10 \cdot 2x = 40$.
$\therefore\ x = 2$ and $y = \sqrt{5}$;
$\therefore$ the depth is $3 - \sqrt{5}$,
or $3 + \sqrt{5}$.





37. The original number is $100h + 10t + u$. The number with digits reversed is $100u + 10t + h$.
(Since $h > u$, to subtract we must add 10 to u, etc.)

$$\begin{array}{lll} \therefore\ 100(h-1) & +\ 10(t+9) & +\ u + 10 \\ \underline{100u\qquad\qquad} & \underline{+\ 10t\qquad\qquad} & \underline{+\ h\qquad} \\ 100(h-1-u) & +\ 90 & +\ 10 + u - h. \end{array}$$

Since $10 + u - h = 4$, $h - 1 - u = 5$ and therefore, the digits from right to left are 9 and 5.

38. Solving the system:

$$\begin{aligned} \tfrac{1}{3}(a+b+c)+d &= 29 \\ \tfrac{1}{3}(b+c+d)+a &= 23 \\ \tfrac{1}{3}(c+d+a)+b &= 21 \\ \tfrac{1}{3}(d+a+b)+c &= 17, \end{aligned}$$

we obtain $a = 12$, $b = 9$, $c = 3$, $d = 21$. Thus (B) is the correct choice.

39. The least value occurs when $x = -p/2$. For $x = -p/2$,
$y = (-p^2/4) + q = 0$; $\therefore\ q = p^2/4$ (See 1950, Problem 41.)

40. A non-unit fraction b/d is changed in value by the addition of the same non-zero quantity x to the numerator and denominator. $\therefore$ (A) is the correct choice.

41. $1 + \frac{1}{2} + \frac{d-R}{(4/5)R} = 1 + 1 + \frac{80}{R} + \frac{1}{2} + \frac{d-R-80}{(4/5)R}$; $\therefore\ R = 20$.

(See also 1954, Problem 48.)

42. If $\sqrt{a + \frac{b}{c}} = a\sqrt{\frac{b}{c}}$ then $a + \frac{b}{c} = a^2\frac{b}{c}$.

$\therefore\ ac = b(a^2 - 1)$; $\therefore\ c = b(a^2 - 1)/a$.

43. $y = (x+1)^2$ and $y(x+1) = 1$; $\therefore\ (x+1)^2 = 1/(x+1)$; and $\therefore\ (x+1)^3 = 1$. This last equation has one real and two imaginary roots. $\therefore$ (E) is the correct choice.

44. $\measuredangle\, OBA = 2y$; $\therefore\ \measuredangle\, OAB = 2y$; and $\therefore\ x = \measuredangle\, OAB + y = 3y$.

45. Let the two series be $a, ar, ar^2, \cdots$ and $0, d, 2d, \cdots$. $a + 0 = 1$; $\therefore a = 1$ and $r + d = 1$, $r^2 + 2d = 2$. $\therefore r = 2$, $d = -1$.

$$S_1 = \frac{a(r^n - 1)}{r - 1} = \frac{1(2^{10} - 1)}{2 - 1} = 1023,$$

$$S_2 = \frac{n}{2}[0 + (n - 1)d] = \frac{10}{2}[0 + 9(-1)] = -45.$$

$\therefore S = 1023 - 45 = 978.$

46. We are required to solve the system of equations:

$$\begin{aligned} 2x + 3y &= 6 \\ 4x - 3y &= 6 \\ x \qquad &= 2 \\ y &= \tfrac{2}{3}. \end{aligned}$$

Solve the first two simultaneously, obtaining $x = 2$, $y = 2/3$. Since these results are consistent with the last two equations, the solution is $(2, 2/3)$ and (B) is the correct choice.

47. For the equality $a + bc = a^2 + ab + ac + bc$ to hold, we must have $a = a^2 + ab + ac$ or $1 = a + b + c$.

48. (A) is true because FH is parallel and equal to AE. (C) is true because when HE, which is parallel to CA, is extended, it meets AB in D; DC and BH are corresponding sides of congruent triangles ACD and HDB. (D) is true because

$$\overline{FG} = \overline{FE} + \overline{EG} = \overline{AD} + \tfrac{1}{2}\overline{DB} = \tfrac{3}{4}\overline{AB}.$$

(E) is true because G is the midpoint of HB. (FE is parallel to AB and E is the midpoint of CB.) (B) cannot be proved from the given information. Challenge: What additional information is needed to prove (B)?

49. Since $y = (x^2 - 4)/(x - 2) = (x - 2)(x + 2)/(x - 2) = x + 2$ (for $x \neq 2$, i.e. $y \neq 4$), then $y = (x^2 - 4)/(x - 2)$ is a straight line with the point $(2, 4)$ deleted. The straight line $y = 2x$ crosses the first line in the deleted point. $\therefore$ (C) is the correct choice.

50. Let r be the increase in the rate of A and d be the distance (in miles) A travels to pass B safely. The times traveled by A, B and C are equal:

$$\frac{d}{50 + r} = \frac{d - (30/5280)}{40}, \qquad \frac{d - (30/5280)}{40} = \frac{(210/5280) - d}{50}.$$

Solve the second equation for d. $\therefore d = 110/5280$ miles.
Solve the first equation for r. $\therefore r = 5$ mph.

SOLUTION KEY

Seventh Annual

MATHEMATICAL CONTEST

1956

Sponsored by

THE MATHEMATICAL ASSOCIATION OF AMERICA

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Part 1

1. $2 + 2(2^2) = 2 + 8 = 10.$

2. $S_1 = C_1 + \frac{1}{5}C_1$ and $S_2 = C_2 - \frac{1}{5}C_2$. $\therefore C_1 = \frac{5}{6}S_1 = 1.00$ and $C_2 = \frac{5}{4}S_2 = 1.50$. $\therefore S_1 + S_2 = 2.40$ and $C_1 + C_2 = 2.50$; $\therefore$ (D) is the correct choice.

3. $5{,}870{,}000{,}000{,}000 = 587 \cdot 10^{10}$, $(587 \cdot 10^{10})100 = 587 \cdot 10^{12}$.

4. $\frac{1}{20} \cdot 4000 + \frac{1}{25} \cdot 3500 + \frac{x}{100} \cdot 2500 = 500$, $25x = 160$, $x = 6.4$.

5. Around a given circle can be placed exactly six circles each equal to the given circle, tangent to it, and tangent to two others. The arc between two successive points of contact of the given circle and the outside circles is one-sixth of the circumference.

6. Let a be the number of cows and b the number of chickens. Then $4a + 2b$ is the total number of legs and

$$4a + 2b = 14 + 2(a + b), \quad 2a = 14, \quad a = 7 \text{ (cows)}.$$

Note: The number of chickens is indeterminate.

7. For reciprocal roots, the product of the roots $c/a = 1$. $\therefore c = a$.

8. $8 \cdot 2^x = 5^0$; $\therefore 2^{3+x} = 5^0 = 1$; $\therefore 3 + x = 0$ and $x = -3$.
Note: $2^0 = 5^0$, but, otherwise, $2^a \neq 5^a$.

9. $a^{9(1/6)(1/3)4} \cdot a^{9(1/3)(1/6)4} = a^2 \cdot a^2 = a^4$.

10. $\measuredangle ADB = \frac{1}{2}\measuredangle ACB = 30°$.

11. $$\frac{1 \cdot (1 + \sqrt{3})(1 - \sqrt{3}) - 1 \cdot (1 - \sqrt{3}) + 1 \cdot (1 + \sqrt{3})}{(1 + \sqrt{3})(1 - \sqrt{3})} = 1 - \sqrt{3}.$$

12. $x^{-1} - 1 = \frac{1}{x} - 1 = \frac{1 - x}{x}$, $\frac{1 - x}{x} \cdot \frac{1}{x - 1} = -\frac{1}{x}$.

13. $y - x$ is the excess of y over x. The basis of comparison is the ratio of the excess to y, namely, $(y - x)/y$. Therefore, the per cent required is $100(y - x)/y$.

14. $\overline{PB} \cdot \overline{PC} = \overline{PA}^2$. $\overline{PB}(\overline{PB} + 20) = 300$; $\therefore \overline{PB} = 10$.

15. Multiply both sides of the equation by $x^2 - 4 = (x + 2)(x - 2)$.

$$15 - 2(x + 2) = x^2 - 4; \qquad \therefore x^2 + 2x - 15 = 0;$$

and $\therefore x = -5$ or $x = 3$. Each of these roots satisfies the original equation.

16. Let $10m$, $15m$, $24m$ be the three numbers. Then

$$10m + 15m + 24m = 98, \qquad \therefore m = 2 \qquad \text{and} \qquad 15m = 30.$$

17. $$\frac{A(2x-3)+B(x+2)}{(x+2)(2x-3)} = \frac{5x-11}{2x^2+x-6};$$

$$A(2x-3)+B(x+2) \equiv 5x-11.$$

Equating the coefficients of like powers of x, we obtain $2A + B = 5$,

$$-3A + 2B = -11; \qquad \therefore A = 3, \quad B = -1.$$

18. $10^{2y} = 5^2$, $10^y = 5$, $10^{-y} = 1/10^y = 1/5$.

19. Let the height in each case be 1. $1 - \frac{1}{4}t = 2(1 - \frac{1}{3}t)$; $\quad \therefore t = 2\frac{2}{5}$.

20. $\left(\frac{2}{10}\right)^x = 2; \qquad \therefore \log\left(\frac{2}{10}\right)^x = x \log\frac{2}{10} = x(\log 2 - \log 10) = \log 2.$

$$\therefore x = \frac{0.3010}{0.3010 - 1} \sim -0.4.$$

21. Each line has 1 or 2 intersection points. Therefore, for both lines, there may be 2, 3, or 4 intersection points.

22. $t_1 = 50/r$, $t_2 = 300/3r = 100/r = 2t_1$; $\quad \therefore$ (B) is the correct choice.

23. Each of the roots is $\sqrt{2}/a$. Since a is real, (C) is necessarily the correct choice.

24. Let $\measuredangle DAE = a$. Then $\measuredangle BCA = \frac{1}{2}(180 - 30 - a) = 75 - (a/2)$ and $\measuredangle DEA = \frac{1}{2}(180 - a) = 90 - (a/2) = \measuredangle ADE$.

$$a + \measuredangle ADE + x + \measuredangle BCA = 180; \qquad \therefore x = 15 \text{ (degrees)}.$$

25. $s = 3 + 5 + 7 + \cdots + (2n+1) = \frac{n}{2}(3 + 2n + 1) = n(n+2)$.

26. Combination (A) determines the shape of the triangle, but not its size. All the other combinations determine both the size and the shape of the triangle.

27. The two triangles are similar and hence $A(\text{new})/A(\text{old}) = (2s)^2/s^2 = 4$. $\therefore$ (C) is the correct choice.

28. Let $4x$ be the daughter's share, $3x$ the son's share. Then $4x + 3x = \frac{1}{2}$ estate and $6x + 500 = \frac{1}{2}$ estate. $\quad \therefore 7x = 6x + 500$; $\therefore x = 500$; and $\therefore 13x + 500 = 7000$ (dollars).

29. $x^2 + y^2 = 25$ is a circle with center at the origin. $xy = 12$ is a hyperbola, symmetric with respect to the line $x = y$. $\therefore$ points of intersection are symmetric with respect to $x = y$. There are either no intersections, two intersections (if the hyperbola is tangent to the circle) or four intersections. Simultaneous solution of the equations reveals that there are four points of intersection, and that they determine a rectangle.

30. $h = \frac{s}{2}\sqrt{3}$; $\quad \therefore s = \frac{2h}{\sqrt{3}}$, $A = \frac{s^2\sqrt{3}}{4} = \frac{4h^2}{3} \cdot \frac{\sqrt{3}}{4} = \frac{6\sqrt{3}}{3} = 2\sqrt{3}$.

31. $20 = 1 \cdot 4^2 + 1 \cdot 4^1 + 0 \cdot 4^0$; $\quad \therefore$ the required number is 110_4;

or

$20 \div 4$ yields 5 and remainder 0;

$5 \div 4$ yields 1 and remainder 1;

$\therefore 20_{10}$ becomes 110_4.

32. After 1½ minutes each was in the center of the pool. After 3 minutes they were at opposite ends of the pool. After 4½ minutes each was again in the center of the pool.

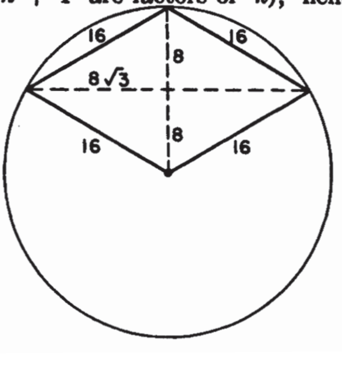
33. It can be shown† that $\sqrt{2}$ is not a rational number. It can also be shown† that repeating non-terminating decimals represent rational numbers, and terminating decimals clearly represent rational numbers with a power of 10 in the denominator. Hence a non-terminating, non-repeating decimal is the only possibility for representing $\sqrt{2}$.

34. By considering the special case $n = 2$ we immediately rule out choices (B), (C), (D), and (E). We now show that (A) holds.

$$n^2(n^2 - 1) = n\{(n - 1)n(n + 1)\} = nk,$$

where k is a product of three consecutive integers, hence always divisible by 3. If n is even, k is divisible also by 2 (since n is a factor of k), hence by 6, and nk by 12; if n is odd, k is divisible also by 4 (since the even numbers $n - 1$ and $n + 1$ are factors of k), hence by 12.

35. $A = 16 \cdot 8\sqrt{3} = 128\sqrt{3}$.



36. $S = K(K+1)/2 = N^2$. The possible values for N^2 are $1^2, 2^2, 3^2, \cdots, 99^2$. For K to be integral, the discriminant $1 + 8N^2$ of the equation $K^2 + K - 2N^2 = 0$ must be a perfect square. This fact reduces the possible values for N^2 to 1^2, 6^2, and 35^2. Hence the values of K are 1, 8, and 49.
Note: There are ways of shortening the number of trials for N^2 still further, but these involve a knowledge of number-theoretic theorems. The shortest way to do this problem is by testing the choices given.

37. The diagonal divides the rhombus into two equilateral triangles. A (rhombus) $= 2\cdot s^2 \cdot \dfrac{\sqrt{3}}{4} = 2\cdot\dfrac{(3/16)^2\sqrt{3}}{4}$. Since the map scale is 400 miles to $1\frac{1}{2}$ inches, 1 inch represents $\frac{2}{3}\cdot 400$ miles and 1 square inch represents $(\frac{2}{3}\cdot 400)^2$ square miles. The area of the estate is thus

$$\frac{2\cdot 3^2\sqrt{3}}{16^2\cdot 4}\cdot\frac{800^2}{3^2} = 1250\sqrt{3} \text{ (square miles).}$$

38. The segments of the hypotenuse are $\dfrac{a^2}{c}$ and $\dfrac{b^2}{c}$ (see 1954, Problem 29).

Using similar triangles, $\dfrac{x}{a} = \dfrac{b^2/c}{b}$, $\dfrac{x}{b} = \dfrac{a^2/c}{a}$.

$$\therefore x^2 = \frac{a^2}{c}\cdot\frac{b^2}{c} = \frac{a^2b^2}{a^2+b^2}; \qquad \text{and} \qquad \therefore \frac{1}{x^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

39. $b^2 = c^2 - a^2 = c^2 - (c-1)^2 = 2c - 1 = c + c - 1 = c + a;$

or

$$b^2 = c^2 - a^2 = (a+1)^2 - a^2 = 2a + 1 = a + 1 + a = c + a;$$

or

$$b^2 = c^2 - a^2 = (c+a)(c-a) = c + a \text{ since } c - a = 1.$$

40. $2S = gt^2 + 2V_0t; \qquad Vt = gt^2 + V_0t; \qquad \therefore 2S - Vt = V_0t;$ and

$$\therefore t = \frac{2S}{V+V_0};$$

or

$$g = \frac{V - V_0}{t},\ S = \frac{1}{2}\frac{V - V_0}{t}\cdot t^2 + V_0t = t\left(\frac{1}{2}V + \frac{1}{2}V_0\right).$$

$$\therefore t = \frac{2S}{V+V_0}.$$

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† See, *e.g.*, Ivan Niven, *Numbers: Rational and Irrational*, also in this series.

40. $2S = gt^2 + 2V_0t$; $Vt = gt^2 + V_0t$; $\therefore 2S - Vt = V_0t$; and

$$\therefore t = \frac{2S}{V + V_0};$$

or

$$g = \frac{V - V_0}{t},\ S = \frac{1}{2}\frac{V - V_0}{t} \cdot t^2 + V_0t = t\left(\frac{1}{2}V + \frac{1}{2}V_0\right).$$

$$\therefore t = \frac{2S}{V + V_0}.$$

41. Since $y = 2x$, $3y^2 + y + 4 = 12x^2 + 2x + 4$.
$\therefore 12x^2 + 2x + 4 = 12x^2 + 4x + 4$; $\therefore 2x = 4x$; $\therefore x = 0$.

42. $\sqrt{x + 4} = \sqrt{x - 3} - 1$; $\therefore x + 4 = x - 3 - 2\sqrt{x - 3} + 1$; $\therefore 3 = -\sqrt{x - 3}$. This is impossible since the left side is positive while the right side is negative.

43. Let the largest side be c. Then $a + b + c \leq 12$. But $c < a + b$. $\therefore 2c < 12$ or $c < 6$. Now try integral combinations such that the triangle is scalene. Since c is the largest side, it cannot be less than 4. $\therefore$ there are 3 combinations:

$c = 5$	$c = 5$	$c = 4$
$a = 4$	$a = 4$	$a = 3$
$b = 3$	$b = 2$	$b = 2$

44. Since $x < a$ and $a < 0$, $x^2 > ax$ and $ax > a^2$. $\therefore x^2 > ax > a^2$.

45. $$N_1 = \frac{1 \text{ mile}}{\pi D},\ N_2 = \frac{1 \text{ mile}}{\pi[D - (1/2)]},\ N_2 - N_1 = \frac{1 \text{ mile}}{\pi} \cdot \frac{1/2}{D[D - (1/2)]}.$$

$$\therefore \frac{N_2 - N_1}{N_1} = \frac{1/2}{D - 1/2} = \frac{1/2}{24\frac{1}{2}} = \frac{1}{49};$$

$\therefore$ the percent increase is $100 \cdot \frac{1}{49} \sim 2\%$.

46. Solving for x we have $x = 1/(2N + 1)$. Since N is positive, $1/(2N + 1)$ is positive and less than 1. $\therefore$ (A) is the correct choice.

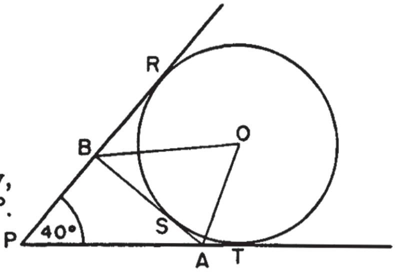
47. In 1 day, x machines can do $\frac{1}{3}$ of the job and $x + 3$ machines $\frac{1}{2}$ of the job. $\therefore$ 3 machines can do $\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$ of the job in 1 day, and 1 machine can do $\frac{1}{18}$ of the job in 1 day. Hence, 18 days are needed for 1 machine to complete the job;

or

$(x + 3)/x = 3/2$; $\therefore x = 6$, that is, 6 machines can do $1/3$ of the job in 1 day, etc.

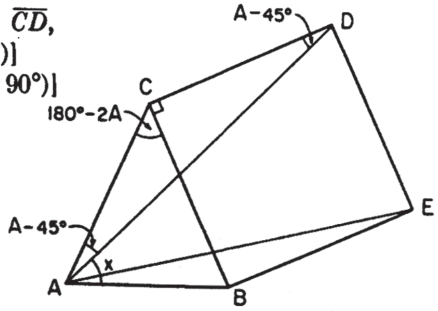
48. Let $(3p + 25)/(2p - 5) = n$, a positive integer. Then, $3p + 25 = kn$, $2p - 5 = k$; $\therefore k(2n - 3) = 65 = 1\cdot 65 = 5\cdot 13$.
$\therefore k = 1, 65, 5,$ or 13 and $2n - 3 = 65, 1, 13,$ or 5 correspondingly.
$\therefore 2p = 5 + 1,\ 5 + 65,\ 5 + 5,$ or $5 + 13$; $\therefore p = 3, 35, 5,$ or 9.
$\therefore$ choice (B) is correct.

49. $\measuredangle P = 40°$; $\therefore \measuredangle PAB + \measuredangle PBA = 180° - 40° = 140°$.
$\measuredangle TAS = 180° - \measuredangle PAB$;
$\measuredangle RBS = 180° - \measuredangle PBA$;
$\measuredangle TAS + \measuredangle RBS = 360° - 140°$
$= 220°$.



Since OA and OB bisect angles TAS and RBS, respectively, $\measuredangle OAS + \measuredangle OBS = \frac{1}{2}(220°) = 110°$.
$\therefore \measuredangle AOB = 180° - 110° = 70°$.
The number of degrees in $\measuredangle AOB$ is independent of the position of tangent ASB.

50. $\measuredangle ACB = 180° - 2A$. Since $\overline{CA} = \overline{CD}$,
$\measuredangle CAD = \frac{1}{2}[180° - (\measuredangle ACB + 90°)]$
$= \frac{1}{2}[180° - (180° - 2A + 90°)]$
$= A - 45°$
$= A - X$.
$\therefore X = 45°$



SOLUTION KEY

Eighth Annual

MATHEMATICAL CONTEST

1957

Sponsored Jointly by

THE MATHEMATICAL ASSOCIATION OF AMERICA
and
THE SOCIETY OF ACTUARIES

In the original American Mathematics Competitions archives, there exist no answers page or solutions booklet for this year's contest. In the interest of providing a complete reference for those using these contests for practice or learning purposes we are providing a set of answers and solutions for this year.

The solutions were taken from:

> **Contest Problem Book No 1: Annual High School Mathematics Examinations 1950-1960** (New Mathematical Library) by Charles T. Salkind (Washington, D.C.: Mathematical Association of America, 1975)

Part 1

1. The altitudes, the medians, and the angle-bisectors are distinct for the legs but coincident for the base. Hence, 7 lines.

2. $h/2 = 4,\ h = 8,\ 2k/2 = -3,\ k = -3.$

3. $1 - (1 - a)/(1 - a + a) = 1 - 1 + a = a$, provided $a \neq 1$.

4. In this case $a = 3x + 2,\ b = x,\ c = -5.$
$\therefore\ ab + ac = (3x + 2)x + (3x + 2)(-5).$

5. $\log a - \log b + \log b - \log c + \log c - \log d$
$- \log a - \log y + \log d + \log x = \log (x/y).$

6. The dimensions are $x,\ 10 - 2x,\ 14 - 2x.$ $\therefore\ V = 140x - 48x^2 + 4x^3.$

7. For the inscribed circle
$r = h/3 = s\sqrt{3}/6.\ p = 3s = 6\sqrt{3}r = (6\sqrt{3})(4\sqrt{3}) = 72.$
$(A = \pi r^2,\ r = \sqrt{48} = 4\sqrt{3}.)$

8. The numbers are easily found to be 20, 30 and 50. $\therefore\ 30 = 20a - 10;$
$\therefore\ a = 2.$

9. $2 - (-2)^4 = 2 - 16 = -14.$

10. For a low or high point $x = -4/4 = -1$ (see 1950, Problem 41). When $x = -1,\ y = 1$. The point $(-1, 1)$ is a low point since the coefficient of x^2 is positive.

11. $90 - (60^\circ + \frac{1}{4}\cdot 30)^\circ = 22\frac{1}{2}^\circ;$
$\therefore$ (E) is correct. (See also 1951, Problem 12.)

12. $10^{-49} = 10\cdot 10^{-50};$ $\therefore\ 10^{-49} - 2\cdot 10^{-50} = 10^{-50}(10 - 2) = 8\cdot 10^{-50}.$

13. (A) and (B) are irrational. Of the remaining choices only (C) is between $\sqrt{2}$ and $\sqrt{3}$.

14. Since $\sqrt{x^2 - 2x + 1} = |x - 1|$ and $\sqrt{x^2 + 2x + 1} = |x + 1|$, (D) is correct.

15. The formula satisfying the table is $s = 10t^2$. When $t = 2.5,\ s = 62.5.$

Part 2

16. Since the number of goldfish n is a positive integer from 1 to 12, the graph is a finite set of distinct points.

17. The longest path covers 8 edges, and $8\cdot 3 = 24$ (inches).

18. ABM is a right triangle. By similar triangles, $\overline{AP}/\overline{AB} = \overline{AO}/\overline{AM}$. $\therefore \overline{AP}\cdot\overline{AM} = \overline{AO}\cdot\overline{AB}$.

19. $10011_2 = 1\cdot 2^4 + 0\cdot 2^3 + 0\cdot 2^2 + 1\cdot 2^1 + 1\cdot 2^0 = 16 + 2 + 1 = 19$. (See also 1956, Problem 31.)

20. $\frac{1}{A} = \frac{1}{2}\left(\frac{1}{50} + \frac{1}{45}\right)$; $\therefore A = 47\frac{7}{19}$. (See 1950, Problem 27.)

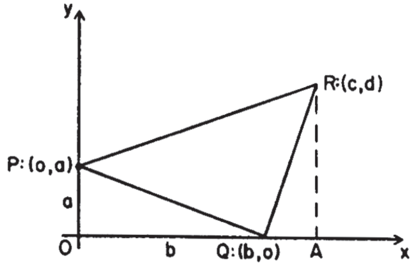
21. (1) is the inverse, (2) is the converse, (3) is the contrapositive, and (4) is an alternative way of stating the theorem. $\therefore$ the combination (3), (4) is correct.

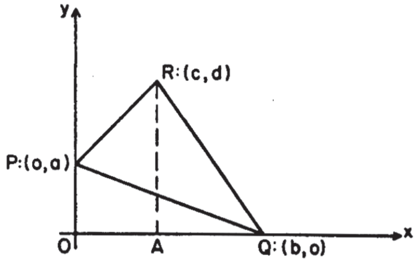
22. $\sqrt{x-1} + 1 = \sqrt{x+1}$; $\therefore x - 1 + 2\sqrt{x-1} + 1 = x + 1$. $\therefore 2\sqrt{x-1} = 1$; $\therefore 4x - 4 = 1$; and $\therefore 4x = 5$. (This value of x satisfies the original equation.)

23. The intersection points are (0, 10) and (1, 9). $d = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$

24. $(10t + u)^2 - (10u + t)^2 = 99(t^2 - u^2)$. This is divisible by 9, 11, $t + u$, $t - u$. $\therefore$ (B) is the correct choice.

25. From $R:(c, d)$, draw line segment RA perpendicular to the x-axis. Let O denote the origin (0, 0).





If $c > b$,

area ΔPQR = area trapezoid $OPRA$ − area ΔQAR − area ΔOPQ

$$= \tfrac{1}{2}c(a + d) - \tfrac{1}{2}d(c - b) - \tfrac{1}{2}ab = \tfrac{1}{2}(ac + bd - ab).$$

If $c < b$,

area ΔPQR = area trapezoid $OPRA$ + area ΔQAR − area ΔOPQ

$$= \tfrac{1}{2}c(a + d) + \tfrac{1}{2}d(b - c) - \tfrac{1}{2}ab = \tfrac{1}{2}(ac + bd - ab).$$

26. Let d be the distance from the point P to any side, say c, of the triangle. Then we want Area $\Delta APB = \frac{1}{3}$ Area ΔABC, or if h is the altitude to the side c, $\frac{1}{2}dc = \frac{1}{3}\cdot\frac{1}{2}ch$. $\therefore d = \frac{1}{3}h$. For this to hold for each side of the triangle, the required point must be the intersection of the medians.

27. $\frac{1}{r} + \frac{1}{s} = \frac{r + s}{rs} = \frac{-p}{q}$.

28. Let $x = b^{\log_b a}$; $\therefore \log_b x = \log_b a \cdot \log_b b = \log_b a$; $\therefore x = a$; and $\therefore$ (B) is correct.

29. First note that the equality $x^2(x^2 - 1) = 0$ is satisfied by $0,\ 0,\ +1,\ -1$. Since x^2 is non-negative, then $x^2(x^2 - 1) > 0$ implies $x^2 - 1 > 0$. $\therefore x^2 > 1$; $\therefore |x| > 1$, that is, $x > 1$ or $x < -1$. Combining these, we have $x = 0,\ x \leq -1,\ x \geq 1$.

30. The formula is $S = n(n + 1)(2n + 1)/6$;

or

first let $S = 1^2$ and then let $S = 1^2 + 2^2$.

$$1^2 = \frac{1(1 + c)(2 + k)}{6}$$

and

$$1^2 + 2^2 = \frac{2(2 + c)(4 + k)}{6}.$$

Solving for c and k, we have $c = k = 1$.

31. Let x be the length of one of the legs of the isosceles triangle. Then a side of the square is given by $1 - 2x$ and this side must equal the hypotenuse of the triangle (to form a regular octagon). $\therefore x\sqrt{2} = 1 - 2x$ $\therefore x = (2 - \sqrt{2})/2$.

32. Since $n^5 - n = (n - 1)n(n + 1)(n^2 + 1) = k(n^2 + 1)$, where k is a product of 3 consecutive integers, k is always divisible by 6 at least (see 1951, Problem 15). We show next that in addition, $n^5 - n$ is always divisible by 5. If we divide n by 5, we get a remainder whose value is either $0, 1, 2, 3,$ or 4; i.e.,

$$n = 5q + r, \qquad r = 0, 1, 2, 3 \text{ or } 4.$$

By the binomial theorem,

$$\begin{aligned} n^5 - n &= (5q + r)^5 - (5q + r) \\ &= (5q)^5 + 5(5q)^4r + 10(5q)^3r^2 + 10(5q)^2r^3 + 5(5q)r^4 + r^5 \\ &\qquad - 5q - r. \\ &= 5N + r^5 - r, \qquad \text{where } N \text{ is an integer.} \end{aligned}$$

If $r = 0$, $n^5 - n = 5N$ is divisible by 5.
If $r = 1$, $n^5 - n = 5N$ is divisible by 5.
If $r = 2$, $n^5 - n = 5N + 30 = 5(N + 6)$ is divisible by 5.
If $r = 3$, $n^5 - n = 5N + 240 = 5(N + 48)$ is divisible by 5.
If $r = 4$, $n^5 - n = 5N + 1020 = 5(N + 204)$ is divisible by 5.
Thus, for all integers n, $n^5 - n$ is divisible by 5 and by 6 hence by 30;

or

$n^5 - n = n(n^4 - 1)$ is divisible by 5 since $n^{p-1} \equiv 1 \pmod{p}$ where p is a prime. $\therefore n^5 - n$ is divisible by 30.

33. $9^{x+2} - 9^x = 240$; $\therefore 9^x(81 - 1) = 240$; $\therefore 3^{2x} = 3$; $\therefore 2x = 1$; and $\therefore x = \frac{1}{2}$.

34. $x^2 + y^2 < 25$ is the set of points interior to the circle $x^2 + y^2 = 25$. Let the straight line $x + y = 1$ intersect the circle in A and B. Then all the points of the straight line segment AB, except A and B, are also interior to the circle. $\therefore$ (C) is correct.

35. All the triangles so formed with A as one vertex are similar. Let h_k be the side parallel to BC at a distance $(k/8)\cdot\overline{AC}$ from $A, k = 1, 2, \cdots, 7$.

$$\therefore \frac{h_k}{(k/8)\cdot\overline{AC}} = \frac{10}{\overline{AC}}; \qquad \therefore h_k = \frac{10k}{8},$$

$$S = h_1 + h_2 + \cdots + h_7 = (10/8)\,(1 + 2 + \cdots + 7) = 35.$$

Part 3

36. If $x + y = 1$, then $P = xy = x(1 - x) = -x^2 + x$. The largest value of P occurs when $x = 1/2$ (see 1950, Problem 41).
$\therefore P$ (maximum) $= 1/4$.

37. By similar triangles $\overline{MN}/x = 5/12$, or $\overline{MN} = 5x/12$. Also, $\overline{NP} = \overline{MC} = 12 - x$. $\therefore y = 12 - x + 5x/12 = (144 - 7x)/12$.

38. $10t + u - (10u + t) = 9(t - u)$. Since both t and u lie between 0 and 9, $t - u$ cannot exceed 9. $\therefore 9(t - u)$ cannot exceed 81. The possible cubes are, therefore, 1, 8, 27, and 64, of which only 27 is divisible by 9. $\therefore 9(t - u) = 27$; $\therefore t = u + 3$ so that the possibilities for $N = 10t + u$ are 96, 85, 74, 63, 52, 41, 30, seven in all.

39. Let t be the time each man walked. Then the distance the second man walked is given by $S = 2 + 2\frac{1}{2} + \cdots + \left(2 + \frac{t - 1}{2}\right) = \frac{7t + t^2}{4}$.

$$\therefore \frac{7t + t^2}{4} + 4t = 72; \qquad \therefore t = 9; \text{ and } \therefore \text{each man walks 36 miles,}$$

so that they meet midway between M and N.

40. When the vertex is on the x-axis, $y = 0$ and the roots of $-x^2 + bx - 8 = 0$ are equal since this parabola touches the x-axi at only one point. $\therefore b^2 - 32 = 0$ and $b = \pm\sqrt{32}$.

41. There is no solution when $a(-a) - (a - 1)(a + 1) = 0$, that is, whe $a = \pm\sqrt{2}/2$ (see 1950, Problem 14);

or

solve for x, obtaining $x = (2a - 1)/(2a^2 - 1)$, which is undefined whe $2a^2 - 1 = 0$, that is, when $a = \pm\sqrt{2}/2$.

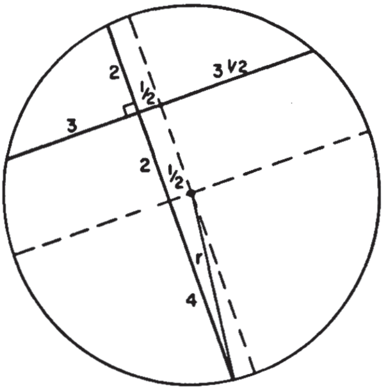
42. For integral values of n, i^n equals either i, -1, $-i$ or $+1$. Whe $i^n = i$, then $i^{-n} = 1/i = -i/1$. $\therefore S$ can be 0, -2, or $+2$.

43. On the line $x = 4$ the integral ordinates are $0, 1, 2, \cdots, 16$; on th line $x = 3$ they are $0, 1, 2, \cdots, 9$; on $x = 2$ they are $0, 1, \cdots, 4$ on $x = 1$ they are 0, 1; and on $x = 0$ the integral ordinate is 0. $\therefore$ The number of lattice points is $17 + 10 + 5 + 2 + 1 = 35$.

44. $\measuredangle BAD = \measuredangle CAB - \measuredangle CAD = \measuredangle CAB - \measuredangle CDA$
$= \measuredangle CAB - (\measuredangle BAD + \measuredangle B)$.
$\therefore 2\measuredangle BAD = \measuredangle CAB - \measuredangle B = 30°$; $\therefore \measuredangle BAD = 15°$.

45. Solving for x, we have $x = y^2/(y - 1)$, $y \neq 1$. Solving for y, w have $y = (x \pm \sqrt{x^2 - 4x})/2$. $\therefore x(x - 4) \geq 0$, so that $x \geq$ or $x \leq 0$ and (A) is the answer. Note that choice (B) was quickly eliminated. The values $x = 4$, $y = 2$ rule out choices (D) and (C). Th values $x = 5$, $y = (5 \pm \sqrt{5})/2$ rule out (E).

46. The center of the circle is the point of intersection of the perpendicular bisectors of the given chords. Draw the radius to the endpoint of one of the chords and use Pythagoras' Theorem.
$\therefore r^2 = 4^2 + (1/2)^2 = 65/4$,
$r = \sqrt{65}/2$,
$d = \sqrt{65}$.



47. XY is the perpendicular bisector of AB. $\therefore \overline{MB} = \overline{MA}$. $\measuredangle BM$ is inscribed in a semi-circle and thus is a right angle.
$\therefore \measuredangle ABM = 45°$; $\therefore \widehat{AD} = 90°$; and $\therefore \overline{AD} = r\sqrt{2}$.

48. Consider first the special case where M coincides with C. Then $\overline{AM} = \overline{AC}$ and $\overline{BM} + \overline{MC} = \overline{BM} = \overline{BC} = \overline{AC}$. $\therefore \overline{AM} = \overline{BM} + \overline{MC}$.

To prove this in general, lay off $\overline{MN}$ (on MA) equal to $\overline{MC}$. Since $\measuredangle CMA = 60°$, ΔMCN is equilateral. We can show that $\Delta ACN \cong \Delta MCB$ because $\overline{AC} = \overline{CB}$, $\overline{CN} = \overline{CM}$, and $\measuredangle ACN = \measuredangle ACM - 60° = \measuredangle MCB$. $\therefore \overline{BM} = \overline{AN}$ and $\overline{AM} = \overline{AN} + \overline{MN} = \overline{BM} + \overline{MC}$.

49. Let x and y be the two upper segments of the non-parallel sides. Then $3 + y + x = 9 + 6 - y + 4 - x$. $\therefore x + y = 8$. Since $x:y = 4:6 = 2:3$, $2y/3 + y = 8$. $y = 24/5$, and $y:(6 - y) = 24/5:6/5 = 24:6 = 4:1$.

50. $A'ABB'$ is a trapezoid. Its median OO' is perpendicular to AB.

$$\overline{OO'} = \frac{1}{2}(\overline{AA'} + \overline{BB'}) = \frac{1}{2}(\overline{AG} + \overline{BG}) = \frac{1}{2}\overline{AB}.$$

$\therefore$ O' is a fixed distance from O on the perpendicular to AB, and therefore the point O' is stationary.

SOLUTION KEY

NINTH ANNUAL

MATHEMATICAL CONTEST

1958

Sponsored Jointly by

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and

THE SOCIETY OF ACTUARIES

USE OF KEY

Since it is established policy of our Committee to invite your considered comments on every phase of our contest, we make a brief prefatory explanation of this first attempt to provide a solution key to the 1958 contest problems.

1. The Key is for the exclusive use of the teachers locally in charge of the contest.
2. Some of the solutions are intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This solution Key validates our statement, at least for the 1958 examination, that no mathematics beyond intermediate algebra is needed to solve the problems posed.

1. $[2 - 3(-1)^{-1}]^{-1} = 5^{-1} = \frac{1}{5}$

2. $\frac{y - x}{xy} = \frac{1}{z};\ z = \frac{xy}{y - x}$

3. $\frac{a^{-1}b^{-1}}{a^{-3} - b^{-3}} \cdot \frac{a^3b^3}{a^3b^3} = \frac{a^2b^2}{b^3 - a^3}$

4. $\dfrac{\frac{x + 1}{x - 1} + 1}{\frac{x + 1}{x - 1} - 1} = \frac{x + 1 + (x - 1)}{x + 1 - (x - 1)} = x$

5. $\frac{(2 + \sqrt{2})^2(\sqrt{2} - 2) + \sqrt{2} - 2 + 2 + \sqrt{2}}{(2 + \sqrt{2})(\sqrt{2} - 2)} = \frac{-4}{-2} = 2$

6. $\frac{1}{2}\left(\frac{x + a}{x} + \frac{x - a}{x}\right) = \frac{1}{2}\left(\frac{2x}{x}\right) = 1$

7. $y = mx + b$ is satisfied by $(-1, 1)$ and $(3, 9)$

$1 = -m + b \qquad m = 2$
$9 = 3m + b \qquad b = 3 \qquad y = 2x + 3$

when $y = 0,\ x = -3/2$

8. $\sqrt{\pi^2} = \pi$, irrational

$\sqrt[3]{.8} = \sqrt[3]{8/10} = \frac{2}{\sqrt[3]{10}}$, irrational

$\sqrt[4]{.00016} = \sqrt[4]{16 \times 10^{-5}} = 2 \cdot 10^{-1} \sqrt[4]{10^{-1}}$, irrational

$\sqrt[3]{-1} \cdot \sqrt{(.09)^{-1}} = -1 \cdot \frac{1}{.3}$, rational

9. $x^2 + b^2 = a^2 - 2ax + x^2$

$$x = \frac{a^2 - b^2}{2a}$$

10. $x = \frac{-k \pm \sqrt{k^2 - 4k^2}}{2}$

For x to be real, $k^2 - 4k^2 \geq 0$, an impossibility

11. Replace $\sqrt{5 - x} = x\sqrt{5 - x}$ by

$$5 - x = x^2(5 - x)$$

The solution set for this equation is $\{5, 1, -1\}$ but the solution set for the original equation is $\{5, 1\}$.

or

$\sqrt{5 - x} = x\sqrt{5 - x}$

If $5 - x \neq 0$, divide by $\sqrt{5 - x}$ and obtain $x = 1$
If $5 - x = 0$, then $x = 5$

12. $\log P = \log s - n \log(1 + k)$

$$n = \frac{\log s - \log P}{\log(1 + k)} = \frac{\log s/P}{\log(1 + k)}$$

3. $\frac{1}{x} + \frac{1}{y} = \frac{y + x}{xy} = \frac{10}{20} = \frac{1}{2}$

14. Set up the following table, a one-to-one correspondence between the elements of the boy set and the girl set:

Boy number	1	2	3	... b
Girls danced with	5	6	7	... b + 4

$\therefore g = b + 4;\ b = g - 4$

or

5, 6, 7, ..., g
$l = a + (n - 1)d$
$g = 5 + (b - 1)(1);\ b = g - 4$

15. Take a special case, the square, where all the inscribed angles are equal; and each inscribed angle

$x = \frac{1}{2}(270°);\ 4x = 540°$

or

Let the angles be a, b, c, d, and the corresponding arcs A, B, C, D. Then $a = 1/2(B + C + D)$, etc. Then $a + b + c + d = 3/2(A + B + C + D) = 3/2 \cdot 360 = 540°$

16. $\pi r^2 = 100\pi;\ r = 10$
$A = 1/2\, ap = 3as;\ a = r = 10;\ s = 2 \cdot \frac{10}{\sqrt{3}}$

$\therefore A = 3 \cdot 10 \cdot \frac{20}{\sqrt{3}} = 200\sqrt{3}$

17. $1/2 \log x \geq \log 2$
$\log x \geq \log 4$
$x \geq 4$

or

$\log x \geq \log 2\sqrt{x}$
$x \geq 2\sqrt{x}$
$\sqrt{x} \geq 2$
$x \geq 4$

18. $A = \pi r^2$
$2A = \pi(r + n)^2$
$2\pi r^2 = \pi(r^2 + 2rn + n^2)$
$r = n(1 + \sqrt{2})$; the value $1 - \sqrt{2}$ is not used

19. $a^2 = cr \qquad b^2 = cs$

$\frac{r}{s} = \frac{a^2}{b^2} = \frac{1}{9}$

20. $4^x - 4^x \cdot 4^{-1} = 24$

$3/4 \cdot 4^x = 24$

$4^x = 32$

$(2^2)^x = 2^5$

$2x = 5;\ x = 5/2;\ (2x)^x = 5^{5/2} = 25\sqrt{5}$

21. Consider the special case with A on C. The answer is obvious

or

$A(\Delta CED) = r^2;\ A(\Delta AOB) = 1/2\, r^2$; ratio 2:1

22. When $y = 6 \qquad x = 4$ or -3
$y = -6 \qquad x = 0$ or 1
Because of symmetry the particle may roll from (4, 6) to (1, −6) or from (−3, 6) to (0, −6). In either case the horizontal distance is 3.

23. $y_1 = x^2 - 3$

$y_2 = (x \pm a)^2 - 3$

$y_2 - y_1 = \pm 2ax + a^2$

24. Let a = rate north = 30 mph

b = rate south = 120 mph

average rate for round trip is $\frac{2ab}{a+b}$

$$\frac{2ab}{a+b} = \frac{2 \cdot 30 \cdot 120}{30 + 120} = 48 \text{ mph}$$

25. $A = \log_k x$ $\qquad B = \log_5 k$

$x = k^A$ $\qquad k = 5^B$

$(\log_k x)(\log_5 k) = AB = 3$

$x = (5^B)^A = 5^3 = 125$

26. $s_1 = a_1 + a_2 + \ldots + a_n$

$s_2 = 5a_1 + 80 + 5a_2 + 80 + \ldots + 5a_n + 80$

$= 5(a_1 + a_2 + \ldots + a_n) + 80n$

$= 5s + 80n$

or

$\sum^n a_i = s$

$\sum^n [5(a_i + 20) - 20] =$

$= 5\sum^n a_i + \sum^n 80 = 5s + 80n$

27. $m = \frac{\Delta y}{\Delta x} = \frac{3+3}{4-2} = \frac{k/2 - 3}{5-4} \qquad k = 12$

where $\Delta y = y_2 - y_1$, etc.

28.

Removals	Water drawn	Water left	Anti-freeze drawn	Antifreeze left
First	4	12	0	4
Second	3	9	1	$3 + 4 = 7$
Third	$2\frac{1}{4}$	$6\frac{3}{4}$	$1\frac{3}{4}$	$5\frac{1}{4} + 4 = 9\frac{1}{4}$
Fourth	$1\frac{11}{16}$	$5\frac{1}{16}$	$2\frac{5}{16}$	$6\frac{15}{16} + 4 = 10\frac{15}{16}$

$\frac{5\frac{1}{16}}{16} = \frac{81}{256}$ $\qquad \frac{10\frac{15}{16}}{16} = \frac{175}{256}$

29. From triangle AEC $\quad x + n + y + w = 180°$

From triangle BED $\quad m + a + w + b = 180°$

$\therefore x + y + n = a + b + m$

30. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{y^2 + x^2}{x^2y^2} = a$

$\therefore a = \frac{x^2 + y^2}{b^2}$ or $x^2 + y^2 = ab^2$

$(x + y)^2 = x^2 + 2xy + y^2$

$= ab^2 + 2b = b(ab + 2)$

31. Let base be 2b $\qquad 2a + 2b = 32;\ a + b = 16$

$a^2 - b^2 = (a - b)(a + b) = 64$

$a - b = 4$

$a + b = 16$ $\qquad b = 6 \qquad$ area $= 8 \cdot 6 = 48$

32. $25s + 26c = 1000$

$s = 40 - \frac{26}{25}c$

The only solution in positive integers is C = 25, s = 14

33. roots p, q where $p = 2q$

$p + q = 3q = \frac{-b}{a};\ 9q^2 = \frac{b^2}{a^2}$

$pq = 2q^2 = \frac{c}{a};\quad 2q^2 = \frac{c}{a}$

$\frac{9}{2} = \frac{b^2}{ac}$ or $2b^2 = 9ac$

34. $6x + 1 > 7 - 4x$

$x > \frac{3}{5}$

$\therefore \frac{3}{5} < x \leq 2$

35. There is no loss of generality by placing one vertex at the origin, O. Let the other two vertices be A(a, c) and B(b, d). Let D and C be the projections of A and B, respectively, on the x-axis.

area OBA = area OCBA − area OCB

= area ODA + area DCBA − area OCB

$= \frac{1}{2}ac + \frac{1}{2}(b - a)(c + d) - \frac{1}{2}bd$

$= \frac{1}{2}(bc - ad)$

This may be integral or fractional

36. Let h be the altitude and x, the smaller segment

$30^2 - x^2 = 70^2 - (80 - x)^2;\ x = 15;\ 80 - x = 65$

37. $s = \frac{n}{2}[2a + (n - 1)d] \qquad a = k^2 + 1,\ n = 2k + 1,\ d = 1$

$s = (2k + 1)(k^2 + k + 1) = 2k^3 + 3k^2 + 3k + 1$

$= k^3 + 3k^2 + 3k + 1 + k^3 = (k + 1)^3 + k^3$

38. $s^2 - c^2 = \frac{y^2 - x^2}{r^2}$

max. val. $= \frac{y^2}{r^2}$ (when $x = 0$) $= \frac{1}{1} = 1$

min. val. $= \frac{-x^2}{r^2}$ (when $y = 0$) $= \frac{-1}{1} = -1$

or

$\sin\theta = s$ $\qquad$ Let $s^2 - c^2 = |a|$

$\cos\theta = c$ $\qquad$ Since $s^2 + c^2 = 1$

$2s^2 = 1 + |a|;\ |a| = 2s^2 - 1$

Since max. value of s is 1, max $|a| = 1$

39. If $x > 0$ $\quad x^2 + x - 6 = 0$

$(x - 2)(x + 3) = 0 \qquad x + 3 \neq 0 \qquad x = 2$

If $x < 0$ $\quad x^2 - x - 6 = 0$

$(x - 3)(x + 2) = 0 \qquad x - 3 \neq 0 \qquad x = -2$

or

$[|x| + 3][|x| - 2] = 0 \quad |x| + 3 \neq 0$

$\therefore |x| = 2$, i.e., $x = 2$ or -2

40. When n = 1 $a_1^2 - a_0a_2 = (-1)^1$; $a_2 = 10$

When n = 2 $a_2^2 - a_1a_3 = (-1)^2$; $a_3 = 33$

41. $r + s = -\frac{B}{A}$ $\qquad rs = \frac{C}{A}$

$r^2 + 2rs + s^2 = \frac{B^2}{A^2} \qquad 2rs = \frac{2C}{A}$

$\therefore r^2 + s^2 = \frac{B^2 - 2AC}{A^2}$

$\therefore p = -(r^2 + s^2) = \frac{2AC - B^2}{A^2}$

42. $\Delta AEB \sim \Delta ABD$

$\frac{AD}{12} = \frac{12}{8} \qquad AD = 18$

43. $16 = a^2 + \frac{b^2}{4}$

$49 = \frac{a^2}{4} + b^2$

$65 = 5/4(a^2 + b^2)$

$a^2 + b^2 = c^2 = 52; \qquad c = 2\sqrt{13}$

44. Let p be the proposition "a is greater than b"
q " " " "c is greater than d"
r " " " "e is greater than f"
From the given $p \rightarrow q$ and $\sim q \rightarrow r$
From the Rule of the Contrapositive $\sim q \rightarrow \sim p$ and $\sim r \rightarrow q$.
Therefore, (A), (B), (C), (D) are not valid conclusions

45. Let x = 10a + b dollars
y = 10c + d cents
face value of check 1000a + 100b + 10c + d cents
cashed value of check 1000c + 100d + 10a + b cents
difference 990c + 99d − (990a + 99b) = 1786
(10c + d) − (10a + b) = 18
y − x = 18
Therefore, y can equal 2x

46. $y = \frac{1}{2} \; \frac{x^2 - 2x + 2}{x - 1} = \frac{1}{2}\left[x - 1 + \frac{1}{x - 1}\right]$

The sum of a number and its reciprocal is numerically least when the number is ± 1.
For $x - 1 = 1$, $x = 2$ which is excluded
$\therefore$ $x - 1 = -1$ and $y = -1$
All other values of x in the interval given yield values of y less than −1.

or

$\frac{dy}{dx} = \frac{(2x-2)(2x-2) - (x^2 - 2x + 2)(2)}{(2x-2)^2} = 0$

x = 0, x = 2 (excluded)

At $x = 1/2$ $\quad \frac{dy}{dx}$ is negative

At $x = -1/2$ $\quad \frac{dy}{dx}$ is positive

Therefore y is a maximum when x = 0

47. $\Delta PTR \sim \Delta ATQ$; $\frac{PR}{AQ} = \frac{PT}{AT}$

PT = AT ($\angle PAT = \angle PBS = \angle APT$)
PR = AQ; PS = QF
PR + PS = AQ + QF = AF

or

$\angle SBP = \angle TPA = \angle TAP$
A, Q, R, P are concyclic
arc PR = arc AQ
PR = AQ
PS = QF; PR + RS = AF

48. Let P′ be the intersection of AB and a perpendicular from P to AB and let P′B = x. Then $(PP')^2 =$
$= x(10 - x)$

$CP = \sqrt{x(10-x) + (6-x)^2} \qquad DP = \sqrt{x(10-x) + (4-x)^2}$

$CP + DP = \sqrt{36 - 2x} + \sqrt{16 + 2x}$

This sum is greatest when $\sqrt{36 - 2x} = \sqrt{16 + 2x}$, that is when x = 5. Hence (E)

or

An ellipse through A and B with C and D as foci is the locus of points such that CP′ + P′D = 10 where P′ is on the ellipse
(A) is untrue since CP + PD > CP′ + P′D = 10
(B) When P is at B, CP + PD = 10
(C) When P is not at B or A, CP + PD > 10
(D) When PD ⊥ AB, CP + PD > 10
(E) is correct

49. $(a + b)^{10}$ has 11 terms
$(a + d)^{10}$ has 11 terms, 10 containing d if d = b + c
There are 2 terms each containing d^1
There are 3 terms each containing d^2 etc.
There are $10/2(2 + 11) = 65$ terms with d
$\therefore$ total number of terms is 65 + 1 = 66

50. $DP = x = a \qquad D'P' = y \qquad \frac{PB}{P'B'} = \frac{1}{4}$

$y = D'B' + B'P' = 1 + BP' = 1 + 4PB$
$PB = 4 - x = 4 - a$
$y = 1 + 4(4 - a) = 17 - 4a$
$x + y = 17 - 3a$

or

$m = \frac{\Delta y}{\Delta x} = \frac{1-5}{4-3} = -4$ since $\begin{array}{ccc} x & 3 & 4 \\ y & 5 & 1 \end{array}$

$y = -4x + b \qquad 5 = -12 + b \qquad b = 17$
$y = -4x + 17 \qquad$ When x = a $\qquad x + y = 17 - 3a$

SOLUTION KEY

TENTH ANNUAL H. S.
MATHEMATICS CONTEST

1959

Sponsored Jointly by

THE MATHEMATICAL ASSOCIATION OF AMERICA

and

THE SOCIETY OF ACTUARIES

USE OF KEY

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1. S (old) = $6x^2$ S (new) = $6(1.5x)^2 = 13.50x^2$

   Increase = $7.5x^2 = \frac{5}{4}(6x^2)$

   Increase (%) = 125

2. Let x be the distance from P to AB. By similar triangles

   $\frac{1}{2} = \frac{(1-x)^2}{1^2}$ $\therefore 1 - x = \pm\frac{1}{\sqrt{2}}$ $\therefore x = \frac{2-\sqrt{2}}{2}$ (negative sq. root rejected)

3. There are a variety of figures satisfying the given conditions, and not falling within the classifications (A), (B), (C), (D). For example, the "kite".

4. $x + \frac{1}{3}x + \frac{1}{6}x = 78.$ $9x = 468.$ $\frac{1}{3}x = 17\frac{1}{3}$

5. $(256)^{.16}(256)^{.09} = (256)^{.25} = (256)^{1/4} = 4$

6. The converse is: If a quadrilateral is a rectangle, then it is a square.

   The inverse is: If a quadrilateral is not a square, then it is not a rectangle.

   Both statements are false.

   or

   Use Venn diagrams to picture the sets mentioned.

7. $a^2 + (a+d)^2 = (a+2d)^2$ $\therefore a = 3d$ $\therefore \frac{a}{d} = \frac{3}{1}$

8. $x^2 - 6x + 13 = (x-3)^2 + 4$. The smallest value for this expression, 4, is obtained when x = 3.

   or

   Graph $y = x^2 - 6x + 3$. The turning point, whose ordinate is 4, is a minimum point.

9. $n = \frac{1}{2}n + \frac{1}{4}n + \frac{1}{5}n + 7.$ $n = 140$

10. Let the altitude from B to AC be $\underline{h}$. Then the altitude from D to AE is $\frac{1}{3}\underline{h}$. Let AE = x. Then

    $\frac{1}{2}\cdot\frac{1}{3}hx = \frac{1}{2}h(3.6)$ $\therefore x = 10.8$

11. Let $\log_2 .0625 = x$ $\therefore 2^x = .0625 = \frac{1}{2^4}$ $\therefore x = -4$

12. Let a = the constant

    $\frac{20+a}{50+a} = \frac{50+a}{100+a}$ $\therefore a = 25$ $\therefore r = \frac{5}{3}$

13. Arith. mean = $\frac{\text{sum of numbers}}{\text{number of numbers}}$

    $\therefore S = 50 \times 38 = 1900$ $\therefore x = \frac{1900 - 45 - 55}{48} = 37.5$

14. Addition, subtraction, and multiplication with even integers always yield even integers.

    Note: This is a good opportunity to underscore operational restrictions imposed by the available domain.

15. $c^2 = 2ab$ $\therefore a^2 + b^2 = 2ab$ $\therefore (a-b)^2 = 0$ $\therefore a = b$

    Since the right triangle is isosceles, one of the acute angles is 45°.

16. $\frac{(x-2)(x-1)}{(x-3)(x-2)} \cdot \frac{(x-3)(x-4)}{(x-4)(x-1)} = 1$

17. $y = a + \frac{b}{x}$ $1 = a - b$

    $5 = a - \frac{1}{5}b$ $\therefore b = 5$ and $a = 6$

18. $\left(\frac{r-s}{s} - \frac{(r+1)-(s+1)}{s+1}\right)100 = \frac{100(r-s)}{s(s+1)}$

19.

Weights used	No. of weighings possible
1. Singly	3
2. two at a time (same pan)	3
3. three at a time (same pan)	1
4. two at a time (diff. pans)	3
5. three at a time (diff. pans)	3
Total	13

or

$111_3 = 1\cdot3^2 + 1\cdot3 + 1 = 13_{10}$, so that the number of weighings equals the sum of the weights, when the weights are related in this manner.

20. $x = \frac{Ky}{Z^2}$ $10 = \frac{K(4)}{14^2}$ $\therefore K = \frac{10\cdot14^2}{4}$

    $\therefore x = \frac{\frac{10\cdot14^2}{4}\cdot16}{7^2} = 160$

21. $R = \frac{2}{3}$ altitude $= \frac{2}{3}\cdot\frac{1}{2}\cdot\frac{p}{3}\sqrt{3} = \frac{p}{3\sqrt{3}}$ $\therefore A = \frac{\pi p^2}{27}$

22. The median of a trapezoid goes through the midpoints of the diagonals. Let x be the length of the shorter base.

    $\therefore$ length of median $= \frac{x}{2} + 3 + \frac{x}{2}$

    $\therefore \frac{1}{2}(x + 97) = \frac{x}{2} + 3 + \frac{x}{2}$ $\therefore x = 91$

23. $\log_{10}(a^2 - 5a) = 2$ $\therefore 10^2 = a^2 - 15a$

    $\therefore a = 20$ or $a = -5$

24. $\frac{\frac{m^2}{100} + x}{m + x} = \frac{2m}{100}$ $\therefore x = \frac{m^2}{100 - 2m}$

25. Let AB be a straight line segment with its midpoint at 3, and its right end at B. Each of the two intervals from A to 3 and from 3 to B is less than 4. Hence B is to the left of 7 and A is to the right of −1.

    or

    $|3 - x| < 4$

    $\therefore 3 - x < 4$ $\therefore -x < 1$ $\therefore x > -1$

    $-4 < 3 - x$ $\therefore -7 < -x$ $\therefore x < 7$

    $\therefore -1 < x < 7$

26. $x^2 + x^2 = 2 \quad \therefore x = 1 \quad \therefore DF = \frac{\sqrt{2}}{2}$

But $\frac{AD}{AE} = \frac{2}{3} \quad \therefore \frac{DF}{EG} = \frac{2}{3}$

$\therefore EG = \frac{3\sqrt{2}}{4} \quad \therefore$ altitude $CDF = \frac{3\sqrt{2}}{2}$

$\therefore A = \frac{1}{2} \cdot \sqrt{2} \cdot \frac{3\sqrt{2}}{2} = \frac{3}{2}$

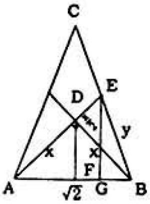


Fig. 1

or

$x^2 + x^2 = 2 \quad \therefore x = 1$

$\therefore y^2 = 1 + \frac{1}{4} \qquad \therefore y = \frac{\sqrt{5}}{2} \qquad \therefore BC = \sqrt{5}$ etc.

27. Sum of roots is $\frac{1}{i} = -i$

28. The bisector of an angle of a triangle divides the opposite sides into segments proportional to the other two sides

$\therefore \frac{AM}{MB} = \frac{b}{a}$ and $\frac{CL}{LB} = \frac{b}{c}$

Since $\frac{c}{a} \cdot \frac{b}{c} = \frac{b}{a}$, $k = \frac{c}{a}$

29. $\frac{15 + \frac{1}{3}(x - 20)}{x} = \frac{1}{2} \qquad x = 50$

30. Let x = B's time in seconds. Letting 1 represent the length of the track, we have $\frac{1}{40}(15) + \frac{1}{x}(15) = 1$

$\therefore x = 24$

31. Let $2s$ be the side of the smaller square.

Then $\frac{r - s}{2s} = \frac{2s}{r + s} \quad \therefore r = \sqrt{50}$

Let S be the side of the larger square

$S = r\sqrt{2} = 10 \qquad \therefore A = 100$

32. The point A, the point of tangency, and the center of the circle, determine a right triangle with one side $\underline{l}$, another side $\underline{r}$, and the hypotenuse $x + r$, where x is the shortest distance from A to the circle.

$\left(\frac{4}{3}r\right)^2 + r^2 = (x + r)^2 \quad \therefore x = \frac{2}{3}r = \frac{1}{2}l$

33. $\frac{1}{3}, \frac{1}{4}, \frac{1}{16}, \frac{1}{12}, 0$ with $d = -\frac{1}{12}$

$\therefore$ the terms of the H.P. are 3, 4 6, 12 only $\therefore S_4 = 25$

34. $x^2 - 3x + 1 = 0 \quad \therefore rs = 1$ and $r + s = 3$
$\therefore (r + s)^2 = r^2 + 2rs + s^2 = 9 \quad \therefore r^2 + s^2 = 7$

35. $(x - m)^2 - (x - n)^2 = (m - n)^2 \quad m > 0 \quad n < 0$
Since $(m - n)^2 > 0 \quad (x - m)^2 > (x - n)^2$
$\therefore |x - m| > (x - n)$

$x - m > x - n \qquad\qquad 2x < m + n$
not usable $\qquad\qquad x < \frac{m + n}{2}$

or

Solving for x we have $x(-m + n) = -mn + n^2 \quad x = n$

36. Let the triangle be ABC, with AB = 80, BC = a, CA = b = 90 − a, $\angle B = 60°$. Let CD be the altitude to AB. Let x = BD
$\therefore CD = \sqrt{3}x \quad a = 2x \quad b = 90 - 2x$
$\therefore 3x^2 + (80 - x)^2 = (90 - 2x)^2$

$\therefore x = \frac{17}{2} \quad \therefore a = 17 \quad b = 73$

37. $\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{4}\right)\left(1 - \frac{1}{5}\right) \ldots \left(1 - \frac{1}{n}\right)$

$= \frac{2}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \frac{\cancel{4}}{\cancel{5}} \cdot \ldots \frac{\cancel{n-2}}{\cancel{n-1}} \frac{\cancel{n-1}}{n} = \frac{2}{n}$

38. $4x \pm \sqrt{2x} = 1 \qquad \therefore 4x - 1 = -\sqrt{2x}$
$16x^2 - 10x + 1 = 0 \qquad \therefore x = \frac{1}{8}$

$x = \frac{1}{2}$ does not satisfy the original equation.

or

Let $y = \sqrt{x} \qquad \therefore 4y^2 + \sqrt{2}y - 1 = 0$

$y = \frac{2\sqrt{2}}{8}, \quad \frac{-4\sqrt{2}}{8}, \; x = y^2 = \frac{1}{8}, \frac{1}{2}$ Reject

39. $x + x^2 + x^3 + \ldots + x^9$

$S_1 = \frac{x(1 - x^9)}{1 - x} = \frac{x^{10} - x}{x - 1}$

$a + 2a + 3a + \ldots + 9a$

$S_2 = \frac{9}{2}(a + 9a) = 45a \qquad S = S_1 + S_2$

40. Let G be a point on EC so that FE = EG. Connect D with G. Then FDGB is a parallelogram.
$\therefore DG = 5 \quad AF = 10 \quad AB = 15$

41.

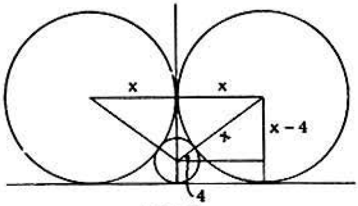


Fig. 2

$x^2 + (x - 4)^2 = (x + 4)^2 \qquad x = 16, 0$

42. Represent a, b, c in terms of their prime factors. Then D is the product of all the common prime factors, each factor taken as often as it appears the least number of times in a or b or c. M is the product of all the non-common prime factors, each factor taken as often as it appears the greatest number of times in a or b or c.

Therefore, MD may be less than abc, but it cannot exceed abc.

Obviously, MD equals abc when there are no common factors.

43. Let h be the altitude to side 39. Then $\frac{2R}{25} = \frac{40}{h}$

But $\frac{1}{2}h \cdot 39 = \text{Area} = \sqrt{s(s-a)(s-b)(s-c)}$

$\therefore h = \frac{2}{39}\sqrt{52 \cdot 13 \cdot 12 \cdot 27} = 24 \qquad \therefore 2R = \frac{125}{3}$

44. Let the roots be $1 + m$ and $1 + n$ with m and n both positive. $\therefore 1 + m + 1 + n = -b$ and $(1+m)(1+n) = c$
$\therefore s = b + c + 1 = mn > 0.$

45. Since $\log_a b = \frac{\log_c b}{\log_c a}$, we have, with base x,

$$\frac{\log x}{\log 3}\ \frac{\log 2x}{\log x}\ \frac{\log y}{\log 2x} = 2$$

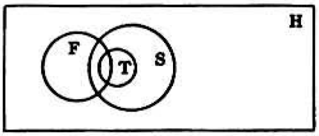
$\log y = 2 \log 3 = \log 9 \qquad \therefore y = 9$

46.

	rainy mornings	non-rainy mornings
rainy afternoons	a	b
non-rainy afternoons	c	e

$d = a + b + c + e \qquad a + b + c = 7 \qquad a = 0$
$c + e = 5 \qquad b + e = 6 \qquad \therefore e = 2 \qquad \therefore d = 9$

47.



48. For $h = 3$ we can have $1x^2, 1x^1 + 1, 1x^1 - 1, 2x^1, 3x^0$

49. Combine the terms in threes, as follows,

$$\frac{1}{4} + \frac{1}{32} + \frac{1}{256}, \dots \qquad \therefore S = \frac{\frac{1}{4}}{1 - \frac{1}{8}} = \frac{2}{7}$$

or

Since there is absolute convergence, the terms may be re-arranged to yield the three series

$$1 + \frac{1}{8} + \frac{1}{64} + \dots, \qquad -\frac{1}{2} - \frac{1}{16} - \frac{1}{128} - \dots,$$

$$-\frac{1}{4} - \frac{1}{32} - \frac{1}{256} - \dots$$

$$S_1 = \frac{1}{1 - \frac{1}{8}} = \frac{8}{7} \qquad S_2 = \frac{-\frac{1}{2}}{1 - \frac{1}{8}} = -\frac{4}{7}$$

$$S_3 = \frac{-\frac{1}{4}}{1 - \frac{1}{8}} = -\frac{2}{7} \qquad \therefore S = \frac{2}{7}$$

50. This problem may be interpreted as a miniature finite geometry of 4 lines and 6 points, so that each pair of lines has only 1 point in common, and each pair of points has only 1 line in common.

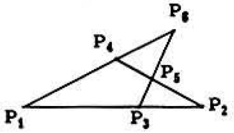


Fig. 3

or

$C(4,2) = \frac{4!}{2!2!} = 6$, displayed as follows

Committees

	A	B	C	D
Members	a	b	d	f
	c	e	a	e
	b	d	f	c

SOLUTION KEY

ELEVENTH ANNUAL H. S.
MATHEMATICS CONTEST

1960

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USE OF KEY

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1. Substituting 2 for x, we have $2^3 + h \cdot 2 + 10 = 0$
$\therefore h = -9$

2. For the six strokes there are five equal intervals so that the time interval between successive strokes is one second. For twelve strokes, then, eleven seconds are required.

3. 40% of \$10,000 is \$4,000
36% of \$10,000 is \$3,600; 4% of (\$10,000 − \$3,600) is \$256.
\$3,600 + \$256 = \$3,856 ∴ Difference is 4000 − 3856 = 144
or
Two successive discounts of 36% and 4% are equivalent to one discount of 38.56% (.36 + .04 −(.36)(.04) = .3856)

4. The triangle is equiangular with side 4.
$A = \frac{s^2\sqrt{3}}{4} = \frac{4^2\sqrt{3}}{4} = 4\sqrt{3}$

5. Substitute $y^2 = 9$ into $x^2 + y^2 = 9$ to obtain $x^2 + 9 = 9$, so that $x = 0$. Therefore, the common solutions, and, hence, the intersection points are (0, 3) and (0, −3)
or
$x^2 + y^2 = 9$ represents a circle with radius 3 and centered at the origin. $y^2 = 9$ represents the pair of lines $y = +3$ and $y = -3$, tangent to the circle at (0, 3) and (0, −3)

6. Since $100 = \pi d$, $d = \frac{100''}{\pi}$. Let s be the side of the inscribed square.
$s = \frac{d}{\sqrt{2}} = \frac{100}{\pi\sqrt{2}} = \frac{50\sqrt{2}}{\pi}$

7. $\frac{\text{Area II}}{\text{Area I}} = \frac{\pi R^2}{\pi r^2} = \frac{(2r)^2}{r^2} = 4$
∴ Area II = 4 Area I = 4 · 4 = 16

8. Let N = 2.52525... 100N = 252.52525... Subtracting the first equality from the second, we obtain
$99N = 250 \quad \therefore N = \frac{250}{99}$. Since this fraction is already in lowest terms, we have 250 + 99 = 349

9. $\frac{a^2 + b^2 - c^2 + 2ab}{a^2 + c^2 - b^2 + 2ac} = \frac{(a+b)^2 - c^2}{(a+c)^2 - b^2} = \frac{(a+b+c)(a+b-c)}{(a+c+b)(a+c-b)}$
$= \frac{a+b-c}{a+c-b}$ with $(a + c)^2 \neq b^2$

10. To negate the given statement we say: It is false that all men are good drivers, that is, at least one man is a bad driver.

11. The product of the roots, here, is equal to $2k^2 - 1$.
$\therefore 2k^2 - 1 = 7$, $k^2 = 4$.
Discriminant $= (-3k)^2 - 4 \cdot 1 \cdot (2k^2 - 1) = k^2 + 4 = 8$; the roots are irrational.

12. The locus is a circle, centered at the given point, having a radius equal to that of the given circles.

13. The given lines intersect in the three points A(0, 2), $B\left(\frac{4}{3}, -2\right)$ and $C\left(-\frac{4}{3}, -2\right)$. These three points are the vertices of a triangle with AC = AB, each of length $\frac{4}{3}\sqrt{10}$, and $BC = \frac{8}{3}$, so that (B) is the correct choice.

14. $3x - 5 + a = bx + 1$, $3x - bx = 6 - a$, $x = \frac{6-a}{3-b}$ if $b \neq 3$

15. Since the triangles are similar, we have $\frac{A}{a} = \frac{P}{p} = \frac{R}{r} = \frac{\sqrt{K}}{\sqrt{k}}$ always, so that (B) is the correct choice.

16. $69 = 2 \cdot 5^2 + 3 \cdot 5 + 4 \cdot 1 = 234_5$ (that is, 234 in the base 5 system). The correct choice is, therefore, (C).

17. We have $N = 8 \cdot 10^8 \cdot x^{-\frac{3}{2}}$ with N = 800; to find x
$\therefore 800 = 8 \cdot 10^8 \cdot x^{-\frac{3}{2}}$, $1 = 10^6 \cdot x^{-\frac{3}{2}}$, $x^{\frac{3}{2}} = 10^6$, $x^{\frac{1}{2}} = 10^2$,
$x = 10^4$

18. $3^{x+y} = 81 = 3^4 \quad \therefore x + y = 4$
$81^{x-y} = 3 = 81^{\frac{1}{4}} \quad \therefore x - y = \frac{1}{4}$
$\therefore x = 2 + \frac{1}{8}$, $y = 2 - \frac{1}{8}$
The correct choice is (E)

19. For (A) we have $n + n+1 + n+2 = 3n + 3 = 46$,
an impossibility for integral n.
For (B) we have $2n + 2n+2 + 2n+4 = 6n + 6 = 46$,
an impossibility for integral n.
For (C) we have $n + n+1 + n+2 + n+3 = 4n + 6 = 46$,
solvable for integral n.
For (D) we have $2n + 2n+2 + 2n+4 + 2n+6$
$= 8n + 12 = 46$,
an impossibility for integral n.
For (E) we have $2n+1 + 2n+3 + 2n+5 + 2n+7$
$= 8n + 16 = 46$,
an impossibility for integral n.

20. $$\left(\frac{x^2}{2} - \frac{2}{x}\right)^8 = \frac{1}{(2x)^8}(x^3 - 4)^8 = \frac{1}{(2x)^8}\left[(x^3)^8 + 8(x^3)^7(-4) + \frac{8 \cdot 7}{1 \cdot 2}(x^3)^6(-4)^2 + \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3}(x^3)^5(-4)^3 + \ldots\right]$$
The required term is $\frac{1}{2^8 \cdot x^8} \cdot \frac{8 \cdot 7 \cdot 6}{1 \cdot 2 \cdot 3}(x^3)^5(-4)^3$; the required coefficient is −14
or

$\frac{1}{2^8x^8}\,C(8,5)(x^3)^5(-4)^3 = \frac{1}{2^8x^8}\cdot\frac{8\cdot 7\cdot 6}{1\cdot 2\cdot 3}(-4)^3x^7 = -14x^7;$

the required coefficient is -14.

21. Let $\underline{s}$ be a side of square I; $\underline{S}$, a side of square II. Area I = s^2, Area II = S^2. Since $S^2 = 2s^2$ $S = s\sqrt{2}$

But $s = \frac{d}{\sqrt{2}} = \frac{a+b}{\sqrt{2}}$ $\therefore S = \frac{a+b}{\sqrt{2}}\cdot\sqrt{2} = a + b$

$\therefore$ Perimeter of II = $4S = 4(a+b)$

22. $(x+m)^2 - (x+n)^2 = (m-n)^2$
Factor the left side of the equation as the difference of two squares.
$(x+m-x-n)(x+m+x+n) = (m-n)^2$
$(m-n)(2x+m+n) = (m-n)^2$
Since $m \neq n$ $2x + m + n = m - n$, $x = -n$, so that (A) is the correct choice.

23. $V = \pi R^2H$, $\pi(R+x)^2H = \pi R^2(H+x)$
$R^2H + 2RxH + x^2H = R^2H + R^2x$
$2Rxh + x^2H = R^2x$
since $x \neq 0$ $2RH + xH = R^2$, $x = \frac{R^2 - 2RH}{H}$

For $R = 8$, $H = 3$ $x = \frac{16}{3}$, so that (C) is the correct choice.

24. $\log_{2x}216 = x$, $(2x)^x = 216$, $2^x \cdot x^x = 2^3\cdot 3^3$

An obvious solution to this equation is $x = 3$, so that (A) is the correct choice.

25. $m = 2r + 1$ where $r = 0, \pm 1, \pm 2, \ldots$
$n = 2s + 1$ where $s = 0, \pm 1, \pm 2, \ldots$
$m^2 - n^2 = 4r^2 + 4r + 1 - 4s^2 - 4s - 1$
$= 4(r-s)(r+s+1)$, a number certainly divisible by 4.
If r and s are both even or both odd, $r - s$ is divisible by 2.
If r and s are one even and one odd, then $r + s - 1$ is divisible by 2.
Thus $m^2 - n^2$ is divisible by $4\cdot 2 = 8$

26. The numerical value of $|5-x|$ must be less than 6 for the inequality to hold. Therefore, x must be greater than -1, and less than 11.

or

$\left|\frac{5-x}{3}\right| < 2$, $-2 < \frac{5-x}{3} < 2$, $-6 < 5 - x < 6$

$\therefore -6 < 5 - x$ and $5 - x < 6$
$x < 11$ $x > -1$ $\therefore -1 < x < 11$

27. Since each interior angle is $7\frac{1}{2}$ times its associated exterior angle, $S = 7\frac{1}{2}$(sum of the exterior angles)

$= \frac{15}{2}\cdot 360 = 2700$

Since $S = (n-2)180$, $(n-2)180 = 2700$, $n = 17$
A 17-sided polygon that is equiangular may or may not be equilateral, and, hence, regular.

28. A formal solution of $x - \frac{7}{x-3} = 3 - \frac{7}{x-3}$ yields a repeated value of 3 for x. But $x = 3$ makes the expression $\frac{7}{x-3}$ meaningless. The correct choice is, therefore, (B).

29. $5a + b > 51$
$3a - b = 21$ $\therefore 8a > 72$, $a > 9$, $3a > 27$
Since $b = 3a - 21$, $b > 6$

30. The locus of points equidistant from the coordinate axes is the pair of lines $y = x$ and $y = -x$.
Solve simultaneously $3x + 5y = 15$ and $y = x$ to obtain $x = y = \frac{15}{8}$. The point $\left(\frac{15}{8}, \frac{15}{8}\right)$ in quadrant I, therefore, satisfies the required condition.
Solve simultaneously $3x + 5y = 15$ and $y = -x$ to obtain $x = -\frac{15}{2}$, $y = \frac{15}{2}$. The point $\left(-\frac{15}{2}, \frac{15}{2}\right)$ in quadrant II, therefore, satisfies the required condition.
There are no other solutions to these pairs of equations, and, therefore, no other points satisfying the required condition.

31. Let the other factor be $x^2 + ax + b$. Then
$(x^2+2x+5)(x^2+ax+b) \equiv x^4 + x^3(2+a) + x^2(5+b+2a)$
$+ x(5a+2b) + 5b \equiv x^4 + px^2 + q$
Match the coefficients of like powers of x
For x^3 we have $2 + a = 0$ $\therefore a = -2$
x $5a + 2b = 0$ $\therefore b = 5$
x^2 $5 + b + 2a = p$ $\therefore p = 6$
x^0 $5b = q$ $\therefore q = 25$

or

Let $y = x^2$ so that $x^4 + px^2 + q = y^2 + py + q$. Let the roots of $y^2 + py + q = 0$ be r^2 and s^2. Since $y = x^2$ the roots of $x^4 + px^2 + q = 0$ must be $\pm r$, $\pm s$. Now $x^2 + 2x + 5$ is a factor of $x^4 + px^2 + q$; consequently, one pair of roots, say $\underline{r}$ and $\underline{s}$, must satisfy the equation $x^2 + 2x + 5 = 0$. It follows that $-r$ and $-s$ must satisfy the equation $x^2 - 2x + 5 = 0$. Therefore, the other factor must be $x^2 - 2x + 5$.
$\therefore (x^2+2x+5)(x^2-2x+5) \equiv x^4 + 6x^2 + 25 \equiv x^4 + px^2 + q$
$\therefore p = 6$, $q = 25$

or

$$
\begin{array}{r|lllll}
 & x^2 & - & 2x & & +\,(p-1) \\
\hline
x^2+2x+5 & x^4 & + & px^2 & & +\,q \\
 & \underline{x^4 + 2x^3} & \underline{+} & \underline{5x^2} & & \\
 & & -2x^3 & +(p-5)x^2 & & \\
 & & \underline{-2x^3 -} & \underline{4x^2 -} & \underline{10x} & \\
 & & & (p-1)x^2 + & 10x + q & \\
 & & & \underline{(p-1)x^2} & \underline{+2(p-1)x} & \underline{+\,5(p-1)} \\
 & & & & (12-2p)x & +\,(q-5p+5)
\end{array}
$$

since the remainder must be zero (why?) $12 - 2p = 0$, $p = 6$ and $q - 5p + 5 = 0$, $q = 25$.

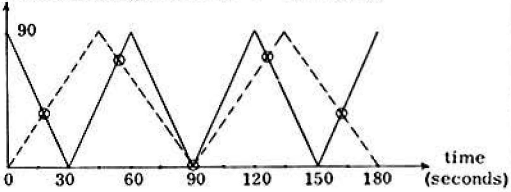
Since $\frac{\overline{AD}}{\overline{AB}} = \frac{\overline{AB}}{\overline{AE}}$, $\overline{AB}^2 = \overline{AD} \cdot \overline{AE}$

But $\overline{AE} = \overline{AD} + 2r = \overline{AD} + \overline{AB}$ $\therefore \overline{AB}^2 = \overline{AD}(\overline{AD}+\overline{AB})$
$= \overline{AD}^2 + \overline{AD} \cdot \overline{AB}$
$\therefore \overline{AD}^2 = \overline{AB}^2 - \overline{AD} \cdot \overline{AB} = \overline{AB}(\overline{AB} - \overline{AD})$
Since AP = AD, $\overline{AP}^2 = \overline{AB}(\overline{AB} - \overline{AP}) = \overline{AB} \cdot \overline{PB}$

When n is prime each member of the sequence is divisible by n. When n is composite each member of the sequence is divisible by all the prime numbers that divide n.

It follows that each member of the sequence is composite, so that (A) is the correct choice.

Graph the position of each swimmer with respect to time [this graph ends with t = 3(minutes)]



At the end of 3 minutes they are back to their original positions, so that in 12 minutes this cycle is repeated four times. Since there are five meetings in this cycle, the total number of meetings is 4 × 5 = 20.

The problem may be viewed as one in periodic phenomena; the period for the faster swimmer is 60 seconds, while the period for the slower swimmer is 90 seconds (the period is the time-interval necessary for the swimmer to return to his starting point). The common period is 180 seconds or 3 minutes.

$\frac{m}{t} = \frac{t}{n}$, $mn = t^2$. Since $m + n = 10$, $t^2 = m(10 - m)$

$\therefore t = \sqrt{m(10 - m)}$

Since m, 10 − m, and t are integers, we have t = 3 when m = 1 and t = 4 when m = 2. The remaining integral values of m, namely m = 3, 4, 6, 7, yield non-integral values of t. The value m = 5 cannot be used, since m and n are unequal.

$s_1 = \frac{n}{2}[2a + (n-1)d]$, $s_2 = \frac{2n}{2}[2a + (2n-1)d]$,

$s_3 = \frac{3n}{2}[2a + (3n-1)d]$

$R = s_3 - s_2 - s_1 = \frac{n}{2}[6a + 9nd - 3d - 4a - 4nd + 2d - 2a$
$- nd + d] = 2n^2d$, so that (B) is the correct choice.

37. Designate the base of the rectangle by y. Then, because of similar triangles, $\frac{h-x}{y} = \frac{h}{b}$, $y = \frac{b}{h}(h-x)$

$\therefore$ Area $= xy = \frac{bx}{h}(h-x)$

38. $b + 60 = a + B$
$a + 60 = c + C$ $\therefore b - a = a - c + B - C$ Since

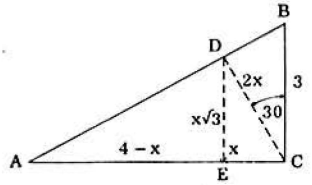
$B = C$, we have $b + c = 2a$, $a = \frac{b+c}{2}$

39. $\frac{a+b}{a} = \frac{b}{a+b}$, $a^2 + 2ab + b^2 = ab$, $a^2 + ab + b^2 = 0$

$\therefore a = \frac{-b \pm \sqrt{b^2 - 4b^2}}{2} = \frac{-b \pm ib\sqrt{3}}{2}$

If b is real, a is not real. If b is not real, a may be real or not real.

40. Since CD trisects right angle C, ∠BCD = 30° and ∠DCA = 60° and ∠CDE = 30°, so that △DEC is a 30°-60°-90° triangle. Let EC = x
$\therefore DE = x\sqrt{3}$ and $DC = 2x$



$\frac{4-x}{x\sqrt{3}} = \frac{4}{3}$, $x = \frac{12}{3 + 4\sqrt{3}}$, $2x = \frac{24}{3 + 4\sqrt{3}} = \frac{32\sqrt{3} - 24}{13}$

or

Area of $\triangle BCD = \frac{1}{2} \cdot 3 \cdot 2x \cdot \sin 30° = \frac{3x}{2}$

Area of $\triangle ACD = \frac{1}{2} \cdot 4 \cdot 2x \cdot \sin 60° = 2\sqrt{3}x$

Area of △BCD + Area of △ACD = Area of △ACB

$\therefore \frac{3}{2}x + 2\sqrt{3}x = 6$, $x = \frac{12}{3 + 4\sqrt{3}} = \frac{16\sqrt{3} - 12}{13}$,

$2x = \frac{32\sqrt{3} - 24}{13}$

SOLUTION KEY
AND ANSWER KEY

TWELFTH ANNUAL H. S.
MATHEMATICS CONTEST

1961

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Note: The letter following the problem number refers to the correct choice of the five listed in the examination.

1\. (C) $\left(-\frac{1}{125}\right)^{-2/3} = (-125)^{2/3} = ((-125)^{1/3})^2 = (-5)^2 = 25$

2\. (E) D (yards) = R (yards per minute) × T (minutes)

$$D = \frac{\frac{1}{3}\cdot\frac{a}{6}}{\frac{r}{60}}\cdot 3 = \frac{10a}{r}$$

3\. (E) Let m_1 be the slope of the line $2y + x + 3 = 0$, and m_2, the slope of the line $3y + ax + 2 = 0$ For perpendicularity $m_1m_2 = -1$

$$\therefore \left(-\frac{1}{2}\right)\left(-\frac{a}{3}\right) = -1 \quad \therefore a = -6$$

4\. (B) u is not closed under addition since, for example, $1 + 4 = 5$ and 5, not being a perfect square of an integer, is not a member of u. Similarly for division and positive integral root extraction. u is closed under multiplication because, if m^2 and n^2 are elements of u, $m^2 \cdot n^2 = (mn)^2$ is a member of u (u contains the squares of all positive integers and mn is an integer.

5\. (C) Consider the binomial expansion, $(a + 1)^4 = a^4 + 4a^3 + 6a^2 + 4a + 1$. Let $x - 1 = a$. Then $S = (x - 1 + 1)^4 = x^4$.

6\. (E) $\log 8 \div \log \frac{1}{8} = \log 8 \div (\log 1 - \log 8) = \log 8 \div (-\log 8) = -1$

7\. (A) Since $(r + s)^6 = r^6 + 6r^5s + 15r^4s^2 + \cdots$, we have, letting $r = \frac{a}{\sqrt{x}}$ and $s = -\frac{\sqrt{x}}{a^2}$ in the third term,

$$15\left(\frac{a}{\sqrt{x}}\right)^4\left(-\frac{\sqrt{x}}{a^2}\right)^2 = \frac{15}{x}$$

or

Use the formula for the general term of $(r + s)^n$, *n* positive integer: $(k + 1)$ th term

$$= \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k}(r)^{n-k}(s)^k$$

For $k = 2$ we have $\frac{6\cdot 5}{1\cdot 2}\left(\frac{a}{\sqrt{x}}\right)^4\left(-\frac{\sqrt{x}}{a^2}\right)^2 = \frac{15}{x}$

8\. (B) Since $B + C_2 = 90°$ and $A + C_1 = 90°$, $B + C_2 = A + C_1$ $\therefore C_1 - C_2 = B - A$

9\. (C) $r = (2a)^{2b} = ((2a)^2)^b = (4a^2)^b$; also $r = a^b \cdot x^b = (ax)^b$ $\therefore (ax)^b = (4a^2)^b$ $\therefore ax = 4a^2$ $\therefore x = 4a$

10\. (D) $\overline{BE}^2 = \overline{BD}^2 + \overline{DE}^2$, $\overline{BD} = 6$, $\overline{DE} = \frac{1}{2}\overline{DA} = \frac{1}{2}\cdot 6\sqrt{3} = 3\sqrt{3}$ $\therefore \overline{BE}^2 = 36 + 27$ $\therefore \overline{BE} = \sqrt{63}$

11\. (C) p (perimeter of triangle APR) = AP + PQ + RQ + RA, and PQ = PB and RQ = RC $\therefore$ p = AP + PB + RC + RA = AB + AC = 20 + 20 = 40.

12\. (A) $l = ar^{n-1}$, $r = \sqrt[3]{2} \div \sqrt{2}$ $\therefore$ the 4th term is

$$\sqrt{2}\left(\frac{\sqrt[3]{2}}{\sqrt{2}}\right)^3 = \frac{\sqrt{2}\cdot 2}{2\sqrt{2}} = 1$$

or

Since $\sqrt{2}$, $\sqrt[3]{2}$, $\sqrt[6]{2}$ can be written as $2^{3/6}$, $2^{2/6}$, $2^{1/6}$, the fourth term is $2^0 = 1$.

13\. (E) $\sqrt{t^4 + t^2} = \sqrt{t^2(t^2 + 1)} = \sqrt{t^2}\sqrt{t^2 + 1} = |t|\sqrt{1 + t^2}$

14\. (E) Let the diagonals be d and 2d. Then $\frac{1}{2}d \cdot 2d = K$ or $d^2 = K$. Since $s^2 = d^2 + \left(\frac{d}{2}\right)^2 = K + \frac{K}{4} = \frac{5K}{4}$, $s = \frac{1}{2}\sqrt{5K}$.

15\. (B) Since x men working x^2 hours produce x articles, one man, working x^2 hours, produces one article. Therefore, one man produces $\frac{1}{x^2}$ part of one article in one hour.
Let *n* be the number of articles produced by y men working y hours a day for each of y days.

Then $n = y \cdot y \cdot y \cdot \frac{1}{x^2} = \frac{y^3}{x^2}$

or

$T = \frac{A}{MDH}$ where T is the time required for 1 article, A, the number of articles, M, the number of men, D, the number of days, H, the number of hours per day. $T = \frac{x}{x\cdot x\cdot x} = \frac{1}{x^2}$

$$\therefore A' = TM'D'H' = \frac{1}{x^2}\cdot y\cdot y\cdot y = \frac{y^3}{x^2}$$

16\. (E) Let *d* be the amount taken from the base *b*. Then $\frac{1}{2}(b - d)(h + m) = \frac{1}{2}\cdot\frac{1}{2}bh$

Solving for *d*, we obtain $d = \frac{b(2m + h)}{2(h + m)}$

17\. (D) We are given that 1000 m.u. − 340 m.u. = 440 m.u. (base r)
$\therefore 1\cdot r^3 + 0\cdot r^2 + 0\cdot r + 0 - (3r^2 + 4r + 0) = 4r^2 + 4r + 0$ $\therefore r^3 - 7r^2 - 8r = 0$
$\therefore r^2 - 7r - 8 = 0$ $\therefore r = 8$

or

More simply, we have $\begin{array}{r} 440 \\ +\ 340 \\ \hline 1000 \end{array}$. Since 4 + 4 terminates in 0 in the system being used, the system has modulus 8, that is, r = 8.

18\. (A) Let P be the original population. Then,
after 1 year the population is 1.25P
after 2 years the population is (1.25)(1.25)P
after 3 years the population is (1.25)(1.25)(.75)P
after 4 years the population is

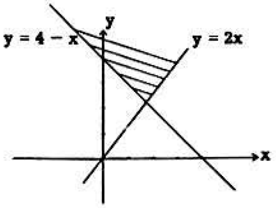
$(1.25)(1.25)(.75)(.75)P = \frac{225}{256}P$

The net percent change is $\frac{\frac{225}{256}P - P}{P} \times 100$

$$= \frac{0.88P - P}{P} \times 100 = -12.$$

19\. (B) Writing $2\log x$ as $\log x^2$, we obtain the abscissas of the intersection points by solving the equation $x^2 = 2x$. Since the possible values of x are restricted to positive numbers, there is but one root, namely 2. Hence, the graphs intersect only in (2, log 4).

20. (A) The set of points satisfying the inequalities $y > 2x$ and $y > 4 - x$ is the hatched area shown.



21. (D) Area ($\triangle$ MNE) = $\frac{1}{2}$ Area ($\triangle$ MAE)
= $\frac{1}{2} \cdot \frac{1}{3}$ Area ($\triangle$ CAE) = $\frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4}$ Area ($\triangle$ ABC)
$\therefore k = \frac{1}{24}$

22. (C) Since $3x^3 - 9x^2 + kx - 12$ is divisible by $x - 3$,
$3 \cdot 3^3 - 9 \cdot 3^2 + k \cdot 3 - 12 = 0 \quad \therefore k = 4$
Divide $3x^3 - 9x^2 + 4x - 12$ by $x - 3$ and obtain $3x^2 + 4$.

or

$3x^3 - 9x^2 + kx - 12 \equiv (x - 3)(Ax^2 + Bx + C)$
$= Ax^3 + (B - 3A)x^2 + (C - 3B)x - 3C$
$\therefore A = 3, C = 4$ and $B - 3A = -9 \quad \therefore B = 0.$
The other factor is, therefore, $3x^2 + 4$.

23. (B) Since P divides AB in the ratio 2 : 3, AP : AB = 2 : 5 and AQ : AB = 3 : 7 $\therefore \frac{AQ - AP}{AB} = \frac{1}{35}$.

Since PQ = AQ − AP = 2, AB = 70.

24. (A) Let the prices, arranged in order from left to right, be $P, P + 2, P + 4, \cdots, P + 58, P + 60$. The price of the middle book is $P + 30$. Then either $P + 30 + P + 32 = P + 60$ or $P + 30 + P + 28 = P + 60$. The first equation yields a negative value for P, an impossibility. The second equation leads to $P = 2$, so that (A) is the correct choice.

25. (A) Represent the magnitude of angle B by m. Then, in order, we obtain angle QPB = m, angle AQP = 2m, angle QAP = 2m, angle QPA = 180 − 4m, angle APC = 3m, angle ACP = 3m, and angle PAC = 180 − 6m. Since angle BAC = angle BCA, 180 − 6m + 2m = 3m $\therefore m = 25\frac{5}{7}$

26. (C) $200 = \frac{50}{2}(a + a + 49d)$ and
$2700 = \frac{50}{2}((a + 50d) + (a + 99d))$
Solve these equations simultaneously to obtain $a = -20.5$.

27. (D) The smallest value for x is 60°. $\therefore kx = 120°$ if $k = 2$ and $kx = 180°$ if $k = 3$. Since each angle of a (convex) polygon must be less than 180°, the second possibility is ruled out. It follows that there is only one permissible value for k, and, hence, only one for the pair (x, k), namely (60°, 2).

28. (D) The final digits for $(2137)^n$ are respectively 1 if $n = 0$, 7 if $n = 1$, 9 if $n = 2$, and 3 if $n = 3$. For larger values of *n* these digits repeat in cycles of four. $\therefore (2137)^{753} = (2137)^{4 \cdot 188 + 1} = (2137^4)^{188} \cdot (2137)^1$. The final digit of $(2137^4)^{188}$ is 1 and the final digit of $(2137)^1$ is 7. The product yields a final digit of $1 \cdot 7 = 7$.

29. (B) We have $r + s = -\frac{b}{a}$ and $rs = \frac{c}{a}$. The equation

with roots $ar + b$ and $as + b$ is
$(x - (ar + b))(x - (as + b)) = 0$
$\therefore x^2 - (a(r + s) + 2b)x + a^2rs + ab(r + s) + b^2 = 0$

$\therefore x^2 - \left(a\left(-\frac{b}{a}\right) + 2b\right)x + a^2\left(\frac{c}{a}\right) + ab\left(-\frac{b}{a}\right) + b^2 = 0$

$\therefore x^2 - bx + ac = 0$

30. (D) Let $\log_5 12 = x \quad \therefore 5^x = 12 \quad \therefore x \log_{10} 5 = \log_{10} 12$

$\therefore x = \frac{\log_{10} 12}{\log_{10} 5} = \frac{2 \log_{10} 2 + \log_{10} 3}{\log_{10} 10 - \log_{10} 2} = \frac{2a + b}{1 - a}$

or

Use the formula $\log_a N = \log_b N \div \log_b a$

31. (D) $\frac{PA}{AB} = \frac{PA}{PB - PA} = \frac{3m}{4m - 3m} = \frac{3}{1}$

32. (C) Area of the regular polygon = $\frac{1}{2}ap$ where

$a = R \cos \frac{180}{n}$ and $p = ns = n \cdot 2R \sin \frac{180}{n}$

$\therefore 3R^2 = \frac{1}{2} R \cos \frac{180}{n} \cdot 2nR \sin \frac{180}{n}$

$\therefore \frac{6}{n} = 2 \sin \frac{180}{n} \cos \frac{180}{n} = \sin \frac{360}{n}$ where n is a

positive integer equal to or greater than 3. Of the possible angles the only one whose sine is a rational number is 30°. $\therefore n = 12$. To check, note that $\frac{6}{12} = \frac{1}{2} = \sin \frac{360}{12} = \sin 30$

or

Area of the polygon = *n* times the area of one triangle whose vertices are the center of the circle and two consecutive vertices of the polygon

$\therefore 3R^2 = n \cdot \frac{1}{2} R^2 \sin \frac{360}{n}$ or, as before, $\frac{6}{n} = \sin \frac{360}{n}$.

33. (B) $2^{2x} - 3^{2y} = (2^x + 3^y)(2^x - 3^y) = 11.5 = 55.1$

$\therefore 2^x + 3^y = 11$	$2^x + 3^y = 55$
$2^x - 3^y = 5$	$2^x - 3^y = 1$
$2 \cdot 2^x = 16$	This system does not yield
$x = 3$	integral values for x and y.
$y = 1$	

34. (A) Let $y = \frac{2x + 3}{x + 2} \quad \therefore y = \frac{2x + 4 - 1}{x + 2} = \frac{2(x + 2) - 1}{x + 2}$

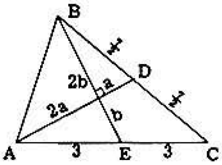
$= 2 - \frac{1}{x + 2}$. From the form $y = 2 - \frac{1}{x + 2}$ we see

that y increases as x increases for all $x \geqq 0$. Thus the smallest value of y is obtained when $x = 0$. $\therefore m = \frac{3}{2}$ and m is in S.
As x continues to increase y approaches 2 but never becomes equal to 2. $\therefore M = 2$ and M is not in S.

35. (D) $695 = a_1 + a_2(2 \cdot 1) + a_3(3 \cdot 2 \cdot 1) + a_4(4 \cdot 3 \cdot 2 \cdot 1) + a_5(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)$
$695 = a_1 + 2a_2 + 6a_3 + 24a_4 + 120a_5$ with $0 \leqq a_k \leqq k$.
Since a_5 must equal 5 (in order to obtain 695), $a_4 \neq 4$ because $5 \cdot 120 + 4 \cdot 24 > 695$. Also a_4

cannot be less than 3, since, for $a_4 = 2$, we have $2 \cdot 24 + 3 \cdot 6 + 2 \cdot 2 + 1 < 95$. $\therefore a_4 = 3$
Check: $5 \cdot 120 + 3 \cdot 24 + 3 \cdot 6 + 2 \cdot 2 + 1 = 695$

36. (B) From the diagram we have $4a^2 + b^2 = 9$ and $a^2 + 4b^2 = \frac{49}{4}$ $\therefore 5a^2 + 5b^2 = \frac{85}{4}$ $\therefore a^2 + b^2 = \frac{17}{4}$
$\overline{AB}^2 = 4a^2 + 4b^2 = 17$ $\therefore \overline{AB} = \sqrt{17}$



37. (C) Let v_A, v_B, v_C be the respective speeds of A, B, C.

$\therefore$ (1) $\frac{d}{v_A} = \frac{d-20}{v_B}$ (2) $\frac{d}{v_B} = \frac{d-10}{v_C}$

(3) $\frac{d}{v_A} = \frac{d-28}{v_C}$ $\therefore$ (4) $\frac{d-20}{v_B} = \frac{d-28}{v_C}$. Divide

equation (4) by equation (2) and obtain

$\frac{d-20}{d} = \frac{d-28}{d-10}$ $\therefore d = 100$

38. (A) Let $AC = h$ and $BC = l$. Then $h^2 + l^2 = 4r^2$
Since $s = h + l$, $s^2 = h^2 + l^2 + 2hl = 4r^2 + 2hl$ $= 4r^2 + 4$ times the area of $\triangle ABC$. The maximum area of $\triangle ABC$ occurs when $h = l = r\sqrt{2}$. $\therefore s^2 \leqq 4r^2 + 4(r^2) = 8r^2$.

39. (B) The greatest distance between two points P_1 and P_2 is the length of a diagonal, $\sqrt{2}$. Place P_1, P_2, P_3, P_4 at the four corners. For the fifth point P_5 to be as far as possible from the other four points, it must be located at the center of the square. The distance from P_5 to any of the other four points is $\frac{1}{2}\sqrt{2}$.
Location of the points inside the square will yield distances between pairs of points, the smallest of which is less than $\frac{1}{2}\sqrt{2}$.

40. (A) The minimum value of $\sqrt{x^2 + y^2}$ is given by the perpendicular OC from O to the line $5x + 12y = 60$.

From similar triangles $\frac{OC}{5} = \frac{12}{13}$ $\therefore OC = \frac{60}{13}$.

SOLUTION KEY
AND ANSWER KEY

THIRTEENTH ANNUAL H. S.

MATHEMATICS CONTEST

1962

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USE OF KEY

1. This Key is for the exclusive use of teachers.
2. Some of the solutions are intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This solution Key validates our statement that no mathematics beyond intermediate algebra is needed to solve the problems posed.

National Office Pan American College, Edinburg, Texas
New York Office Polytechnic Institute of Brooklyn, Brooklyn 1, N. Y.

Note: The letter following the problem number refers to the correct choice of the five listed in the examination.

1. (D) The numerator becomes 1, the denominator becomes $\frac{1}{5} + \frac{1}{3} = \frac{8}{15}$, and $1 \div \frac{8}{15} = \frac{15}{8}$

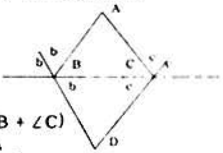
2. (A) $\sqrt{\frac{4}{3}} - \sqrt{\frac{3}{4}} = \frac{2\sqrt{3}}{3} - \frac{\sqrt{3}}{2} = \frac{4\sqrt{3} - 3\sqrt{3}}{6} = \frac{\sqrt{3}}{6}$

3. (B) Let $\underline{d}$ be the common difference
$d = (x + 1) - (x - 1) = 2$, and
$d = (2x + 3) - (x + 1) = x + 2$
$\therefore x + 2 = 2 \quad \therefore x = 0$
or
$2(x + 1) = (x - 1) + (2x + 3)$
$2x + 2 = 3x + 2$
$x = 0$

4. (B) $8^x = 32$, $(2^3)^x = 2^5$, $2^{3x} = 2^5$
$\therefore 3x = 5$. $x = \frac{5}{3}$

5. (C) The ratio of the circumference of any circle to its diameter is a constant represented by the symbol π.

6. (D) $A\,(\text{triangle}) = 9\sqrt{3} = \frac{s^2\sqrt{3}}{4} \quad \therefore s = 6$
$P\,(\text{triangle}) = 18 = P\,(\text{square})$
$\therefore$ side of the square, $x = \frac{9}{2}$ $\therefore$ diagonal of the square $= \sqrt{2x^2} = \frac{9\sqrt{2}}{2}$



7. (C)
$2\angle b = 180° - \angle B$
$2\angle c = 180° - \angle C$
$\therefore \angle b + \angle c = 180° - \frac{1}{2}(\angle B + \angle C)$
But $\angle B + \angle C = 180° - \angle A$
$\therefore \angle b + \angle c = 180° - \frac{1}{2}(180° - \angle A)$
$\angle BDC = 180° - (\angle b + \angle c)$
$= 180° - [180° - \frac{1}{2}(180° - \angle A)]$
$= \frac{1}{2}(180° - \angle A)$

8. (D) Let s be the sum and $M\left(= \frac{s}{n}\right)$ be the arithmetic mean of the given numbers. Since the number of 1's is $n - 1$, $s = (n - 1)(1) + 1 - \frac{1}{n} = n - \frac{1}{n}$
$\therefore M = \frac{n - \frac{1}{n}}{n} = 1 - \frac{1}{n^2}$

9. (B) $x^9 - x = x(x^8 - 1) = x(x^4 + 1)(x^4 - 1)$
$= x(x^4 + 1)(x^2 + 1)(x + 1)(x - 1)$

10. (A) Let R be the average rate for the trip going
Then $R = \frac{150}{3\frac{1}{3}} = 45$ (miles per hour)
$r = \frac{300}{7\frac{1}{2}} = 40$ (miles per hour)
$R - r = 45 - 40 = 5$ (miles per hour)

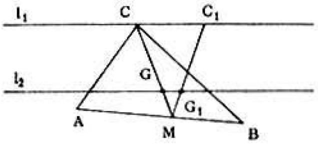
11. (B) The roots are $\frac{p + 1}{2}$ and $\frac{p - 1}{2}$. The difference between the larger and the smaller root is 1.

12. (C) The last three terms of the expansion of $\left(1 - \frac{1}{a}\right)^6$ correspond in inverse order to the first three terms in the expansion of $\left(-\frac{1}{a} + 1\right)^6$. These are $1 - 6 \cdot \frac{1}{a} + 15\left(-\frac{1}{a}\right)^2$, and the sum of these coefficients is $1 - 6 + 15 = 10$
or
Expanding completely we have $1 - \frac{6}{a} + \frac{15}{a^2} - \frac{20}{a^3} + \frac{15}{a^4} - \frac{6}{a^5} + \frac{1}{a^6}$; $15 - 6 + 1 = 10$

13. (B) The relation between R, S, and T can be represented in several ways, to wit,
(1) $R = k\frac{S}{T}$, k a constant (2) $\frac{RT}{S} = k$, k a constant
(3) $\frac{R_1 T_1}{S_1} = \frac{R_2 T_2}{S_2}$
Using form 3 we have $\frac{\frac{4}{3} \cdot \frac{9}{14}}{\frac{3}{7}} = \frac{\sqrt{48} \cdot \sqrt{75}}{S_2} \quad \therefore S_2 = 30$

14. (B) Since $1 + a + a^2 + \ldots = \frac{1}{1 - a}$ when $|a| < 1$, the required limiting sum equals $\frac{4}{1 - (-\frac{2}{3})} = 2.4$

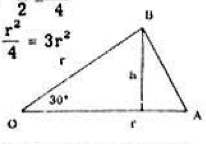
15. (D) Let the vertex C move along the straight line l_1 to position C_1. Let G be the location of the centroid (the intersection point of the three medians) of ΔCAB; let G_1 be the location of the centroid of ΔC_1AB.
Since $\overline{CG} = \frac{2}{3}\,\overline{CM}$ and $\overline{C_1G_1} = \frac{2}{3}\,\overline{C_1M_1}$ the line GG_1 is parallel to line CC_1. (If a line divides two sides of a triangle proportionally, it is parallel to the third side.) Therefore, as C moves along l_1, G moves along l_2 which is parallel to l_1.



16. (C) The sides of R_1 are 2 inches and 6 inches and the square of its diagonal, d^2, equals $6^2 + 2^2 (=40)$ Since the rectangles are similar, we have
$\frac{\text{area } R_2}{\text{area } R_1} = \frac{D^2}{d^2} \quad \therefore \text{area } R_2 = \frac{225}{40} \cdot 12 = \frac{135}{2}$

17. (B) Since $a = \log_8 225$, $8^a = 225$, $2^{3a} = 15^2$
$b = \log_2 15$, $2^b = 15$, $2^{2b} = 15^2$
$\therefore 2^{3a} = 2^{2b}$, $3a = 2b$, $a = \frac{2b}{3}$

18. (A) The dodecagon can be decomposed into 12 congruent triangles like triangle OAB shown. (AB is a side of the polygon).
area $(\Delta OAB) = \frac{1}{2} rh = \frac{1}{2} r \cdot \frac{r}{2} = \frac{r^2}{4}$
$\therefore$ area (dodecagon) $= 12 \cdot \frac{r^2}{4} = 3r^2$



19. (C) By substituting into the given equation the three sets of x,y-values given, we obtain three equations in a, b, and c, namely, $a - b + c = 12$,

$c = 5$, and $4a + 2b + c = -3$. Solve the first and third equations for a and b, using for c the value 5, from the second equation. The values obtained are $a = 1$, $b = -6$, $c = 5$, and the sum of these is zero.

20. (A) Let the degree measurement of the angles be represented by $a - 2d$, $a - d$, a, $a + d$, $a + 2d$. Then $5a = 540^\circ$ and $a = 108^\circ$

21. (E) Since the given equation has coefficients which are real numbers, the second root must be $3 - 2i$. Therefore, the product of the roots,

$$\frac{S}{2} = (3 + 2i)(3 - 2i) \quad \therefore S = 26$$

or

Substitute $3 + 2i$ for x in the given equation.

$2(3 + 2i)^2 + r(3 + 2i) + S = 0$

$10 + 3r + S + i(24 + 2r) = 0 + 0 \cdot i$

$\therefore 24 + 2r = 0$, $r = -12$ and $10 + 3r + S = 0$, $S = 26$

22. (D) $121_b = 1 \cdot b^2 + 2 \cdot b + 1 = b^2 + 2b + 1 = (b + 1)^2$

The expression $(b + 1)^2$ is, of course, a square number for any value of b. Since, however the largest possible digit appearing in a number written in the integral base b is $b - 1$, the possible values for b do not include 1 and 2. Therefore, $b > 2$ is the correct answer.

23. (E) When angles A and B are acute and angle C is either acute, obtuse, or right, triangles ABE and CBD are similar. Since $\overline{AB}$, $\overline{CD}$, and $\overline{AE}$ are known, $\overline{CB}$ can be found. Now, by applying the Pythagorean theorem to triangle CBD, where $\overline{CB}$ and $\overline{CD}$ are known, we can find $\overline{DB}$.

When angle A is obtuse the same analysis holds.

When angle B is obtuse, triangles ABE and ADC are similar. From this fact and the given lengths, $\overline{CA}$ can be found. Next $\overline{AD}$ can be found with the aid of the Pythagorean theorem. Finally, $DB = \overline{AD} - \overline{AB}$.

When either angle A or angle B is right, the problem is trivial. In the former case $\overline{DB} = \overline{AB}$; in the latter, $\overline{DB} = 0$.

24. (A) Since $\frac{1}{x}$ represents the fractional part of the job done in 1 hour when the three machines operate together, and so forth,

$$\frac{1}{x+6} + \frac{1}{x+1} + \frac{1}{x+x} = \frac{1}{x} \text{ or, more simply,}$$

$$\frac{1}{x+6} + \frac{1}{x+1} - \frac{1}{2x} = 0$$

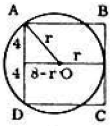
$\therefore 2x(x + 1) + 2x(x + 6) - (x + 6)(x + 1) = 0$

$\therefore 3x^2 + 7x - 6 = 0$, $x = \frac{2}{3}$

25. (C) $r^2 = 4^2 + (8 - r)^2$

$16r = 80$

$r = 5$



26. (E) Transform the expression $8x - 3x^2$ into

$$3\left(\frac{8}{3}x - x^2\right) = 3\left(\frac{16}{9} - \frac{16}{9} + \frac{8}{3}x - x^2\right)$$

$$= 3\left[\frac{16}{9} - \left(x - \frac{4}{3}\right)^2\right] = \frac{16}{3} - 3\left(x - \frac{4}{3}\right)^2.$$

The maximum value of this last expression occurs when the term $-3\left(x - \frac{4}{3}\right)^2$ is zero.

Therefore, the maximum value of $8x - 3x^2$ is $\frac{16}{3}$

or

Let $y = -3x^2 + 8x$. When graphed this equation represents a parabola with a highest point at $(\frac{4}{3}, \frac{16}{3})$. Therefore, the maximum value of $-3x^2 + 8x$ is $\frac{16}{3}$.

27. (E) The correctness of rule (1) is obvious. To establish rule (2) take three numbers n_1, n_2, n_3 such that $n_1 > n_2 > n_3$. First select the larger of n_2 and n_3, which is n_2; then select the larger of n_1 and n_2, which is n_1. We thus obtain n_1 from the left side of the rule. From the right side of the rule we also obtain n_1 as follows: First select the larger of n_1 and n_2, which is n_1, then select the larger of n_1 and n_3, which is n_1.

For rule 3 we proceed as follows: For the left side we first select the larger of n_2 and n_3 (this is n_2), then select the smaller of n_1 and n_2 (this is again n_2). For the right side, select the smaller of n_1 and n_2 (this is n_2) then select the smaller of n_1 and n_3 (this is n_3), and, finally, select the larger of n_2 and n_3 (this is n_2).

28. (D) Take logarithms of both sides to the base 10.

$(\log_{10}x)(\log_{10}x) = 3\log_{10}x - \log_{10}100$ or

$(\log_{10}x)^2 - 3\log_{10}x + 2 = 0$

Solve this equation as a quadratic equation letting $y = \log_{10}x$.

$y^2 - 3y + 2 = 0$, $y = 2$ or 1 $\therefore \log_{10}x = 2$ or 1

$\therefore x = 10^2 = 100$ or $x = 10^1 = 10$

29. (A) $2x^2 + x < 6$, $x^2 + \frac{x}{2} < 3$, $x^2 + \frac{x}{2} + \frac{1}{16} < 3 + \frac{1}{16}$,

$\left(x + \frac{1}{4}\right)^2 < \frac{49}{16}$, $\left|x + \frac{1}{4}\right| < \frac{7}{4}$, that is, $x + \frac{1}{4} < \frac{7}{4}$ and

$x + \frac{1}{4} > -\frac{7}{4}$ $\therefore x < \frac{3}{2}$ and $x > -2$.

30. (D) Form I. Since $\sim(p \wedge q) = \sim p \vee \sim q$, statements (2), (3), and (4) are each correct.

Form II. The negation of the statement "p and q are both true" means that either p is true and q is false, or p is false and q is true, or p is false and q is false. Therefore, the statements (2), (3), and (4) are each correct.

31. (C) Let N and n, respectively, represent the numbers of the sides of the two polygons. Then

$$\frac{(N - 2)\,130}{N} = \frac{3}{2}\,\frac{(n - 2)\,180}{n} \quad \therefore N = \frac{4n}{6 - n}$$

To find N we need try for n only the values 3, 4, 5 (Why?) We thus obtain $n = 3$, $N = 4$; $n = 4$, $N = 8$; $n = 5$, $N = 20$, three pairs in all.

32. (E) Since $x_{k+1} = x_k + \frac{1}{2}$ and $x_1 = 1$, $x_2 = x_1 + \frac{1}{2} = \frac{3}{2}$,

$x_3 = x_2 + \frac{1}{2} = \frac{4}{2}$, $x_4 = x_3 + \frac{1}{2} = \frac{5}{2}$, and so forth with

$x_n = \frac{n+1}{2}$. The required sum is, therefore,

$$\frac{2}{2} + \frac{3}{2} + \frac{4}{2} + \ldots + \frac{n+1}{2} = \frac{1}{2}n\left(\frac{2}{2} + \frac{n+1}{2}\right)$$

$$= \frac{1}{2}n\left(\frac{n+3}{2}\right) = \frac{n^2+3n}{4}$$

33. (A) The inequality $2 \leq |x - 1| \leq 5$ means that, when $x - 1$ is positive, then $x - 1 \leq 5$ and $x - 1 \geq 2$, and that, when $x - 1$ is negative, then $x - 1 \geq -5$ and $x - 1 \leq -2$. Solve each of these four inequalities.

$x \leq 6$ and $x \geq 3$ or $3 \leq x \leq 6$, and
$x \geq 4$ and $x \leq -1$ or $-4 \leq x \leq -1$

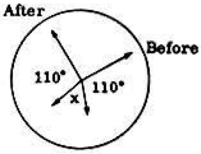
34. (E) Since $x = K^2(x^2 - 3x + 2)$, $K^2x^2 - x(3K^2 + 1) + 2K^2 = 0$.

For x to be a real number the discriminant must be greater than or equal to zero

$\therefore K^4 + 6K^2 + 1 \geq 0$. This is true for all values of K.

35. (B) During the interval the large hand has moved $220 + x$ degrees while the small hand has moved x degrees.

$\therefore 220 + x = 12x$, $x = 20$
$\therefore$ the interval equals $\frac{20}{30} \times 60 = 40$ (minutes)



36. (C) $(x - 8)(x - 10) = 2^y$, $x^2 - 18x + 80 - 2^y = 0$

$$\therefore x = \frac{18 \pm \sqrt{(18)^2 - 4(80 - 2^y)}}{2} = 9 \pm \sqrt{1 + 2^y}$$

For x to be an integer, $1 + 2^y$ must be the square of an integer. This is true only for $y = 3$, and for $y = 3$, $x = 12$ or 6.

There are, therefore, two solutions, namely, (12, 3) and (6, 3).

37. (D) Let x be the common length of AE and AF

$$\text{Area (EGC)} = \frac{1}{2}(1 - x)(1)$$

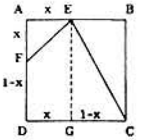
$$\text{Area (DFEG)} = x\left[\frac{(1 - x) + 1}{2}\right] = x\left(1 - \frac{x}{2}\right)$$

$$\therefore \text{Area (CDFE)} = \frac{1}{2}(1 + x - x^2)$$

$$= \frac{1}{2}\left[\frac{5}{4} - \left(x^2 - x + \frac{1}{4}\right)\right]$$

$$= \frac{5}{8} - \frac{1}{2}\left(x - \frac{1}{2}\right)^2$$

The maximum value of this expression is $\frac{5}{8}$. See problem 26.



38. (B) Let N be the original population. Then $N = x^2$, $N + 100 = y^2 + 1$, and $N + 200 = Z^2$.

Subtract the first equation from the second to obtain $100 = y^2 - x^2 + 1$ or $y^2 - x^2 = 99$
$\therefore (y + x)(y - x) = 99 \cdot 1$ or $33 \cdot 3$ or $11 \cdot 9$.
Try $y + x = 99$ and $y - x = 1$; to satisfy these equations $y = 50$, $x = 49$. $\therefore N = 49^2 = 2401$, a multiple of 7. Furthermore, $N + 100 = 2501 = 50^2 + 1$ and $N + 200 = 2601 = 51^2$.
With either of the other two sets of factors you obtain a value of N which satisfies the first and second conditions of the problem, but not the third.

39. (C) Figure ADBG is a parallelogram, so that $AD = GB = 4$ and triangle ADM ≅ triangle BGM
Area △ ADG = area △ AMG + area △ MGB

$$= \text{area } \triangle ABG = \frac{1}{3} \text{ area } \triangle ABC$$

$$\therefore \frac{1}{3} \cdot 3\sqrt{15} = \sqrt{(3 + x)(3 - x)(x + 1)(x - 1)},$$

$15 = (9 - x^2)(x^2 - 1)$, $x^4 - 10x^2 + 24 = 0$
$x^2 = 4$ or 6, $x = 2$ or $\sqrt{6}$. The value $x = 2$ makes $CM = 6$ and must be rejected since the given triangle is scalene. $\therefore CM = 3\sqrt{6}$.

40. (B) Write the given series as:

$$\frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \ldots$$
$$+ \frac{1}{10^2} + \frac{1}{10^3} + \frac{1}{10^4} + \ldots$$
$$+ \frac{1}{10^3} + \frac{1}{10^4} + \ldots \text{ and so forth.}$$

Let s_1 be the limiting sum of the first-row series, s_2, that of the second-row series, and so forth.

$$s_1 = \frac{\frac{1}{10}}{1 - \frac{1}{10}} = \frac{1}{9},\ s_2 = \frac{\frac{1}{10^2}}{1 - \frac{1}{10}} = \frac{1}{90},\ s_3 = \frac{\frac{1}{10^3}}{1 - \frac{1}{10}} = \frac{1}{900},$$

and so forth.

Therefore, the required limiting sum equals

$$\frac{1}{9} + \frac{1}{90} + \frac{1}{900} + \ldots = \frac{\frac{1}{9}}{1 - \frac{1}{10}} = \frac{10}{81}$$

SOLUTION KEY
AND ANSWER KEY

FOURTEENTH ANNUAL H. S.
MATHEMATICS CONTEST

1963

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USE OF KEY

1. This Key is for the exclusive use of teachers.
2. Some of the solutions are intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This solution Key validates our statement that no mathematics beyond intermediate algebra is needed to solve the problems posed.

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Note: The letter following the problem number refers to the correct choice of the five listed in the examination.

1. (D) Since $\frac{x}{x+1}$ is not defined when the denominator is zero, and since the denominator $x + 1$ equals zero when $x = -1$, any point whose abscissa is -1, such as $(-1,1)$ can not be on the graph.

2. (A) $n = 2-(-2)^{2-(-2)} = 2-(-2)^4 = 2- 16 = -14$

3. (E) Since $\frac{1}{x+1} = x-1$, $x^2-1 = 1$, $x^2 = 2$, $x = +\sqrt{2}$ or $-\sqrt{2}$

4. (D) For $x^2 = 3x + k$ to have two equal roots, the discriminant $9 + 4k$ must equal zero. $\therefore k = -\frac{9}{4}$.

5. (E) Choices (A), (B), and (D) must be rejected since the logarithm of a negative real number is not defined (on this level of mathematical study). Choice (C) must be rejected since $\log 1 = 0$. Choice (E) is the correct one since $\log x$ for $0 < x < 1$ exists, and is a negative real number.

or

Since it is given that $\log_{10} x < 0$, then $x < 10^0 (=1)$. This result, together with the fact that (on this level) $\log x$ is defined only for $x > 0$, leads to choice (E).

6. (B) BC is the median to the hypotenuse AD, and is, therefore, equal to one-half the hypotenuse. $\therefore \overline{AB} = \overline{BC} = \overline{AC}$, and, consequently, the magnitude of angle DAB is 60°.

7. (A) Two lines are perpendicular when the product of their slopes is -1. Since the slopes are, respectively, $\frac{2}{3}$, $-\frac{2}{3}$, $-\frac{2}{3}$, $-\frac{3}{2}$, and, since $(\frac{2}{3})(-\frac{3}{2}) = -1$, lines (1) and (4) are perpendicular.

or

Two lines, $a_1x + b_1y = c_1$ and $a_2x + b_2y = c_2$, are perpendicular if $a_1a_2 + b_1b_2 = 0$. In (1) $a_1 = -2$, $b_1 = 3$ and in (4) $a_2 = 3$, $b_2 = 2$, so that $a_1a_2 + b_1b_2 = -6 + 6 = 0$.

8. (D) $1260x = (2 \cdot 2 \cdot 3 \cdot 3 \cdot 5 \cdot 7)x$. For the smallest integer cube the exponent of each different prime factor must be 3. Therefore, $x = 2 \cdot 3 \cdot 5^2 \cdot 7^2 = 7350$.

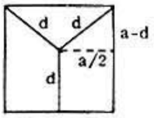
9. (C) $\left(a - \frac{1}{a^{1/2}}\right)^7 = a^7 - 7a^{11/2} + 21a^4 - 21a^{5/2} + 35a - 21a^{-1/2} + 7a^{-2} - a^{-7/2}$

or

$(r+1)$th term $= \binom{7}{r}a^{7-r}\left(-a^{-1/2}\right)^r = \pm\binom{7}{r}a^{7-r-r/2}$. We need $a^{7-r-r/2} = a^{-1/2}$, $7 - \frac{3r}{2} = -\frac{1}{2}$, $r = 5$

$\therefore$ 6th term $= \frac{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5} a^2(-a)^{-5/2} = -21a^{-1/2}$.

10. (B) $\left(\frac{a}{2}\right)^2 + (a-d)^2 = d^2$, $d = 5a/8$



11. (B) Since the arithmetic mean of the 50 numbers is 38, the total of the 50 numbers is 50 x 38 = 1900. Removing the two numbers 45 and 55 leaves 48 numbers with a total of 1800. Therefore, the new arithmetic mean $= \frac{1800}{48} = 37.5$

or

$a_1 + a_2 + \cdots + a_{48} + 100 = 50 \cdot 38 = 50(36 + 2) = 50 \cdot 36 + 50 \cdot 2$.

$\therefore a_1 + a_2 + \cdots + a_{48} = 50 \cdot 36 = 48$ A.M.(new). $\therefore$ A.M.(new) $= \frac{50 \cdot 36}{48} = 37.5$

12. (E) The sum of the x-coordinates of P and R equals the sum of the x-coordinates of Q and S, and, similarly, for the y-coordinates. $\therefore x + 1 = 6$, $x = 5$ and $y - 5 = -1$, $y = 4$ $\therefore x + y = 9$

or

Let S_x be the x-coordinate of S and S_y, the y-coordinate of S, and, similarly, for P, Q, and R. Then, by congruent triangles, $Q_x - R_x = P_x - S_x$, $1 - 9 = -3 - S_x$, $S_x = 5$. Similarly $S_y = 4$. $\therefore S_x + S_y = 9$.

13. (E) Suppose one of a, b, c, d is a negative integer, say d; then, we have $2^a + 2^b - 3^c = 3^{-d}$, an impossibility since the left side is an integer while the right side is a non-integer. Similar reasoning shows that no two, and no three, of the exponents can be negative integers.

If all four are negative integers, then we may write $\frac{1}{2^a} + \frac{1}{2^b} = \frac{1}{3^c} + \frac{1}{3^d}$, with a, b, c, d positive integers. Then $\frac{2^b + 2^a}{2^a \cdot 2^b} = \frac{3^d + 3^c}{3^c \cdot 3^d}$

If $a < b$, divide the numerator and the denominator of the left fraction by 2^a, and obtain $\frac{2^{b-a} + 1 (= N_1)}{2^b (= D_1)} = \frac{3^c + 3^d (= N_2)}{3^c \cdot 3^d (= D_2)}$, or, equivalently, $N_1D_2 = N_2D_1$. But N_1 is odd, D_2 is odd, N_1D_2 is odd, N_2 is even, D_1 is even, N_2D_1 is even—a contradiction.

If $b < a$, use instead the divisor 2, and prove that the same contradiction arises.

If $b = a$, the left fraction can be reduced to $\frac{1}{2^{a-1}}$, and, again, the same contradiction arises.

14. (A) Let the roots of the first equation be r and s. Then $r + s = -k$ and $rs = 6$. Then, for the second equation, $r + 5 + s + 5 = k$ and $(r + 5)(s + 5) = 6$. $\therefore rs + 5(r + s) + 25 = 6$, $6 + 5(-k) + 25 = 6$, $k = 5$.

or

Let r be one of the roots of the first equation; then $r + 5$ is the associated root of the second equation. $\therefore (r + 5)^2 - k(r + 5) + 6 = 0$, $r^2 + (10 - k)r + 31 - 5k = 0$. But r satisfies the first equation so that $r^2 + kr + 6 = 0$. $\therefore 10 - k = k$ and $31 - 5k = 6$. Each of these equations leads to the result $k = 5$.

15. (C) Let S, s, r, respectively, represent the magnitudes of a side of the equilateral triangle, a side of the square, and a radius of the circle. Since $S = 2r\sqrt{3}$ and $s = r\sqrt{2}$, the area of the triangle is $\frac{(2r\sqrt{3})^2\sqrt{3}}{4}$ $(= 3\sqrt{3}r^2)$ and the area of the square is $(r\sqrt{2})^2$ $(= 2r^2)$. The required ratio is, therefore, $3\sqrt{3} : 2$.

16. (C) Since a, b, c are in A.P., $2b = a + c$. Since $a + 1$, b, c are in G.P., $b^2 = c(a + 1)$. Similarly, $b^2 = a(c + 2)$. $\therefore c = 2a$, $a = \frac{2}{3}b$, $b^2 = \frac{8}{9}b^2 + \frac{4}{3}b$. $\therefore b = 12$.

17. (A) To avoid division by zero, $y \neq a$ and $y \neq -a$. With these values excluded we may simplify the fraction by multiplying numerator and denominator by $(a + y)(a - y)$, obtaining

$$\frac{a^2 - ay + ay + y^2}{ay - y^2 - a^2 - ay} = \frac{a^2 + y^2}{-(a^2 + y^2)} = -1$$

18. (A) Triangle EFA is a right triangle (since EF is a diameter) with angle E as one of its acute angles. Triangle EUM is a right triangle with angle E as one of its acute angles. Therefore, triangle EFA $\sim$ triangle EMU.

19. (B) $\frac{49 + \frac{7}{8}(n - 50)}{n} \geqq \frac{9}{10}$. $\therefore n \leqq 210$.

20. (D) Let h = number of hours in which they meet. Then $\frac{9}{2}h + \frac{1}{2}h\left[2 \cdot \frac{13}{4} + (h - 1)\frac{1}{2}\right] = 76$, $h^2 + 30h - 304 = 0$, $h = 8$. The meeting point is, therefore, 36 miles from R and 40 miles from S, so that $x = 4$.

21. (E) $x^2 - y^2 - z^2 + 2yz + x + y - z = x^2 - (y^2 - 2yz + z^2) + x + y - z = x^2 - (y - z)^2 + x + y - z$
$= (x + y - z)(x - y + z) + x + y - z = (x + y - z)(x - y + z + 1)$

22. (D) The magnitude of angle BOE is 156°. Therefore, the magnitude of angle OBE $= \frac{1}{2}(180° - 156°) = 12°$. Since the magnitude of angle BAC = 36°, the required ratio is $\frac{12}{36}$ or $\frac{1}{3}$.

23. (B) If a, b, c, respectively, represent the initial amounts of A, B, C, then the given conditions lead to the following:

After Transaction	A has	B has	C has
I	$a - b - c$	$2b$	$2c$
II	$2(a - b - c)$	$2b - (a - b - c) - 2c$ $(= 3b - a - c)$	$4c$
III	$4(a - b - c)$	$2(3b - a - c)$	$4c - 2(a - b - c) - (3b - a - c)$ $(= 7c - a - b)$

Consequently $4(a - b - c) = 16$, $6b - 2a - 2c = 16$, $7c - a - b = 16$. Solving this set of equations, you obtain $a = 26$, $b = 14$, $c = 8$

or

Working from the last condition to the first, we may set up the following table:

Amounts	Step 4	Step 3	Step 2	Step 1
a	16	8	4	26 (required)
b	16	8	28	14
c	16	32	16	8

24. (B) For real roots $b^2 - 4ac \geqq 0$, or $b^2 \geqq 4ac$. Tabulating the possibilities, we have 19 in all as follows:

or

When c is selected to be:	6	6	5	5	4	4	4	3	3	3	2	2	2	2	1	1	1	1	1
The possible values for b are:	5	6	5	6	4	5	6	4	5	6	3	4	5	6	2	3	4	5	6
The no. of possible values for b are:	2		2		3			3			4				5				

25. (A) $\Delta CDF \backsim \Delta CBE$ ($\underline{CD} = \underline{CB}$, angle DCF = angle BCE), $\therefore CF = CE$.
Area $(\Delta CEF) = \frac{1}{2}\,\overline{CE} \cdot \overline{CF} = \frac{1}{2}\,\overline{CE}^2 = 200$. $\therefore \overline{CE}^2 = 400$.
Area (square) $= \overline{CB}^2 = 256$. Since $\overline{BE}^2 = \overline{CE}^2 - \overline{CB}^2 = 400 - 256 = 144$, $\overline{CE} = 12$.

26. (E) The implication $(p \rightarrow q) \rightarrow r$ is true when (i) the consequent, r, is true and the antecedent, $p \rightarrow q$, is either true or false, and (ii) the consequent is false and the antecedent is false.

The implication $p \rightarrow q$ is true when (i) the consequent, q, is true and the antecedent, p, is either true or false, and (ii) the consequent is false and the antecedent is false.
In (1) $p \rightarrow q$ is false and r is true so that $(p \rightarrow q) \rightarrow r$ is true. In (2) $p \rightarrow q$ is true and r is true, so that $(p \rightarrow q) \rightarrow r$ is true. In (3) $p \rightarrow q$ is false and r is false so that $(p \rightarrow q) \rightarrow r$ is true. In (4) $p \rightarrow q$ is true and r is true, so that $(p \rightarrow q) \rightarrow r$ is true.

27. (C) Let l be the number of lines and r, the number of regions. It is not too difficult to discover the rule that $r = \frac{1}{2}l(l + 1) + 1$. For $l = 6$, $r = \frac{1}{2} \cdot 6 \cdot 7 + 1 = 22$. Hint: (1) for one line there are two regions, that is, one more than the number of lines; (2) note what happens to the regions when a line is added.

28. (D) The product of two numbers whose sum is a fixed quantity is maximized when each of the numbers is one-half the sum. Since the sum of the roots is $\frac{4}{3}$, the maximum product, $\frac{4}{9}$, is obtained when each of the roots is $\frac{2}{3}$. Therefore, $k/3 = \frac{4}{9}$, so that $k = \frac{4}{3}$.

or

Solving the given equation for k/3 (the product of the roots), we have $k/3 = \frac{4}{3}x - x^2 = \frac{4}{9} - (\frac{2}{3} - x)^2$. The right side of this equation is a maximum when $x = \frac{2}{3}$, so that k/3 is a maximum when $x = \frac{2}{3}$.
$\therefore k/3\ (\max) = \frac{4}{3} \cdot \frac{2}{3} - \frac{4}{9} = \frac{4}{9}$. $\therefore k(\max) = \frac{4}{3}$

or

The product of the roots is k/3. We seek the largest possible $\underline{k}$ consistent with real roots. For real roots $16 - 12\,k \geqq 0$, so that $\frac{4}{3} \geqq k$. Hence, the desired $k = \frac{4}{3}$.

29. (C) The abscissa of the maximum (or minimum) point on the parabola $ax^2 + bx + c$ is $-b/2a$. For the given parabola $160t - 16t^2$, $-b/2a = -160/-32 = 5$, and the value of s when $t = 5$ is 400.

or

Since $s = 160t - 16t^2$, $s = 400 - 400 + 160t - 16t^2 = 400 - 16(5 - t)^2$. The right side of this equation is a maximum when $5 - t = 0$. Therefore, $s(\max) = 400$.

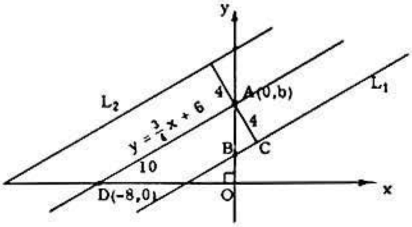
30. (C) Since $\dfrac{1 + \dfrac{3x + x^3}{1 + 3x^2}}{1 - \dfrac{3x + x^3}{1 + 3x^2}} = \dfrac{1 + 3x + 3x^2 + x^3}{1 - 3x + 3x^2 - x^3} = \dfrac{(1 + x)^3}{(1 - x)^3} = \left(\dfrac{1 + x}{1 - x}\right)^3$, we have

$$F(\text{new}) = \log\left(\frac{1 + x}{1 - x}\right)^3 = 3 \log \frac{1 + x}{1 - x} = 3F(\text{original})$$

31. (D) Solving $2x + 3y = 763$ for x, we have $x = \dfrac{763 - 3y}{2}$. Since x is a positive integer, $763 - 3y$ must be a positive even number, so that y must be a positive odd integer, such that $3y \leqq 763$. There are 254 multiples of 3 less than 763, half of which are even multiples and half, odd multiples. Therefore, there are 127 possible solutions to the given equation under the stated conditions.

32. (A) From the given conditions we have (1) $3(x + y) = a + b$ and (2) $3\,x\,y = ab$. Divide equation (2) by equation (1): $\frac{1}{y} + \frac{1}{x} = \frac{1}{b} + \frac{1}{a}$. This is impossible since both x and y are less than $\underline{a}$ and less than $\underline{b}$.

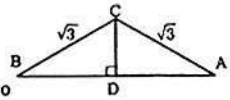
33. (A) There are two possibilities, L_1 and L_2. The methods for finding the equation of L_2 are similar to the methods for finding the equation of L_1 shown here: Since L_1 is parallel to the line $y = \frac{3}{4}x + 6$, its equation is $y = \frac{3}{4}x + b$, where b, the y-intercept, is to be determined. Since $\Delta ABC \sim \Delta DAO$, $\frac{\overline{AB}}{\overline{AD}} = \frac{\overline{AC}}{\overline{DO}}$, so that $\frac{\overline{AB}}{10} = \frac{4}{8}$ and $\overline{AB} = 5$. Hence, $b = 1$ and the equation of L_1 is $y = \frac{3}{4}x + 1$.



or

Let d_1 be the distance from the origin to L_1 and let d_2 be the distance from the origin to the given line. Then $d_1 = \frac{4b}{5}$ and $d_2 = \frac{24}{5}$. $\therefore d_2 - d_1 = \frac{24}{5} - \frac{4b}{5} = 4$, and $b = 1$.

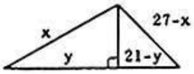
34. (B) Consider the triangle ABC where $a = \sqrt{3}$, $b = \sqrt{3}$, and $c = 3$, and CD is the altitude to AB. $\therefore \overline{BD} = 1\frac{1}{2}$ and $\overline{AD} = 1\frac{1}{2}$. Then $\overline{CD}^2 = 3 - (\frac{3}{2})^2 = \frac{3}{4}$, and $\overline{CD} = \frac{\sqrt{3}}{2}$. $\therefore \angle B = \angle A = 30^\circ$ and $\angle C = 120^\circ$. Since side c is given greater than 3, $\angle C$ exceeds 120°. (If two triangles have two sides of one equal to two sides of the other, and the third side of the first is greater than the third side of the second, the included angle of the first is greater than the included angle of the second.)



or

$c^2 = a^2 + b^2 - 2ab\cos C$, $\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{3 + 3 - c^2}{2\sqrt{3}\sqrt{3}}$ where $c^2 > 9$. $\therefore \cos C < -\frac{1}{2}$, so that $\angle C > 120^\circ$.

35. (B) $x^2 - y^2 = (27 - x)^2 - (21 - y)^2$. Simplifying, we obtain $48 - 9\,x + 7y = 0$. $\therefore x = 5 + \frac{3 + 7y}{9}$. The smallest integer y that makes x an integer is $y = 6$, and, for this value, $x = 10$. For $y = 15$, $x = 17$, so that $27 - x = 10$. For larger values of y, we obtain impossible triangles.



36. (C) Let M_0 represent the number of cents at the start. Suppose the first bet results in a win; then the amount at the end of the first bet is $\frac{M_0}{2} + M_0 = \frac{3}{2}M_0$. Let $M_1 = \frac{3}{2}M_0$. Suppose the second bet results in a win; then the amount at the end of the second bet is $\frac{M_1}{2} + M_1 = \frac{3}{2}M_1 = (\frac{3}{2})^2 M_0$. Hence, a win at any stage results in an amount $\frac{3}{2}$ times the amount at hand at that stage.

If the first bet results in a loss, the amount at the end of the first bet is $\frac{M_0}{2}$. Let $m = \frac{1}{2}M_0$. Suppose the second bet results in a loss, then the amount at the end of the second bet is $\frac{m}{2} = (\frac{1}{2})^2 M_0$. Hence, a loss at any stage results in an amount $\frac{1}{2}$ times the amount at hand at that stage.

For three wins and three losses, in any order, the amount left is $(\frac{3}{2})^3(\frac{1}{2})^3 M_0 = \frac{27}{64}M_0$. Consequently, the amount lost is $M_0 - \frac{27}{64}M_0 = \frac{37}{64}M_0$. Since $M_0 = 64$ cents, the amount lost is 37 cents.

37. (D) Consider first a single segment P_1P_2; in this case the point P can be selected anywhere in the segment since, for any such selection, $s = PP_1 + PP_2 = P_1P_2$. Now consider the case of two segments with end-points P_1, P_2, P_3; the smallest value of s is obtained by choosing P to coincide with P_2 since, for this selection, $s = P_1P_2 + P_2P_3$ whereas, for any other selection, $s > P_1P_2 + P_2P_3$.

These two cases are typical, respectively, of odd numbers of segments (that is, even numbers of points) and of even numbers of segments (that is, odd numbers of points). Since the given problem is a 7-point system, the selection of P should be at point P_4.

For a 6-point system the selection of P is anywhere in segment P_3P_4.

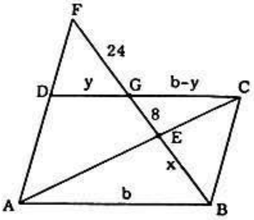
38. (E) Let $\overline{BE} = x$, $\overline{DG} = y$, $\overline{AB} = b$. Since $\Delta BEA \sim \Delta GEC$,

$\frac{8}{x} = \frac{b-y}{b}$, $b - y = \frac{8b}{x}$, $y = b - \frac{8b}{x} = \frac{b(x-8)}{x}$

Since $\Delta FDG \sim \Delta BCG$, $\frac{24}{x+8} = \frac{y}{b-y}$,

$\frac{24}{x+8} = \frac{b(x-8)}{x \cdot \frac{8b}{x}} = \frac{x-8}{8}$, $x^2 - 64 = 192$, $x = 16$

Note: Try to prove generally that $\frac{1}{\overline{BE}} = \frac{1}{\overline{BG}} + \frac{1}{\overline{BF}}$



39. (D) Draw DR ∥ AB. $\frac{\overline{CR}}{\overline{RE}} = \frac{\overline{CD}}{\overline{DB}} = \frac{3}{1}$, $\frac{\overline{RD}}{\overline{EB}} = \frac{\overline{CD}}{\overline{CB}} = \frac{3}{4}$

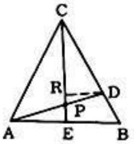
$\therefore \overline{CR} = 3\overline{RE} = 3\overline{RP} + 3\overline{PE}$ and $\overline{RD} = \frac{3}{4}\ \overline{EB}$

$\therefore \overline{CP} = \overline{CR} + \overline{RP} = 4\overline{RP} + 3\overline{PE}$.

Since $\Delta RDP \sim \Delta EAP$, $\frac{\overline{RP}}{\overline{PE}} = \frac{\overline{RD}}{\overline{AE}}$

$\therefore \overline{RD} = \frac{\overline{RP} \times \overline{AE}}{\overline{PE}}$. But $\overline{AE} = \frac{3}{2}\ \overline{EB}$. $\therefore \overline{RD} = \frac{\overline{RP}}{\overline{PE}} \cdot \frac{3}{2}\ \overline{EB}$

$\therefore \frac{3}{4}\ \overline{EB} = \frac{3}{2}\ \overline{EB} \cdot \frac{\overline{RP}}{\overline{PE}}$, $\overline{RP} = \frac{1}{2}\ \overline{PE}$, $\overline{CP} = 4 \cdot \frac{1}{2}\ \overline{PE} + 3\overline{PE} = 5\overline{PE}$ $\therefore \frac{\overline{CP}}{\overline{PE}} = 5$



40. (C) $\sqrt[3]{x+9} - \sqrt[3]{x-9} = 3$. Cube both sides and obtain

$x + 9 - 3(x+9)^{2/3}(x-9)^{1/3} + 3(x+9)^{1/3}(x-9)^{2/3} - x + 9 = 27$

Simplify to $9 = -3(x+9)^{1/3}(x-9)^{1/3}\left[(x+9)^{1/3} - (x-9)^{1/3}\right] = -3(x+9)^{1/3}(x-9)^{1/3} \cdot 3$

$\therefore (x^2 - 81)^{1/3} = -1$, $x^2 = 80$.

or

Let $f(x) = \sqrt[3]{x+9} - \sqrt[3]{x-9}$. We want to find x, so that $f(x_1) = 3$. Since $f(9) = \sqrt[3]{18} < 3$ and $f(8) = \sqrt[3]{17} + 1 > 3$, then $8 < x_1 < 9$. We refine our guess by trying next $x_1 = 8\frac{7}{8}$; $f(8\frac{7}{8}) = \sqrt[3]{17\frac{7}{8}} + \frac{1}{2} > 3$. $\therefore 8\frac{7}{8} < x_1 < 9$ $\therefore (8\frac{7}{8})^2 < x_1^2 < 81$, $78 < x_1^2 < 81$.

SOLUTION KEY
AND ANSWER KEY
FIFTEENTH ANNUAL H. S.
MATHEMATICS CONTEST

1964

Sponsored Jointly by

THE MATHEMATICAL ASSOCIATION OF AMERICA

and

THE SOCIETY OF ACTUARIES

USE OF KEY

1. This Key is prepared for the convenience of teachers.

2. Some of the solutions are intentionally incomplete; crucial steps are shown.

3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.

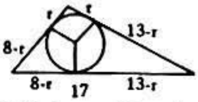
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.

5. This solution Key validates our statement that no mathematics beyond intermediate algebra is needed to solve the problems posed.

National Office Pan American College, Edinburg, Texas
New York Office Polytechnic Institute of Brooklyn, Brooklyn 1, N. Y.

Note: The letter following the problem number refers to the correct choice of the five listed for that problem in the 1964 examination.

1. (E) $[\log_{10}(5\log_{10}100)]^2 = [\log_{10}(5\cdot 2)]^2 = 1^2 = 1$
2. (C) $x^2 - 4y^2 = (x + 2y)(x - 2y) = 0 \therefore x + 2y = 0$ or $x - 2y = 0$. The graph of each equation is a straight line, so that the correct answer is (C).
3. (D) $\frac{x + 2uy}{y} = \frac{x}{y} + 2u$. Since $\frac{x}{y} = u + \frac{v}{y}$, $\frac{x + 2uy}{y} = 3u + \frac{v}{y}$, so that the remainder is v.
4. (A) Since $P = x + y$ and $Q = x - y$, $P + Q = 2x$, $P - Q = 2y$. $\therefore \frac{P + Q}{P - Q} - \frac{P - Q}{P + Q} = \frac{2x}{2y} - \frac{2y}{2x} = \frac{x^2 - y^2}{xy}$
5. (A) $\frac{y_1}{x_1} = \frac{y_2}{x_2}$ $\therefore \frac{8}{4} = \frac{y_2}{-8}$, $y_2 = -16$
6. (B) The common ratio is $\frac{2x + 2}{x} = \frac{3x + 3}{2x + 2} = \frac{3}{2}$, $x \neq 0$, $x \neq -1$
   $\therefore 2x + 2 = \frac{3}{2}x$, $x = -4$ $\therefore$ the 4th term is $(-4)(\frac{3}{2})^3 = -13\frac{1}{2}$
7. (C) For equal roots the discriminant, $p^2 - 4p = 0$. This equation is satisfied for $p = 0$ or $p = 4$, so that (C) is the correct answer.
8. (C) We can re-write the equation as $(x - \frac{3}{4})[(x - \frac{3}{4}) + (x - \frac{1}{2})] = 0$
   $\therefore (x - \frac{3}{4})(2x - \frac{5}{4}) = 0$, $x = \frac{3}{4}$ or $x = \frac{5}{8}$. Since $\frac{5}{8} < \frac{3}{4}$, the correct answer is (C).
9. (E) The cost of the article is $\frac{7}{8} \times \$24 = \$21$. A gain of $33\frac{1}{3}\%$ of the cost is $\frac{1}{3} \times \$21 = \$7$. The article must, therefore, be sold for $21 + 7 = 28$ (dollars). Let M be the marked price, in dollars. $M - \frac{1}{5}M = 28$ $\therefore M = 35$, so that the correct answer is (E).
10. (A) For the square the side is s, the diagonal is $s\sqrt{2}$, and the area is s^2.
    $\therefore \frac{1}{2}(s\sqrt{2})h = s^2$ where h is the altitude to the base of the triangle. $\therefore h = s\sqrt{2}$
11. (D) $2^x = 8^{y+1} = 2^{3(y+1)}$ $\therefore x = 3y + 3$, $3^{x-9} = 9^y = 3^{2y}$ $\therefore x - 9 = 2y$
    The solution of the two linear equations is $x = 21$, $y = 6$ $\therefore x + y = 27$
12. (E) To negate the statement "all members of a set have a given property" we write "some member of the set does not have the given property". Therefore, the negation is: For some x, $x^2 \not> 0$, that is, for some x, $x^2 \leq 0$
13. (A) $(8 - r) + (13 - r) = 17$, $r = 2$
    $s = 8 - r = 6$ $\therefore r : s = 1 : 3$



14. (C) Let P be the amount paid for the 749 sheep. Therefore, the selling price per sheep is P/700, and the complete profit is 49 × P/700. Therefore, the gain is $\frac{49 \times P/700}{P} \times 100 = 7$ (per cent).
15. (B) Let the line cut the x-axis in $(-a, 0)$ and the y-axis in $(0, h)$. Then $\frac{1}{2}ah = T$ and $h = 2T/a$. The slope of the line is $\frac{h}{a} = \frac{2T}{a}$. Therefore, $y = \frac{2T}{a^2}x + \frac{2T}{a}$ $\therefore 2Tx - a^2y + 2aT = 0$
16. (E) $x^2 + 3x + 2 \underset{6}{\equiv} (x + 2)(x + 1) \underset{6}{\equiv} (x + 4)(x + 5) \underset{6}{\equiv} 0$. If $x + 2 \underset{6}{\equiv} 0$, $x = 4$; if $x + 1 \underset{6}{\equiv} 0$, $x = 5$; if $x + 4 \underset{6}{\equiv} 0$, $x = 2$; if $x + 5 \underset{6}{\equiv} 0$, $x = 1$. Additional values of x are, respectively, 10, 11, 8, 7, and so forth.
17. (D) If $x_2 = kx_1$ and $y_2 = ky_1$, $k \neq 0$, $x_1y_1 \neq 0$, the points P, Q, R lie in a straight line through θ. Otherwise, a parallelogram is formed with $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ the midpoint of diagonal PQ and $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$ the midpoint of diagonal OR (if the diagonals of a quadrilateral bisect each other, the figure is a parallelogram).
18. (C) When two equations have the same graph, one can be transformed into the other by multiplying by a constant $k \neq 0$.
    $\therefore 3kx + kby + kc = cx - 2y + 12$, so that $3k = c$, $kb = -2$, $kc = 12$. The solution of these three equations is $c = 6$, $b = -1$ or $c = -6$, $b = 1$.
19. (A) $2x - 3y = z$
    $x + 3y = 14z$ $\therefore x = 5z$, $y = 3z$ $\therefore \frac{x^2 + 3xy}{y^2 + z^2} = \frac{70z^2}{10z^2} = 7$
20. (B) Since the expansion represents the binomial for all values of x and y, it represents the binomial for $x = 1$ and $y = 1$. For these values of x and y each term in the expansion has the same value as the numerical coefficient of the term. Therefore, the sum of the numerical coefficients of all the terms in the expansion equals $(1 - 2)^{18} = (-1)^{18} = 1$.
21. (D) Let $\log_{b^2} x = m$ and let $\log_{x^2} b = n$. Then $x = b^{2m}$ and $b = x^{2n}$.
    $\therefore (x^{2n})^{2m} = b^{2m}$ $\therefore x^{4mn} = x$ $\therefore 4mn = 1$ $\therefore n = \frac{1}{4m}$
    $\therefore m + \frac{1}{4m} = 1$ $\therefore m = \frac{1}{2}$ $\therefore x = b$

22. (C) Since $\overline{DF} = \frac{1}{3}\overline{DA}$, area ($\triangle$DFE) = $\frac{1}{3}$ area ($\triangle$DEA). Since E is the midpoint of DB, area ($\triangle$DEA) = $\frac{1}{2}$ area ($\triangle$DBA). Therefore, area ($\triangle$DFE) = $\frac{1}{3}\cdot\frac{1}{2}$ area ($\triangle$DBA) $\therefore$area (quad. ABEF) = $\frac{5}{6}$ area ($\triangle$DBA) $\therefore$area ($\triangle$DFE) : area (quad. ABEF) = 1 : 5.

23. (D) Let the numbers be represented by x and y. Then $x + y = 7(x - y)$ and $xy = 24(x - y)$. Therefore, $8y = 6x$ and $xy = 24x - 24y = 24x - 18x = 6x$
$\therefore x(y - 6) = 0$. The value $x = 0$ is not usable, so that $y = 6$. Since $6x = 8y$, $x = 8$ $\therefore xy = 48$.

24. (A) $y = (x - a)^2 + (x - b)^2 = 2[x^2 - (a + b)x] + a^2 + b^2$

$$\therefore y = 2\left[x^2 - (a + b)x + \left(\frac{a + b}{2}\right)^2\right] + a^2 + b^2 - 2\left(\frac{a + b}{2}\right)^2$$

$\therefore y = 2\left[x - \frac{a + b}{2}\right]^2 + \frac{(a - b)^2}{2}$. For y to be a minimum $x = \frac{a + b}{2}$

or

The graph of $y = 2x^2 - 2(a + b)x + a^2 + b^2$ is a parabola concave up with respect to the x-axis. It has a minimum point on the axis of symmetry, the equation of which is $x = -\frac{-2(a + b)}{4}$, that is, $x = \frac{a + b}{2}$

25. (B) Let the linear factors be $x + ay + b$ and $x + cy + d$. When multiplied out and equated to the given expression, we obtain these equations between the coefficients: ① $a + c = 3$ ② $b + d = 1$ ③ $ad + bc = m$ ④ $bd = -m$ ⑤ $ac = 0$.
From equations ① and ⑤ , either $c = 0$, $a = 3$ or $a = 0$, $c = 3$. With the first set of values equation ③ becomes $3d = m$. Substituting into equation ④ , we have $3d = -bd$. If $d = 0$, $m = 0$. If $d \neq 0$, $b = -3$. $\therefore d = 4$ and $m = -12$. The second set of values, $a = 0$, $c = 3$, yields the same results because of the symmetry of the equations.

26. (C) From the given information the ratio of the distances covered by First, Second, and Third, in that order, is 10 : 8 : 6. Since the speeds remain constant the ratio of the distances remains constant, so that, when Second completes an additional 2 miles, Third completes an additional $1\frac{1}{2}$ miles since $\frac{2}{x} = \frac{8}{6}$ where x is the additional distance completed by Third. Therefore, when Second completes 10 miles, Third completes $7\frac{1}{2}$ miles, that is, Second beats Third by $2\frac{1}{2}$ miles.

27. (E) When $x \geqq 4$, $|x - 4| + |x - 3| = x - 4 + x - 3 \geqq 1$
When $x \leqq 3$, $|x - 4| + |x - 3| = 4 - x + 3 - x \geqq 1$
When $3 < x < 4$, $|x - 4| + |x - 3| = 4 - x + x - 3 = 1$
Since $a > |x - 4| + |x - 3|$, then $a > 1$.

28. (D) Let a be the first term of the progression. Then $\frac{n}{2}[a + (a + 2(n - 1))] = 153$, or $n^2 + n(a - 1) - 153 = 0$. Pairs of factors of -153 are $-1, 153; 1, -153, -9, 17; 9, -17; -3, 51; 3, -51$. Therefore, the possible values of $a - 1$ are $+152, -152, +8, -8, +48, -48$ and the possible values of n are 1, 153, 9, 17, 3, and 51. The value $n = 1$ is rejected, so that there are five permissible values.

29. (E) $\triangle$RFD $\sim$ $\triangle$RSF (an angle of one triangle equal to an angle of the other triangle and the including sides in proportion).

$\therefore \frac{\overline{RS}}{\overline{RF}} = \frac{\overline{SF}}{\overline{RD}}$, $\frac{RS}{5} = \frac{7\frac{1}{2}}{6}$, $RS = 6\frac{1}{4}$

or

by the law of cosines, $5^2 = 4^2 + 6^2 - 2\cdot4\cdot6\cos\angle D$
$\therefore \cos\angle D = \frac{27}{48} = \cos\angle RFS$
$\therefore RS^2 = 5^2 + (7\frac{1}{2})^2 - 2(7\frac{1}{2})(5)(\frac{27}{48})$ $\therefore RS = 6\frac{1}{4}$

30. (D) Let the roots be r and s with $r \geqq s$.

$\therefore r - s = \frac{\sqrt{(2 + \sqrt{3})^2 + 8(7 + 4\sqrt{3})}}{7 + 4\sqrt{3}}$. Since $7 + 4\sqrt{3} = (2 + \sqrt{3})^2$, $r - s = \frac{3}{2 + \sqrt{3}} = 6 - 3\sqrt{3}$

31. (B) $$f(n + 1) - f(n - 1) = \frac{5 + 3\sqrt{5}}{2}\left(\frac{1 + \sqrt{5}}{2}\right)^{n+1} + \frac{5 - 3\sqrt{5}}{2}\left(\frac{1 - \sqrt{5}}{2}\right)^{n+1} - \frac{5 + 3\sqrt{5}}{2}\left(\frac{1 + \sqrt{5}}{2}\right)^{n-1} - \frac{5 - 3\sqrt{5}}{2}\left(\frac{1 - \sqrt{5}}{2}\right)^{n-1}$$

$$\therefore f(n + 1) - f(n - 1) = \frac{5 + 3\sqrt{5}}{2}\left(\frac{1 + \sqrt{5}}{2}\right)^{n}\left[\frac{1 + \sqrt{5}}{2} - \frac{2}{1 + \sqrt{5}}\right] + \frac{5 - 3\sqrt{5}}{2}\left(\frac{1 - \sqrt{5}}{2}\right)^{n}\left[\frac{1 - \sqrt{5}}{2} - \frac{2}{1 - \sqrt{5}}\right]$$

$$\therefore f(n + 1) - f(n - 1) = \frac{5 + 3\sqrt{5}}{2}\left(\frac{1 + \sqrt{5}}{2}\right)^{n}(1) + \frac{5 - 3\sqrt{5}}{2}\left(\frac{1 - \sqrt{5}}{2}\right)^{n}(1) = f(n)$$

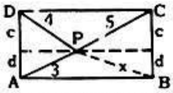
32. (C) Since $\frac{a+b}{b+c} = \frac{c+d}{d+a}$, then $\frac{a+b}{c+d} = \frac{b+c}{d+a}$ and $\frac{a+b}{c+d} + 1 = \frac{b+c}{d+a} + 1$

$\therefore \frac{a+b+c+d}{c+d} = \frac{a+b+c+d}{a+d}$. If $a+b+c+d \neq 0$, then $a = c$.

If $a+b+c+d = 0$, then a may or may not equal c, so that the correct choice is (C).

33. (B) $16 - c^2 = 9 - d^2$

$25 - c^2 = x^2 - d^2 \quad \therefore x^2 - 9 = 9 \quad \therefore x = 3\sqrt{2}$



or

Since x depends only upon the position of P, we may consider the special case where P is on side DA. We then have $x^2 - 9 = 5^2 - 4^2$, $x^2 = 18$, $x = 3\sqrt{2}$

34. (C) $s = 1 + 2i + 3i^2 + \ldots + (n+1)i^n$

is $= \quad i + 2i^2 + \ldots + ni^n + (n+1)i^{n+1}$

$s(1-i) = 1 + i + i^2 + \ldots + i^n - (n+1)i^{n+1}$

$s(1-i) = \frac{1 - i^{n+1}}{1-i} - (n+1)i^{n+1} = 1 - (n+1)i$ since $i^n = 1$

$\therefore s = \frac{1-(n+1)i}{1-i} \cdot \frac{1+i}{1+i} = \frac{1}{2}(n + 2 - ni)$

or

$s = 1 + 3i^2 + 5i^4 + 7i^6 + \ldots + (n+1)i^n + 2i + 4i^3 + \ldots + ni^{n-1}$

$s = (1-3) + (5-7) + \ldots ((n-3) - (n-1)) + n + 1 + i\,[(2-4) + (6-8) + \ldots + (n-2-n)]$

$s = \frac{n}{4}(-2) + n + 1 + \frac{n}{4}(-2)i = \frac{1}{2}(n + 2 - ni)$

35. (B) $14^2 - q^2 = 13^2 - p^2$, $27 = q^2 - p^2$, $15 = q + p \quad \therefore q = \frac{42}{5}$, $p = \frac{33}{5}$

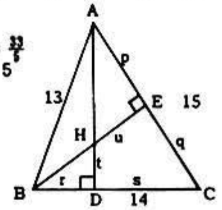
$13^2 - r^2 = 15^2 - s^2$, $56 = s^2 - r^2$, $14 = s + r \quad \therefore s = 9$, $r = 5$

$\overline{AD}^2 = 13^2 - 5^2$, $\overline{AD} = 12$, $\overline{BE}^2 = 13^2 - (\frac{33}{5})^2$, $\overline{BE} = \frac{56}{5}$

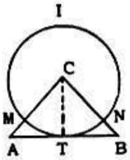
$\triangle HDB \sim \triangle HEA \quad \therefore \frac{t}{u} = \frac{r}{p} = \frac{\frac{56}{5} - u}{12 - t}$

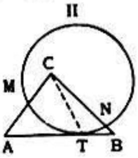
Since $r = 5$, $p = \frac{33}{5}$, $u = \frac{33}{25}t$, $12 - t = \frac{33 \cdot 156}{125} - \frac{33 \cdot 33}{25 \cdot 25}t$

$\therefore t = \frac{435}{116}$, $12 - t = \frac{957}{116} \quad \therefore \overline{HD} : \overline{HA} = 435 : 957 = 5 : 11$



36. (E) Consider position I: $\angle C$ (60°) is measured by $\widehat{MT} + \widehat{TN}$: so that $\widehat{MTN} = 60°$. In any other position, such as II, $\angle C$ is still measured by $\widehat{MT} + \widehat{TN}$. Since $\angle C$ remains unchanged at 60°, $\widehat{MTN}$ remains unchanged at 60°.





37. (D) We first prove that the A.M., $\frac{b+a}{2}$, is greater than the G.M., $\sqrt{ab}$, $b > a$.

$b - a > 0$, $b^2 - 2ab + a^2 > 0$, $b^2 + 2ab + a^2 > 4ab$, $b + a > 2\sqrt{ab}$, $\frac{b+a}{2} > \sqrt{ab}$.

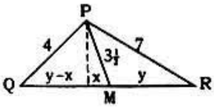
$\therefore \frac{b+a}{2} - a > \sqrt{ab} - a$, $\frac{b-a}{2} > \sqrt{a}\,(\sqrt{b} - \sqrt{a})$

$\therefore \frac{(b-a)^2}{4} > a\,(b + a - 2\sqrt{ab}) \quad \therefore \frac{(b-a)^2}{8a} > \frac{b+a}{2} - \sqrt{ab}$, that is, the difference of the A.M. and the G.M. is less than $\frac{(b-a)^2}{8a}$.

38. (D) $16 - (y-x)^2 = (\frac{7}{2})^2 - x^2 \quad \therefore (y-x)^2 - x^2 = \frac{15}{4}$

$16 - (y-x)^2 = 7^2 - (y+x)^2 \quad \therefore (y+x)^2 - (y-x)^2 = 33$

$\therefore y^2 - 2xy = \frac{15}{4}$ and $2xy = \frac{33}{2} \quad \therefore y^2 = \frac{81}{4}$, $y = \frac{9}{2} \quad \therefore QR = 9$



or

By the law of cosines, $(\frac{7}{2})^2 + y^2 - 2 \cdot \frac{7}{2} \cdot y \cos \angle PMQ = 16$ and $(\frac{7}{2})^2 + y^2 - 2 \cdot \frac{7}{2} \cdot y \cos (180° - \angle PMQ)$ $= (\frac{7}{2})^2 + y^2 + 7y \cos \angle PMQ = 49 \quad \therefore \frac{49}{2} + 2y^2 = 65$, $y = \frac{9}{2}$, $2y = 9$

39. (A) The length of each of the lines through P must be less than the length of the larger of the two sides of the triangle adjacent to the line through P. To see this let BB′ be fixed and allow point P to take a position very close to B′. Then A′ takes a position very close to C so that $\overline{AA'}$ is very nearly equal to b, but less than b. Similarly, $\overline{CC'}$ remains smaller than a and $\overline{BB'}$ remains smaller than a for all interior positions of P.

$\therefore \overline{AA'} + \overline{BB'} + \overline{CC'} < b + a + a \quad \therefore s < 2a + b$.

40. (A) First we note that the number of minutes in a day is 24 × 60 = 1440, while the number of minutes as recorded by the watch in a day is $2\frac{1}{2}$ less or $1437\frac{1}{2}$, so that the correction factor for this watch is $1440/1437\frac{1}{2}$ or, more simply, 576/575.

Designate a minute recorded according to the watch as a "watch-minute" and a day recorded by the watch as a "watch-day." Since the time interval given is $5\frac{5}{6}$ watch-days, or $5\frac{5}{6} \times 24 \times 60$ watch-minutes, we have $5\frac{5}{6} \times 24 \times 60 + n = 5\frac{5}{6} \times 24 \times 60 \times \frac{576}{575} \quad \therefore n = 140 \cdot 60\,(\frac{576}{575} - 1) = 14\frac{14}{23}$.

SOLUTION KEY
AND ANSWER KEY

SIXTEENTH ANNUAL H. S.
MATHEMATICS EXAMINATION

1965

Sponsored Jointly by

THE MATHEMATICAL ASSOCIATION OF AMERICA

and

THE SOCIETY OF ACTUARIES

USE OF KEY

1. This Key is prepared for the convenience of teachers.
2. Some of the solutions are intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This solution Key validates our statement that no mathematics beyond intermediate algebra is needed to solve the problems posed.

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Note: The letter following the problem number refers to the correct choice of the five listed for that problem in the 1965 examination.

1. (C) $2^{2x^2-7x+5} = 1 = 2^0 \quad \therefore 2x^2 - 7x + 5 = 0$; the solution set of this equation has two members.

2. (D) Let r be the magnitude of the radius of the circle; then the length of a side of the hexagon is r and the length of the shorter arc intercepted by the side is $2\pi r/6$. The required ratio is $r:(2\pi r/6)$ or $3:\pi$.

3. (B) $81^{-(2^{-2})} = 81^{-1/4} = \frac{1}{81^{1/4}} = \frac{1}{3}$. Query: Is there another permissible value not listed in the five choices?

4. (C) Let l_4 be the set of points such that each point is equidistant from l_1 and l_3. Designate the distance by d. Let l_5 and l_6 be the set of points at the distance d from l_2 (l_5 and l_6 are parallel to l_2) The intersection set of l_4, l_5, and l_6, containing two points, is the required set of points

5. (A) $0.363636\ldots = \frac{36}{10^2} + \frac{36}{10^4} + \ldots = \frac{36/10^2}{1-1/10^2} = \frac{36}{99} = \frac{4}{11}$, so that the correct answer is (A).

or

Let $F = 0.363636\ldots \quad 100F = 36.363636\ldots \quad 99F = 36 \quad F = \frac{36}{99} = \frac{4}{11}$

6. (B) Since $10^{\log_{10}9} = 8x + 5$, $(\log_{10}9)(\log_{10}10) = \log_{10}(8x + 5) \quad \therefore 9 = 8x + 5 \quad \therefore x = 1/2$

7. (E) Let the roots be r and s; then $r + s = -b/a$ and $rs = c/a$
$\therefore \frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs} = \frac{-b/a}{c/a} = -b/c,\ c \neq 0,\ a \neq 0$

8. (A) Let s be the length of the required segment; then $s^2/18^2 = 2/3$ so that $s = 6\sqrt{6}$ (the areas of two similar triangles are to each other as the squares of corresponding sides).

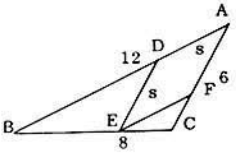
9. (E) $y = x^2 - 8x + C = x^2 - 8x + 16 + C - 16 = (x - 4)^2 + C - 16$ (a parabola). For its vertex to be on the x-axis, its coordinates must be (4,0) so that C must have the value 16.

10. (A) $x^2 - x - 6 < 0,\ x^2 - x < 6,\ x^2 - x + \frac{1}{4} < 6 + \frac{1}{4},\ \left(x - \frac{1}{2}\right)^2 < \left(\frac{5}{2}\right)^2$
$\therefore |x - \frac{1}{2}| < \frac{5}{2}$ or $-\frac{5}{2} < x - \frac{1}{2} < \frac{5}{2}$, that is $-2 < x < 3$

or

$x^2 - x - 6 < 0$, $(x - 3)(x + 2) < 0$. This inequality is satisfied if $x - 3 < 0$ and $x + 2 > 0$ or if $x - 3 > 0$ and $x + 2 < 0$. The first set of inequalities implies $-2 < x < 3$; the second set is impossible to satisfy.

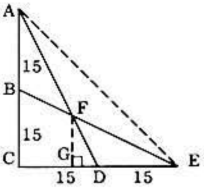
11. (B) $\sqrt{-4} = 2\sqrt{-1}$, $\sqrt{-16} = 4\sqrt{-1} \quad \therefore (\sqrt{-4})(\sqrt{-16}) = 8(-1) = -8$ but $\sqrt{(-4)(-16)} = \sqrt{64} = 8$
$\therefore$ Statement I is incorrect.



12. (D) Since $\Delta BDE \sim \Delta BAC$, $s/6 = (12 - s)/12$, $s = 4$

13. (E) Let the line $5y - 3x = 15$ intersect the circle $x^2 + y^2 = 16$ in points R and S. All the points on segment RS (which is a chord of the circle) satisfy the given system, so that the correct answer is (E).

14. (A) $(x^2 - 2xy + y^2)^7 = ((x - y)^2)^7 = (x - y)^{14}$. By setting $x = y = 1$ in the expansion of this binomial we obtain the sum s of the integer coefficients. Consequently $s = (1 - 1)^{14} =$ zero



15. (B) $52_b = 5b + 2$, $25_b = 2b + 5$. Since $52_b = 2(25_b)$, $5b + 2 = 2(2b + 5) \therefore b = 8$

16. (C) Draw AE and the altitude FG to the base DE of triangle DEF. Since F is the intersection point of the medians of triangle ACE, $\overline{FD} = \frac{1}{3}\overline{AD}$.
$\therefore \overline{FG} = \frac{1}{3} AC = \frac{1}{3} \cdot 30 = 10 \quad \therefore \text{area } (\Delta DEF) = \frac{1}{2} \cdot 15 \cdot 10 = 75$

17. (E) The given statement may be rephrased as "If the picnic on Sunday is not held, then the weather is not fair." Its contrapositive is "If the weather is fair, then the picnic is held on Sunday." This is statement (E)

18. (B) Since $\frac{1}{1+y} = 1 - y + \frac{y^2}{1+y}$, the error in using the approximation $1 - y$ is $\frac{y^2}{1+y}$.
The ratio of the error to the correct value is, therefore, $\left(\frac{y^2}{1+y}\right)\Big/\left(\frac{1}{1+y}\right) = y^2$

19. (C) Let $x^4 + 4x^3 + 6px^2 + 4qx + r = (x + a)(x^3 + 3x^2 + 9x + 3) = x^4 + (a + 3)x^2 + (3a + 9)x^2 + (9a + 3)x + 3a$. Equating the coefficients of like powers of x, we have $a = 1$, $p = 2$, $q = 3$, $r = 3$ so that $(p + q)r = (2 + 3)(3) = 15$

or

Divide $x^4 + 4x^3 + 6px^2 + 4qx + r$ by $x^3 + 3x^2 + 9x + 3$; the quotient is $x + 1$ and the remainder is the second-degree polynomial $(6p - 12)x^2 + (4q - 12)x + r - 3$. This remainder equals zero since the division is exact. Therefore $p = 2$, $q = 3$, $r = 3$ and $(p + q)r = 15$

20. (C) The r th term equals $S_r - S_{r-1}$ and $S_r = 2r + 3r^2$ and $S_{r-1} = 2(r - 1) + 3(r - 1)^2 = 3r^2 - 4r + 1$
$\therefore$ the r th term equals $6r - 1$

or

Let the r th term be u_r; then $S_r = \frac{r}{2}(a + u_r) = 2r + 3r^2 \quad \therefore a + u_r = 4 + 6r$

But $a = S_1 = 2 \cdot 1 + 3 \cdot 1^2 = 5 \quad \therefore u_r = 4 + 6r - 5 = 6r - 1$

or

$a = S_1 = 5,\ S_2 = 16 \quad \therefore u_2 = S_2 - S_1 = 11$, but $u_2 = a + d \quad \therefore d = 6$
$\therefore u_r = a + (r - 1)d = 5 + (r - 1)(6) = 6r - 1$

21. (D) $\log_{10}(x^2 + 3) - 2\log_{10} x = \log_{10}\frac{x^2 + 3}{x^2} = \log_{10}\left(1 + \frac{3}{x^2}\right)$. For a sufficiently large value of x, $\frac{3}{x^2}$ may be made less than a specified positive number N and so $\log_{10}\left(1 + \frac{3}{x^2}\right)$ may be made less than the specified positive number $\log_{10}(1 + N)$. Challenge: If, in the problem, the condition $x > \frac{1}{2}$ were given instead of $x > \frac{2}{3}$, show that then (B) would also be an acceptable answer.

22. (A) $a_2x^2 + a_1x + a_0 = a_2\left(x^2 + \frac{a_1}{a_2}x + \frac{a_0}{a_2}\right) = a_2(x^2 - (r + s)x + rs)$
$= a_2(r - x)(s - x) = a_2rs\left(1 - \frac{x}{r}\right)\left(1 - \frac{x}{s}\right),\ rs \neq 0$ Since $rs = \frac{a_0}{a_2}$ and $rs \neq 0$ and $a_2 \neq 0$, then $a_0 \neq 0$
$\therefore a_2x^2 + a_1x + a_0 = a_2\left(\frac{a_0}{a_2}\right)\left(1 - \frac{x}{r}\right)\left(1 - \frac{x}{s}\right) = a_0\left(1 - \frac{x}{r}\right)\left(1 - \frac{x}{s}\right)$ for all values of x provided $a_0 \neq 0$

23. (D) Since $|x - 2| < 0.01$, $|x^2 - 4| = |x - 2||x + 2|$. But $|x + 2| \leqq |x| + 2 < 2.01 + 2 = 4.01$,
since $|x - 2| < 0.01$ implies that $x < 2.01 \quad \therefore |x^2 - 4| < (.01)(4.01) = .0401$

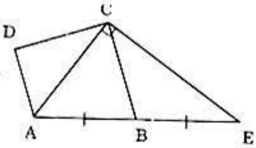
or

$|x - 2| < 0.01$ implies $1.99 < x < 2.01 \quad \therefore 3.9599 < 3.9601 < x^2 < 4.0401$
$\therefore -0401 < x^2 - 4 < .0401$, that is, $|x^2 - 4| < .0401$

24. (E) Let $P = 10^{1/11} \cdot 10^{2/11} \cdots 10^{n/11} = 10^s$ where $s = \frac{1 + 2 + \ldots + n}{11} = \frac{1}{11} \cdot \frac{1}{2}n(n + 1)$

For $P > 100000$, $10^s > 10^5$, that is, $s > 5 \quad \therefore \frac{n^2 + n}{22} > 5,\ n^2 + n - 110 > 0,\ (n + 11)(n - 10) > 0,\ n > 10$

25. (D) Since CB is the median to the hypotenuse AE of right triangle AEC,
$\overline{CB} = \frac{1}{2}\overline{AE} = \overline{AB}$



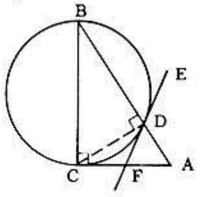
26. (E) From the given information $m = \frac{a + b + c + d + e}{5}$, $2k = a + b$, $3l = c + d + e$,
and $p = \frac{k + l}{2} \quad \therefore m = \frac{2k + 3l}{5}$ If $m = p$, $\frac{2k + 3l}{5} = \frac{k + l}{2}$ so that $l = k$;
if $m > p$, then $l > k$; if $m < p$, then $l < k$.
Since, for random selections of a, b, c, d, e, it is possible for l to equal, to be greater than, or to be less than k, so it follows that m can be equal, can be greater than, or can be less than p.

27. (A) Since $y^2 + my + 2 = (y - 1)f(y) + R_1$ is true for all values of y, we have, letting $y = 1, 3 + m = R_1$.
Similarly, since $y^2 + my + 2 = (y + 1)g(y) + R_2$ is true for all values of y, we have, letting $y = -1$,
$3 - m = R_2$. Since $R_1 = R_2$, $3 + m = 3 - m. \quad \therefore m = 0$.

28. (B) Let the time needed for a full step, at any given level of the escalator, to reach the next level, be designated as unit time. Then Z's time to travel the escalator may be represented as $1 + k \cdot 1$ where k is the number of escalator steps covered by Z in unit time. Similarly A's time may be represented as $1 + 2k \cdot 1$.
Since the rates of A and Z are inversely proportional to their negotiated number of steps,
$\frac{n}{1 + 2k} : \frac{n}{1 + k} = 18:27$ so that $k = 1$. Setting $\frac{n}{1 + 2k} = 18$, we have $\frac{n}{3} = 18$, $n = 54$. Alternatively, we set $\frac{n}{1 + k} = 27$ and obtain $\frac{n}{2} = 27$. $n = 54$.

29. (A) Let n_1 ($n_1 \geqq 0$) represent the number of students taking Mathematics and English only. Let n_2 ($n_2 > 0$, n_2 even) represent the number of students taking all three subjects.
From the given information we have, then, $n_1 + n_1 + 5n_2 + n_2 + 6 = 28$, that is, $n_1 + 3n_2 = 11$. Under the restrictions given in the problem, this equation is satisfied only by $n_2 = 2$ and $n_1 = 5$

30. (B) $\angle A = \angle BCD$ (Why?), $DF = CF$ (Why?) $\quad \therefore \angle DCF = \angle CDF$
$\angle BCD = 90° - \angle DCF$, $\angle FDA = 90° - \angle CDF$
$\therefore \angle FDA = \angle BCD = \angle A \quad \therefore DF = FA = CF$, that is, DF bisects CA.
Also $\angle CFD = \angle FDA + \angle A = 2\angle A$.
The given information is, therefore, sufficient to prove choices (A), (C), (D), and (E). For choice (B) to be true, segments CD and AD would have to be equal, but such equality can not hold for all possible positions of point D.



31. (C) Let $\log_a x = y \quad \therefore x = a^y \quad \therefore \log_b x = \log_b a^y = y\log_b a$
$\therefore \{(\log_a x)(\log_b x) = \log_a b\}$ implies $\{y \cdot y\log_b a = \log_a b\}$
Since $\log_b a = 1/\log_a b$, $y^2 = (\log_a b)^2$, $y = \log_a b$ or $y = -\log_a b$
$\therefore \log_a x = \log_a b$ or $\log_a x = \log_a b^{-1} \quad \therefore x = b$ or $x = b^{-1} = 1/b$

or

Let $\log_b x = y \quad \therefore x = b^y \quad \therefore (\log_a b^y)(\log_b b^y) = \log_a b$
$(y\log_a b)(y\log_b b) = \log_a b$, $y^2\log_a b = \log_a b \quad \therefore y^2 = 1 \quad \therefore y = 1$ or $y = -1 \quad \therefore x = b$ or $x = b^{-1}$

or

$(\log_b x)(\log_a x) = \log_a x^{(\log_b x)} = \log_a b \quad \therefore x^{\log_b x} = b$
$\therefore (\log_b x)(\log_b x) = \log_b b = 1 \quad \therefore \log_b x = 1$ or $\log_b x = -1 \quad \therefore x = b$ or $x = b^{-1}$

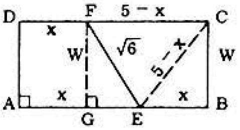
32. (C) $100 = C - x,\ C = 100 + x$
$\therefore S' = C + 1\frac{1}{9} = 100 + x + \frac{10}{9} = 100 + \frac{x}{100}\left(100 + x + \frac{10}{9}\right)$
$\therefore \frac{x^2}{100} + \frac{x}{90} - \frac{10}{9} = 0 \quad \therefore x = 10$

33. (D) $15! = 10^3(3 \cdot 14 \cdot 13 \cdot 12 \cdot 11 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 3 \cdot 2 \cdot 1)$ and $15! = 12^5(15 \cdot 14 \cdot 13 \cdot 11 \cdot 5 \cdot 7 \cdot 5)$
$\therefore k = 3$ and $h = 5 \quad \therefore k + h = 8$

34. (B) Let $y = \frac{4x^2 + 8x + 13}{6(1 + x)} \quad \therefore 4x^2 + (8 - 6y)x + 13 - 6y = 0$, a quadratic equation in x with discriminant $D = 36(y^2 - 4)$. Since $x \geqq 0$, D must be non-negative.
$\therefore y \geqq 2$, that is, the minimum value of the given fraction is 2.

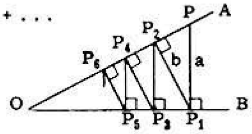
or

Let $y = \frac{4x^2 + 8x + 13}{6(1 + x)} = \frac{2(x + 1)}{3} + \frac{3}{2(x + 1)}$ so that y is of the form $N + \frac{1}{N}$. The minimum value of such an expression occurs when $N = \frac{1}{N}$, that is, when $\frac{2(x + 1)}{3} = \frac{3}{2(x + 1)}$, in other words, when $x = \frac{1}{2}$. With this value of x, $y = 2$.

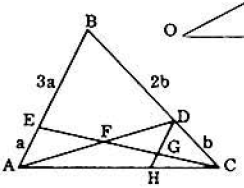


35. (D) Let $EB = x$; then $AE = EC = 5 - x$ and $DF = x$, so that $AG = x$ and $GE = 5 - 2x$
$\therefore w^2 = (\sqrt{6})^2 - (5 - 2x)^2$ and, also, $w^2 = (5 - x)^2 - x^2 \quad \therefore 6 - 25 + 20x - 4x^2 = 25 - 10x \quad \therefore x = 2$ and $w = \sqrt{5}$. The value $x = 11/2$ is rejected.

36. (E) $\Delta P_3P_2P_1 \sim \Delta P_2P_1P \quad \therefore \overline{P_3P_2}:b = b:a,\ \overline{P_3P_2} = b^2/a$
$\Delta P_4P_3P_2 \sim \Delta P_3P_2P_1 \quad \therefore \overline{P_4P_3}:b^2/a = b^2/a:b,\ \overline{P_4P_3} = b^3/a^2$
Similarly $\overline{P_5P_4} = b^4/a^3$ and so forth. The limiting sum is $a + b + \frac{b^2}{a} + \frac{b^4}{a^2} + \frac{b^4}{a^3} + \ldots$
$= a + \frac{b}{1 - b/a} = a + \frac{ab}{a - b} = \frac{a^2}{a - b}$



37. (C) Draw DGH || AB $\therefore \overline{DG}:3a = b:3b;\ \overline{DG} = a = \overline{EA}$
$\therefore \overline{EF} = \overline{FG}$ and $\overline{AF} = \overline{FD}$ so that $\overline{AF}/\overline{FD} = 1$
Also $\overline{DH}:4a = b:3b,\ \overline{DH} = 4a/3$ and $\overline{GH} = \overline{DH} - \overline{DG} = a/3$
$\therefore \overline{GC} = \frac{1}{3}\overline{EC}$ and $\overline{EG} = \frac{2}{3}\overline{EC}$, and, since $\overline{EF} = \overline{FG},\ \overline{FC} = \frac{2}{3}\overline{EC}$
$\therefore EF/FC = \frac{1}{2} \quad \therefore \overline{EF}/\overline{FC} + \overline{AF}/\overline{FD} = \frac{1}{2} + 1 = \frac{3}{2}$

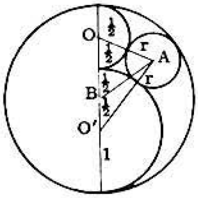


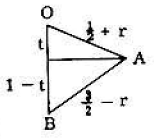
38. (E) $\frac{1}{A} = \frac{1}{m}\left(\frac{1}{B} + \frac{1}{C}\right),\ \frac{1}{B} = \frac{1}{n}\left(\frac{1}{C} + \frac{1}{A}\right),\ \frac{1}{C} = \frac{1}{x}\left(\frac{1}{A} + \frac{1}{B}\right)$
$\therefore \frac{m}{A} - \frac{n}{B} = \frac{1}{B} - \frac{1}{A}$ and $\frac{m}{A} + \frac{n}{B} = \frac{1}{B} + \frac{1}{A} + \frac{2}{C} = \frac{1}{B} + \frac{1}{A} + \frac{2}{x}\left(\frac{1}{A} + \frac{1}{B}\right)$
$\therefore \frac{1}{A}\left(m + 1\right) = \frac{1}{B}(n + 1)$ and $\frac{1}{A}\left(m - 1 - \frac{2}{x}\right) = \frac{1}{B}\left(1 - n + \frac{2}{x}\right)$
$\therefore \frac{A}{B} = \frac{m + 1}{n + 1}$ and $\frac{A}{B} = \frac{m - 1 - 2/x}{1 - n + 2/x} \quad \therefore \frac{m + 1}{n + 1} = \frac{m - 1 - 2/x}{1 - n + 2/x} \quad \therefore x = \frac{m + n + 2}{mn - 1}$

39. (B) $\overline{OA} = \frac{1}{2} + r,\ \overline{O'A} = 1 + r,\ \overline{BA} = 1\frac{1}{2} - r.$
Area $(\Delta OBA) = 2$ Area $(\Delta BO'A)$
Semiperimeter $(\Delta OBA) = \frac{1}{2}(\frac{1}{2} + r + 1 + \frac{3}{2} - r) = \frac{3}{2}$
Semiperimeter $(\Delta BO'A) = \frac{1}{2}(\frac{3}{2} - r + \frac{1}{2} + 1 + r) = \frac{3}{2}$
$\therefore \sqrt{\frac{3}{2}(1 - r)(\frac{1}{2})r} = 2\sqrt{\frac{3}{2}(r)(1)(\frac{1}{2} - r)} \quad \therefore 7r = 3,\ r = \frac{3}{7},\ d = \frac{6}{7} \approx .86$

or

Draw the altitude h from A to OB. We have, then
$(\frac{1}{2} + r)^2 - t^2 = (\frac{3}{2} - r)^2 - (1 - t)^2,\ t = 2r - \frac{1}{2},\ h^2 = 3r - 3r^2$
Since Area $(\Delta OAO') = \frac{1}{2}h \cdot \frac{3}{2} = \sqrt{(\frac{3}{2} + r)(r)(1)(\frac{1}{2})}$, we have
$\frac{9}{16}h^2 = \frac{9}{16}(3r - 3r^2) = \frac{3}{4}r + \frac{1}{2}r^2 \quad \therefore r = \frac{3}{7}$ and $d = \frac{6}{7}$





40. (D) Let $P = x^4 + 6x^3 + 11x^2 + 3x + 31 = (x^2 + 3x + 1)^2 - 3(x - 10) = y^2$
$\therefore (x^2 + 3x + 1)^2 - y^2 = 3(x - 10)$. When $x = 10$, $P = (x^2 + 3x + 1)^2 = 131^2 = y^2$. To prove that 10 is the only possible value we use the following lemma: If $|N| > |M|$, N, M integers, then $N^2 - M^2 \geqq 2|N| - 1$
(This lemma is easy to prove, try it)
Case I If $x > 10$, then $3(x - 10) = (x^2 + 3x + 1)^2 - y^2 \geqq 2|x^2 + 3x + 1| - 1$, an impossibility
Case II If $x < 10$, then $3(10 - x) = y^2 - (x^2 + 3x + 1)^2 \geqq 2|y| - 1 > 2|x^2 + 3x + 1| - 1$. This inequality holds for the integers $x = 2, 1, 0, -1, -2, -3, -4, -5, -6$, but none of these values makes P the square of an integer.

SOLUTION KEY
AND ANSWER KEY
SEVENTEENTH ANNUAL H. S.
MATHEMATICS EXAMINATION

1966

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USE OF KEY

1. This Key is prepared for the convenience of teachers.

2. Some of the solutions are intentionally incomplete; crucial steps are shown.

3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.

4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.

5. This solution Key validates our statement that no mathematics beyond intermediate algebra is needed to solve the problems posed.

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Note: The letter following the problem number refers to the correct choice of the five listed for that problem in the 1966 examination.

1. (C) $3x - 4 = k(y + 15)$, $6 - 4 = k(3 + 15)$ $\quad\therefore k = 1/9$ $\quad\therefore 3x - 4 = 1/9\ (12 + 15) = 3$, $3x = 7$, $x = 7/3$

2. (E) $K_0 = \frac{1}{2}bh$, $K_n = \frac{1}{2}(b + .1b)(h - .1h)$ $\quad\therefore K_n = \frac{1}{2}bh + .05bh - .05bh - \frac{1}{2}(.01)\ bh = (1 - .01)K_0$. Therefore, the change is a decrease of 1% of the original area.

3. (D) Let r and s be the roots of the required equation. Since $6 = \frac{r + s}{2}$, $r + s = 12$ and since $10 = \sqrt{rs}$, $rs = 100$, so that, in the required equation, the sum of the roots is 12 and the product of the roots is 100. Hence, $x^2 - 12x + 100 = 0$.

4. (B) Let s be the length of a side of the square. The radius of circle I is $\frac{1}{2}s\sqrt{2}$ and its area $K_1 = \frac{1}{2}\pi s^2$. The radius of circle II is $\frac{1}{2}$ s and its area $K_2 = \frac{1}{4}\pi s^2$. $\quad\therefore r = \frac{K_1}{K_2} = 2$

5. (A) For $x \neq 0$, $x \neq 5$, the left side of the equation reduces to 2. $\quad\therefore 2 = x - 3$, $x = 5$, but $x = 5$ is unacceptable. The equation is, therefore, satisfied by no value of x.

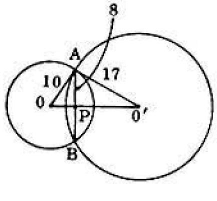
6. (C) Triangle ABC is a right triangle with hypotenuse AB = 5 inches and leg BC equal to a radius of the circle, $\frac{5}{2}$. We, therefore, have $\overline{AC}^2 = \overline{AB}^2 - \overline{BC}^2 = 25 - \frac{25}{4} = \frac{75}{4}$ $\quad\therefore AC = \frac{5\sqrt{3}}{2}$ (inches).

7. (A) $35x - 29 \equiv N_1(x - 2) + N_2(x - 1)$ is an identity in x. Letting $x = 1$ we find $N_1 = -6$ and letting $x = 2$ we find $N_2 = 41$. $\quad\therefore N_1N_2 = -246$

8. (B) From the diagram we have $\overline{OP}^2 = \overline{OA}^2 - \overline{AP}^2 = 10^2 - 8^2$, $OP = 6$

$\overline{O'P}^2 = \overline{O'A}^2 - \overline{AP}^2 = 17^2 - 8^2$, $O'P = 15$

$\therefore OO' = OP + O'P = 6 + 15 = 21$



9. (A) $\log_2 8 = 3$ and $\log_8 2 = \frac{1}{3}$ $\quad\therefore x = \left(\frac{1}{3}\right)^3$ $\quad\therefore \log_3 x = 3 \log_3 \frac{1}{3} =$
$3(0 - 1) = -3$

or

Let $y = \log_2 8 (= 3)$ $\quad\therefore 2^y = 8 (= 2^3)$ $\quad\therefore y \log_8 2 = \log_8 8 = 1$
$\therefore \log_8 2 = \frac{1}{y}$ $\quad\therefore x = \frac{1}{y}^{y} = \frac{1}{y^y} = y^{-y}$
$\therefore \log_3 x = -y \log_3 y = -3(1) = -3$

10. (E) Let the numbers be x and y. Then $x^3 + y^3 = (x + y)^3 - 3xy(x + y)$
$\therefore x^3 + y^3 = 1^3 - 3(1)(1) = -2$

or

Consider the equation $Z^2 - Z + 1 = 0$ with roots $Z_1 = \frac{1 + i\sqrt{3}}{2}$, $Z_2 = \frac{1 - i\sqrt{3}}{2}$.
The sum of these roots is 1 and their product is 1, so that we may take for our numbers the numbers Z_1 and Z_2. Therefore $Z_1^3 + Z_2^3 = \left(\frac{1 + i\sqrt{3}}{2}\right)^3 + \left(\frac{1 - i\sqrt{3}}{2}\right)^3 =$

$$\frac{1 + 3i\sqrt{3} + 3i^2 \cdot 3 + i^3 \cdot 3\sqrt{3} + 1 - 3i\sqrt{3} + 3i^2 \cdot 3 - i^3 \cdot 3\sqrt{3}}{8} = \frac{2 - 18}{8} = -2$$

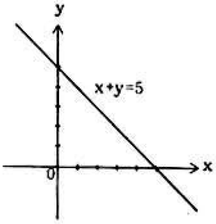
11. (C) Since the bisector of an angle of a triangle divides the opposite side into segments proportional to the sides including the given angle, taken in the proper order, we have
$\frac{10 - x}{x} = \frac{3m}{3m} = \frac{3}{4}$ $\quad\therefore 40 = 7x$, $x = 5\frac{5}{7}$, $10 - x = 4\frac{2}{7}$ $\quad\therefore$ The longer segment is $5\frac{5}{7}$.

12. (E) The given equation is equivalent to $(2^{6x+3})(2^{2(3x+6)}) = 2^{3(4x+5)}$
The left side of this equation equals $2^{6x+3} \cdot 2^{6x+12} = 2^{12x+15}$ and the right side is also equal to 2^{12x+15}, so that the given equality is an identity in x. It is, therefore, satisfied by any real value of x.

13. (E) Since $x + y \leqq 5$, $y \leqq 5 - x$; for any rational value of x, $5 - x$ and, hence, y is rational.

or

Consider the graph of $x + y \leqq 5$; it is the half-plane below the line $x + y = 5$, including the line $x + y = 5$. Since this half-plane contains an infinite set of rational points (points with rational coordinates), choice (E) is the correct one.



14. (C) The area of triangle ABC equals $\frac{1}{2} \cdot 5 \cdot 3 = \frac{15}{2}$. Since AE = EF = FC, the area of triangle BEF equals $\frac{1}{3}$ of the area of triangle ABC, that is, $\frac{1}{3} \cdot \frac{15}{2} = \frac{5}{2}$.

15. (D) Since $x - y > x$, $y < 0$ and since $x + y < y$, $x < 0$.
$\therefore$ Choice (D) is correct.

16. (B) $\frac{4^x}{2^{x+y}} = \frac{2^{2x}}{2^{x+y}} = 2^{x-y} = 8 = 2^3$ $\therefore$ $x - y = 3$ $\therefore x = 4, y = 1$

$\frac{9^{x+y}}{3^{5y}} = \frac{3^{2x+2y}}{3^{5y}} = 3^{2x-3y} = 243 = 3^5$ $\therefore 2x - 3y = 5$ $\therefore xy = 4$

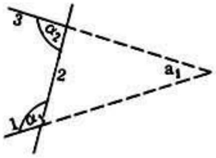
17. (C) Both curves are ellipses with the centers at the origin. The first, $\frac{x^2}{1^2} + \frac{y^2}{(\frac{1}{2})^2} = 1$, has x-intercepts +1 and −1, and y-intercepts $+\frac{1}{2}$ and $-\frac{1}{2}$. The second, $\frac{x^2}{1^2} + \frac{y^2}{2^2} = 1$, has x-intercepts +1 and −1, and y-intercepts +2 and − 2. The number of distinct points of intersection is 2.

18. (A) $s = \frac{n}{2}(a + l)$ where $a = 2$, $l = 29$, and $s = 155$ $\therefore 155 = \frac{n}{2}(2 + 29) = \frac{31n}{2}$, $n = 10$.
But $l = a + (n - 1)d$; so that $29 = 2 + 9d$, $9d = 27$, $d = 3$

19. (B) $s_1 = \frac{n}{2}(16 + (n - 1)\,4)$, $s_2 = \frac{n}{2}(34 + (n - 1)\,2)$. But $s_1 = s_2$ implies $\frac{n}{2}(12 + 4n) = \frac{n}{2}(32 + 2n)$
$\therefore 12 + 4n = 32 + 2n$, $n = 10$ so that choice (B) is correct.

20. (C) To negate the proposition "$p \rightarrow q$", where p and q themselves are propositions, we form the proposition "p and not-q". In this case p is the proposition "$a = 0$" and q is the proposition "$ab = 0$". The negation is, therefore, "$a = 0$ and $ab \neq 0$", corresponding to "if $a = 0$, then $ab \neq 0$".

21. (E) Let the measures of the angles at the n points be $a_1, a_2, \ldots, a_n$ and let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be the measures of the interior angles of the polygon, with $\alpha_n = \alpha_1$. We have $a_1 = 180 - (180 - \alpha_1) - (180 - \alpha_2) = \alpha_1 + \alpha_2 - 180$,
$a_2 = \alpha_2 + \alpha_3 - 180, \ldots, a_n = \alpha_{n-1} + \alpha_n - 180$. Summing, we have
$a_1 + a_2 + \ldots + a_n = 2(\alpha_1 + \alpha_2 + \ldots + \alpha_n) - n \cdot 180$
$\therefore S = 2\,((n - 2)180) - n \cdot 180 = 180(n - 4)$



22. (A) I is satisfied when $b = a\sqrt{-1}$, II is satisfied when $b = \frac{a}{\sqrt{a^2 - 1}}$, $a \neq 1$, and III and IV are both satisfied when $b = 0$ and a is chosen arbitrarily. Therefore, (A) is the correct choice.

23. (A) We treat the equation as a quadratic equation in y for which the discriminant $D = 16x^2 - 16(x + 6) = 16(x^2 - x - 6) = 16(x - 3)(x + 2)$. For y to be real $D \geqq 0$. This inequality is satisfied when $x \leqq -2$ or $x \geqq 3$.

24. (B) Let $\log_M N = x$; then $\log_N M = \frac{1}{\log_M N} = \frac{1}{x}$. $\therefore x^2 = 1$, $x = +1$ or -1. If $x = 1$ then $M = N$, but this contradicts the given $M \neq N$. If $x = -1$, then $N = M^{-1}$ $\therefore MN = 1$

or

Let $\log_M N = x = \log_N M$ $\therefore N = M^x$ and $M = N^x$ $\therefore (M^x)^x = N^x = M$
$\therefore x \cdot x = 1$ $\therefore x = 1$ (rejected) or $x = -1$ $\therefore N = M^{-1}$ $\therefore NM = 1$

25. (D) $2F(n + 1) = 2F(n) + 1$
$2F(n) = 2F(n - 1) + 1$
$\vdots \qquad \vdots$
$2F(2) = 2F(1) + 1$
$\therefore 2F(n + 1) = 2F(1) + n \cdot 1$
$\therefore F(n + 1) = F(1) + \frac{1}{2}n$ $\therefore F(101) = 2 + \frac{1}{2} \cdot 100 = 52$

or

$2\sum_{1}^{100} F(n + 1) = 2F(1) + 100 = 2 \cdot 2 + 100 = 104$ $\therefore F(101) = 52$.

26. (C) $13x + 11\,(mx - 1) = 700$, $x(13 + 11m) = 711$, $x = \frac{711}{13 + 11\,m}$, with m integral and x integral. It is easy to see that m must be even. Try 2, 4, 6 in turn. Neither 2 nor 4 produce an integral x but for $m = 6$, we have $x = 9$. Since two straight lines can intersect in at most one point (these lines do not coincide), $m = 6$ is the only possible value.

27. (A) Let m (miles per hour) be the man's rate in still water, and let c (miles per hour) be the rate of the current. Then

$\frac{15}{m + c} = \frac{15}{m - c} - 5$ and $\frac{15}{2m + c} = \frac{15}{2m - c} - 1$

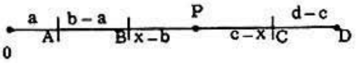
$15m - 15c = 15m + 15c - 5m^2 + 5c^2$ and $30m - 15c = 30m + 15c - 4m^2 + c^2$

$\therefore -5m^2 + 5c^2 = -30c$ and $-4m^2 + c^2 = -30c$

$\therefore m^2 - 4c^2 = 0$, $m = 2c$, $3c^2 = 6c$, $c = 2$

28. (B) Since $\frac{AP}{PD} = \frac{BP}{PC}$, $\frac{x-a}{d-x} = \frac{x-b}{c-x}$

$\therefore x(a-b+c-d) = ac - bd$, $x = \frac{ac-bd}{a-b+c-d}$



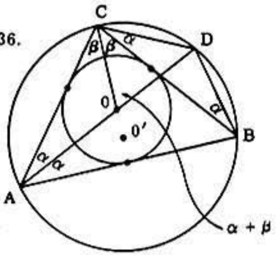
29. (B) The required number n is $999 - N_1(5) - N_2(7) + N_3(35)$ where $N_1(5)$ is the number of multiples of 5, namely 199, $N_1(7)$ is the number of multiples of 7, namely 142, and $N_1(35)$ is the number of multiples of 35, namely 28. $\therefore n = 999 - 199 - 142 + 28 = 686$

30. (D) $1 + 2 + 3 + r_4 = 0$, $r_4 = -6$. Since $-a$ represents the sum of the roots taken two at a time and c represents the product of the roots, we have $-a = -2 - 3 + 6 - 6 + 12 + 18 = 25$ and $c = (1)(2)(3)(-6) = -36$
$\therefore a + c = -25 - 36 = -61$

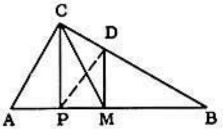
or

Solve the system $1 + a + b + c = 0$, $16 + 4a + 2b + c = 0$, $81 + 9a + 3b + c = 0$ to obtain $a = -25$, $b = 60$, $c = -36$. $\therefore a + c = -61$

31. (D) Quadrilateral ABDC is inscriptible
$\therefore \widehat{CD} = \widehat{BD}$, $\therefore CD = BD$
$\angle DCO = \alpha + \beta$, $\angle DOC = \alpha + \beta$, $\therefore OD = CD = BD$



32. (B) Since M is the midpoint of AB, the area of triangle BMC = $\frac{1}{2}$ the area of triangle BAC. But the area of $\triangle BMC$ = the area of $\triangle BMD$ + the area of $\triangle MDC$, and $\triangle MDC = \triangle MDP$ in area (they have the same base MD and equal altitudes to this base since MD ∥ PC). Therefore, in area $\triangle BMC = \triangle BMD + \triangle MDP = \triangle BPD$, $\therefore r = \frac{1}{2}$ independent of the position of P between A and M. Query: Is the theorem true when P is to the left of A?



33. (D) $a(x-a)^2(x-b) + b(x-b)^2(x-a) = ab^2(x-b) + a^2b(x-a)$
$a(x-a)[(x-a)(x-b) - ab] = -b(x-b)[(x-a)(x-b) - ab]$
$\therefore [(x-a)(x-b) - ab][ax - a^2 + bx - b^2] = 0.$
$x^2 - (a+b)x = 0$ or $(a+b)x = a^2 + b^2$ $\quad \therefore x = 0,\ x = a + b$, or $x = \frac{a^2+b^2}{a+b}$

34. (B) $r = \frac{11}{t} \cdot \frac{1}{5280} \cdot 3600$ where t is given in seconds $\quad \therefore \frac{11}{t - \frac{1}{4}} \cdot \frac{1}{5280} \cdot 3600 = \frac{11}{t} \cdot \frac{1}{5280} \cdot 3600 + 5$
$8t^2 - 2t - 3 = (4t-3)(2t+1) = 0$, $t = \frac{3}{4}$, $r = 10$

35. (C)

$$\begin{array}{l} OA + OC > AC \\ OC + OB > BC \\ \underline{OB + OA > AB} \\ 2s_1 > s_2 \\ s_1 > \frac{1}{2}s_2 \end{array} \qquad \begin{array}{l} OB + OC < AB + AC \\ OC + OA < BC + AB \\ \underline{OA + OB < AC + BC} \\ 2s_1 < 2s_2 \\ s_1 < s_2 \end{array}$$

36. (E) $(1 - x + x^2)^n \equiv a_0 - a_1x + a_2x^2 - a_3x^3 + \ldots \qquad (1 + x + x^2)^n \equiv a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots$
$\therefore 2(a_0 + a_2x^2 + a_4x^4 + \ldots + a_{2n}x^{2n}) \equiv (1 - x + x^2)^n + (1 + x + x^2)^n$
Let $x = 1$ $\quad \therefore 2(a_0 + a_2 + \ldots + a_{2n}) = 1^n + 3^n \quad \therefore 2s = 1^n + 3^n \quad \therefore s = \frac{3^n + 1}{2}$

37. (C) Let a, b, c be the number of hours needed, respectively, by Alpha, Beta, and Gamma to do the job when working alone. Then $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{a-6} = \frac{1}{b-1} = \frac{1}{c/2}$ $\quad \therefore b = a - 5$ and $c = 2a - 12$
$\therefore \frac{1}{a} + \frac{1}{a-5} + \frac{1}{2(a-6)} = \frac{1}{a-6} \qquad \therefore a = \frac{20}{3}$; the value $a = 3$ is rejected.
$\therefore b = \frac{5}{3}$ and $c = \frac{4}{3} \qquad \therefore \frac{1}{a} + \frac{1}{b} = \frac{1}{20/3} + \frac{1}{5/3} = \frac{3}{4} \qquad \therefore h = \frac{4}{3}$

38. (D) In area $\triangle COM = \frac{1}{2}\triangle COB = \frac{1}{2} \cdot \frac{1}{3}\triangle ABC = \frac{1}{6}\triangle ABC = \triangle CQM + n = \frac{1}{8}\triangle ABC + n$

$\therefore \frac{1}{6}\triangle ABC = \frac{1}{8}\triangle ABC + n$

$\therefore n = \frac{1}{24}\triangle ABC \qquad \therefore \triangle ABC = 24n$

39. (E) $F_1 = \frac{3R_1 + 7}{R_1^2 - 1} = \frac{2R_2 + 5}{R_2^2 - 1}$ and $F_2 = \frac{7R_1 + 3}{R_1^2 - 1} = \frac{5R_2 + 2}{R_2^2 - 1} \qquad \therefore \frac{3R_1 + 7}{2R_2 + 5} = \frac{R_1^2 - 1}{R_2^2 - 1} = \frac{7R_1 + 3}{5R_2 + 2}$

$\therefore R_2 = \frac{29R_1 + 1}{R_1 + 29}$ Knowing that R_1 and R_2 must each be integral, and that $R_1 \geqq 8$ (why?), we solve for R_2 with permissible values of R_1. For $R_1 = 8$, R_2 is not integral; for $R_1 = 9$ or 10, R_2 is not integral; for $R_1 = 11$, $R_2 = 8$; for $R_1 = 12$, R_2 is not integral; for $R_1 = 13$, $R_2 = 9$. The values $R_1 = 13$, $R_2 = 9$ do not satisfy the conditions of the problem; the values $R_1 = 11$, $R_2 = 8$ do.
$\therefore R_1 + R_2 = 19$

40. (A) $\frac{x}{2a} = \frac{AE}{AC} = \frac{CD}{AC}$ and $\overline{BC}^2 = CD \cdot CA$ and $\frac{x}{y} = \frac{2a}{BC}$ so that $\overline{BC}^2 = \frac{4a^2y^2}{x^2}$. Since $\frac{x}{2a} \cdot \overline{BC}^2 = \frac{CD}{AC} \cdot CD \cdot CA$, $\frac{x}{2a} \cdot \frac{4a^2y^2}{x^2} = \overline{CD}^2 \qquad \therefore \frac{2ay^2}{x} = \overline{CD}^2 = \overline{AE}^2 = x^2 + y^2 \qquad \therefore y^2(2a - x) = x^3 \qquad \therefore y^2 = \frac{x^3}{2a - x}$

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Note: The letter following the problem number refers to the correct choice of the five listed for that problem in the 1967 examination.

1. (C) Since 5b9 is divisible by 9, $b = 4$ $\therefore a + 2 = 4$, $a = 2$, and $a + b = 6$.

2. (D) The given expression equals $\left(x + \frac{1}{x}\right)\left(y + \frac{1}{y}\right) + \left(x - \frac{1}{x}\right)\left(y - \frac{1}{y}\right)$

   $= xy + \frac{y}{x} + \frac{x}{y} + \frac{1}{xy} + xy - \frac{y}{x} - \frac{x}{y} + \frac{1}{xy} = 2xy + \frac{2}{xy}$.

   or

   The given expression equals $\frac{x^2y^2 + x^2 + y^2 + 1 + x^2y^2 - x^2 - y^2 + 1}{xy} = \frac{2x^2y^2 + 2}{xy} = 2xy + \frac{2}{xy}$.

3. (B) Let r be the radius; h, the altitude; and x, the side of the square.

   $r = \frac{1}{3}h = \frac{1}{3} \cdot \frac{1}{2}\, s\sqrt{3} = \frac{s\sqrt{3}}{6}$ $\therefore$ Area (square) $= x^2 = 2r^2 = \frac{s^2}{6}$.

4. (C) Since $\frac{\log a}{p} = \log x$, $a = x^p$. Similarly $b = x^q$ and $c = x^r$.

   $\therefore \frac{b^2}{ac} = \frac{x^{2q}}{x^p \cdot x^r} = x^{2q-p-r}$. Since, also, $\frac{b^2}{ac} = x^y$, $y = 2q - p - r$.

5. (D) Let the sides of the triangle, in inches, be a, b, and c. Then

   $K = \frac{1}{2}ra + \frac{1}{2}rb + \frac{1}{2}rc = \frac{1}{2}rP$. $\therefore P/K = \frac{2}{r}$.

6. (D) $f(x + 1) - f(x) = 4^{x+1} - 4^x = 4^x(4 - 1) = 3 \cdot 4^x = 3f(x)$.

7. (E) If $bd > 0$ then $a < \frac{-bc}{d}$. When $-c > 0$, $\frac{-bc}{d} > 0$, so that a is less than a positive quantity.

   $\therefore$ a may be positive, zero, or negative.

8. (A) $\frac{m \cdot \frac{m}{100}}{m + x} = \frac{m - 10}{100}$ $\therefore x = \frac{10m}{m - 10}$. The solution is valid for $m > 10$. It, therefore, holds for $m > 25$.

9. (E) The shorter base, the altitude, and the longer base may be represented by $a - d$, a, and $a + d$, respectively. $\therefore K = \frac{1}{2}a(a - d + a + d) = a^2$. Since a^2 may be the square of an integer, a non-square integer, the square of a rational fraction, a non-square rational fraction, or irrational, the correct choice is (E).

10. (A) $a(10^x + 2) + b(10^x - 1) = 2 \cdot 10^x + 3$ $\therefore a + b = 2$ and $2a - b = 3$ $\therefore a = \frac{5}{3}$ and $b = \frac{1}{3}$ $\therefore a - b = \frac{4}{3}$.

11. (B) Let the dimensions be x and $10 - x$ and let the diagonal be d. Then $d^2 = x^2 + (10 - x)^2 = 2x^2 - 20x + 100$ $= 2(x^2 - 10x + 25) + 50 = 2(x - 5)^2 + 50$. The least value of d^2 (and, hence, of d) occurs when $x = 5$. $\therefore$ d (least) $= \sqrt{50}$.

    or

    The graph of $2x^2 - 20x + 100$ is a parabola with the low point (minimum) at vertex (5, 50).

    $\therefore d^2$ (least) $= 50$ and d (least) $= \sqrt{50}$.

    or

    Of all rectangles with a fixed perimeter the one with least diagonal is the square. Since the perimeter is 20, the side of the square is 5, and, therefore, $d = \sqrt{50}$:

12. (B) The area of the trapezoid thus formed is $\frac{1}{2}(3)(m + 4 + 4m + 4) = 7$. $\therefore 5m = -\frac{10}{3}$, $m = -\frac{2}{3}$.

13. (E) The given parts are not independent since $h_c = a \sin B$. There is no triangle if $h_c < a \sin B$ or $h_c > a \sin B$. When $h_c = a \sin B$, we are free to choose vertex A anywhere on BD, including right and left extensions, where D is the foot of h_c.

14. (C) Since $y = \frac{x}{1 - x}$, $y - yx = x$ and $x = \frac{y}{1 + y}$ $\therefore x = -\frac{-y}{1 - (-y)} = -f(-y)$.

15. (D) Let T_1, T_2 represent the larger, smaller areas, respectively. Since the triangles are similar and 3 is a side of the smaller triangle, we have $\frac{T_1}{T_2} = \frac{x^2}{3^2}$. $\therefore \frac{T_2 + 18}{T_2} = \frac{x^2}{3^2} = k^2$, k a positive integer $\therefore T_2 = \frac{18}{k^2 - 1}$; since T_2 is an integer, 18 must be divisible by $k^2 - 1$. Only $k^2 = 4$ is acceptable. $\therefore \frac{x^2}{3^2} = 4$ and $x = 6$.

or

Since $\frac{T_2 + 18}{T_2} = 1 + \frac{18}{T_2} = k^2$, T_2 must equal 6 and k must equal 2. But k is the ratio of corresponding sides. It follows that the side of the larger triangle corresponding to side 3 of the smaller triangle is 6.

16. (B) Let $P = (b + 2)(b + 5)(b + 6) = 3146$ (base b) $\therefore b^3 + 13b^2 + 52b + 60 = 3b^3 + b^2 + 4b + 6$. $\therefore 0 = b^3 - 6b^2 - 24b - 27$ and $b = 9$. But $s = (b + 2) + (b + 5) + (b + 6) = 3b + 9 + 4$. $\therefore s = 3 \cdot 9 + 9 + 4 = 4 \cdot 9 + 4 = 44$ (base b).

17. (A) Since r_1 and r_2 are real and distinct, $p^2 - 32 > 0$ $\therefore |p| > 4\sqrt{2}$. But $r_1 + r_2 = -p$ $\therefore |r_1 + r_2| = |p|$ $\therefore |r_1 + r_2| > 4\sqrt{2}$.

18. (B) Since $x^2 - 5x + 6 = (x - 3)(x - 2) < 0$, we have $2 < x < 3$. Since $P = x^2 + 5x + 6$, then $P < 3^2 + 5 \cdot 3 + 6 = 30$ and $P > 2^2 + 5 \cdot 2 + 6 = 20$, that is, $20 < P < 30$.

19. (E) If l and w represent the dimensions of the rectangle, then $\left(l + \frac{5}{2}\right)\left(w - \frac{2}{3}\right) = \left(l - \frac{5}{2}\right)\left(w + \frac{4}{3}\right) = lw$ $\therefore l = \frac{15}{2}$ and $w = \frac{8}{3}$, and the area is 20.

20. (A) The radius of the first circle is $\frac{1}{2}m$; the side of the second square is $\frac{m}{2}\sqrt{2}$; the radius of the second circle is $\frac{1}{2}\left(\frac{m\sqrt{2}}{2}\right) = \frac{m}{2\sqrt{2}}$; and so forth. If S is the limiting value of S_n, then $S = \pi\left(\frac{m}{2}\right)^2 + \pi\left(\frac{m}{2\sqrt{2}}\right)^2 + \pi\left(\frac{m}{4}\right)^2 + \ldots = \pi m^2\left(\frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots\right) = \pi m^2\left(\frac{1}{2}\right)$.

21. (B) Let $BA_1 = x$ $\therefore \frac{x}{4 - x} = \frac{5}{3}$, $x = \frac{5}{2}$ and $4 - x = \frac{3}{2}$. $\therefore PR = 4 - x = \frac{3}{2}$ and $PQ = x = \frac{5}{2}$.

$\therefore \triangle PQR \sim \triangle BAC$, the ratio of the sides being 1:2. In $\triangle BAC$, $\overline{AA_1}^2 = 3^2 + \left(\frac{3}{2}\right)^2 = \frac{45}{4}$

$\therefore AA_1 = \frac{3\sqrt{5}}{2}$ But $PP_1 : AA_1 = 1 : 2$ $\therefore PP_1 = \frac{1}{2} \cdot \frac{3\sqrt{5}}{2} = \frac{3\sqrt{5}}{4}$.

22. (A) $P = QD + R$ and $Q = Q'D' + R'$ $\therefore P = Q'DD' + R'D + R$, which means that when P is divided by DD' the quotient is Q' and the remainder is $R + R'D$.

23. (B) $\text{Log}_3 (6x - 5) - \log_3(2x + 1) = \log_3 \frac{6x - 5}{2x + 1} = \log_3 \frac{6 - \frac{5}{x}}{2 + \frac{1}{x}}$ for $x \neq 0$. As x grows beyond all bounds, the last expression approaches $\log_3 \frac{6}{2} = \log_3 3 = 1$.

24. (A) $3x = 501 - 5y$. For x to be a positive integer $\frac{501 - 5y}{3} > 0$ $\therefore 5y < 501$ and $y \leq 100$. Also $x = 167 - y - \frac{2y}{3}$; for integral x, y must be a multiple of 3, that is, $y = 3k$. Since $y \leq 100$, $k = 1, 2, \ldots, 33$.

or

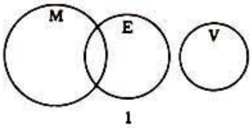
In Number Theory it is shown that if x_0, y_0 is one solution of $3x + 5y = 501$, then other solutions are $x = x_0 - \frac{5}{d}t$, $y = y_0 + \frac{3}{d}t$ where t is an integer and d is the greatest common divisor of 3 and 5, so that, in this case, $d = 1$. An obvious solution of the given equation is $x = 167$, $y = 0$. Therefore, other solutions are $x = 167 - 5t$, $y = 0 + 3t$. Since $x = 167 - 5t > 0$, $t < \frac{167}{5}$ so that $t = 1, 2, \ldots, 33$.

25. (A) Since p is odd and $p > 1$, then $\frac{1}{2}(p - 1) \geq 1$. In every case one factor of $(p - 1)^{\frac{1}{2}(p-1)} - 1$ will be $[(p - 1) - 1] = p - 2$. The other choices are either possible only for special permissible values of p or not possible for any permissible values of p.

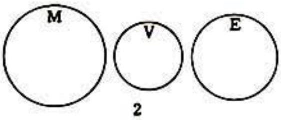
26. (C) Since $2^{10} = 1024 > 10^3 = 1000$, $10 \log_{10} 2 > 3$ so that $\log_{10} 2 > \frac{3}{10}$. Since $2^{13} = 8192 < 10^4 = 10000$, $13 \log_{10} 2 < 4$ so that $\log_{10} 2 < \frac{4}{13}$.

27. (C) Let t represent the number of hours before 4 P. M. necessary to obtain the desired result. Then, representing the original length by L, we have $L - \frac{t}{4}L = 2\left(L - \frac{t}{3}L\right)$ or, more simply, $1 - \frac{t}{4} = 2\left(1 - \frac{t}{3}\right)$ $\therefore t = 2\frac{2}{5}$ and $4 - 2\frac{2}{5} = 1\frac{3}{5}$. Therefore, the candles should be lighted $1\frac{3}{5}$ hours after noon, that is, at 1:36 P. M.

28. (E) The given hypotheses are satisfied by any one of these six Venn diagrams:



1



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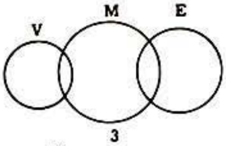

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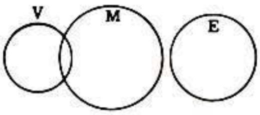

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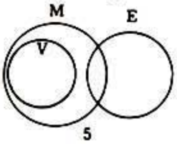

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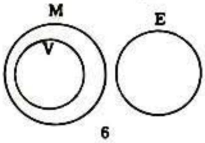
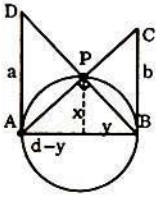


6

Choice (A) is contradicted by diagrams 3, 4, 5, 6. Choice (B) is contradicted by diagrams 5, 6. Choice (C) is contradicted by diagrams 3, 4, 5, 6. Choice (D) is contradicted by diagrams 1, 2.

or

Suppose there is only one Mem and that it is not an En, and suppose there are no Vees. Then the hypotheses are satisfied but (B) and (D) are contradicted. If there are Vees and the Mem is one of them, then again the hypotheses are satisfied but (A) and (C) are contradicted.



29. (C) From similar triangles $\frac{x}{y} = \frac{a}{d}$ and $\frac{x}{d-y} = \frac{b}{d}$

$\therefore \frac{x^2}{y(d-y)} = \frac{ab}{d^2}$. But $x^2 = y(d-y)$. $\therefore 1 = \frac{ab}{d^2}$ $\therefore d = \sqrt{ab}$.

or

Let the degree measure of arc AP be m.

Then $\angle D = 90 - \frac{m}{2}$ and $\angle C = \frac{m}{2}$.

$\therefore \frac{d}{b} = \tan\frac{m}{2}$ and $\frac{d}{a} = \tan\left(90 - \frac{m}{2}\right) = \frac{1}{\tan\frac{m}{2}}$

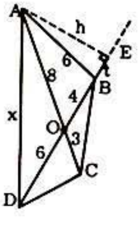
$\therefore \tan^2\frac{m}{2} = \frac{a}{b}$ $\therefore \frac{d^2}{b^2} = \frac{a}{b}$, $d^2 = ab$, $d = \sqrt{ab}$.

30. (D) $(n-2)\left(\frac{d}{n} + 8\right) + 2\cdot\frac{d}{2n} = 72 + d$, $8n^2 - 88n - d = 0$, $n^2 - 11n - \frac{d}{8} = 0$. Since n is a (least positive) integer, d must be such that $n^2 - 11n - \frac{d}{8}$ yields linear factors with integer coefficients. Hence $d = 96$ and $\frac{d}{8} = 12$ so that $n^2 - 11n - 12 = (n-12)(n+1) = 0$ and n (least) = 12.

or

Since $8n^2 - 88n - d = 0$, $88 = 8n - \frac{d}{n}$ whence n can be any integer > 11.

31. (C) $D = a^2 + b^2 + c^2 = a^2 + (a+1)^2 + (a(a+1))^2 = a^4 + 2a^3 + 3a^2 + 2a + 1 = (a^2 + a + 1)^2$. $\therefore \sqrt{D} = a^2 + a + 1$, an odd integer for any integer a.



32. (E) $8^2 = h^2 + (4+t)^2$, $6^2 = h^2 + t^2$, $t = \frac{3}{2}$

$\therefore h = \frac{3\sqrt{15}}{2}$ $\therefore x^2 = h^2 + \overline{DE}^2 = \frac{135}{4} + \frac{529}{4} = 166$ $\therefore x = \sqrt{166}$.

or

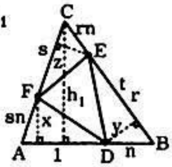
$\Delta BOC \sim \Delta AOD$ with side ratio 1 : 2 $\therefore BC = \frac{1}{2}x$

$\Delta AOB \sim \Delta DOC$ $\therefore CD = 4\frac{1}{2}$. Since ABCD is inscriptible (why?) we may use Ptolemy's Theorem.

$\therefore x\cdot\frac{x}{2} + 6\cdot 4\frac{1}{2} = (6+4)(8+3)$ $\therefore x = \sqrt{166}$.

33. (D) Shaded area $= \frac{1}{2}\left(\frac{\pi}{4}\overline{AB}^2 - \frac{\pi}{4}\overline{AC}^2 - \frac{\pi}{4}\overline{CB}^2\right)$. Since AB = AC + CB, shaded area $= \frac{1}{2}\cdot\frac{\pi}{4}\left(\overline{AC}^2 + 2AC\cdot CB + \overline{CB}^2 - \overline{AC}^2 - \overline{CB}^2\right) = \frac{\pi}{4}(AC)(CB)$. But the area of the required circle equals $\pi\overline{CD}^2$, and since $\overline{CD}^2 = (AC)(CB)$, the area of the circle equals $\pi(AC)(CB)$. Therefore, the required ratio is 1 : 4.

34. (A) Designate AD by 1, DB by n, BE by r, EC by rn, CF by s, and FA by sn. Let h_1 be the altitude from C, h_2, the altitude from A, and h_3, the altitude from B. By similar triangles $\frac{x}{h_1} = \frac{sn}{sn+s} = \frac{n}{1+n}$ $\therefore x = \frac{n}{1+n}h_1$ $\therefore$ area ($\triangle$ADF) $= \frac{1}{2}(1)(x) = \frac{1}{2}\frac{nh_1}{1+n}$. In a like manner $y = \frac{nh_2}{1+n}$ and area ($\triangle$BDE) $= \frac{1}{2}\frac{rnh_2}{1+n}$, and $z = \frac{nh_3}{1+n}$ and area ($\triangle$CFE) $= \frac{1}{2}\frac{snh_3}{1+n}$.



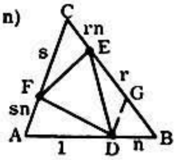
But the area ($\triangle$ABC) $= \frac{1}{3}\cdot\frac{1}{2}[h_1(1+n) + h_2(1+n)r + h_3(1+n)s] = \frac{1}{6}(1+n)(h_1 + h_2r + h_3s)$

$\therefore$ area ($\triangle$DEF) $= \frac{1}{6}(1+n)(h_1+h_2r+h_3s) - \frac{1}{2}\frac{n}{1+n}(h_1+h_2r+h_3s) = \frac{n^2-n+1}{6(1+n)}(h_1+h_2r+h_3s)$

$\therefore$ the required ratio is $\frac{n^2-n+1}{(n+1)^2}$.

or

Draw DG ∥ AC. By similar triangles $\frac{BG}{BC} = \frac{n}{1+n}$ $\therefore BG = \frac{n}{1+n}BC = \frac{n}{1+n}\cdot r(1+n)$ $= nr$. Also, letting K be the area of $\triangle$ABC, we have $\frac{\text{area}(\triangle DBG)}{K} = \frac{n^2}{(1+n)^2}$ $\therefore$ area ($\triangle$DBG) $= \frac{n^2}{(1+n)^2}K$. Since $\triangle$BDE has base r and altitude equal to that of $\triangle$BDG, $\frac{\text{area}(\triangle BDE)}{\text{area}(\triangle BDG)} = \frac{r}{BG} = \frac{r}{rn} = \frac{1}{n}$ $\therefore$ area ($\triangle$BDE) $= \frac{1}{n}\cdot\frac{n^2}{(1+n)^2}K$ $= \frac{n}{(1+n)^2}K$. In like manner ($\triangle$ECF) $= \frac{n}{(1+n)^2}K$ and area ($\triangle$ADF) $= \frac{n}{(1+n)^2}K$.

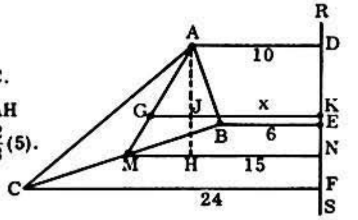


$\therefore$ area ($\triangle$DEF) $= K - \frac{3n}{(1+n)^2}K = \frac{n^2-n+1}{(n+1)^2}K$.

35. (B) Let the roots be a − d, a, and a + d. Then $[x-(a-d)](x-a)[x-a+d] = x^3 - 3ax^2 + x(3a^2-d^2) + (a^3 - ad^2) = 0$. But $64x^3 - 144x^2 + 92x - 15 = 64\left(x^3 - \frac{9}{4}x^2 + \frac{23}{16}x - \frac{15}{64}\right) = 0$ $\therefore x^3 - \frac{9}{4}x^2 + \frac{23}{16}x - \frac{15}{64} = 0$ $= x^3 - 3ax^2 + x(3a^2-d^2) - (a^3 - ad^2)$ $\therefore -3a = -\frac{9}{4}$, $a = \frac{3}{4}$ and $3a^2 - d^2 = \frac{23}{16}$, $d^2 = \frac{4}{16}$, $d = +\frac{1}{2}$ or $-\frac{1}{2}$. $\therefore$ the roots are $\frac{5}{4}$, $\frac{3}{4}$, and $\frac{1}{4}$, and the required difference is 1.

36. (C) $S = a(1 + r + r^2 + r^3 + r^4) = 211$. If r is an integer a must equal 1 since 211 is prime. However, neither a = 1, r = 1 nor a = 1, r = 2, nor a = 1, r = 3 is satisfactory. $\therefore 2 < r < 3$ or $1 < r < 2$. Noting that when $r = \frac{m}{n}$, m, n integers, a must equal n^4 since 211 is prime, we try $r = \frac{3}{2}$, a = 16. $\therefore s = 16\left(1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 + \left(\frac{3}{2}\right)^4\right) = 211$, and the combination a = 16, $r = \frac{3}{2}$ is usable. By symmetry the combination a = 81, $r = \frac{2}{3}$ is also usable. In either case, the odd-numbered terms are squares of integers, and their sum is 16 + 36 + 81 = 133.

37. (A) Let M be the midpoint of CB and let G be the intersection point of the three medians. Draw MN perpendicular to RS. Then AD, x, BE, MN, and CF are parallel. MN is the median of trapezoid BEFC. $\therefore MN = \frac{1}{2}(6 + 24) = 15$. In trapezoid ADNM draw AH ⊥ MN and let AH intersect GK in J. Then HN = 10 and MH = 5 and JK = 10 and $GJ = \frac{2}{3}(5)$. $\therefore x = GK = \frac{10}{3} + 10 = \frac{40}{3}$.

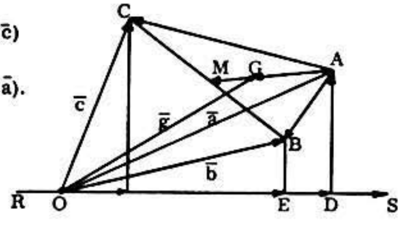


or

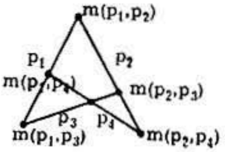
Start with MN = 15 and use the Principle of Weighted Means: since the ratio AG : GM = 2 : 1, we have $GK = \frac{2\cdot 15 + 1\cdot 10}{3} = \frac{40}{3}$.

or

Using vectors with O as origin, we first show that $\bar{g} = \frac{1}{3}(\bar{a} + \bar{b} + \bar{c})$ where $\bar{g} = \overrightarrow{OG}$, $\bar{a} = \overrightarrow{OA}$, $\bar{b} = \overrightarrow{OB}$, $\bar{c} = \overrightarrow{OC}$. We have $\overrightarrow{AB} = \bar{b} - \bar{a}$, $\overrightarrow{AC} = \bar{c} - \bar{a}$, $\overrightarrow{AM} = \frac{1}{2}(\overrightarrow{AB} + \overrightarrow{AC})$ $\therefore \overrightarrow{AM} = \frac{1}{2}(\bar{b} + \bar{c} - 2\bar{a})$. Since $\overrightarrow{AG} = \frac{2}{3}\overrightarrow{AM}$, $\overrightarrow{AG} = \frac{1}{3}(\bar{b} + \bar{c} - 2\bar{a})$. But $\overrightarrow{AG} = \bar{g} - \bar{a}$ $\therefore \frac{1}{3}(\bar{b} + \bar{c} - 2\bar{a}) = \bar{g} - \bar{a}$ $\therefore \bar{g} = \frac{1}{3}(\bar{a} + \bar{b} + \bar{c})$.



Let unit vector $\hat{\imath}$ be in the direction RS and unit vector $\hat{\jmath}$ be in the direction DA. Then $\bar{g} = \hat{\imath}x + \hat{\jmath}y$, $\bar{a} = \hat{\imath}x_1 + 10\hat{\jmath}$. $\bar{b} = \hat{\imath}x_2 + 6\hat{\jmath}$, $\bar{c} = \hat{\imath}x_3 + 24\hat{\jmath}$ $\therefore \hat{\jmath}y = \frac{1}{3}(10 + 6 + 24)\hat{\jmath}$ $\therefore y = \frac{40}{3}$.

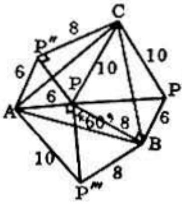


38. (E) One method of establishing theorems for a finite geometry is to construct a model. In the one shown here the maa, $m(p_1, p_2)$, for example, is common to pib 1 and pib 2. The maa common to pib 3 and pib 4 should be labeled $m(p_3, p_4)$. In this model each of the four postulates is satisfied. Since there must be a maa for every pair of pibs, there must be a maa for each of the pairs (p_1, p_2), (p_1, p_3), (p_1, p_4), (p_2, p_3), (p_2, p_4), and (p_3, p_4), six in all. This establishes T_1. Each pib consists of exactly three maas. For example, p_3 consists of $m(p_1, p_3)$, $m(p_4, p_3)$, and $m(p_2, p_3)$. This establishes T_2. For $m(p_1, p_2)$ there is $m(p_3, p_4)$ not in p_1 or p_2, and, similarly, for each of the other maas. This establishes T_3.

or

By P_2 and P_3 there is a one - one correspondence between the set of maas and the set of pairs of pibs. The four pibs yield six pairs (listed above) and so T_1 is true. Each pib belongs to three of the six pairs and so T_2 is true. Each pair of pibs is disjoint from one other pair (for example, the pair p_1, p_2 is disjoint from the pair p_3, p_4) and so T_3 is true.

39. (B) The first element in the nth set is 1 more than the sum of the number of elements in the preceding $n - 1$ sets, that is, $\{1 + 2 + \ldots + n - 1\} + 1 = \frac{(n-1)(n)}{2} + 1 = \frac{n^2 - n + 2}{2}$. Since the nth set contains n elements its last element is $\frac{n^2 - n + 2}{2} + n - 1 = \frac{n^2 + n}{2}$. Therefore, $S_n = \frac{n}{2}\left(\frac{n^2 - n + 2}{2} + \frac{n^2 + n}{2}\right) = \frac{n}{2}(n^2 + 1)$ $\therefore S_{21} = \frac{21}{2}(21^2 + 1) = 4641$.



40. (D) Rotate CP through 60° to position CP′. Draw BP′. This is equivalent to rotating $\triangle CAP$ into position CBP′. In a similar manner rotate $\triangle ABP$ into position ACP″ and $\triangle BCP$ into position BAP‴.

On the one hand hexagon AP‴BP′CP″ consists of $\triangle ABC$ and $\triangle CBP'$, ACP″, BAP‴. Letting K represent area, we have, from the congruence relations, $K(ABC) = K(CBP') + K(ACP'') + K(BAP''')$ $\therefore K(ABC) = \frac{1}{2}K$ (hexagon).

On the other hand the hexagon consists of three quadrilaterals PCP′B, PBP‴A, and PAP″C, each of which consists of the 6-8-10 right triangle and an equilateral triangle. $\therefore K(\text{hexagon}) = 3\left(\frac{1}{2} \cdot 6 \cdot 8\right) + \frac{1}{4} \cdot 10^2\sqrt{3} + \frac{1}{4} \cdot 8^2\sqrt{3} + \frac{1}{4} \cdot 6^2\sqrt{3}$ $= 72 + 50\sqrt{3}$ $\therefore K(ABC) = 36 + 25\sqrt{3} \approx 79$.

or

Applying the Law of Cosines to $\triangle APB$ wherein $\angle APB = 150°$, we have $s^2 = 6^2 + 8^2 - 2 \cdot 6 \cdot 8 \cos 150°$ $= 100 + 48\sqrt{3}$. Therefore, $K(ABC) = \frac{s^2\sqrt{3}}{4} = 25\sqrt{3} + 36 \approx 79$.

SOLUTION-ANSWER KEY

NINETEENTH ANNUAL H. S.
MATHEMATICS EXAMINATION

1968

19

Sponsored Jointly by the

MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

USE OF KEY

1. This Key is prepared for the convenience of teachers.
2. Some of the solutions may be intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This Solution-Answer Key validates our statement that no mathematics beyond intermediate algebra is needed to solve the problems posed.

National Office Pan American College, Edinburg, Texas 78539
New York Office Polytechnic Institute of Brooklyn, Brooklyn, N. Y. 11201

Note: The letter following the problem number refers to the correct choice of the five listed for that problem in the 1968 examination.

1. (D) Let C, d, respectively, represent the measures of the circumference and diameter. Then $C + P = \pi(d + \pi) = \pi d + \pi^2$. Since $C = \pi d$, $P = \pi^2$.

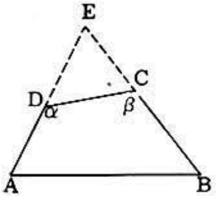
2. (B) $64^{x-1} \div 4^{x-1} = 4^{3x-3} \div 4^{x-1} = 4^{2x-2}$. Since $256^{2x} = 4^{8x}$, $4^{2x-2} = 4^{8x}$ $\therefore 2x - 2 = 8x$, $x = -\frac{1}{3}$.

3. (A) Method I. $y = m_2x + b$, $4 = m_2 \cdot 0 + b$ $\therefore b = 4$. Also $m_2m_1 = \frac{1}{3}m_2 = -1$ $\therefore m_2 = -3$
$\therefore y = -3x + 4$ or $y + 3x - 4 = 0$

Method II. $\frac{y - 4}{x - 0} = -3$ $\therefore y - 4 = -3x$ or $y + 3x - 4 = 0$.

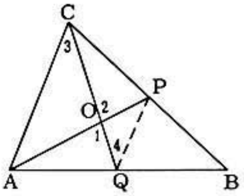
4. (C) $4 * 4 = \frac{4 \cdot 4}{4 + 4} = 2$ $\therefore 4 * (4 * 4) = \frac{4 \cdot 2}{4 + 2} = \frac{4}{3}$.

5. (A) $f(r) - f(r - 1) = \frac{1}{3}r(r + 1)(r + 2) - \frac{1}{3}(r - 1)(r)(r + 1) = \frac{1}{3}r(r + 1)(r + 2 - r + 1)$
$= r(r + 1)$.



6. (E) Method I. $S = \angle CDE + \angle DCE = 180° - \alpha + 180° - \beta$
$= 360° - (\alpha + \beta)$
$S' = \angle BAD + \angle ABC = 360 - (\alpha + \beta)$ $\therefore r = S/S' = 1$.

Method II. $S = \angle CDE + \angle DCE = 180° - \angle E$
$S' = \angle BAD + \angle ABC = 180° - \angle E$ $\therefore r = S/S' = 1$.



7. (E) Since $OQ = 3$, $CQ = 9$. But $OP = \frac{1}{3}AP$ and we can establish an arbitrary length for AP and, hence, for OP, as follows:
Draw $CQ = 9$ with $OQ = 3$. Through O draw OP of any length and extend PO through O to A so that $AP = 3(OP)$. Point B is then the intersection of AQ and CP and, thus, $\triangle ABC$ has AP and CQ as medians:

(1) $\frac{AO}{OP} = \frac{CO}{OQ}$, $\angle 1 = \angle 2$, $\triangle AOC \sim \triangle POQ$

(2) $\therefore \frac{AC}{QP} = \frac{CO}{OQ} = \frac{2}{1}$, $\angle 3 = \angle 4$, $QP \parallel AC$

(3) $\therefore \frac{AB}{QB} = \frac{CB}{PB} = \frac{AC}{QP} = \frac{2}{1}$ so that Q, P are midpoints.

8. (B) Let N represent the original number; the result obtained is $\frac{1}{6}N$; the correct result is 6N. The error, then, is $6N - \frac{1}{6}N$. Therefore, the per cent error is
$\frac{6N - N/6}{6N} \times 100 = \frac{3500}{36} \approx 97$.

9. (E) Since $2 + 2 \neq 2(2 - 2)$, $x \neq 2$ and since $-2 + 2 \neq -2(-2 - 2)$, $x \neq -2$.
For $x > 2$, $x + 2 = 2(x - 2)$, $x = 6$
For $-2 < x < 2$, $x + 2 = -2(x - 2)$, $x = 2/3$
For $x < -2$, $-(x + 2) = -2(x - 2)$, $x = 6$, a contradiction since $6 \not< -2$.
$\therefore$ the sum is $6 + 2/3 = 6\,2/3$

10. (C)

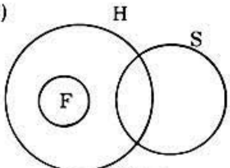


fig. 1

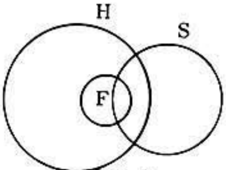


fig. 2

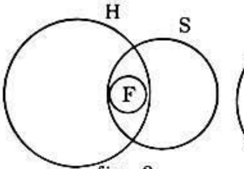


fig. 3

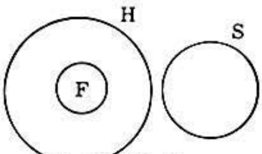


fig. 4

(A) is invalid by figures 1, 4
(B) is invalid by figure 3
(C) is valid in all cases (since some students exist who are not honest and, hence, they can't be fraternity members all of whom are honest)
(D) is invalid by figures 2, 3
(E) is invalid by figures 2, 3

11. (B) $\frac{60}{360} \times 2\pi r_1 = \frac{45}{360} \times 2\pi r_2 \therefore \frac{r_1}{r_2} = \frac{3}{4} \therefore \frac{\text{Area (I)}}{\text{Area (II)}} = \left(\frac{3}{4}\right)^2 = \frac{9}{16}$

12. (C) Method I. Since $(7\tfrac{1}{2})^2 + 10^2 = (12\tfrac{1}{2})^2$, the triangle is right with hypotenuse $12\tfrac{1}{2}$. Therefore, $D = 2R = 12\tfrac{1}{2}$, $R = {}^{25}\!/_4$.

Method II. $R = \frac{abc}{4K} = \frac{abc}{4\sqrt{s(s-a)(s-b)(s-c)}} = \frac{(12\tfrac{1}{2})(10)(7\tfrac{1}{2})}{4\sqrt{15(2\tfrac{1}{2})(5)(7\tfrac{1}{2})}} = \frac{25}{4}$.

13. (B) $m + n = -m$ and $mn = n \therefore m = 1,\ n = -2 \therefore m + n = -1$.

14. (E) Method I. By subtraction we obtain $x - y = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$

$\therefore (x - y)\left(1 - \frac{1}{xy}\right) = 0$, $x = y$ (The result $y = \frac{1}{x}$ is rejected. Why?)

Method II. $xy = y + 1$ and $xy = x + 1 \therefore y + 1 = x + 1,\ x = y$.

Method III. Since $y = 1 + \frac{1}{x}$ and $x = 1 + \frac{1}{y}$, $y = 1 + \frac{1}{1 + \frac{1}{y}}$ and $x = 1 + \frac{1}{1 + \frac{1}{x}}$.

Therefore, $y^2 - y - 1 = 0$ and $x^2 - x - 1 = 0$. Let the roots of $z^2 - z - 1 = 0$ be r and s. Then the given equations imply that, when $x = r$, so does $y = r$ or that, when $x = s$, so does $y = s$. $\therefore y = x$.

Note. For all three methods, since $y = x \neq 0$, y cannot equal any of the other choices shown.

15. (D) Method I. Set $P = (2k - 1)(2k + 1)(2k + 3)$. If $2k - 1 = 3m$ then $3 | P$. If $2k - 1 = 3m + 1$, then $2k + 1 = 3m + 3$ and so $3 | P$. If $2k - 1 = 3m - 1$, then $2k + 3 = 3m + 3$ and so $3 | P$.

Method II. Set $P = (2k - 1)(2k + 1)(2k + 3) = 8k^3 + 12k^2 - 2k - 3$. $\therefore P = (9k^3 + 12k^2 - 3k - 3) - k^3 + k = 3Q - (k - 1)(k)(k + 1)$. Since the product of three consecutive integers is divisible by 3, $P = 3Q - 3R$. Hence, $3 | P$.

Method III. Consider the three consecutive integers k, $k + 1$, $k + 2$; one of these is divisible by 3 (see Method II). If $k + 1$ is divisible by 3, then so is $k + 1 + 3 = k + 4$. Therefore, one of k, $k + 2$, $k + 4$ is divisible by 3. Therefore, if $P = k(k + 2)(k + 4)$ with k odd, P is divisible by 3.

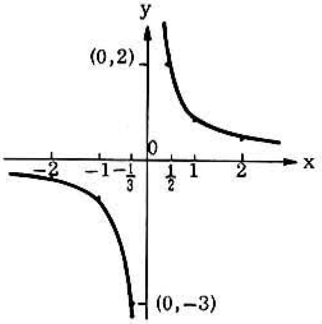
Note. For all three methods, since the greatest common factor of $1 \cdot 3 \cdot 5$ and $7 \cdot 9 \cdot 11$ is 3, no number > 3 is an exact divisor for all P.

16. (E) Method I. Let $y = \frac{1}{x}$. Then (a) For $y > 0$, when $y < 2$, $\frac{1}{y} > \frac{1}{2} \therefore x > \frac{1}{2}$.

(b) For $y < 0$, when $y > -3$, $-y < 3$, $-\frac{1}{y} > \frac{1}{3}$, $\frac{1}{y} < -3$

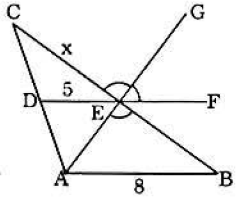
$\therefore x < -\frac{1}{3}$.

Method II. Let $y = \frac{1}{x}$ $\therefore xy = 1$; the graph is the two-branched hyperbola shown. At the point $x = \frac{1}{2}$, $y = 2$. When $y < 2$, x is to right of $\frac{1}{2}$, that is, $x > \frac{1}{2}$. At the point $x = -\frac{1}{3}$, $y = -3$. When $y > -3$, x is to the left of $-\frac{1}{3}$, that is, $x < -\frac{1}{3}$.



17. (C) Since $x_k = (-1)^k$, $x_1 = -1$, $x_2 = 1, \ldots, x_{2r-1} = -1$, $x_{2r} = 1$. Therefore, for $n = 2r$, $x_1 + x_2 + \ldots + x_{2r} = 0$ and for $n = 2r - 1$, $x_1 + x_2 + \ldots + x_{2r-1} = -1$. Therefore, $f(n) = 0$ in the former case and $f(n) = -\frac{1}{n}$ in the latter case.

18. (D) $\angle FEG = \angle CEG$. But $\angle BAE = \angle FEG$, and $\angle BEA = \angle CEG$. $\therefore \angle BAE = \angle BEA$. $\therefore BE = AB = 8$ $\therefore \frac{8 + x}{x} = \frac{8}{5}$, $x = \frac{40}{3}$.



19. (E) $25q + 10d = 1000$, $q = 40 - \frac{2d}{5}$ $\therefore 40 > \frac{2d}{5}$, $d < 100$. But $d = 5k$ $\therefore 5k < 100$ $\therefore k \leq 19$ $\therefore n = 19$

20. (A) Method I. $a + (n-1)5 = 160$, $a = 160 - (n-1)5$
$(n-2)180 = \frac{1}{2}n[2a + (n-1)5] = \frac{1}{2}n[320 - (n-1)5]$
$n^2 + 7n - 144 = 0 = (n+16)(n-9)$ $\therefore n = 9$

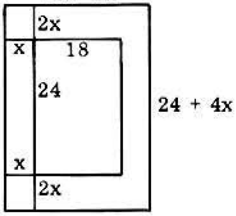
Method II. The exterior angles are 20, 25, 30, . . . ; their sum is 360°
$\therefore 360 = \frac{1}{2}n[40 + (n-1)5]$ $\therefore n^2 + 7n - 144 = 0$ $\therefore n = 9$

21. (D) Each of 5!, 6!, . . . , 99! ends in zero, and, in consequence, their sum ends in zero. Therefore, the units' digit of S is determined by $1! + 2! + 3! + 4! = 1 + 2 + 6 + 24 = 33$. Therefore, the units' digit of S is 3.

22. (E) Let the sides be s_1, s_2, s_3, s_4. Since it is given that $s_1 + s_2 + s_3 + s_4 = 1$, $s_1 + s_2 + s_3 = 1 - s_4$. But $s_1 + s_2 + s_3 > s_4$. Therefore, $0 < 1 - 2s_4$, $s_4 < \frac{1}{2}$. Similar reasoning applied to s_1, s_2, s_3, in turn, shows that each side is less than $\frac{1}{2}$. Conversely, a quadrilateral exists if $s_1 + s_2 + s_3 > s_4$, $s_1 + s_2 + s_4 > s_3$, $s_1 + s_3 + s_4 > s_2$, and $s_2 + s_3 + s_4 > s_1$. It is given that $s_1 + s_2 + s_3 + s_4 = 1$ and $s_1 < \frac{1}{2}$, $s_2 < \frac{1}{2}$, $s_3 < \frac{1}{2}$, $s_4 < \frac{1}{2}$. Therefore, $s_2 + s_3 + s_4 > \frac{1}{2}$ and so $s_2 + s_3 + s_4 > s_1$. In a similar manner we prove the other three cases.

23. (B) $\log(x+3) + \log(x-1) = \log(x^2 - 2x - 3)$ $\therefore \log(x+3)(x-1) = \log(x^2 - 2x - 3)$. $\therefore x^2 + 2x - 3 = x^2 - 2x - 3$, $x = 0$. But when $x = 0$, neither $\log(x-1)$ nor $\log(x^2 - 2x - 3)$ is a real number.

24. (C) $(24 + 4x)(18 + 2x) = 2(24 \times 18)$
$\therefore x^2 + 15x - 54 = 0$, $x = 3$
$\therefore 24 + 4x = 36$, $18 + 2x = 24$, $24 : 36 = 2 : 3$



25. (C) Let z represent the number of yards Ace runs when he is caught. Let a represent Ace's speed; then xa represents Flash's speed. Then

$$\frac{y+z}{xa} = \frac{z}{a},\ z = \frac{y}{x-1} \quad \therefore y + z = y + \frac{y}{x-1} = \frac{xy}{x-1}$$

26. (E) $S = 2 + 4 + 6 + \ldots + 2N = 2(1 + 2 + 3 + \ldots + N) = N(N+1)$. Since $S > 10^6$, $N(N+1) > 10^6$, $N \approx 10^3$. When $N = 10^3 - 1$, $S < 10^6$.
$\therefore N \geq 10^3$ $\therefore N(\text{smallest}) = 10^3 = 1000$; the sum of its digits is 1.

27. (B) $S_n = 1 - 2 + 3 - 4 + \ldots + (-1)^{n-1} n,\ n = 1, 2, 3, \ldots$

For n even, $S_n = 1 + 2 + 3 + \ldots + n - 2(2 + 4 + \ldots + n) = \frac{1}{2} n(n+1) - 4 \cdot \frac{1}{2} \cdot \frac{n}{2}\left(1 + \frac{n}{2}\right)$

$$\therefore S_n = \tfrac{1}{2} n(n+1) - \tfrac{1}{2} n(n+2) = -\tfrac{1}{2} n$$

For n odd, $S_n = 1 + 2 + 3 + \ldots + n - 2\big(2 + 4 + \ldots + (n-1)\big)$

$$= \frac{1}{2} n(n+1) - 4 \cdot \frac{1}{2} \cdot \frac{n-1}{2}\left(1 + \frac{n-1}{2}\right)$$

$$\therefore S_n = \tfrac{1}{2} n(n+1) - \tfrac{1}{2}(n-1)(n+1) = \tfrac{1}{2}(n+1)$$

$\therefore S_{17} = 9,\ S_{33} = 17,\ S_{50} = -25 \quad \therefore S_{17} + S_{33} + S_{50} = 1.$

28. (D) $\frac{a+b}{2} = 2\sqrt{ab},\ a + b = 4\sqrt{ab},\ a^2 + 2ab + b^2 = 16ab$

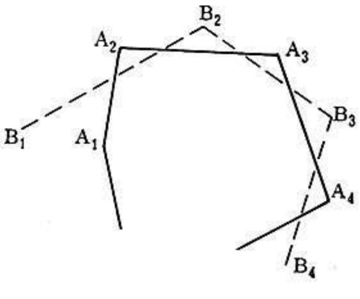
$$\therefore a^2 - 14ab + b^2 = 0,\ a = \frac{14b + b\sqrt{192}}{2} \quad \therefore \frac{a}{b} = \frac{14 + \sqrt{192}}{2} \approx \frac{14 + 14}{2} = 14$$

Note. The value $\frac{a}{b} = \frac{14 - \sqrt{192}}{2} \approx \frac{14 - 14}{2} = 0$ is not listed

29. (A) Method I. Since $0 < x < 1$, x to any positive power is less than 1.
$\therefore \frac{x}{y} = x^{1-x} < 1$ so that $x < y$, $\frac{z}{y} = x^{y-x} < 1$ so that $z < y$, and $\frac{x}{z} = x^{1-y} < 1$ so that $x < z$. $\therefore x < z < y$.

Method II. Since $0 < x < 1$, $\log x < 0$. It follows that $\log x < x \log x$ or $\log x < \log x^x$ so that $x < x^x = y$. Since $z = x^{(x^x)} = x^y$, $\log z = y \log x$, and, since $y > x$, $y \log x < x \log x$ or $\log z < \log y$, so that $z < y$. However, since $0 < y < 1$ and $\log x < 0$, $y \log x > \log x$ or $\log z > \log x$, so that $x < z$. Finally $x < z < y$.

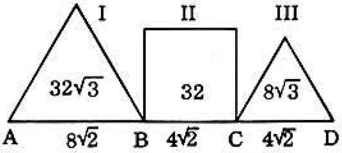
30. (A) Each side of P_1 can (1) fail to enter the boundary or interior of P_2, or (2) it can enter and terminate, or (3) it can continue on to the exterior. Therefore, at most, each side of P_1 meets two sides of P_2, so that the maximum number of intersections is $2n_1$. This maximum is attained as follows:



31. (D) $L = 16\sqrt{2}$, $L' = 16\sqrt{2} - 2\sqrt{2} = 14\sqrt{2} = 8\sqrt{2} + x + 4\sqrt{2}$

$\therefore x = 2\sqrt{2}$ $\therefore II' = (2\sqrt{2})^2 = 8$

$\therefore II - II' = 32 - 8 = 24$ (decrease), $\frac{24}{32} \times 100 = 75\%$



32. (C) Let r_A represent the speed of A, and let r_B represent the speed of B.

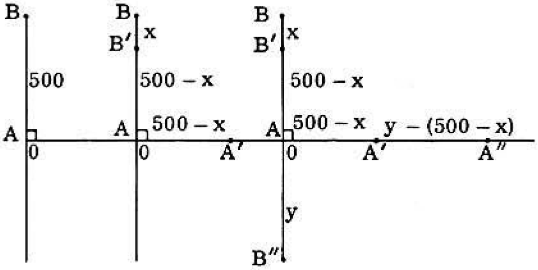
$\frac{x}{r_B} = 2$, $\frac{500 - x}{r_A} = 2$, $x = 2r_B$

$\therefore \frac{500 - 2r_B}{r_A} = 2$, $r_A + r_B = 250$

$\frac{500 - x + y}{r_B} = 8$, $\frac{y - 500 + x}{r_A} = 8$

$$\left.\begin{array}{l} y - 500 + x = 8r_A \\ y + 500 - x = 8r_B \end{array}\right\} \begin{array}{l} 2y = 8(r_A + r_B) = 8 \cdot 250 \\ y = 1000 \end{array}$$

$8r_B = 1000 + 500 - 2r_B$, $r_B = 150$ $\therefore r_A = 100$ $\therefore r_A : r_B = 2 : 3$



33. (A) $a_1 \cdot 7^2 + a_2 \cdot 7 + a_3 = a_3 \cdot 9^2 + a_2 \cdot 9 + a_1$, $48a_1 = 80a_3 + 2a_2$, $3a_1 = 5a_3 + \frac{2a_2}{16}$

$\therefore 2a_2$ must be divisible by 16. But $a_2 \neq 8$ $\therefore a_2 = 0$

Check $3a_1 = 5a_3$ $\therefore a_1 = 5$, $a_3 = 3$

$5 \cdot 7^2 + 0 \cdot 7 + 3 = 248 = 3 \cdot 9^2 + 0 \cdot 9 + 5$

34. (B) Let y_1, n_1, respectively, represent the numbers voting for, against the bill originally, and let y_2, n_2, respectively, represent the numbers voting for, against the bill on the re-vote.

$y_1 + n_1 = 400$, $y_2 + n_2 = 400$, $y_2 = \frac{12}{11}n_1$, $y_2 - n_2 = 2(n_1 - y_1)$

$\therefore 2(y_1 + n_1) = 2(y_2 + n_2)$ and $2(-y_1 + n_1) = y_2 - n_2$

$\therefore 4n_1 = 3y_2 + n_2 = 3 \cdot \frac{12}{11}n_1 + n_2$ $\therefore n_1 = \frac{11}{8}n_2$

$\therefore y_2 = \frac{12}{11} \cdot \frac{11}{8}n_2 = \frac{3}{2}n_2$ so that $\frac{3}{2}n_2 + n_2 = 400$, $n_2 = 160$, $n_1 = 220$

$\therefore n_1 - n_2 = 60$ or, since $y_2 = 240$ and $y_1 = 180$, $y_2 - y_1 = 60$.

35. (D) $CD = 2GD = 2\sqrt{a^2 - (a - 2x)^2} = 2\sqrt{4ax - 4x^2}$

$FE = 2HF = 2\sqrt{a^2 - (a - x)^2} = 2\sqrt{2ax - x^2}$

$$\frac{K}{R} = \frac{\frac{1}{2}x\,[2\sqrt{4ax - 4x^2} + 2\sqrt{2ax - x^2}]}{x\,[2\sqrt{2ax - x^2}]} = \frac{\frac{1}{2} \cdot 2\sqrt{x}\,(\sqrt{4a - 4x} + \sqrt{2a - x})}{2\sqrt{x}\,(\sqrt{2a - x})}$$

$$= \frac{2\sqrt{a - x} + \sqrt{2a - x}}{2\sqrt{2a - x}} = \frac{\sqrt{a - x}}{\sqrt{2a - x}} + \frac{1}{2}$$

As OG increases toward the value a, x becomes arbitrarily close to zero, so that $\frac{\sqrt{a - x}}{\sqrt{2a - x}}$ becomes arbitrarily close to $\frac{\sqrt{a}}{\sqrt{2a}} = \sqrt{\frac{1}{2}}$. Therefore, as OG increases toward the value a, $\frac{K}{R}$ becomes arbitrarily close to $\sqrt{\frac{1}{2}} + \frac{1}{2} = \frac{1}{\sqrt{2}} + \frac{1}{2}$

SOLUTION-ANSWER KEY

TWENTIETH ANNUAL H. S.
MATHEMATICS EXAMINATION

1969

20

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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

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1. This Key is prepared for the convenience of teachers.
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Note: The letter following the problem number refers to the correct choice of the five listed for that problem in the 1969 examination.

1\. (B) $\frac{a+x}{b+x} = \frac{c}{d}$ $\therefore ad + xd = bc + xc$ $\therefore ad - bc = x(c-d)$ $\therefore x = \frac{ad-bc}{c-d}$

Comment. If $\frac{c}{d} = \frac{b}{a}$, $x = -(a+b)$.

2\. (A) Let C represent the cost (in dollars). Then $x = C - .15C = .85C$, and $y = C + .15C = 1.15C$. Therefore, $y : x = (1.15C) : (.85C) = 23 : 17$.

3\. (E) Method I. $N - 1 = 11\,000_2 - 1 = 1 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2 + 0 - 1$
$\therefore N - 1 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 1 \cdot 2 + (2-1) = 10\,111_2$

Method II. $N = 11\,000_2 = 24_{10}$ and $24_{10} - 1 = 23_{10}$
Since $23_{10} = 10\,111_2$, $N - 1 = 10\,111_2$

4\. (E) By definition $(3,2) * (0,0) = (3-0, 2-0) = (3,2)$ and $(x,y) * (3,2) = (x-3, y-2)$
$\therefore x - 3 = 3 - 0$ and $y - 2 = 2 - 0$. $\therefore x = 6$ (and $y = 4$).

5\. (B) $N - 4 \cdot \frac{1}{N} = R$ $\therefore N^2 - RN - 4 = 0$. Let the values of N satisfying this equation be N_1 and N_2. Therefore, $N_1 + N_2 = R$. For example, if $R = 3$, then N_1, N_2 are $4, -1$ and the sum of 4 and -1 equals 3.

6\. (C) Let R be the larger radius, let r be the smaller radius, and let L (inches) be the length of the chord. Then $R^2 - r^2 = \left(\frac{L}{2}\right)^2$. Since $\pi R^2 - \pi r^2 = \frac{25\pi}{2}$, $R^2 - r^2 = \frac{25}{2}$.
$\therefore \left(\frac{L}{2}\right)^2 = \frac{25}{2}$ and $L = \sqrt{50} = 5\sqrt{2}$.

7\. (A) $y_1 = a + b + c$ and $y_2 = a - b + c$ $\therefore y_1 - y_2 = 2b$. Since $y_1 - y_2 = -6$, $2b = -6$, $b = -3$.

8\. (D) $x + 75 + 2x + 25 + 3x - 22 = 360$ $\therefore x = 47$, $\widehat{AB} = 122°$, $\widehat{BC} = 119°$, $\widehat{CA} = 119°$.

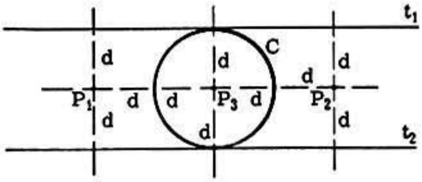
The interior angles of ΔABC are, in degrees, 61, $59\frac{1}{2}$, $59\frac{1}{2}$.

9\. (C) Method I. We have an arithmetic sequence with the first term $a = 2$, the common difference $d = 1$, and the last term $l = a + (n-1)d = 2 + (52-1)(1) = 53$. Since $S_{52} = \frac{52}{2}(2 + 53)$, AM. $= \frac{52/2(2+53)}{52} = \frac{55}{2} = 27\frac{1}{2}$.

Method II. Designate the terms of the arithmetic sequence by $u_1, u_2, \ldots, u_n$. Then A.M. $= \frac{1}{2}(u_1 + u_n)$ since $\frac{1}{2}(u_1 + u_2) = \frac{1}{2}(a + a + (n-1)d = \frac{1}{2} \cdot \frac{n(2a + (n-1)d)}{n} = \frac{1}{2} \cdot \frac{S_n}{n}$. $\therefore$ A.M. $= \frac{1}{2}(55) = 27\frac{1}{2}$.

Comment. Generally, A. M. $= \frac{u_i + u_{n+1-i}}{2}$, $i = 1, 2, \ldots, n$.

10\. (C) Points P_1, P_2, and P_3 are each d distance from tangent lines t_1 and t_2 and circle C.



11\. (B) Method I. Since PR + RQ is a minimum, points P_1, R_1 and Q are collinear. Therefore, $\frac{-2-m}{-1-1} = \frac{-2-2}{-1-4}$ so that $m = -\frac{2}{5}$.

Method II. Since PR + RQ is a minimum, points P_1, R_1 and Q are collinear. Therefore, PR + RQ = PQ so that $\sqrt{(1-(-1))^2 + (m+2)^2} + \sqrt{(4-1)^2 + (2-m)^2}$
$= \sqrt{(4-(-1))^2 + (2+2)^2}$

$\therefore\ \sqrt{4 + (m+2)^2} + \sqrt{9 + (2-m)^2} = \sqrt{25 + 16} = \sqrt{41}$
$\therefore\ \sqrt{4 + m^2 + 4m + 4} - \sqrt{41} = -\sqrt{9 + 4 - 4m + m^2}$
$\therefore\ 4 + m^2 + 4m + 4 - 2\sqrt{41}\ \sqrt{m^2 + 4m + 8} + 41 = 9 + 4 - 4m + m^2$
$\therefore\ 8m + 36 = 2\sqrt{41}\ \sqrt{m^2 + 4m + 8},\ 4m + 18 = \sqrt{41}\ \sqrt{m^2 + 4m + 8}$
$\therefore\ 16m^2 + 144m + 324 = 41m^2 + 164m + 328$
$\therefore\ 25m^2 + 20m + 4 = 0,\ 5m + 2 = 0,\ m = -\frac{2}{5}$

12. (A) Method I. $F = x^2 + \frac{8x}{3} + \frac{m}{2} = (x + r)^2 = x^2 + 2rx + r^2$

$\therefore 2r = \frac{8}{3}$ and $r^2 = \frac{m}{2}$. $\therefore r = \frac{4}{3}$ and $m = \frac{32}{9} = 3\frac{5}{9}$ and $3 < 3\frac{5}{9} < 4$.

Comment. When $m = \frac{32}{9}$, $F = (x + \frac{4}{3})^2$.

Method II. By completing the square we find that the trinomial $x^2 + \frac{8x}{3} + \frac{16}{9} = \left(x + \frac{4}{3}\right)^2$. Therefore, $x^2 + \frac{8x}{3} + \frac{16}{9} = x^2 + \frac{8x}{3} + \frac{m}{2}$, so that $\frac{m}{2} = \frac{16}{9}$ and $m = \frac{32}{9}$.

13. (B) $\pi R^2 = \frac{a}{b}(\pi R^2 - \pi r^2)$ $\therefore r^2\frac{a}{b} = R^2(\frac{a}{b} - 1)$ so that $\frac{R^2}{r^2} = \frac{a/b}{a/b - 1} = \frac{a}{a-b}$ and, hence, $R : r = \sqrt{a} : \sqrt{a-b}$.

14. (A) Method I. Since $\frac{x^2-4}{x^2-1} > 0$, $(x^2 - 1 > 0) \Rightarrow (x^2 - 4 > 0)$. Therefore, all real values of x such that $x > 2$ or $x < -2$ satisfy the inequality. Also $(x^2 - 1 < 0) \Rightarrow (x^2 - 4 < 0)$. Therefore, all real values of x such that $-1 < x < 1$ satisfy the inequality.

Method II. Since $\frac{x^2-4}{x^2-1} > 0$, $\frac{x^2-1}{x^2-1} - \frac{3}{x^2-1} > 0$. Therefore, $1 > \frac{3}{x^2-1}$. This latter inequality is satisfied by all x such that $x^2 - 1 > 3$, that is, by $x > 2$ or $x < -2$, and by all x such that $x^2 - 1 < 0$, that is, by $-1 < x < 1$.

15. (D) Since $AB = r$, the measure of angle A is 60 (degrees) and $AM = \frac{1}{2}r$. In right triangle MDA, since $AM = \frac{1}{2}r$ and the measure of angle AMD is 30 (degrees), $AD = \frac{1}{4}r$ and $MD = \frac{1}{4}r\sqrt{3}$.

Therefore, the area of $\triangle MDA = \frac{1}{2}(AD)(MD) = \frac{1}{2}\cdot\frac{r}{4}\cdot\frac{r\sqrt{3}}{4} = \frac{r^2\sqrt{3}}{32}$.

16. (E) $(a - b)^n = a^n - na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 - \dots$

$\therefore -n(kb)^{n-1}b + \frac{n(n-1)}{2}(kb)^{n-2}b^2 = 0$

$\therefore -k^{n-1} + \frac{n-1}{2}k^{n-2} = 0$. $\therefore n - 1 = \frac{2k^{n-1}}{k^{n-2}} \therefore n = 2k + 1$.

17. (D) $2^{2x} - 8\cdot 2^x + 12 = (2^x - 6)(2^x - 2) = 0$ $\therefore 2^x - 6 = 0$, $2^x = 6$.

$\therefore x \log 2 = \log 6 = \log 2 + \log 3$ $\therefore x = 1 + \frac{\log 3}{\log 2}$.

Comment. The equation is also satisfied by $x = 1$.

18. (B) Each of the graphs is a pair of non-parallel straight lines. Each line of the first pair intersects the second pair in two distinct points, giving a total of four points.

Comment. The intersection points are $(0, 2)$, $(-1, 1)$, $(1, 1)$, $(\frac{13}{17}, \frac{29}{17})$.

19. (B) $x^4y^4 - 10x^2y^2 + 9 = (x^2y^2 - 1)(x^2y^2 - 9) = 0$ $\therefore x^2y^2 = 1$ or $x^2y^2 = 9$. $\therefore xy = +1$ or -1, $xy = +3$ or -3. Ordered pairs of positive integral values satisfying these equations are $(1, 1)$, $(1, 3)$, $(3, 1)$.

20. (C) Let N_1 represent the first factor of P and let N_2 represent the second factor. Then $(4)(10)^{18} > N_1 > (3)(10)^{18}$ and $(\frac{1}{2})(10)^{15} > N_2 > (\frac{1}{3})(10)^{15}$. Therefore, $(2)(10)^{33} > N_1N_2 > (1)(10)^{33}$ so that the number of digits in $P(= N_1N_2)$ is 34.

21. (E) Method I. Let OA be the x-intercept of the straight line $x + y = \sqrt{2m}$ and let OB be its y- intercept. Then $OA = OB = \sqrt{2m}$ so that AOB is a 45°-45°-90° triangle.

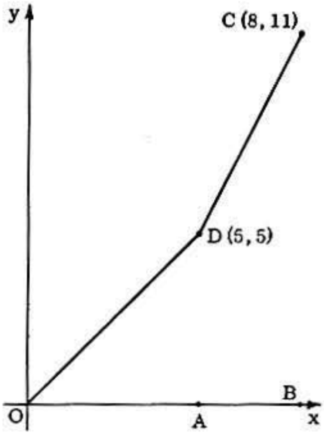
Let the point of tangency be labeled C. Since the graph of $x^2 + y^2 = m$ is a circle, OC, the radius r of the circle is perpendicular to AB; that is, OC is the median to hypotenuse AB. Therefore, $OC = r = \frac{1}{2} \cdot 2\sqrt{m} = \sqrt{m}$, and this is the same value of the radius given by the equation $x^2 + y^2 = m = (\sqrt{m})^2$. Consequently, m may be any non-negative real number.

Method II. Using the distance formula from a line to a point, we have $d = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$ where $Ax + By + C = 0$ is the given line and (x_1, y_1) is the given point. The given line is $x + y - \sqrt{2m} = 0$ and the given point is (0, 0). Therefore, the required $d = \frac{|0 + 0 - \sqrt{2m}|}{\sqrt{1^2 + 1^2}} = \sqrt{m}$. But d = r, the radius of the circle $x^2 + y^2 = m = (\sqrt{m})^2$. Hence, m may be any non-negative real number.

Method III. Let (x, y) be the intersection point of the graphs of $x^2 + y^2 = m$ and $x + y = \sqrt{2m}$. This pair of equations yields $xy = \frac{m}{2}$ and $x + y = \sqrt{2m}$.

Thus x and y may be taken as the roots of $t^2 - \sqrt{2m}\ t + \frac{m}{2} = 0$. For tangency the discriminant of this last equation must equal zero. Therefore $2m - 4 \cdot \frac{m}{2} = 0$, an identify in m. Hence, m may be any non-negative real number.

22. (C) The total area consists of the triangular region OAD and the trapezoidal region ABCD.
$\therefore K = \frac{1}{2}(5)(5) + \frac{1}{2}(3)(5 + 11) = 36.5$



23. (A) Consider the integer $n! + k$ where $1 < k < n$. Since n! contains each of the factors 1, 2, 3, . . . , n, it contains the factor k. Since $n! + k$ can be written as the product of two factors, one of which is k, it is composite. Hence, there are no primes between $n! + 1$ and $n! + n$.

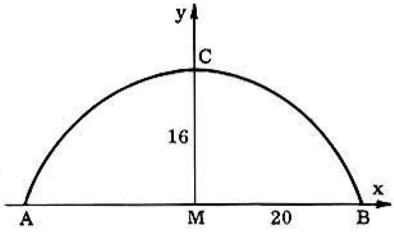
Comment. The conclusion holds for $n \geqq 1$.

24. (E) We may write $P = Q_1D + R$ where $0 < R < D$, and $P' = Q_2D + R'$ where $0 \leqq R' < D$. Therefore, $PP' = (Q_1D + R)(Q_2D + R') = Q_1Q_2D^2 + Q_2DR + Q_1DR' + RR'$. But $RR' = Q_4D + r'$. Therefore, $PP' = Q_1Q_2D^2 + Q_2DR + Q_1DR' + Q_4D + r' = D(Q_1Q_2D + Q_2R + Q_1R' + Q_4) + r'$. But $PP' = Q_5D + r$. Therefore $Q_5 = Q_1Q_2D + Q_2R + Q_1R' + Q_4$ and $r = r'$.

25. (D) Since $\log_2 a + \log_2 b \leqq 6$, $\log_2 ab \leqq 6$. $\therefore ab \leqq 2^6 = 64$. Since ab is a maximum, ab = 64. Therefore, the least value that can be taken on by a + b is 8 + 8 = 16. [If P = ab,

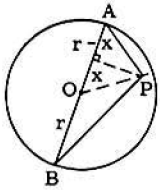
the least value of a + b is $\sqrt{P} + \sqrt{P} = 2\sqrt{P}$. One way of proving this theorem is to consider the minimum value of $f = a^2 - aS + P$ where $S = a + b$.

Hint. $S = a + b = a + \frac{P}{a}$.]



26. (B) Choose axes so that the x-axis coincides with the span AB and the y-axis coincides with the height MC. We may then write the equation for the parabola as $y = ax^2 + 16$. Since $0 = a \cdot 20^2 + 16$, $a = -\frac{1}{25}$, so that $y = -\frac{x^2}{25} + 16$. When $x = 5$, $y = -1 + 16 = 15$.

27. (E) Let v_n be the speed for the n-th mile. Then $v_n = \frac{k}{n-1}$, $n \geqq 2$. $\therefore v_2 = \frac{k}{2-1} = k$. Since $T = \frac{D}{R}$, $2 = \frac{1}{v_2} = \frac{1}{k}$ so that $k = \frac{1}{2}$. Therefore, $v_n = \frac{1}{2(n-1)}$. $\therefore T_n = \frac{1}{v_n} = 2(n-1)$.

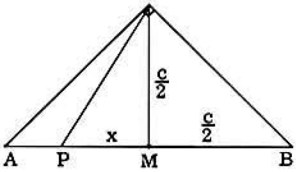


28. (E) Method I. $AP^2 - (r-x)^2 = OP^2 - x^2$, $BP^2 - (r+x)^2 = OP^2 - x^2$.
$\therefore AP^2 + BP^2 - 2r^2 - 2x^2 = 2OP^2 - 2x^2$. $\therefore 3 = 2 + 2OP^2$.
$\therefore OP = \frac{1}{\sqrt{2}}$, that is P may be any point of a circle with radius $\frac{1}{\sqrt{2}}$.

Method II. Since $AP^2 + BP^2 = 3 < 4 = AB^2$, angle APB > 90° so that P is interior to the circle. Therefore, there are many points P satisfying the given conditions such that $0 < AP \leqq \sqrt{\frac{3}{2}}$.

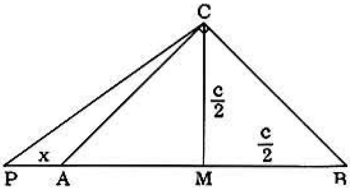
29. (C) Method I. $x = t^{\frac{1}{t-1}}$, $y = t^{\frac{t}{t-1}} = t^{1+\frac{1}{t-1}} = t \cdot t^{\frac{1}{t-1}}$ $\therefore y = tx$. Also $y = t^{\left(\frac{1}{t-1}\right)t} = x^t$
$\therefore y^x = (x^t)^x = x^{tx}$ $\therefore y^x = x^y$.

Method II. $x = t^{\frac{1}{t-1}}$, $y = t^{\frac{t}{t-1}}$ $\therefore \frac{y}{x} = t^{\frac{t}{t-1} - \frac{1}{t-1}} = t$ $\therefore x = \left(\frac{y}{x}\right)^{\frac{1}{y/x-1}} = \left(\frac{y}{x}\right)^{\frac{x}{x-y}}$
$\therefore x^{y-x} = \frac{y^x}{x^x}$ $\therefore x^y = y^x$

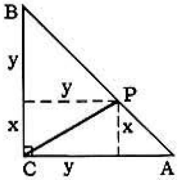


30. (D) Method I. (see fig. 1) Let M be the midpoint of hypotenuse c. Then $S = AP^2 + PB^2 = \left(\frac{c}{2} - x\right)^2 + \left(\frac{c}{2} + x\right)^2 = \frac{c^2}{2} = 2x^2$. But $x^2 = CP^2 - \left(\frac{c}{2}\right)^2$

$$\therefore S = \frac{c^2}{2} + 2CP^2 - \frac{c^2}{2} = 2CP^2$$

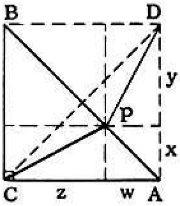


(see fig. 2) $S = AP^2 + PB^2 = x^2 + (c+x)^2 = c^2 + 2cx + 2x^2$. But $CP^2 = \left(x + \frac{c}{2}\right)^2 + \left(\frac{c}{2}\right)^2 = x^2 + cx + \frac{c^2}{2}$. $\therefore S = 2CP^2$

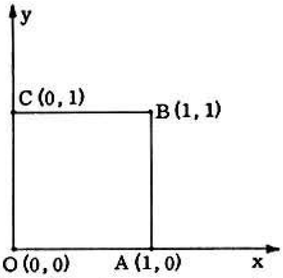


Method II. (for P interior to AB, fig. 3)
$S = AP^2 + PB^2$, $AP^2 = x^2 + x^2 = 2x^2$, $PB^2 = y^2 + y^2 = 2y^2$. $\therefore S = 2x^2 + 2y^2 = 2(x^2 + y^2) = 2CP^2$

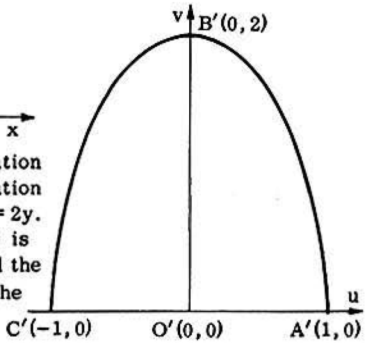
Method III. (for P interior to AB, fig. 4)
Complete ΔACB to square ACBD. Draw PD and draw perpendiculars from P to AC, CB, BD, DA. $AP^2 = x^2 + w^2$, $BP^2 = y^2 + z^2$, $CP^2 = x^2 + z^2$, $DP^2 = y^2 + w^2$. $\therefore AP^2 + BP^2 = x^2 + y^2 + z^2 + w^2$, $CP^2 + DP^2 = x^2 + y^2 + z^2 + w^2$ $\therefore S = AP^2 + BP^2 = CP^2 + DP^2$. But DP = CP (Why?) $\therefore S = 2CP^2$



31. (D) The image of A(1, 0) in the xy-plane is A′ (1, 0) in the uv-plane since $u = 1^2 - 0^2 = 1$ and $v = 2(1)(0) = 0$. The image of B(1, 1) is B′ (0, 2), the image of C (0, 1) is C′(−1, 0), and the image of O (0, 0) is O′(0, 0).



The image of the straight line AB whose xy− equation is x = 1 is the parabolic arc A′B′ whose uv− equation is $v^2 = 4(1 - u)$, a parabola, since $u = 1 - y^2$ and $v = 2y$. The image of the straight line OA (equation y = 0) is the straight line O′A′ (equations $v = 0$, $u = x^2$), and the image of the straight line OC (equation x = 0) is the straight line O′C′ (equations $v = 0$, $u = -y^2$).



32. (C)
$$\left.\begin{array}{l} u_{n+1} - u_n = 3 + 4(n-1)\\ u_n - u_{n-1} = 3 + 4(n-2)\\ \quad\cdot \qquad\qquad \cdot\\ \quad\cdot \qquad\qquad \cdot\\ \quad\cdot \qquad\qquad \cdot\\ u_3 - u_2 = 3 + 4(2-1)\\ u_2 - u_1 = 3 \end{array}\right\}$$
By "telescopic" addition we obtain $u_{n+1} - u_1 = 3n + 4 \cdot \frac{1}{2}(n-1)(n-1+1)$. Since $u_1 = 5$, $u_{n+1} = 2n^2 + n + 5$. Therefore, $u_n = 2(n-1)^2 + (n-1) + 5 = 2n^2 - 3n + 6$.

Since 2 − 3 + 6 = 5, the correct answer is (C).

33. (A) Let a_1 and d_1 be the first term and the common difference, respectively, of the first series, and let a_2 and d_2 be the first term and the common difference, respectively, of the second series.
Then $\frac{Sn}{Tn} = \frac{2a_1 + (n-1)d_1}{2a_2 + (n-1)d_2} = \frac{7n+1}{4n+27}$.
Let u_{11} and v_{11}, respectively, be the eleventh terms of the two series whose sums are Sn and Tn. Then
$\frac{u_{11}}{v_{11}} = \frac{a_1 + 10d_1}{a_2 + 10d_2} = \frac{2a_1 + 20d_1}{2a_2 + 20d_2}$. For n = 21, $\frac{Sn}{Tn} = \frac{2a_1 + 20d_1}{2a_2 + 20d_2}$.
Therefore, $\frac{u_{11}}{v_{11}} = \frac{7(21)+1}{4(21)+27} = \frac{148}{111} = \frac{4}{3}$.

34. (B) $x^{100} = Q(x)(x^2 + 3x + 2) + R(x) = Q(x)(x-2)(x-1) + R(x)$ where $R(x) = ax + b$.
$\therefore R(1) = 1^{100} = 1 = a + b$ and $R(2) = 2^{100} = 2a + b$
$\therefore a = 2^{100} - 1$, $b = 2 - 2^{100}$. $\therefore R(x) = x(2^{100} - 1) + 2 - 2^{100} = x \cdot 2^{100} - x + 2 - 2^{100}$.
$\therefore R(x) = 2^{100}(x-1) - (x-2)$.

35. (B) $$r = \frac{L(-m) - L(m)}{m} = \frac{-\sqrt{6-m} + \sqrt{6+m}}{m}$$
$$\therefore r = \frac{-\sqrt{6-m} + \sqrt{6+m}}{m} \cdot \frac{-\sqrt{6-m} - \sqrt{6+m}}{-\sqrt{6-m} - \sqrt{6+m}} = \frac{-2m}{-m(\sqrt{6-m} + \sqrt{6+m})}$$
$$\therefore r = \frac{2}{\sqrt{6-m} + \sqrt{6+m}}.\ \text{As } m \to 0,\ r \to \frac{2}{\sqrt{6} + \sqrt{6}} = \frac{1}{\sqrt{6}}.$$

SOLUTION-ANSWER KEY

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MATHEMATICS EXAMINATION

1970

21

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USE OF KEY

1. This Key is prepared for the convenience of teachers.
2. Some of the solutions may be intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This Solution-Answer Key validates our statement that no mathematics beyond intermediate algebra is needed to solve the problems posed.

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Note: The letter following the problem number designates the correct choice of the five listed for the problem in the 1970 examination.

1. (E) If x denotes the given expression, then $x^2 = 1 + \sqrt{2}$ and $x^4 = 3 + 2\sqrt{2}$.

2. (A) Let the common perimeter be p. The area of the circle is $\frac{1}{4}p^2/\pi$ and the area of the square is $p^2/16$, so the ratio is $4/\pi$.

3. (C) $x - 1 = 2^p$, $y - 1 = 2^{-p}$ $\therefore (x-1)(y-1) = 1 \therefore y = x/(x-1)$.

4. (B) Let the consecutive integers be $(n-1)$, n, $(n+1)$. Then the sum of their squares is $3n^2 + 2$ which is never divisible by 3 but is 77 when n = 5 and is then divisible by 11 as required in (B). Choices (A), (C), and (D) are eliminated by taking the middle integer n equal to 0, 4, and 2 respectively.

5. (D) Since i^2 and i^4 are −1 and 1, $f(i) = (1-1)/(1+i) = 0/(1+i) = 0$.

6. (B) $x^2 + 8x = (x+4)^2 - 16$ which is least (−16) when $(x+4)^2 = 0$ or when $x = -4$.

7. (E) Triangle ABX is equilateral with altitude $\frac{1}{2}s\sqrt{3}$. The required distance is $s - \frac{1}{2}s\sqrt{3}$ or $\frac{1}{2}s(2 - \sqrt{3})$.

8. (B) $225 = 8^a = 2^{3a} \therefore 15 = 2^{3a/2}$. Also $15 = 2^b \therefore 3a/2 = b \therefore a = 2b/3$.

9. (C) $AP = \frac{2}{5}AB$ and $AQ = \frac{3}{7}AB \therefore PQ = \left(\frac{3}{7} - \frac{2}{5}\right)AB = AB/35 = 2 \therefore AB = 70$.

10. (D) $F = .4 + .081 + .00081 + \cdots = .4 + .081(1 + .01 + .0001 + \cdots)$

$$= .4 + .081 \times \frac{1}{1 - .01} = (4/10) + (81/1{,}000)(1/.99) = 53/110.$$

Denominator − Numerator = 110 − 53 = 57.

11. (E) Write $f(x) = 2x^3 - hx + k$. Then $f(-2) = -16 + 2h + k = 0$ and $f(1) = 2 - h + k = 0$. Hence $h = 6$, $k = 4$ and $|2h - 3k| = 0$.

12. (C) The length of side AD is 2r as are the distances from the midpoint of the diagonal AC to AD and BC. Hence the length of AB is 4r and the area of the rectangle is $(2r)(4r) = 8r^2$.

13. (D) $(a * b)^n = a^{bn}$ and $a * (bn) = a^{bn}$.

14. (A) The roots are $\frac{1}{2}(-p + \sqrt{p^2 - 4q})$ and $\frac{1}{2}(-p - \sqrt{p^2 - 4q})$. The difference is $\sqrt{p^2 - 4q} = 1$. Hence $p^2 = 4q + 1$ and $p = \sqrt{4q + 1}$.

15. (E) The trisection points are (−1,3) and (2,1). The slopes of lines joining these points to the point (3,4) are $\frac{1}{4}$ and 3. Only the line of (E) has slope $\frac{1}{4}$ and none of the lines has slope 3.

16. (C) Substitution yields F(4) = 2, F(5) = 3, and finally F(6) = 7.

17. (E) Choices (A) and (B) are both false when p and q are 2 and 1 respectively, and choices (C) and (D) are both false when p and q are 1 and −2 respectively. Hence none of these holds for all values of p and q.

18. (A) The required difference is 2 because it is positive and its square is 4.

19. (C) Let a be the first term of the infinite series. Using the formula for the sum, we have $a/(1-r) = 15$. Also $a^2/(1-r^2) = 45$. Dividing gives $a/(1+r) = 3$. $\therefore a = 3 + 3r$ and $a = 15 - 15r \therefore a = 5$.

20. (A) Regardless of how the diagram is drawn, M lies on the perpendicular bisector of HK so that MH = MK always.

21. (B) The distances read are inversely proportional to the radii of the tires so that

$450 \times 15 = 440(15 + d)$ where d is the increase. Hence the increase $d = 15/44 = .34$ inches.

22. (A) Let S_m denote the sum of the first m positive integers. Using the formula for the sum of an arithmetic progression, $S_{3n} - S_n = 3n(3n + 1)/2 - n(n + 1)/2 = 4n^2 + n = 150 \therefore 4n^2 + n - 150 = 0 \therefore (n - 6)(4n + 25) = 0 \therefore n = 6 \therefore S_{4n} = S_{24} = 12(24 + 1) = 300$.

23. (D) The number $10! = 7 \times 5^2 \times 12^4$ when written in the base 12 system, ends with exactly 4 zeros.

24. (B) If the side of the triangle is 2s and hence its area is $s^2\sqrt{3} = 2$, then the side and area of the hexagon are s and $3s^2\sqrt{3}/2 = 3$ respectively.

25. (E) If the number of ounces is $W = w + p$ where w is a non-negative integer and $0 < p \leq 1$, then the postage is

$$6(w + 1) = -6(-w - 1) = -6[-w - 1] = -6[-w - p] = -6[-(w + p)]$$
$$= -6[-W] \text{ cents.}$$

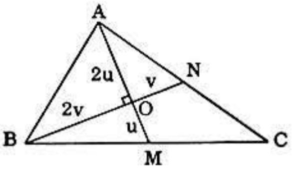
26. (B) The two lines which are the graph of the first equation intersect at the point (2,3) and so do the lines which are the graph of the second equation which are distinct from those of the first. Hence the two graphs have only the one point (2,3) in common.

27. (A) If r is the radius and p the perimeter, then the area of the triangle is $\frac{1}{2}pr = p$. Therefore $r = 2$.

28. (A) Let the medians be AM and BN which intersect at O and let AO and BO have lengths 2u and 2v so that OM and ON have lengths u and v respectively. From right triangles AON and BOM, we obtain

$$4u^2 + v^2 = (6/2)^2 = 36/4$$
$$u^2 + 4v^2 = (7/2)^2 = 49/4$$

Four-fifths of the sum of these two equations gives $4u^2 + 4v^2 = 17$ which is the square of the hypotenuse AB of right triangle AOB. Hence $AB = \sqrt{17}$.



29. (D) Let x be the number of minutes after 10 o'clock now. Then equating vertical angles (measured in minute spaces) formed by a line through 6 and 12 and one in the "exactly opposite" directions, we have $x + 6 = 20 + (x - 3)/12$ so that $x = 15$. Therefore the time is 10:15 now.

30. (E) Let the bisector of angle D be drawn and intersect AB at P. Then PDCB is a parallelogram and PAD an isosceles triangle. The measures of $AP = a$, $PB = b$, and hence $AB = a + b$.

31. (B) To total 43, five digits must be one 7 and four 9's or two 8's and three 9's. There are five and ten or a total of 15 such numbers three of which are divisible by 11. Hence the probability is 3/15 or 1/5. Divisibility by 11 requires that the sum of the first, third, and fifth digits minus the sum of second and fourth, be divisible by 11; only 98,989 and 97,999 and 99,979 satisfy this requirement.

32. (C) Let 2C be the circumference. The ratio of the distances travelled by A and B remains constant because their speeds are uniform. Therefore $(C - 100)/100 = (2C - 60)/(C + 60)$ so that $C^2 = 240C$, $C = 240$ and the circumference is 480 yards.

33. (A) Omit 10,000 for the moment. In the sequence 0000, 0001, 0002, 0003, $\cdots$, 9999 each digit appears the same number of times. There are (10,000)(4) digits in all, each digit appearing 4,000 times. The sum of the digits is $4{,}000(0 + 1 + 2 + \cdots + 9) = 4{,}000(45) = 180{,}000$. Now add the 1 in 10,000 to get 180,001.

34. (C) A number that divides two numbers and leaves the same remainder, divides their difference exactly. We seek the greatest integer that will divide the difference $(13{,}903 - 13{,}511) = 392 = 7^2 \cdot 2^3$ and the difference $(14{,}589 - 13{,}903) = 686 = 7^3 \cdot 2$. The required common factor is $7^2 \cdot 2$ or 98.

35. (D) Let X be the amount of the annual pension and y the number of years of service. Then with constant of proportionality k,

$$X = k\sqrt{y} \qquad X + p = k\sqrt{y + a} \qquad X + q = k\sqrt{y + b}$$
$$X^2 = k^2y \qquad (X + p)^2 = k^2(y + a) \qquad (X + q)^2 = k^2(y + b)$$

Replacing k^2y by X^2 in the last two equations and simplifying,

$$2pX + p^2 = k^2a \qquad \text{and} \qquad 2qX + q^2 = k^2b$$

Hence

$$\frac{2pX + p^2}{2qX + q^2} = \frac{a}{b} \qquad \therefore\ X = \frac{aq^2 - bp^2}{2(bp - aq)}\ .$$

The student should verify that bp − aq can not be zero.

SOLUTION-ANSWER KEY

TWENTY SECOND ANNUAL H. S.
MATHEMATICS EXAMINATION

1971

22

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NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS

USE OF KEY

1. This Key is prepared for the convenience of teachers.
2. Some of the solutions may be intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This Solution-Answer Key validates our statement that no mathematics beyond intermediate algebra is needed to solve the problems posed.

UES-MAA-7212

Note: The letter following the problem number designates the correct choice of the five listed for the problem in the 1971 examination.

1. (B) Rearranging the factors, $N = 2^4(2^8 \times 5^8) = 16 \times 10^8 = 1{,}600{,}000{,}000$ which is a ten digit number.

2. (D) If x, y, and z represent the number of men, days, and bricks respectively, and k is the constant of proportionality, then $z = kxy$ and the given data yields $k = f/(bc)$. With this k, the number of days $y = bcz/(fx) = bcb/(fc) = b^2/f$.

3. (E) Equate the reciprocals of the slopes of the segments joining $(x,-4)$ with $(0,8)$ and $(-4,0)$ with $(0,8)$ to get $-x/12 = 4/8$. Hence $x = -6$.

4. (A) The principal at the beginning of the 2 months was $P = \$255.31/(1 + .05/6) = \$253.20$ so that the interest credited was \$2.11 and the number of cents was 11.

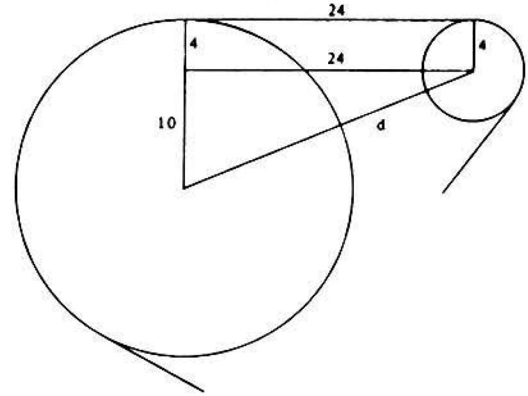
5. (C) $\measuredangle P + \measuredangle Q = 360^\circ - (\measuredangle PAQ + \measuredangle PCQ)$
$= 360^\circ - (180^\circ - 21^\circ) - (180^\circ - 19^\circ) = 40^\circ$.

6. (E) $a * \frac{1}{2a} = 1$ which is not the identity element as required.

7. (C) The given expression $= 2^{-2k}(2^{-1} - 2 + 1) = -2^{-2k}/2 = -2^{-(2k+1)}$

8. (B) The given inequality is equivalent to $(3x + 4)(2x - 1) < 0$ which requires $0 < 3x + 4$ and $2x - 1 < 0 \therefore -\frac{4}{3} < x < \frac{1}{2}$.

9. (D) A right triangle with legs 24 and 10 inches is formed by the line of centers, a radius, and a line parallel to the belt (See figure). $\therefore$ the required length d satisfies $d^2 = 24^2 + 10^2 = 26^2$, $d = 26$ inches.



10. (E) There are $50 - 31 = 19$ blondes and $\therefore 19 - 14 = 5$ brown eyed blondes. $\therefore 18 - 5 = 13$ are brown eyed brunettes.

11. (D) Both a and b must be greater than 7 and $4a + 7 = 7b + 4 \therefore 7b - 4a = 3 \therefore (a,b) = (15,9) \therefore a + b = 24 =$ XXIV.

12. (B) The difference of any two congruent integers must be divisible by N. Since $90 - 69 = 21$ and $125 - 90 = 35$, $N = 7$. Since $81 - 4 = 77$, 81 is congruent to 4.

13. (E) The binomial expansion gives $(1.0025)^{10} = (1 + .0025)^{10} = 1 + 10(.0025) + 45(.0025)^2 + 120(.0025)^3 + 210(.0025)^4 + \cdots = 1.025 + .00028125 + .000001875 +$ terms less than $10^{-8} = 1.02528$ accurate to 5 decimal places.

14. (C) Two factors are 63 and 65 because $2^{48} - 1 = (2^{24} - 1)(2^{24} + 1) = (2^{12} - 1)(2^{12} + 1)(2^{24} + 1) = (2^6 - 1)(2^6 + 1)(2^{12} + 1)(2^{24} + 1) = 63 \times 65\,(2^{12} + 1)(2^{24} + 1)$.

15. (B) Let u and h be the length of the bottom and depth of water when the bottom is level. Then the volume V of the water is $V = 10hu = 10 \cdot \frac{1}{2} \cdot 8 \cdot \frac{3}{4}u$, $h = 3''$.

16. (A) Let the 35 scores be denoted by $x_1, x_2, x_3, \cdots, x_{35}$ and their average by $\bar{x}$. Then $35\bar{x} = x_1 + x_2 + x_3 \cdots + x_{35}$. The average of the 36 numbers is $(x_1 + x_2 + x_3 + \cdots + x_{35} + \bar{x})/36 = (35\bar{x} + \bar{x})/36 = 36\bar{x}/36 = \bar{x}$. Hence the ratio is 1 : 1.

17. (E) A secant can cut across $(n + 1)$, but no more, of the $2n$ equal sectors, dividing each into two parts. The total number of distinct areas is then $2n + (n + 1) = 3n + 1$.

18. (D) Let v = the boat's speed in still water in miles per hour. Then $1 = \dfrac{4}{v+3} + \dfrac{4}{v-3}$

$\therefore v^2 - 9 = 4(v - 3) + 4(v + 3) \therefore v^2 - 8v - 9 = 0, (v - 9)(v + 1) = 0 \therefore v = 9$

$\dfrac{\text{Speed downstream}}{\text{Speed upstream}} = \dfrac{v+3}{v-3} = \dfrac{12}{6} = \dfrac{2}{1}$. The ratio is 2 : 1.

19. (C) Exactly one intersection requires the roots of the quadratic $x^2 + 4(mx + 1)^2 = 1$ to be equal and hence the discriminant to be zero. i.e. $(8m)^2 - 4(4m^2 + 1) \cdot 3 = 0$. Hence $m^2 = \dfrac{3}{4}$.

20. (E) If the roots are r and s, then $r + s = -2h$ and $rs = -3$. Hence $(r + s)^2 = r^2 + s^2 + 2rs = 10 - 6 = 4h^2 \therefore |h| = 1$.

21. (C) Since the antilog of 0 is 1 regardless of base, $\log_3(\log_4 x) = \log_4(\log_2 y) = \log_2(\log_3 z) = 1$ and hence $\log_4 x = 3$, $\log_2 y = 4$ and $\log_3 z = 2$. $\therefore x + y + z = 4^3 + 2^4 + 3^2 = 89$.

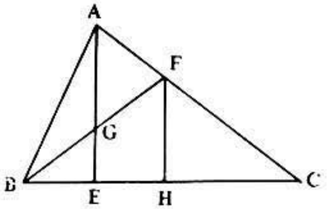
22. (A) Factoring the given equation $x^3 - 1 = 0$ gives $(x - 1)(x^2 + x + 1) = 0$. The imaginary root w satisfies $w^2 + w + 1 = 0$. Hence $1 + w^2 = -w$, $1 + w = -w^2$ $\therefore (1 - w + w^2)(1 + w - w^2) = (-2w)(-2w^2) = 4w^3 = 4$ because $w^3 = 1$.

23. (A) The four sequences of wins (each followed by its probability) for Team A to lose the series are BBB(1/8), ABBB(1/16), BABB(1/16), BBAB(1/16). The total probability for Team A to lose is 5/16 so that, to win is 11/16. The odds favoring Team A to win are 11 to 5.

24. (D) In the first n rows, there are $(2n - 1)$ 1's and $\frac{1}{2}(n - 2)(n - 1) = \frac{1}{2}(n^2 - 3n + 2)$ other numbers. The quotient is $\dfrac{n^2 - 3n + 2}{4n - 2}$.

25. (D) Let b and f denote the boy's and father's age respectively. Then $13 \leq b \leq 19$. Also $99f + 2b = 4289$. Equating the remainders in the division of both members of this equation by 9, (casting out nines), $2b = 32$, $b = 16$. Now $99f = 4289 - 32 = 4257$ and $f = 43$. $\therefore f + b = 59$.

26. (B) Draw FH parallel to AE. Then BE = EH because BG = GF. Also 2EH = HC because 2AF = FC. ∴ 3BE = EH + HC = EC. ∴ E divides BC in the ratio 1 : 3.



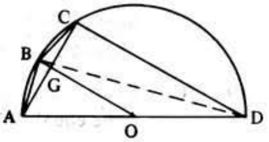
27. (E) Let w, b, and r denote the numbers of white, blue, and red chips respectively. Then $w \leq 2b$ and $3b \leq r$. Now $w + b \geq 55 \therefore 2b + b = 3b \geq 55 \therefore b \geq 18\frac{1}{3} \therefore 19 \leq b \therefore 57 \leq 3b \leq r$. The minimum number of red chips is 57.

28. (C) Let b and h denote the length of the base and altitude respectively. Since 38 = $\frac{1}{2}(b + .9b)(.1h)$, the area $\frac{1}{2}bh = 200$.

29. (E) Since 100,000 = 10^5, the sum of the first n exponents must exceed 5. i.e. $n(n + 1)/22 > 5 \therefore n(n + 1) > 110 \therefore n = 11$.

30. (D) If g(x) is the inverse of the transformation $f_1(x)$, then $g(f_{n+1}(x)) = f_n(x)$. Since $f_{35}(x) = f_5(x)$, successive application of this formula yields $f_{31}(x) = f_1(x) \therefore f_{30}(x) = g(f_1(x)) = x \therefore f_{29}(x) = g(x) \therefore f_{28}(x) = g(g(x))$. But $g(x) = \frac{x + 1}{2 - x} \therefore f_{28}(x) = g(g(x)) = \frac{1}{1 - x}$.

31. (A) Draw radius OB which is perpendicular to and bisects chord AC at G. Side CD is parallel to and has length twice that of segment GO. Similar right triangles BGA and ABD yield



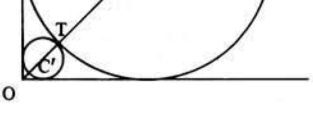
$$\frac{BG}{1} = \frac{1}{AD} = \frac{1}{4} \therefore GO = BO - BG = 2 - \frac{1}{4} = \frac{7}{4}. \quad \therefore CD = 2\,GO = \frac{7}{2}.$$

32. (A) Let $x = 2^{-\frac{1}{32}}$. Then $s = (1 + x)(1 + x^2)(1 + x^4)(1 + x^8)(1 + x^{16})$
Now $(1 - x)s = 1 - x^{32} = \frac{1}{2}. \therefore s = \frac{1}{2}(1 - x)^{-1} = \frac{1}{2}(1 - 2^{-\frac{1}{32}})^{-1}$

33. (B) Let the progression be a, ar, $ar^2, \cdots ar^{n-1}$, then $P = a^n r^{\frac{1}{2}(n-1)n}$, $S = a\frac{1 - r^n}{1 - r}$,
$$S' = \frac{1}{a} \cdot \frac{1 - r^{-n}}{1 - r^{-1}} = \frac{r^{-(n-1)}}{a} \cdot \frac{1 - r^n}{1 - r} \therefore S/S' = a^2 r^{(n-1)} \therefore (S/S')^{\frac{1}{2}n} = a^n r^{\frac{1}{2}(n-1)n} = P.$$

34. (B) 12 hours by the slow clock = 69 × 11 minutes = 12 hrs. + 39 min. ∴ 8 hours by the slow clock = 8 hrs. + 26 min. Overtime of 26 min. @ \$6 per hr. or 10¢ per min. gives \$2.60.

35. (C) Let O denote the vertex of the right angle, C and C′ the centers, r and r′ ($r > r'$) the radii of any two consecutive circles. If T is the point of contact of the circles, then $OT = OC' + r' = (\sqrt{2} + 1)r'$ and $OT = OC - r = (\sqrt{2} - 1)r$. Equating these expressions for OT yields the ratio of consecutive radii $r'/r = (\sqrt{2} - 1)/(\sqrt{2} + 1) = (\sqrt{2} - 1)^2$. If r is the radius of the first circle in the sequence, then πr^2 is its area, and the sum of the areas of all the other circles, which form a geometric series, is



$$\pi r^2 [(\sqrt{2} - 1)^4 + (\sqrt{2} - 1)^8 + \cdots] = \frac{\pi(\sqrt{2} - 1)^4 r^2}{1 - (\sqrt{2} - 1)^4} = \frac{\pi r^2}{(\sqrt{2} + 1)^4 - 1}$$

The quotient of areas = $(\sqrt{2} + 1)^4 - 1 = 16 + 12\sqrt{2}$ and the required ratio is $(16 + 12\sqrt{2}) : 1$.

SOLUTION-ANSWER KEY

TWENTY THIRD ANNUAL H. S.
MATHEMATICS EXAMINATION

1972

23

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USE OF KEY

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Note: The letter following the problem number designates the correct choice of the five listed for the problem in the 1972 examination.

1\. (D) The Pythagorean Theorem applies to show that IV is not but I, II, and III are right triangles.

2\. (B) Let C be the present cost, hence .92C the discounted cost, and x the present profit, in percentage, then equating the fixed proceeds of sale, gives

$C(1 + .01x) = .92C[1 + .01(x + 10)]$, $.08(.01x) = (.92)(1.1) - 1 = .012$, $x = 15$.

3\. (B) $x^2 - x = \frac{1}{4}(1 - i\sqrt{3})^2 - \frac{1}{2}(1 - i\sqrt{3}) = \frac{1}{4}(-2 - 2i\sqrt{3}) - \frac{1}{2}(1 - i\sqrt{3}) = -1$ and the reciprocal requested is also -1.

4\. (D) The union of $\{1,2\}$ with each of the 8 distinct subsets of $\{3,4,5\}$ results in a different solution X and there are no other solutions.
<u>Remark</u>: In making the count, one need only enumerate the 8 subsets of $\{3,4,5\}$ but it is not difficult to write down all 8 solutions. This is left to the student.

5\. (A) First note that $(2^{1/2})^6 = 8 < (3^{1/3})^6 = 9$ so that $2^{1/2} < 3^{1/3}$. Also $(2^{1/2})^{18} = 2^9 = 512 > (9^{1/9})^{18} = 81$ so that $2^{1/2} > 9^{1/9}$, and $2^{1/2} > 8^{1/8} = 2^{3/8}$. Hence $3^{1/3}$, $2^{1/2}$ have the greatest and next to the greatest values in that order.

6\. (C) Let $y = 3^x$ and the given equation is equivalent to $y^2 - 10y + 9 = 0$ or $(y - 9)(y - 1) = 0$. Hence $y = 3^x = 9$ or 1, $x = 2$ or 0 so that $x^2 + 1 = 5$ or 1.

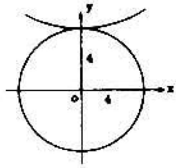
7\. (E) The ratio $\frac{x}{yz} : \frac{y}{zx} = \frac{zx}{yz} : \frac{yz}{zx} = 2 : \frac{1}{2} = 4 : 1$ because it is given that $yz : zx = 1 : 2$ and hence $2yz = zx$ so that $\frac{zx}{yz} = 2$ and $\frac{yz}{zx} = \frac{1}{2}$.

8\. (D) If $(x - \log y)$ is nonnegative, then the given equation requires that $x - \log y = x + \log y$ so that $-\log y = \log y = 0$ and $y = 1$. On the other hand, if $(x - \log y)$ is negative $-(x - \log y) = x + \log y$ so that $2x = 0$. We can write $x(y - 1) = 0$ to say that $x = 0$ or $y = 1$ or both.

9\. (A) Let x and y denote the number of sheets of paper and of envelopes respectively in each box. Then $x - y = 50$ and $y - \frac{1}{3}x = 50$ give $\frac{2}{3}x = 100$, $x = 150$ sheets of paper in each box.

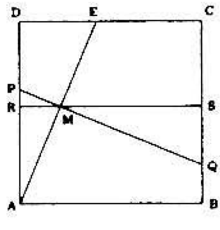
10\. (D) First when $x - 2 \geq 0$, than $1 \leq x - 2 \leq 7$, $3 \leq x \leq 9$. Again when $x - 2 \leq 0$, then $1 \leq 2 - x \leq 7$, $-1 \leq -x \leq 5$ gives $-5 \leq x \leq 1$ as the other possibility as stated in choice (D).

11\. (A) Graphing the circle of radius 4 about the origin and the parabola with y - intercept 4, it becomes apparent that they intersect only when $y = 4$. Alternately, we may subtract the second from the first equation to get $y^2 + 3y - 28 = 0$, $y = -7$ or 4. For $y = 4$, $x = 0$, but there is no real x for $y = -7$.



12\. (B) Let an edge be f feet and hence 12f inches long. Then $f^3 = 6(12f)^2$ so that $f = 6(12)^2 = 6 \times 144 = 864$ ft.

13\. (C) Let R and S be the intersections with sides AD and BC of the line through M parallel to AB (See figure). Then $RM = \frac{1}{2}DE = 2\frac{1}{2}$ inches and hence MS has length $9\frac{1}{2}$ inches. Since PMR and SMQ are similar right triangles $PM : MQ = RM : MS = 2\frac{1}{2} : 9\frac{1}{2} = 5 : 19$.



14\. (B) Let s denote the required side. Then the Law of Sines gives

$$\frac{s}{\sin 30^\circ} = \frac{8}{\sin 45^\circ}, \; s = \frac{8 \sin 30^\circ}{\sin 45^\circ} = \frac{8(\frac{1}{2})}{\frac{1}{2}\sqrt{2}} = 4\sqrt{2}.$$

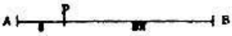
Alternately, the altitude between sides s and 8 has length 4 and is one leg of an isosceles right triangle with hypothenuse $s = 4\sqrt{2}$.

15\. (C) If x is the number of bricks in the wall, then this reduced number of bricks for 5 hours of work together, is

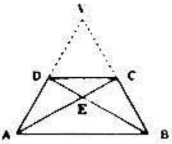
$$5\left(\frac{x}{9} + \frac{x}{10} - 10\right) = x, \; x = 900 \text{ bricks}.$$

16. (B) Let 3, x, y be the first three numbers. Then $\frac{x}{3} = \frac{y}{x}$, $x^2 = 3y$. The last three numbers are x, y, 9 so that $y - x = 9 - y$, $x + 9 = 2y$. Eliminate y getting $2x^2 - 3x - 27 = 0$. Factoring, $(x + 3)(2x - 9) = 0$, $x = 4\frac{1}{2}$, $y = \frac{x^2}{3} = 6\frac{3}{4}$. $\therefore x + y = 11\frac{1}{4}$.

17. (E) Let AB represent the string (See figure) and let P be the point on it such that AP : PB = 1 : x. If AP has length a inches, then the length of PB is ax. The probability that the cut lie on AP is $\frac{a}{a + ax} = \frac{1}{1 + x}$. Since the cut is equally likely to lie within the same distance from the other end B of the string, the probability of either is $\frac{2}{x + 1}$ of choice (E).



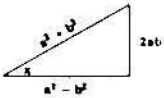
18. (A) Let sides AD and BC of the quadrilateral intersect at V. Then diagonals AC and BD are medians from A and B of triangle ABV intersecting at E which divides AC in the ratio 2 : 1 so that the length of EC is $\frac{11}{3}$ or $3\frac{2}{3}$ units.



19. (D) The k^{th} term of the given sequence is equal to $2^k - 1$ so that the sum of the first n terms may be written as

$$(2^1 - 1) + (2^2 - 1) + (2^3 - 1) + \cdots + (2^n - 1)$$
$$= (2 + 2^2 + 2^3 + \cdots + 2^n) - (1 + 1 + 1 + \cdots \text{ to n terms})$$
$$= (2^{n+1} - 2) - n = 2^{n+1} - n - 2.$$

20. (E) Angle x may be taken as the acute angle opposite the side of length 2ab in a right triangle (See figure) whose other leg then has length $(a^2 - b^2)$. The hypothenuse is then the square root of $(2ab)^2 + (a^2 - b^2)^2 = a^4 + 2a^2b^2 + b^4$ or $a^2 + b^2$. Hence $\sin x = 2ab/(a^2 + b^2)$ by the definition of sine.



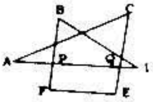
21. (C) Let P and Q denote the intersections of AD with BF and CE respectively. Then 3 sums of angles in degrees are

$$\angle F + \angle FPD + \angle EQA + \angle E = 360^\circ$$
$$\angle B + (180^\circ - \angle FPD) + \angle D = 180^\circ$$
$$\angle A + (180^\circ - \angle EQA) + \angle C = 180^\circ$$

Adding these equations member by member gives

$$\angle A + \angle B + \angle C + \angle D + \angle E + \angle F + 360^\circ = 720^\circ$$

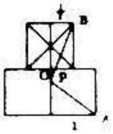
and the required sum is $360^\circ = 90n^\circ$ so that $n = 4$ as in choice (C).



22. (E) Let the third root of the given equation be s. The sum of the roots is zero, $2a + s = 0$, $s = -2a$. The sum of the products of the roots taken two at a time is q,

$$(a + bi)(a - bi) + (a + bi)s + (a - bi)s = q$$
$$(a^2 + b^2) + a(-2a) + bi(-2a) + a(-2a) - bi(-2a) = b^2 - 3a^2 = q$$ which is choice (E).

23. (D) Let P be the center of the circumscribing circle. (See figure). We must have $\overline{AP}^2 = \overline{PB}^2$ so that $(1 - \overline{OP})^2 + 1^2 = (1 + \overline{OP})^2 + (\frac{1}{2})^2$ and $-2\overline{OP} + 1 = 2\overline{OP} + \frac{1}{4}$, $OP = \frac{3}{16}$. Hence $AP^2 = (1 - OP)^2 + 1^2 = \left(\frac{13}{16}\right)^2 + 1 = \frac{169 + 256}{256} = \frac{425}{256} = \frac{25(17)}{256}$ and $AP = \frac{5\sqrt{17}}{16}$.



24. (B) Let D, R, T denote the distance (miles), rate (miles per hour), time (hours) respectively. Then

$D = RT$, $D = \frac{4}{5}(R + \frac{1}{2})T$, $D = (R - \frac{1}{2})(T + 2\frac{1}{2})$.

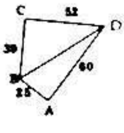
The first and second equations give $\frac{1}{5}D = \frac{2}{5}T$. $\therefore D = 2T$. $\therefore 2T = RT$ and $R = 2$. Replacing T by $\frac{1}{2}D$ and R by 2 in the third equation gives $D = \frac{3}{2}(\frac{1}{2}D + 2\frac{1}{2})$, $D = 15$ miles.

25. (C) Since angles A and C are supplementary, (See figure) one suspects that each may be a right angle. This turns out to be the case with diagonal BD of length 65 as the common hypothenuse and the diameter of the circumscribing circle.

or

The same result may be obtained using the Law of Cosines on triangles ABD and CBD. Thus

$$\overline{BD}^2 = 39^2 + 52^2 - 2x39x52 \cos C$$
$$\overline{BD}^2 = 25^2 + 60^2 - 2x25x60 \cos A$$



Since C is the supplement of A, replacing cos C by its equal (−cos A) and subtracting, gives

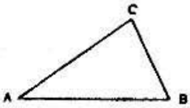
$0 = 0 + (2 \times 39 \times 52 + 2 \times 25 \times 60) \cos A. \ \therefore \cos A = 0$

and A and its supplement C are both right angles. The common hypothenuse BD has length 65 and is the diameter of the circumscribing circle as before.

26. (E) Draw NQ perpendicular to AB at Q where chord MN has measure x. Since arcs BN and AM are equal, PQMN is a rectangle and PQ has measure x. Hence PB = PQ + QB has measure $x + (x+1) = 2x + 1$.

27. (D) The area of $\triangle ABC$ (See figure) is $64 = \frac{1}{2} AB \cdot AC \sin A$.

Now $AB \cdot AC = 144$. Hence $\sin A = \frac{2 \times 64}{144} = \frac{8}{9}$.



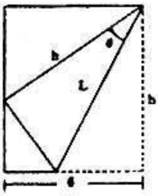
28. (E) All border checkerboard squares are not entirely covered by the disc. Only the 4 corner squares of the 6 × 6 "checkerboard" of the 36 remaining interior squares are not entirely covered. Hence 36 − 4 = 32 squares are entirely covered by the disc.

29. (C) $f\left(\frac{3x + x^3}{1 + 3x^2}\right) = \log \frac{1 + \frac{3x + x^3}{1 + 3x^2}}{1 - \frac{3x + x^3}{1 + 3x^2}} = \log \frac{1 + 3x^2 + 3x + x^3}{1 + 3x^2 - 3x - x^3} = \log \frac{(1 + x)^3}{(1 - x)^3} = \log \left(\frac{1 + x}{1 - x}\right)^3 = 3 \log \frac{1 + x}{1 - x} = 3 f(x)$.

30. (A) Let h denote the length of the sheet. Then $h = \frac{6}{\cos(90^\circ - 2\theta)} = \frac{6}{\sin 2\theta}$

$= 3 \sec \theta \csc \theta.$

Also $L = h \sec \theta = 3 \sec^2 \theta \csc \theta$.



31. (C) The remainder in the division of 2^{12} by 13 is 1 because that of $2^6 = 64$ is −1. Hence $2^{1000} = 2^{996} \times 2^4 = (2^{12})^{83} \times 16$ leaves a remainder of $1^{83} \times 3 = 3$ when divided by 13.

or

Using congruences, $2^6 \equiv -1 \bmod 13$. Hence $2^{1000} \equiv 2^{996} \times 2^4 \equiv (-1)^{166} \times 16 \equiv 1 \times 3 \equiv 3 \bmod 13$.

32. (B) If AC and BD be drawn, then inscribed angles B and C subtend the same arc AD and hence right triangles ACE and DBE are similar. Hence CE : AE = BE : DE and the measure of segment CE is 4. The center of the circle at the intersection of the perpendicular bisectors of chords AB and CD is 4 units to the right of and $\frac{1}{2}$ unit above point A. Hence $(\text{Radius})^2 = (\frac{1}{2} \text{Diameter})^2 = 4^2 + (\frac{1}{2})^2 = (\frac{1}{2}\sqrt{65})^2$. $\therefore$ The length of the diameter is $\sqrt{65}$.

33. (C) Let the units, tens, and hundreds digits be denoted by U, T, and H respectively so that the quotient to be minimized is $\frac{U + 10T + 100H}{U + T + H} = 1 + \frac{9(T + 11H)}{U + T + H}$ which is least when U = 9 regardless of the values of T and H. Now $\frac{T + 11H}{T + H + 9} = 1 + \frac{10H - 9}{T + H + 9}$ which is least when T = 8 and H = 1. Hence the number is 189 and the minimum quotient $\frac{189}{18} = 10.5$.

34. (A) Let T, D, and H denote the ages of Tom, Dick, and Harry respectively. Then $3D + T = 2H$ and $2H^3 = 3D^3 + T^3$ or equivalently $2(H - D) = D + T$ and $2(H^3 - D^3) = D^3 + T^3$. Since, $D + T \neq 0$, $\frac{2(H^3 - D^3)}{2(H - D)} = \frac{D^3 + T^3}{D + T}$ or $H^2 + HD + \cancel{D^2} = \cancel{D^2} - DT + T^2$. Hence $T^2 - H^2 = D(T + H)$ so that $T - H = D$. Eliminating T from the last and very first equation gives H = 4D so that D = 1 and H = 4 because H and D are relatively prime integers. Also T = H + D = 5 and the required sum of squares is $T^2 + D^2 + H^2 = 5^2 + 1^2 + 4^2 = 42$.

35. (D) The point P returns to its original position after 24 = 8 × 3 moves. In 8 of these moves, the rotation is about vertex P with no path traversed by P. In the other 16 moves, 8 are about a mid-point of a side of the square with the radius to P sweeping through 120° or $\frac{2}{3}\pi$ radians, and 8 are about a corner of the square with the radius to P sweeping through 30° or $\frac{1}{6}\pi$ radians. The total angle swept through by the radius to P is $8(\frac{2}{3}\pi + \frac{1}{6}\pi) = \frac{20}{3}\pi$ radians. Hence the length of the path traversed by P is $\frac{40}{3}\pi$ inches. The general solution is:

$$\frac{8}{3}\pi(2 + 5k) \text{ or } \frac{40}{3}\pi(1 + k), \ k=0,1,2, \ldots$$

SOLUTION-ANSWER KEY

TWENTY FOURTH ANNUAL H. S.
MATHEMATICS EXAMINATION
1973

24

Sponsored Jointly by the

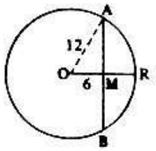
MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
MU ALPHA THETA
NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS
CASUALTY ACTUARIAL SOCIETY

USE OF KEY

1. This Key is prepared for the convenience of teachers.
2. Some of the solutions may be intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, there is also shown a more elementary procedure.
5. This Solution-Answer Key validates our statement that no mathematics beyond intermediate algebra and trigonometry is needed to solve the problems posed.

Note: The letter following the problem number refers to the correct choice of the five listed for that problem in the 1973 examination.

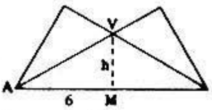
1.(D) Let O denote the center of the circle and OR and AB the radius and the chord (See figure) which are perpendicular bisectors of each other at M. Using the Pythogorean theorem on right ΔOMA, $AM^2 = OA^2 - OM^2 = 12^2 - 6^2 = 108$, $AM = 6\sqrt{3}$. The required chord has length $AB = 2AM = 12\sqrt{3}$.



2.(C) The $8 \times 8 \times 8$ cube of interior cubes contains all 512 of the unpainted cubes. $10^3 - 8^3 = 1,000 - 512 = 488$ cubes have at least one face painted.

3.(B) 13 is the smallest prime p with $126 - p$ also being prime. Thus the largest difference is $113 - 13 = 100$.

4.(D) Let M and V (See figure) denote the ends of the altitude of length h and the bisector of the isoceles triangular common area where A denotes one end of the base. Then $h\sqrt{3} = 6$. $\therefore h = 2\sqrt{3}$ and the required common area is $\frac{1}{2}h(2AM) = \frac{1}{2}(2\sqrt{3}) \times 12 = 12\sqrt{3}$.



5.(D) Denoting the binary operation of averaging by *, we have for any two numbers a and b,

$a * b = \frac{1}{2}(a + b)$.

Accordingly, II and IV are the only valid statements because in II,

$a * b = \frac{1}{2}(a + b) = \frac{1}{2}(b + a) = b * a$ and in IV,
$a + (b * c) = a + \frac{1}{2}(b + c)$ equals
$(a + b) * (a + c) = \frac{1}{2}[a + b + a + c] = a + \frac{1}{2}(b + c)$.

In I, on the other hand,

$a * (b * c) = \frac{1}{2}a + \frac{1}{4}b + \frac{1}{4}c$ and $(a * b) * c = \frac{1}{4}a + \frac{1}{4}b + \frac{1}{2}c$ are not always equal so that * is not associative.

In III, $a * (b + c) = \frac{1}{2}a + \frac{1}{2}b + \frac{1}{2}c$ is not equal to $(a * b) + (a * c) = a + \frac{1}{2}b + \frac{1}{2}c$ as needed.

In V, there exists no one number e such that for every number a, $e * a = a$ as required for an identity element e because $a * e = \frac{1}{2}(a + e)$.

6.(C) Let $b > 5$ be the base. Since $(24_b)^2 = (2b + 4)^2 = 4b^2 + 16b + 16$ and $554 = 5b^2 + 5b + 4$ we must have $5b^2 + 5b + 4 = 4b^2 + 16b + 16$. Solving for b we get $b = -1$ or 12.

7.(A) Using the formulas $S = \frac{1}{2}n(a + \ell)$ and $\ell = a + (n - 1)d$ for the sum S of n terms and the nth term ℓ where $a = 51$ and $d = 10$, we get

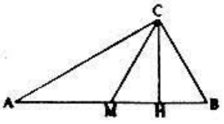
$S = 51 + 61 + \ldots + 341 = \frac{1}{2}n(51 + 341) = 196n$ where
$\ell = 341 = 51 + (n - 1) \times 10 \therefore n = 30 \therefore S = 15 \times 392 = 5880$.

8.(E) The number of pints P of paint used varies jointly as the square of the height h of each and the the number n of statues painted which is to say $P = knh^2$. When $n = 1$ and $h = 6$, $P = 1$ gives the constant of variation

$k = \frac{1}{36}$ so that $P = \frac{1}{36}nh^2$. Now when $n = 540$ and $h = 1$, we get $P = \frac{1}{36} \times 540 \times 1^2 = 15$ pints.

9.(E) Triangles ABC, AMC, MBC, MHC and HBC all have the same altitude HC so that their areas (denoted below by parentheses) are proportional to their bases. We have

(AMC) = (MBC) because AM = MB. Also (MBC) = (MHC) + (HBC) = K + K = 2K because



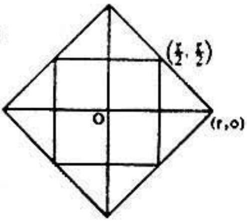
ΔMHC and ΔHBC are congruent 30° – 60° right triangles.
$\therefore (ABC) = (AMC) + (MBC) = 2K + 2K = 4K$.

10.(A) If $n \neq -1$, then elementary operations on the system yields an equivalent system $(x,y,z) = \left(\frac{1}{n+1}, \frac{1}{n+1}, \frac{1}{n+1}\right)$ which is a solution of the given system. If $n = -1$, adding the equations we get the inconsistency $0 = 3$. Alternately, the general condition that the matrix of the coefficients of the system and the augmented matrix must have equal or unequal rank for consistency or inconsistency applies when $n \neq -1$ or $n = -1$ respectively.

11.(B) Consider the figure for some positive integer r. The equation of the outer diamond is $|x| + |y| = r$. Points on or inside the diamond satisfy (1) $|x| + |y| = r$. The equations of the circle is $x^2 + y^2 = \frac{r^2}{2}$, so points on the circle satisfy (2) $\sqrt{2(x^2 + y^2)} = r$.

The equation of the inner square is $\text{Max}\,(|x|, |y|) = \frac{r}{2}$, so points on or outside the square satisfy (3) $r \leq 2\,\text{Max}\,(|x|, |y|)$

Thus from (1), (2), and (3) it is seen that points on the circle satisfy $|x| + |y| \leq r = \sqrt{2(x^2 + y^2)} \leq 2\,\text{Max}\,(|x|, |y|)$.



12.(D) Let m and n denote the number of doctors and lawyers respectively. Then $40(m + n) = 35m + 50n$, $5m = 10n$, $m/n = 2$ and the required ratio $m : n = 2 : 1$.

13.(D) Multiplying the numerator and denominator of the given fraction by $\sqrt{2}$ gives

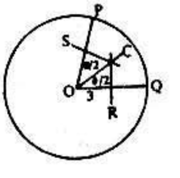
$$\frac{2(\sqrt{2} + \sqrt{6})}{3\sqrt{2 + \sqrt{3}}} = \frac{4(1 + \sqrt{3})}{3\sqrt{4 + 2\sqrt{3}}} = \frac{4(1 + \sqrt{3})}{3\sqrt{(1 + \sqrt{3})^2}} = \frac{4}{3}.$$

14.(C) Let x, y, and z denote the number of tankfuls of water delivered by valves A, B, and C respectively in one hour. Then

$$x + y + z = 1,\; x + z = \frac{1}{1.5},\; y + z = \frac{1}{2}.$$

Subtracting the sum of the last two equations from twice the first, gives $x + y = 5/6$ so that $(6/5)(x + y) = 1$ tankful will be delivered by valves A and B in $6/5 = 1.2$ hours

15.(D) The center of the circle which circumscribes sector POQ is at C, the intersection of the perpendicular bisectors SC and RC. From triangle ORC we see that $\sec\frac{\theta}{2} = \frac{OC}{3}$ or $OC = 3\sec\frac{\theta}{2}$.



16.(B) Let n denote the number sides (angles) of the given convex polygon and x the number of degrees in the excepted angle. Then $180(n - 2) = 2190 + x$ so that

$$n - 2 = \frac{2160 + x}{180} = 12 + \frac{30}{180} + \frac{x}{180} = 12 + 1$$

$\therefore n = 15$ and incidentally the excepted angle $x° = 150°$.

17(E) Using the formula for the cosine of twice an angle $\frac{1}{2}\theta$,

$$\cos\theta = \cos 2\,\tfrac{1}{2}\theta = 1 - 2\sin^2\tfrac{1}{2}\theta = 1 - 2\left(\frac{x-1}{2x}\right) = \frac{1}{x}.$$ Since

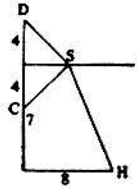
$\sec\theta = \frac{1}{\cos\theta} = x$, $\tan^2\theta = \sec^2\theta - 1 = x^2 - 1$ and $\tan\theta = \sqrt{x^2-1}$.

18.(C) Since the factors $(p-1)$ and $(p+1)$ of $(p^2 - 1)$ are consecutive even integers, both are divisible by 2 and one of them also by 4 so that their product is divisible by 8. Again $(p - 1)$, p, and $(p+1)$ are three consecutive integers so that one of them (but not the prime $p \geq 5$) is divisible by 3. Hence the product $(p - 1)(p+1) = p^2-1$ is always divisible both 3 and 8 and hence by 24.

19.(D) $72_8! = 72 \times 64 \times 56 \times 48 \times 40 \times 32 \times 24 \times 16 \times 8 = 8^9 \times 9!$
$18_2! = 18 \times 16 \times 14 \times 12 \times 10 \times 8 \times 6 \times 4 \times 2 = 2^9 \times 9!$
The quotient $72_8!/18_2! = 8^9/2^9 = 4^9$

20.(C) Let S (See figure) denote any point on the stream SE and C, H, and D be the position of the cowboy, his cabin and the point 8 miles north of C. Then the distance (CS + SH) equals (DS + SH) which is least when DSH is a straight line and then

$CSH = DSH = \sqrt{8^2 + 15^2} = \sqrt{289} = 17$ miles.



21.(B) Either the number of consecutive integers in a set will be odd or even. If odd, let their number be $(2n + 1)$ and their average x, the middle one. Then $(2n + 1)x = 100$. $\therefore x = 100/(2n + 1)$ so that $(2n + 1)$ can only be 5 or 25. If $2n + 1 = 5$, then $x = 20$ and $n = 2$ so that the integers are 18, 19, 20, 21, 22. If $(2n + 1) = 25$, $x = 4$ and $n = 12$ which is impossible because the integers must be positive. If the number of consecutive integers is an even number 2n, let their average (half way between the middle pair) be denoted by x. Then $2nx = 100$, $x = 50/n$. For $n = 4$ $x = 12\frac{1}{2}$ and the middle pair are $x - \frac{1}{2} = 12$, $x + \frac{1}{2} = 13$ so that 9, 10, 11, 12, 13, 14, 15, 16 are the $2n = 8$ integers. For no other integer n, are the middle pair $(x - \frac{1}{2})$ and $(x + \frac{1}{2})$ positive integers. The two displayed are the only sets of positive integers whose sum is 100.

22.(A) 1. If $-2 < x < 1$, then $x - 1 < 0$ and $x + 2 > 0$ so that the given inequality requires $-x + 1 + x + 2 < 5$ or $3 < 5$ which is always true. The inequality is true also when $x = 1$ and $x = -2$.

2. If $x > 1$, then $x - 1 > 0$ and $x + 2 > 0$ so that the given inequality becomes $(x - 1) + (x + 2) = 2x + 1 < 5$. $x < 2$. $\therefore 1 < x < 2$.

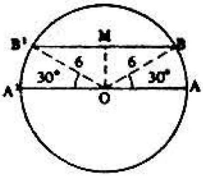
3. If $x < -2$, then $x + 2 < 0$ and $x - 1 < 0$ and the given inequality becomes $-(x - 1) - (x + 2) = 2x - 1 < 5$. $\therefore -2x < 6$, $x > -3$. We have accordingly $-3 < x < 2$ as the set of all real solutions.

23.(D) Let the sides of the first card (both red) be numbered 1 and 2. Let the red and blue sides of the second card be numbered 3 and 4 respectively. On draw, any one of the four sides 1, 2, 3, 4 could come up. But side 4 does not, so 1, 2, or 3 must. Of these three, two undersides are red and one blue so that the probability of red is two out of three or $\frac{2}{3}$.

24.(D) Let s, c, and p denote cost in dollars of 1 sandwich, 1 cup of coffee, and 1 piece of pie respectively. Then $3s + 7c + p = 3.15$ and $4s + 10c + p = 4.20$. Subtracting twice the second of these two equations from three times the first gives $s + c + p = 1.05$ so that \$1.05 is the required cost.

25.(E) Let O denote the center of the plot (See figure) and M the midpoint of the other side of the walk. If AB denotes one arc cut off by the walk, then one half the area of the walk OMBA consists of a 30° sector OAB and a 30°-60° right $\triangle$OMB. Denoting areas by parentheses,

$$\begin{aligned}(\text{Required Area}) &= (\text{Plot}) - (\text{Walk}) \\ &= \pi \times 6^2 - 2\,(\text{Sectors}) - 2(30^\circ - 60^\circ\ \triangle\text{'s}) \\ &= 36\pi - 2 \times \frac{1}{2} \times \frac{\pi}{6} \times 6 \times 6 - 2 \times \frac{1}{2} \times 3 \times 3\sqrt{3} \\ &= 30\pi - 9\sqrt{3} \text{ or choice (E).}\end{aligned}$$



26.(E) Let a, d, and 2n denote the first term, common difference and the even number of terms. If S_o and S_e denote the sums of all odd and all even numbered terms respectively, then

$$S_e = \frac{n}{2}\left[2(a + d) + (n - 1) \times 2d\right] = 30 \text{ and}$$

$$S_o = \frac{n}{2}\left[2a + (n - 1) \times 2d\right] = 24.$$

The difference $S_e - S_o$ equals $\frac{n}{2}[2d] = 6$, $nd = 6$.

$\therefore\ l - a = a + (2n - 1)\, d - a = 10.5$ where l denotes the last term.
$\therefore\ 2nd - d = 10.5,\ 12 - d = 10.5,\ d = 1.5.\ \therefore\ n = 6/d = 6/1.5 = 4,\ 2n = 8$ or choice (E).

27.(A) Let s denote the distance. The time t for car A is $t = \frac{s}{2u} + \frac{s}{2v}$ and the average speed is

$$\frac{s}{t} = \frac{s}{\frac{s}{2u} + \frac{s}{2v}} = \frac{2uv}{u + v} = x.$$

For car B, let s_1, s_2 denote the distance at speed u, v respectively so that if the time for car B is T, $\frac{s_1}{T/2} = u$ and $\frac{s_2}{T/2} = v$. Adding these $\frac{s}{T/2} = \frac{s_1 + s_2}{T/2} = u + v$ and the average speed of car B is $s/T = \frac{u + v}{2} = y$.

The fact that $x \leq y$ for positive u and v follows from $4uv \leq (u + v)^2$.

28.(C) Let $r > 1$ denote the ratio of the G.P. a, b, c so that $b = ar$ and $c = ar^2$ and $\log b = \log a + \log r$, $\log c = \log a + 2 \log r$. Now

$$\log_a n = \log n\left(\frac{1}{\log a}\right),\ \log_b n = \log n\left(\frac{1}{\log b}\right) = \log n\left(\frac{1}{\log a + \log r}\right), \text{ and}$$

$$\log_c n = \log n\left(\frac{1}{\log c}\right) = \log n\left(\frac{1}{(\log a + 2 \log r)}\right)$$

have reciprocals which form an A.P. as required.

29.(A) The boys meet for the nth time after the faster has traveled $\frac{9}{14}$ n laps and the slower $\frac{5}{14}$ n laps. Both of these are first whole numbers of laps when n = 14; there are 13 meetings in between.

30.(B) For any fixed $t \geq 0$, $0 \leq T < 1$. Hence S, the interior of the circle with center (T,0) and radius T, has an area between 0 and Π.

31.(C) The integer $TTT = T\cdot(111) = T\cdot 3\cdot 37$ with T equal to the final digit in E^2. Now $ME > YE$ must contain the factor 37 and so is 37 or 74. The latter possibility is ruled out because then the smallest possible $YE = 14$ gives $14 \times 74 = 1036$ exceeds $999 = TTT$. $ME = 37$ and $YE = Y7$ is divisible by 3 and is therefore 27. We note that $27 \times 37 = 999$ makes the required sum

$E + M + T + Y = 7 + 3 + 9 + 2 = 21$ or choice (C).

32.(A) The volume of any pyramid is equal to one third the area of the base times the altitude. In the present problem, the area of the equilateral base of side 6 is $3 \times 3\sqrt{3} = 9\sqrt{3}$. The altitude h is one leg of a right triangle with hypotenuse an edge of the pyramid of length $\sqrt{15}$ and the other leg the distance from the centroid to a vertex of the base of $2\sqrt{3}$. Then $h^2 = (\sqrt{15})^2 - (2\sqrt{3})^2 = 3$, $h = \sqrt{3}$. The required volume is

$$V = \frac{1}{3}(\text{Area of Base})(\text{Altitude}) = \frac{1}{3}(9\sqrt{3})\sqrt{3} = 9.$$

33.(C) Let (x,y) denote the number of ounces of (water, acid) originally. After addition of 1 oz. water and 1 oz. acid we have respectively y and (y + 1) oz. of acid in mixtures totaling (x + y + 1) and (x + y + 2) oz. which means that

$\frac{y}{x+y+1} = \frac{1}{5}$ and $\frac{y+1}{x+y+2} = \frac{1}{3}$. These yield $(x,y) = (3,1)$ from which the percentage of acid originally was $\frac{y}{x+y} = \frac{1}{3+1} = \frac{1}{4} = 25\%$.

34.(C) Let d, v, w, denote the distance between the towns, speed against, speed with the wind respectively. Then since $d/v = 84$, $d = 84v$, we have

$$\frac{d}{w} = \frac{d}{\frac{1}{2}(w+v)} - 9 \text{ or } 84\,\frac{v}{w} = \frac{84v}{\frac{1}{2}(w+v)} - 9.$$

Letting $x = v/w$, the last equation reduces to $28x = \frac{56}{\frac{1}{x}+1} - 3$ which is equivalent to

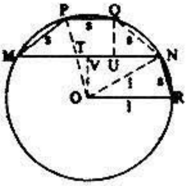
$$28x^2 - 25x + 3 = 0. \quad (4x-3)(7x-1) = 0, x = \frac{3}{4} \text{ or } \frac{1}{7}.$$

The number of minutes required for distance d at speed w with the wind is

$$\frac{d}{w} = 84\,\frac{v}{w} = 84\,\frac{3}{4} \quad \text{or } 84\,\frac{1}{7} \quad = 63 \text{ or } 12 \text{ minutes.}$$

35.(E) Chord MN has length d (See figure) and each chord of length s subtends an arc of $\frac{180^\circ}{5} = 36^\circ$. Draw a radius OP meeting MN at T. Also draw perpendiculars OV and QU to MN from O and Q respectively. Triangles OVN and QUN are similar so that

$$\frac{UN}{QN} = \frac{VN}{1} \text{ or } \frac{d}{2} - \frac{s}{2} = s\,\frac{d}{2}. \text{ Therefore } d - s = ds$$



To see that $d - s = 1$, note that $MT = s$ because $\angle MPT = \angle MTP = 72^\circ$. Also $\angle OTN = \angle ORN$ so that ORNT is a $\parallel$ogram with opposite sides $TN = OR = 1$. $MN - MT = d - s = 1$. Now $(d-s)^2 + 4ds = 1 + 4 = 5$ or $d^2 + 2\,ds + s^2 = 5$ and $d + s = \sqrt{5}$. $\therefore d^2 - s^2 = (d+s)(d-s) = \sqrt{5} \times 1$. All three of the equations I, II, and III are necessarily true as stated in choice (E).

or

Alternately, $d - s = 1$ follows from $MT = s$ and $MN = d$ because OTNR is a $\parallel$ogram with side $TN = MN - MT = d - s = 1$. Now let P′ (not shown) denote the other end of the diameter through P. Then chords PP′ and MN intersect at T and the products of their segments are equal.

$$(PT)\cdot(TP') = (MT)\cdot(TN) \text{ or } (1-s)(1+s) = s\cdot 1$$

This quadratic equation has the positive root $s = \frac{1}{2}(\sqrt{5}-1)$ from which relations II and III can be verified. Therefore all three equations I, II and III are necessarily true.

SOLUTION-ANSWER KEY

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USE OF KEY

1. This Key is prepared for the convenience of teachers.
2. Some of the solutions may be intentionally incomplete; crucial steps are shown.
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1. (D) Multiplying both sides of the given equation by the least common denominator $2xy$ yields $4x + 6y = xy$ or, equivalently, $4x = xy - 6x$. Factoring x from the right side of the last equation gives $4y = x(y - 6)$. Since $y \neq 6$ we can divide both sides of this equation by $y - 6$ to obtain (D).

2. (B) Since x_1 and x_2 are the two roots of the quadratic equation $3x^2 - hx - b = 0$, the sum of the roots is $\frac{h}{3}$.

3. (A) The coefficient of x^7 in $(1 + 2x - x^2)^4$ is the coefficient of the sum of four identical terms $2x(-x^2)^3$, which sum is $-8x^7$.

4. (D) By the remainder theorem $x^{51} + 51$ divided by $x + 1$ leaves a remainder of $(-1)^{51} + 51 = 50$. This can also be seen quite easily by long division.

5. (B) $\angle EBC = \angle ADC$ since both angles are supplements of $\angle ABC$. Note the fact that $\angle BAD = 92°$ is not needed in the solution of the problem.

6. (D) $x * y = \dfrac{xy}{x + y} = \dfrac{yx}{y + x} = y * x$

and

$$(x * y) * z = \frac{xy}{x + y} * z = \frac{\dfrac{xyz}{x + y}}{\dfrac{xy}{x + y} + z} = \frac{xyz}{xy + xz + yz}.$$

Similarly $x * (y * z) = \dfrac{xyz}{xy + xz + yz}$, so "$*$" is both commutative and associative.

7. (D) Let x be the original population, then

$x + 1{,}200 - 0.11(x + 1{,}200) = x - 32.$

Solving for x gives (D).

8. (A) Since 3^{11} and 5^{13} are both odd their sum must be even.

9. (B) All multiples of 8, including 1,000, fall in the second column.

10. (B) Putting the quadratic in its standard form:

$(2k - 1)x^2 - 8x + 6 = 0,$

we see that the discriminant is $64 - 4(2k - 1)6 = 88 - 48k$. A quadratic equation has no real roots if and only if its discriminant is negative.

Since $88 - 48k < 0$ when $k > 11/6$, the smallest integral value of k for which the equation has no real roots is 2.

11. (A) Since (a,b) and (c,d) are on the same line, $y = mx + k$, they satisfy the same equation. Therefore,

$b = ma + k$
$d = mc + k,$

Now the distance between (a,b) and (c,d) is $\sqrt{(a - c)^2 + (b - d)^2}$. We obtain from the above two equations $(b - d) = m(a - c)$, so that

$\sqrt{(a - c)^2 + (b - d)^2} = \sqrt{(a - c)^2 + m^2(a - c)^2} = |a - c|\sqrt{1 + m^2}$

Note we are using the fact that $\sqrt{x^2} = |x|$ for all real x.

12. (B) Since $g(x) = 1/2$ is satisfied by $x = \sqrt{1/2}$,

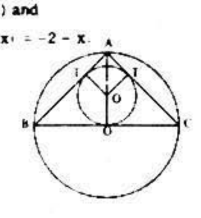
$f(1/2) = f(g(\sqrt{1/2})) = 1.$

13. (D) Statement (D) is the contrapositive of the given one and the only one of the statements (A) through (E) equivalent to the given statement.

14. (A) Since $x^2 > 0$ for all $x \neq 0$, $x^2 > 0 > x$ is true if $x < 0$. Counterexamples to the other statements are easy to construct.

15. (B) By definition $|a| = \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$. If $x < -2$, then $1 + x < 0$ and $|1 + x| = -(1 + x)$ and $|1 - |1 + x|| = |1 + 1 + x| = |2 + x|$. Again if $x < -2$, then $2 + x < 0$ and $|2 + x| = -2 - x$.

16. (A) In the adjoining figure, ΔABC is a right isosceles triangle, with $\angle BAC = 90°$ and $AB = AC$, inscribed in a circle with center O and radius R. The line segment AO has length R and bisects line segment BC and $\angle BAC$. A circle with center O′ lying on AO and radius r is inscribed in ΔABC. The sides AB and AC are tangent to the inscribed circle with points of tangency T and T′, respectively. Since $\Delta ATO'$ has angles 45°-45°-90° and $O'T = r$, $AT = r$ and $O'A = r\sqrt{2}$, then $R = r + r\sqrt{2}$ and $R/r = 1 + \sqrt{2}$.



17\. (C) Since $i^2 = -1$, $(1+i)^2 = 2i$ and $(1-i)^2 = 2i$. Writing $(1+i)^{20} - (1-i)^{20} = ((1+i)^2)^{10} - ((1-i)^2)^{10}$, we have $(1+i)^{20} - (1-i)^{20} = (2i)^{10} - (-2i)^{10} = 0$.

18\. (D) By hypothesis we have $3 = 8^p = 2^{3p}$ and $5 = 3^q$ so $5 = (2^{3p})^q = 2^{3pq}$. Therefore,

$$\log_{10}5 = \log_{10}2^{3pq} = 3pq\log_{10}2 = 3pq\log_{10}\frac{10}{5}.$$

Since $\log_{10}\frac{10}{5} = \log_{10}10 - \log_{10}5 = 1 - \log_{10}5$, we have $\log_{10}5 = 3pq(1 - \log_{10}5)$ and therefore, solving for $\log_{10}5$, we obtain (D).

19\. (A) Let $DM = NB = x$; then $AM = AN = 1 - x$. Denoting the length of each side of the equilateral triangle CMN by y and using the Pythagorean Theorem we see

$$x^2 + 1^2 = y^2 \quad \text{and} \quad (1-x)^2 + (1-x)^2 = y^2.$$

Substituting the first equation into the second we get

$$2(1-x)^2 = x^2 + 1 \quad \text{or} \quad x^2 - 4x + 1 = 0.$$

The roots of this equation are $2 - \sqrt{3}$ and $2 + \sqrt{3}$. Since $2 + \sqrt{3} > 1$ we must choose $x = 2 - \sqrt{3}$ Now,

$$\begin{aligned}\text{area } \triangle CMN &= \text{area } \square ABCD - \text{area } \triangle ANM - \text{area } \triangle NBC - \text{area } \triangle CDM\\ &= 1 - \frac{1}{2}(1-x)^2 - \frac{x}{2} - \frac{x}{2} = \frac{1}{2}(1-x^2).\end{aligned}$$

Substituting $x = 2 - \sqrt{3}$ we obtain area $\triangle CMN = 2\sqrt{3} - 3$.

20\. (D) By rationalizing the denominator of each function, we see

$$T = (3 + \sqrt{8}) - (\sqrt{8} + \sqrt{7}) + (\sqrt{7} + \sqrt{6}) - (\sqrt{6} + \sqrt{5}) + (\sqrt{5} + 2) = 3 + 2 = 5$$

21\. (B) The sum of the first five terms of the geometric series with initial term a and common ratio r is

$$S_5 = a + ar + ar^2 + ar^3 + ar^4 = \frac{a(1 - r^5)}{1 - r}.$$

By hypothesis $ar^4 - ar^3 = 576$ and $ar - a = 9$. Dividing the last equation into the first yields

$$\frac{r^4 - r^3}{r - 1} = 64 \text{ so } r^3 = 64 \text{ and } r = 4.$$ Since $ar - a = 9$ and $r = 4$, $a = 3$ and therefore

$$S_5 = \frac{3(1 - 4^5)}{-3} = 1023.$$

22\. (E) Writing

$$\begin{aligned}\sin\frac{A}{2} - \sqrt{3}\cos\frac{A}{2} = 2\left[\frac{1}{2}\sin\frac{A}{2} - \frac{\sqrt{3}}{2}\cos\frac{A}{2}\right]\\ = 2\left[\cos 60^\circ \sin\frac{A}{2} - \sin 60^\circ \cos\frac{A}{2}\right]\\ = 2\sin\left(\frac{A}{2} - 60^\circ\right)\end{aligned}$$

we see that the last expression is minimum when $\sin\left(\frac{A}{2} - 60^\circ\right) = -1$ or when

$$\frac{A}{2} - 60^\circ = 270^\circ + (360\,m)^\circ,\ m = 0, \pm 1, \pm 2, \cdots.$$

Solving for A we get a minimum when

$$A = 660^\circ + (720\,m)^\circ,\ m = 0, \pm 1, \pm 2, \cdots.$$

None of (A) through (D) satisfy this equation.

23\. (B) Since $TP = T'P$, $OT = OT'' = r$, and $\angle PT'O = \angle PTO = 90^\circ$, we have $\triangle OTP \cong \triangle OT'P$. Similarly $\triangle OT'Q \cong \triangle OT''Q$. Letting $x = \angle TOP = \angle POT''$, and $y = \angle T'OQ = \angle QOT'$ we obtain $2x + 2y = 180^\circ$ But this implies that $\angle POQ = x + y = 90^\circ$. Therefore $\triangle POQ$ is a right triangle with altitude OT . Since the altitude drawn to the hypotenuse of a right triangle is the mean proportion of the segments it cuts, we have

$$\frac{4}{r} = \frac{r}{9} \text{ or } r = 6.$$

24\. (A) Let A be the event of rolling at least a five; then the probability of A is $\frac{2}{6} = \frac{1}{3}$. In six rolls of a die the probability of event A happening six times is $\left(\frac{1}{3}\right)^6$. The probability of getting exactly five successes and one failure of A in a specific order is $\left(\frac{1}{3}\right)^5 \cdot \frac{2}{3}$. Since there are six ways to do this, the probability of getting five successes and one failure of A in any order is $6\left(\frac{1}{3}\right)^5 \cdot \frac{2}{3} = \frac{12}{729}$. The probability of getting all successes <u>or</u> five successes and one failure in any order is thus

$$\frac{12}{729} + \left(\frac{1}{3}\right)^6 = \frac{13}{729}.$$

25. (C) Since DM = AM, $\angle QMA = \angle DMC$ and $\angle CDM = \angle QAM$ we have $\Delta QAM \cong \Delta MCD$. Similarly $\Delta BPN \cong \Delta DNC$. Now,

$$\text{area } \Delta QPO = \text{area } \square ABCD + \text{area } \Delta DOC$$

and

$$\text{area } \Delta DOC = \frac{1}{4}\left(\frac{1}{2} \text{ area } \square ABCD\right) = \frac{k}{8},$$

so that

$$\text{area } \Delta QPO = k + \frac{k}{8} = \frac{9k}{8}.$$

26. (C) Writing 30 as a product of prime factors, $30 = 2 \cdot 3 \cdot 5$, we obtain

$$(30)^4 = 2^4 \cdot 3^4 \cdot 5^4.$$

The divisor of $(30)^4$ are exactly the numbers of the form $2^i \cdot 3^j \cdot 5^k$, where i, j, k are non-negative integers between zero and four inclusively, so there are $(5)^3 = 125$ distinct divisors of $(30)^4$; excluding 1 and $(30)^4$ there are 123 divisors.

27. (A) Consider $|f(x) + 4| = |3x + 2 + 4| = 3|x + 2|$. Now whenever $|x + 2| < \frac{a}{3}$, then $|f(x) + 4| < a$. Consequently whenever $|x + 2| < b$ and $b \leq \frac{a}{3}$, we have $|f(x) + 4| < a$.

28. (D) Using the formula for the sum of a geometric series

$$0 \leq x < \sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n = \frac{\frac{2}{3}}{1 - \frac{1}{3}} = 1.$$

If $a_1 = 0$, then

$$0 \leq x < \sum_{n=2}^{\infty}\left(\frac{2}{3}\right)^n = \frac{\frac{2}{9}}{1 - \frac{1}{3}} = \frac{1}{3}.$$

If $a_1 = 2$, then

$$\frac{2}{3} + 0 + 0 + \cdots \leq x < 1.$$

So either $0 \leq x < \frac{1}{3}$ or $\frac{2}{3} \leq x < 1$.

29. (B) For each $p = 1, \cdots, 10$,

$$\begin{aligned} S_p &= p + p + (2p - 1) + p + 2(2p - 1) + \cdots + p + 39(2p - 1) \\ &= 40p + \frac{(40)(39)}{2}(2p - 1) \\ &= (40 + 40 \cdot 39)p - 20 \cdot 39 \\ &=((40)^2 p - 20 \cdot 39 \end{aligned}$$

Therefore,

$$\begin{aligned} \sum_{p=1}^{10} S_p &= (40)^2 \sum_{p=1}^{10} p - (10)(20)(39) \\ &= (40)^2 \frac{(10)(11)}{2} - (10)(20)(39) \\ &= 80{,}200. \end{aligned}$$

30. (A) Consider the line segment AB cut by a point D with AD = x, DB = y, y < x and $\frac{y}{x} = \frac{x}{x + y}$. Since $\frac{y}{x} = R$ we can choose x = 1 and therefore y = R, thus $R = \frac{1}{1 + R}$ and $R^2 + R - 1 = 0$. We can therefore write $\frac{1}{R} = R + 1$ so that

$$R^2 + \frac{1}{R} = R^2 + R + 1 = (R^2 + R - 1) + 2 = 2, \text{ and}$$

$$R^{\left[R^{R^2} + \frac{1}{R}\right] + \frac{1}{R}} + \frac{1}{R} = R^{\left[R^2 + \frac{1}{R}\right]} + \frac{1}{R} = R^2 + \frac{1}{R} = 2$$

SOLUTION-ANSWER KEY

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Rutgers University, New Brunswick, N.J. 08903

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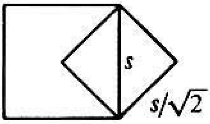
1. (B) $$2-\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2}}}\;\text{(reciprocal)}$$

$$\cfrac{1}{2-\cfrac{1}{2-\cfrac{1}{2-\frac{1}{2}}}}=\cfrac{1}{2-\cfrac{1}{2-\frac{2}{3}}}=\cfrac{1}{2-\frac{3}{4}}=\frac{4}{5}.$$

2. (D) The equations have a solution unless the lines obtained by plotting them are parallel (and not coincident); i.e., unless $m = 2m - 1$.

3. (A) None of the inequalities are satisfied if a, b, c, x, y, z are chosen to be $1, 1, -1, 0, 0, -10$, respectively.

4. (A) In the adjoining figure, if s is the length of a side of the first square, then $s/\sqrt{2}$ is the length of a side of the second square. Thus the ratio of the areas is $s^2/(s/\sqrt{2})^2 = 2$.



5. (B) $(x+y)^9 = x^9 + 9x^8y + 36x^7y^2 + \cdots + y^9$;
$9p^8q = 36p^7q^2$ and $p + q = 1$;
$p = 4q$ and $p + q = 1$; $p = 4/5$.

6. (E) Grouping the terms of the difference as $(2-1)+(4-3)+\cdots+(160-159)$, one obtains 80.

7. (E) $\dfrac{|x-|x||}{x}$ is -2 if x is negative and 0 if x is positive.

8. (D) II and IV are the only negations of the given statement.

9. (C) Let d denote the common difference of the progression $a_1 + b_1, a_2 + b_2, \ldots$. Then $99d = (a_{100} + b_{100}) - (a_1 + b_1) = 0$; Thus $d = 0$, and $100(a_1 + b_1) = 10{,}000$ is the desired sum.

10. (A) Let k be any positive integer. Then $10^k + 1 = 100\cdots01$, and

$$\begin{array}{r}
100\cdots01\\
\underline{100\cdots01}\\
100\cdots01\\
\underline{100\cdots01}\\
100\cdots0200\cdots01.
\end{array}$$

The sum of the digits is therefore $1 + 2 + 1 = 4$.

11. (E) Suppose P is the given point, O is the center of circle K, and M is the midpoint of a chord AB passing through P. Since $\measuredangle OMP$ is 90°, M lies on a circle having OP for the diameter. Conversely, if M is any point on the circle with diameter OP, then the chord of the given circle passing through P and M (the chord of the given circle tangent to circle OP at P if $M = P$) is perpendicular to OM. Hence M is the midpoint of this chord and therefore belongs to the locus.

12. (B) If $a \neq b$, $a^3 - b^3 = 19x^3$ and $a - b = x$, then

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) = x(a^2 + ab + b^2) = 19x^3$$

Dividing by x and substituting $b = a - x$ into the last equality above, we obtain

$$18x^2 + 3ax - 3a^2 = 0$$
$$-3(a - 3x)(a + 2x) = 0.$$

So $a = 3x$ or $a = -2x$.

13. (D) If r is a root, then $r^6 + 8 = 3r^5 + 6r^3 + r$; if r were negative, the left side of this equation would be positive and the right side would be negative. Also, the polynomial has a positive value at $x = 0$ and a negative value at $x = 1$. (The fact that the polynomial has no negative roots also follows directly from Descartes' rule of signs.)

14. (E) Let W, H, I and S denote *whatsis*, *whosis*, *is* and *so*, respectively. Then $H = I$ and $IS = 2S$ imply $W = S$, or equivalently, since $S > 0$, $H = I = 2$ implies $W = S$. Now if $H = S$, $2S = S^2$ and $I = 2$, or equivalently $H = I = S = 2$, then $W = S$, so that $HW = 4 = S + S$.

15. (A) The first eight terms of the sequence are $1, 3, 2, -1, -3, -2, 1, 3$. Since the seventh and eighth terms are the same as the first and second, the ninth term will be the same as the third, etc.; i.e., the sequence repeats every six terms. Hence the sum of the first 96 terms is zero, and the sum of the first one hundred terms is $0 + 1 + 3 + 2 - 1 = 5$.

16. (C) If a is the first term of the series and $1/n$ is its common ratio, then

$$\frac{a}{1-(1/n)} = 3 \text{ or } a = 3 - (3/n).$$ Since a and n are integers and $0 < 1/n < 1$, $n = 3$ and $a = 2$. The sum of the first two terms is $2 + 2(1/3) = 8/3$.

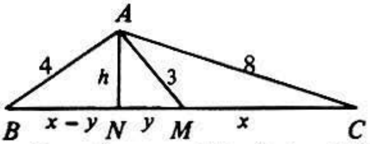
17. (D) The total number of trips is $2x$, so $2x = 9 + (8 + 15)$ and $x = 16$.

18. (D) There are 900 three digit numbers, and three of them (128, 256 and 512) have logarithms base two which are integral. So $3/900 = 1/300$ is the desired probability.

19. (D) For any fixed positive value of x distinct from one, let $a = \log_3 x$, $b = \log_x 5$ and $c = \log_3 5$. Then $x = 3^a$, $5 = x^b$ and $5 = 3^c$. These last equalities imply $3^{ab} = 3^c$ or $ab = c$. Note that $\log_x 5$ is not defined for $x = 1$.

20. (B) In the adjoining figure let h be the length of altitude AN drawn to BC, let $x = BM$ and let $y = NM$. Then

$$\begin{aligned} h^2 + (x+y)^2 &= 64 \\ h^2 + y^2 &= 9 \\ h^2 + (x-y)^2 &= 16. \end{aligned}$$



Subtracting twice the second equation from the sum of the first and third equations yields $2x^2 = 62$. Thus $x = \sqrt{31}$ and $BC = 2\sqrt{31}$.

OR

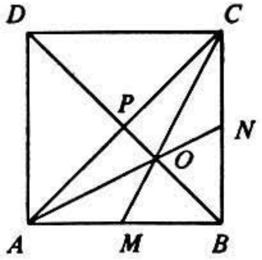
Applying the parallelogram law to the parallelogram having AB and AC as adjacent sides yields

$$\begin{aligned} 4^2 + 8^2 &= 2(3^2) + 2x^2 \\ x &= \sqrt{31}. \end{aligned}$$

21. (D) Letting $a = 0$ in the equation $f(a)f(b) = f(a+b)$ (called a *functional equation*) yields $f(0)f(b) = f(b)$, or $f(0) = 1$; letting $b = -a$ in the functional equation yields $f(a)f(-a) = f(0)$, or $f(-a) = 1/f(a)$; and $f(a)f(a)f(a) = f(a)f(2a) = f(3a)$, or $f(a) = \sqrt[3]{f(3a)}$. The function $f(x) = 2^{-x}$ satisfies the functional equation, but does not satisfy condition IV.

22. (E) Since the product of the positive integral roots is the prime integer q, q must be positive and the roots must be 1 and q. Since $p = 1 + q$ is also prime, $q = 2$ and $p = 3$. Hence all four statements are true.

23. (C) In the adjoining figure diagonals AC and DB are drawn. Since $ABCD$ is a square AC and DB bisect each other at P at right angles. If s denotes the length of a side of the square, then $AC = DB = s\sqrt{2}$. Now O is the intersection of the medians of triangle ABC, so that $OP = (1/3)(s\sqrt{2}/2)$. Thus the area of ΔAOC is $s^2/6$. The area of $AOCD$ is then $s^2/2 + s^2/6 = 2s^2/3$, and the required ratio is 2/3.



OR

Introduce coordinates with respect to which AB is the unit inverval on the positive x-axis and AD is the unit interval on the positive y-axis, and find the coordinates of O by solving simultaneously the equations for the lines AN and MC. Calculate the area of $AOCD$ by drawing perpendiculars from O to AD and CD.

24. (E) If $0° < \theta < 45°$, then (see Figure 1) an application of a theorem on exterior angles of triangles to ΔEAC yields $2\theta = \measuredangle EAC + \theta$. Therefore $\measuredangle EAC = \theta$ and ΔEAC is isosceles. Hence $EC = AE = AD$.

If $\theta = 45°$, then ΔABC is a $45°$–$45°$–$90°$ triangle and $E = B$. Then $EC = BC = AB = AD$.

If $45° < \theta < 60°$, then (see Figure 2)

$$\begin{aligned}\measuredangle EAC &= \measuredangle EAB + \measuredangle BAC \\ &= [180° - 2(180° - 2\theta)] \\ &\qquad + [180° - 3\theta] \\ &= \theta.\end{aligned}$$

Thus ΔEAC is isosceles and $EC = EA = AD$.

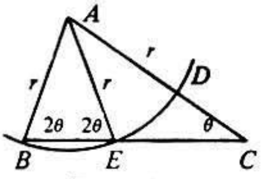


Figure 1

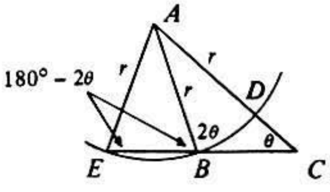


Figure 2

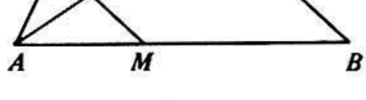
25. (B) If the son is the worst player, the daughter must be his twin. The best player must then be the brother. This is consistent with the given information, since the brother and the son could be the same age. The assumption that any of the other players is worst leads to a contradiction:

If the woman is the worst player, her brother must be her twin and her daughter must be the best player. But the woman and her daughter cannot be the same age.

If the brother is the worst player, the woman must be his twin. The best player is then the son. But the woman and her son cannot be the same age, and hence the woman's twin, her brother, cannot be the same age as the son.

If the daughter is the worst player, the son must be the daughter's twin. The best player must then be the woman. But the woman and her daughter cannot be the same age.

26. (C) In the adjoining figure $\frac{BD}{CD} = \frac{AB}{AC}$, since the bisector of an angle of a triangle divides the opposite side into segments which are proportional to the two adjacent sides. Since $CN = CD$ and $BM = BD$, we have $\frac{BM}{CN} = \frac{AB}{AC}$, which implies MN is parallel to CB. Statements (A), (B), (D) and (E) are false if $\measuredangle A = 90^\circ - \theta$, $\measuredangle B = 60^\circ$ and $\measuredangle C = 30^\circ + \theta$, where θ is any sufficiently small positive angle.



27. (E) Substituting the identity

$$p^2 + q^2 + r^2 = (p + q + r)^2 - 2(pq + qr + rp)$$

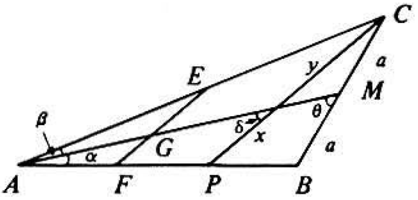
into the identity

$$p^3 + q^3 + r^3 = (p + q + r)(p^2 + q^2 + r^2 - pq - qr - rp) + 3pqr$$

yields

$$\begin{aligned} p^3 + q^3 + r^3 &= (p + q + r)[(p + q + r)^2 - 3(pq + qr + rp)] + 3pqr \\ &= 1[1^2 - 3(1)] + 3(2) = 4. \end{aligned}$$

28. (A) Construct line CP parallel to EF and intersecting AB at P. By proportionality $AP = 8$. Let $a, x, y, \alpha, \beta, \delta$ and θ be as shown in the adjoining diagram. Then



$$\frac{a}{\sin\alpha} = \frac{12}{\sin\theta}\,;\quad \frac{a}{\sin\beta} = \frac{16}{\sin(180^\circ - \theta)}\,;\quad \frac{\sin\beta}{\sin\alpha} = \frac{3}{4}\,.$$

Therefore,

$$\frac{x}{\sin\alpha} = \frac{8}{\sin\delta}\,;\quad \frac{y}{\sin\beta} = \frac{16}{\sin(180^\circ - \delta)}\,;$$

and

$$\frac{EG}{GF} = \frac{y}{x} = 2\frac{\sin\beta}{\sin\alpha} = \frac{3}{2}.$$

OR

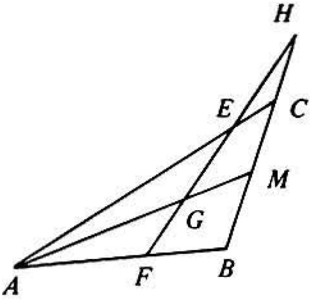
In the adjoining figure side BC extended through C intersects EF extended through E at H. Applying Menelaus' theorem to ΔFBH and ΔHCE with AM as the transversal, we obtain

$$\frac{GH}{GF}\frac{AF}{AB}\frac{MB}{MH} = 1,\quad \frac{GH}{GE}\frac{AE}{AC}\frac{MC}{MH} = 1,$$

respectively. Since $MC = MB$ and $AE = 2AF$, dividing the first equation by the second yields

$$\frac{GE}{GF} = 2\frac{AB}{AC} = 2\cdot\frac{12}{16} = \frac{3}{2}.$$

Menelaus' theorem: The six segments determined by a transversal on the sides of a triangle are such that the product of three non-consecutive segments is equal to the product of the remaining three.



29. (C) Since $a^6 + b^6 = (a^2 + b^2)(a^4 - a^2b^2 + b^4)$ and $\sqrt{3} - \sqrt{2} < 1$,

$$(\sqrt{3} + \sqrt{2})^6 + (\sqrt{3} - \sqrt{2})^6 = (10)(97) = 970,$$

and 970 is the smallest integer larger than $(\sqrt{3} + \sqrt{2})^6$.

30. (B) Let $w = \cos 36^\circ$ and let $y = \cos 72^\circ$. Applying the identities

$$\cos 2\theta = 2\cos^2\theta - 1 \quad \text{and} \quad \cos 2\theta = 1 - 2\sin^2\theta,$$

with $\theta = 36^\circ$ in the first identity and $\theta = 18^\circ$ in the second identity, yields

$$y = 2w^2 - 1 \quad \text{and} \quad w = 1 - 2y^2.$$

Adding these last two equations and dividing the result by $w + y$ yields the desired result.

SOLUTION-ANSWER KEY

TWENTY SEVENTH ANNUAL H. S.

MATHEMATICS EXAMINATION

1976

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1. This key is prepared for the convenience of teachers.
2. Some of the solutions may be intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, a more elementary procedure is also shown.
5. This solution-answer key validates our statement that no mathematics beyond intermediate algebra and trigonometry is needed to solve the problems posed.

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1. (B) $1 - \frac{1}{1-x} = \frac{1}{1-x}$; $1 - x - 1 = 1$; $x = -1$.

2. (B) If $x + 1 \neq 0$ then $-(x+1)^2 < 0$ and $\sqrt{-(x+1)^2}$ is not real; if $x + 1 = 0$ then $\sqrt{-(x+1)^2} = 0$. Thus $x = -1$ is the only value of x for which the given expression is real.

3. (E) The distance to each of the two closer midpoints is one; the distance to each of the other midpoints is $\sqrt{1^2 + 2^2}$.

4. (C) The sum of the terms in the new progression is

$$1 + \frac{1}{r} + \ldots + \frac{1}{r^{n-1}} = \frac{r^{n-1} + r^{n-2} + \ldots + 1}{r^{n-1}} = \frac{s}{r^{n-1}}.$$

5. (C) Let t and u be the tens' digit and units' digit, respectively, of a number which is increased by nine when its digits are reversed. Then $9 = (10u + t) - (10t + u) = 9(u - t)$ and $u = t + 1$. The eight solutions are $\{12, 23, \ldots, 89\}$.

6. (C) Let r be a solution of $x^2 - 3x + c = 0$ such that $-r$ is a solution of $x^2 + 3x - c = 0$. Then

$$r^2 - 3r + c = 0$$
$$r^2 - 3r - c = 0,$$

which implies $2c = 0$. The solutions of $x^2 - 3x = 0$ are 0 and 3.

7. (E) The quantity $(1 - |x|)(1 + x)$ is positive if and only if either both factors are positive or both factors are negative. Both factors are positive if and only if $-1 < x < 1$, while both factors are negative if and only if $x < -1$.

8. (A) The points whose coordinates are integers with absolute value less than or equal to four form a 9×9 array, and 13 of these points are at distance less than or equal to two units from the origin.

9. (D) Since F is the midpoint of BC, the altitude of ΔAEF from F to AE (extended if necessary) is one half the altitude of ΔABC from C to AB (extended if necessary). Base AE of ΔAEF is 3/4 of base AB of ΔABC. Therefore, the area of ΔAEF is $(1/2)(3/4)(96) = 36$.

10. (D) For any real number x, the following equations are equivalent:

$$\begin{aligned} f(g(x)) &= g(f(x)) \\ m(px + q) + n &= p(mx + n) + q \\ mpx + mq + n &= mpx + np + q \\ n(1 - p) &= q(1 - m). \end{aligned}$$

11. (B) The statements "P implies Q," "Not Q implies not P" and "Not P or Q" are equivalent. The given statement, statement III and statement IV are of these forms, respectively.

12. (C) There are 25 different possibilities for the number of apples a crate can contain. If there were no more than five crates containing any given number of apples, there could be at most $25(5) = 125$ crates. Since there are 128 crates, $n \geqslant 6$. But also, $n \leqslant 6$, since it is possible that there are six crates containing k apples for $k = 120, 121, 122$ and five crates containing k apples for $123 \leqslant k \leqslant 144$.

13. (A) A cow gives $\dfrac{(x+1)}{x(x+2)}$ cans per day. Hence $(x + 3)$ cows give $\dfrac{(x+1)(x+3)}{x(x+2)}$ cans per day, and $(x + 3)$ cows give milk at the rate of $\dfrac{x(x+2)}{(x+1)(x+3)}$ days per can. To fill $(x + 5)$ cans takes $\dfrac{x(x+2)(x+5)}{(x+1)(x+3)}$ days.

14. (A) Let n be the number of sides the polygon has. The sum of the interior angles of a convex polygon with n sides is $(n - 2)180°$, and the sum of n terms of an arithmetic progression is $n/2$ times the sum of the first and last terms. Therefore

$$(n - 2)180 = \frac{n}{2}(100 + 140).$$

Solving this equation for n yields $n = 6$.

15. (B) Since d divides $2312 - 1417 = (5)179$ and $1417 - 1059 = (2)179$ (and 179 is prime), $d = 179$, $r = 164$ and $d - r = 15$.

16. (E) Let G and H be the points at which the altitudes from C and F intersect sides AB and DE, respectively. Right triangles AGC and DHF are congruent, since side AG and side DH have the same length, and hypotenuse AC and hypotenuse DF have the same length. Therefore,

$$\measuredangle ACB + \measuredangle DFE = 2\measuredangle ACG + 2\measuredangle DFH = 180^\circ$$

and

$$(\text{area } \Delta ABC) = 2(\text{area } \Delta ACG) = 2(\text{area } \Delta DFH) = (\text{area } \Delta DEF).$$

17. (A) By virtue of the trigonometric identities,

$$\begin{aligned}(\sin\theta + \cos\theta)^2 &= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta\\ &= 1 + \sin 2\theta\\ &= 1 + a.\end{aligned}$$

Since θ is acute, $\sin\theta + \cos\theta > 0$ and $\sin\theta + \cos\theta = \sqrt{1+a}$.

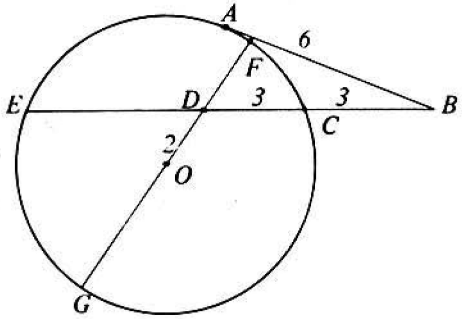
18. (E) In the adjoining figure, E is the point of intersection of the circle and the extension of DB, and FG is the diameter passing through D. Let r denote the radius of the circle.

Then

$$\begin{aligned}(BC)(BE) &= (AB)^2\\ 3(DE + 6) &= 36\\ DE &= 6.\end{aligned}$$

Also

$$\begin{aligned}(DE)(DC) &= (DF)(DG)\\ 18 &= r^2 - 4\\ r &= \sqrt{22}.\end{aligned}$$



19. (B) Let $ax + b$ be the remainder when $p(x)$ is divided by $(x-1)(x-3)$, and let $q(x)$, $r(x)$ and $t(x)$ be the quotients when $p(x)$ is divided by $(x-1)$, $(x-3)$ and $(x-1)(x-3)$, respectively. Then

$$\begin{aligned}p(x) &= (x-1)q(x) + 3\\ p(x) &= (x-3)r(x) + 5\\ p(x) &= (x-1)(x-3)t(x) + ax + b.\end{aligned}$$

Substituting $x = 1$ into the first and third equations and then substituting $x = 3$ into the second and third equations yields

$$\begin{aligned}3 &= a + b\\ 5 &= 3a + b.\end{aligned}$$

Therefore, $ax + b = x + 2$.

20. (E) The given equation may be written in the form

$$4(\log_a x)^2 - 8(\log_a x)(\log_b x) + 3(\log_b x)^2 = 0;$$
$$(2\log_a x - \log_b x)(2\log_a x - 3\log_b x) = 0;$$
$$\log_a x^2 = \log_b x \text{ or } \log_a x^2 = \log_b x^3.$$

Let $r = \log_a x^2$. Then

$$a^r = x^2 \text{ and } b^r = x, \text{ or } a^r = x^2 \text{ and } b^r = x^3;$$
$$a^r = b^{2r} \text{ or } a^{3r} = b^{2r};$$
$$a = b^2 \text{ or } a^3 = b^2.$$

21. (B) Since $2^{1/7} \ldots 2^{(2n+1)/7} = 2^{(n+1)^2/7}$ and $2^{10} = 1024$, we consider values of n for which $(n+1)^2/7$ is approximately 10:

$$2^{(7+1)^2/7} = 2^{9+\frac{1}{7}} < 2^9 2^{1/2} < 1000 < 2^{10} < 2^{(9+1)^2/7},$$

and $n = 9$.

22. (A) Let point P have coordinates (x, y) in the coordinate system in which the vertices of the equilateral triangle are $(0, 0)$, $(s, 0)$ and $(s/2, s\sqrt{3}/2)$. Then P belongs to the locus if and only if

$$a = x^2 + y^2 + (x-s)^2 + y^2 + (x - s/2)^2 + (y - s\sqrt{3}/2)^2,$$

or, equivalently, if and only if

$$a = (3x^2 - 3sx) + (3y^2 - s\sqrt{3}y) + 2s^2$$
$$\frac{a - 2s^2}{3} = (x - s/2)^2 + (y - s\sqrt{3}/6)^2 - s^2/3$$
$$\frac{a - s^2}{3} = (x - s/2)^2 + (y - s\sqrt{3}/6)^2.$$

Thus the locus is the empty set if $a < s^2$; the locus is $\{(s/2, s\sqrt{3}/6)\}$ if $a = s^2$; and the locus is a circle if $a > s^2$.

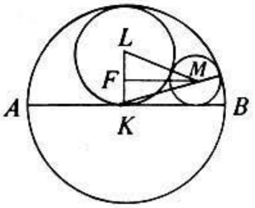
23. (A) Since C_k^n is always an integer the quantity

$$\begin{aligned}\left(\frac{n-2k-1}{k+1}\right)C_k^n &= \left(\frac{n-k}{k+1} - 1\right)C_k^n \\ &= \frac{n!(n-k)}{k!(n-k)!(k+1)} - C_k^n \\ &= \frac{n!}{(k+1)!(n-k-1)!} - C_k^n \\ &= C_{k+1}^n - C_k^n\end{aligned}$$

is always an integer.

24. (C) In the adjoining figure, MF is parallel to AB and intersects KL at F. Let r, $s(=r/2)$ and t be the radii of the circles with centers K, L and M, respectively. Applying the Pythagorean theorem to ΔFLM and ΔFKM yields

$$(MF)^2 = (\frac{r}{2}+t)^2 - (\frac{r}{2}-t)^2$$
$$(MF)^2 = (r-t)^2 - t^2.$$

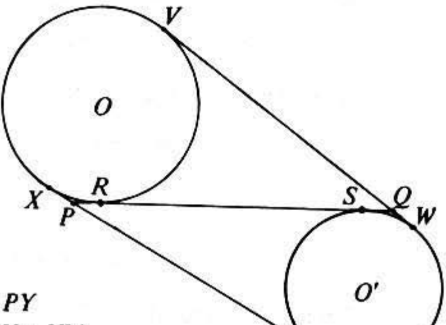


Equating the right members of these equalities yields $r/t = 4$. Therefore the desired ratio is 16.

25. (D) For all positive integers n,

$$\Delta^1(u_n) = (n+1)^3 + (n+1) - n^3 - n = 3n^2 + 3n + 2$$
$$\Delta^2(u_n) = 3(n+1)^2 + 3(n+1) + 2 - 3n^2 - 3n - 2 = 6n + 6$$
$$\Delta^3(u_n) = 6$$
$$\Delta^4(u_n) = 0.$$

26. (C) In the adjoining figure, X, Y, V and W are the points of tangency of the external common tangents; and R and S are the points of tangency of the internal common tangent.



From the fact that the tangents to a circle from an external point are equal, we obtain:

$$PR = PX$$
$$PS = PY$$
$$QS = QW$$
$$QR = QV.$$

So $PR + PS + QS + QR = PX + PY + QW + QV$, and thus $2PQ = XY + VW$. Since $XY = VW$, $PQ = XY = VW$.

27. (A) A direct calculation shows that

$$\left(\frac{\sqrt{\sqrt{5}+2}+\sqrt{\sqrt{5}-2}}{\sqrt{\sqrt{5}+1}}\right)^2 = 2,$$

and

$$3 - 2\sqrt{2} = 2 - 2\sqrt{2} + 1 = (\sqrt{2}-1)^2.$$

Since a radical sign denotes the positive square root,

$$N = \sqrt{2} - (\sqrt{2} - 1) = 1.$$

28. (B) One hundred lines intersect at most at $C_2^{100} = \frac{100(99)}{2} = 4950$ points. But lines $L_4, L_8, \ldots, L_{100}$ are parallel; hence $C_2^{25} = 300$ intersections are lost. Also, lines $L_1, L_5, \ldots, L_{97}$ intersect only at point A, so that $C_2^{25} - 1 = 299$ more intersections are lost. The maximum number of points of intersection is $4950 - 300 - 299 = 4351$.

29. (B) Let A be Ann's present age, and let B be Barbara's present age. Then $B = A - t$ for t such that "$B - t = A - T$ for T such that $B - T = \frac{1}{2}A$." Since, also, $A + B = 44$, the quantities A, B, t and T are solutions to the equations

$$\begin{aligned} A + B \qquad &= 44 \\ A - B - t \quad &= 0 \\ A - B + t - T &= 0 \\ A - 2B + 2T \ &= 0. \end{aligned}$$

These equations may be conveniently solved by using the first two equations to find B and t in terms of A and substituting the results into the third and fourth equations; $A = 24$.

30. (E) Let $u = x/2$, $v = y$ and $w = 2z$. Then

$$\begin{aligned} u + v + w &= 6 \\ uv + vw + uw &= 11 \\ uvw &= 6. \end{aligned}$$

Consider the polynomial $p(t) = (t - u)(t - v)(t - w)$, where (u, v, w) is a solution to the simultaneous equations above. Then $p(t) = t^3 - 6t^2 + 11t - 6$ and (u, v, w) are the solutions of $p(t) = 0$. Conversely if the solutions of $p(t) = 0$ are listed as a triple in any order, this triple is a solution to the simultaneous equations above. Since $p(t) = (t - 1)(t - 2)(t - 3)$, the six permutations of $(1, 2, 3)$ are the solutions to the simultaneous equations in u, v and w. Therefore, the given equations in x, y and z have six solutions.

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NCTM

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1. This key is prepared for the convenience of teachers.
2. Some of the solutions may be intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, a more elementary procedure is also shown.
5. This solution-answer key validates our statement that nothing beyond pre-calculus mathematics is needed to solve the problems posed.

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1. (D) $x + y + z = x + 2x + 2y = x + 2x + 4x = 7x.$

2. (D)

3. (E) If n is the number of coins the man has of each type, then $91n = 273$ and $n = 3$; three each of five types of coins is 15 coins.

4. (C) $\measuredangle B = \measuredangle C = 50^\circ$; $\measuredangle CDE = \measuredangle FDB = 65^\circ$; $\measuredangle EDF = 180^\circ - 2(65^\circ) = 50^\circ$.
Note: The measure of $\measuredangle EDF$ is 50° even if $AB \neq AC$:

$$\begin{aligned}\measuredangle EDF &= 180^\circ - \measuredangle CDE - \measuredangle FDB\\ &= 180^\circ - \tfrac{1}{2}(180^\circ - \measuredangle C) - \tfrac{1}{2}(180^\circ - \measuredangle B)\\ &= \tfrac{1}{2}(\measuredangle B + \measuredangle C) = \tfrac{1}{2}(180^\circ - \measuredangle A) = 50^\circ.\end{aligned}$$

5. (A) If P is on line segment AB, then $AP + PB = AB$; otherwise $AP + PB > AB$.

6. (D) $$\left(2x + \frac{y}{2}\right)^{-1}\left[(2x)^{-1} + \left(\frac{y}{2}\right)^{-1}\right] = \frac{1}{2x + \frac{y}{2}}\left(\frac{1}{2x} + \frac{1}{y/2}\right)$$

$$= \frac{1}{2x + \frac{y}{2}}\,\frac{\left(\frac{y}{2} + 2x\right)}{xy} = \frac{1}{xy} = (xy)^{-1}.$$

7. (E) $$\frac{1}{1 - \sqrt[4]{2}} = \frac{1}{1 - \sqrt[4]{2}} \cdot \frac{1 + \sqrt[4]{2}}{1 + \sqrt[4]{2}} = \frac{1 + \sqrt[4]{2}}{1 - \sqrt{2}} \cdot \frac{1 + \sqrt{2}}{1 + \sqrt{2}} = -(1 + \sqrt[4]{2})(1 + \sqrt{2}).$$

8. (B) If a, b and c are all positive (negative), then 4 (respectively, -4) is formed; otherwise 0 is formed.

9. (B) Let $\widehat{AB} = x^\circ$ and $\widehat{AD} = y^\circ$. Then

$$\begin{aligned}3x + y &= 360\\ \tfrac{1}{2}(x - y) &= 40.\end{aligned}$$

Doubling both sides of the second equation and adding the resulting equation to the first equation yields $x = 110$ and $\measuredangle C = 15^\circ$.

10. (E) The sum of the coefficients of a polynomial $p(x)$ is equal to $p(1)$; $(3 \cdot 1 - 1)^7 = 128$.

11. (B) If $n \leqslant x < n + 1$, then $n + 1 \leqslant x + 1 < n + 2$. Choosing $x = y = 2.5$ shows that II and III are false.

12. (D) Let a, b and c denote Al's age, Bob's age and Carl's age, respectively. Then

$$a + b + c = \frac{(a + b + c)(a - (b + c))}{a - (b + c)}$$
$$= \frac{a^2 - (b + c)^2}{a - (b + c)} = \frac{1632}{16} = 102.$$

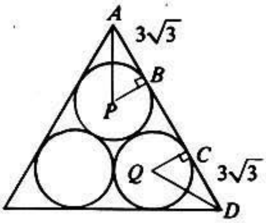
13. (E) The second through the fifth terms of $\{a_n\}$ are $a_2, a_1a_2, a_1a_2^2, a_1^2a_2^3$. If these terms are in geometric progression, then the ratios of successive terms must be equal: $a_1 = a_2 = a_1a_2$. Since a_1 and a_2 are positive, it is necessary that $a_1 = a_2 = 1$. Conversely, if $a_1 = a_2 = 1$, then $\{a_n\}$ is the geometric progression 1, 1, 1, ...

14. (B) If $m + n = mn$, then

$$m + n - mn = 0$$
$$m + n(1 - m) = 0$$
$$n = \frac{m}{m - 1},$$

for $m \neq 1$. There are no solutions for which $m = 1$. The solutions $\left(m, \dfrac{m}{m - 1}\right)$ are pairs of integers only if m is 0 or 2.

15. (D) In the adjoining figure, PB and QC are radii drawn to common tangent AD of circle P and circle Q. Since $\measuredangle PAB = \measuredangle QDC = 30^\circ$, $AB = CD = 3\sqrt{3}$ and $AD = 6 + 6\sqrt{3}$. Therefore the perimeter is $18 + 18\sqrt{3}$.



16. (D) Since

$$i^n \cos(45 + 90n)^\circ = \begin{cases} \dfrac{\sqrt{2}}{2}, & n \text{ even} \\ \dfrac{-i\sqrt{2}}{2}, & n \text{ odd}, \end{cases}$$

the sum is $\frac{\sqrt{2}}{2}(21 - 20i)$.

17. (B) The successful outcomes of the toss are the permutations of $(1,2,3)$, $(2,3,4)$, $(3,4,5)$, $(4,5,6)$. The probability that one of these outcomes will occur is $\frac{6 \cdot 4}{6^3} = \frac{1}{9}$.

18. (B) Let y be the desired product. By definition of logarithm, $2^{\log_2 3} = 3$. Raising both sides of this equation to the $\log_3 4$ power yields $2^{(\log_2 3)(\log_3 4)} = 4$. Continuing in this fashion, one obtains $2^y = 32$. thus $y = 5$.

19. (A) The center of a circle circumscribing a triangle is the point of intersection of the perpendicular bisectors of the sides of the triangle. Therefore, *P, Q, R* and *S* are the intersections of the perpendicular bisectors of line segments *AE, BE, CE* and *DE.* Since line segments perpendicular to the same line are parallel (and since lines *AE* and *CE* coincide and lines *BE* and *DE* coincide), *PQRS* is a parallelogram.

20. (E) All admissible paths must end at the bottom "*T*" of the diagram. Proceeding backwards from this point and considering only those paths which include the central column and that are to the left of the central column, there are two possible paths which lead to the bottom "*T*" in each row (excluding the first row at the top from which there is one admissible path which was already counted). Thus there are 2^6 possible paths. Similarly, there are 2^6 possible paths including the central column to the right of the central column. Therefore, since the central column was counted twice there are $2^6 + 2^6 - 1$ or $2^7 - 1$ possible paths.

21. (B) Subtracting the second given equation from the first yields

$$ax + x + (1 + a) = 0$$

or, equivalently,

$$(a + 1)(x + 1) = 0.$$

Hence, $a = -1$ or $x = -1$. If $a = -1$, then the given equations are identical and have (two complex but) no real solutions; $x = -1$ is a common solution to the given equations if and only if $a = 2$. Therefore, two is the only value of a for which the given equations have a common real solution.

22. (C) Choosing $a = b = 0$ yields

$$2f(0) = 4f(0)$$
$$f(0) = 0.$$

Choosing $a = 0$ and $b = -x$ yields

$$f(x) + f(-x) = 2f(0) + 2f(-x)$$
$$f(x) = f(-x).$$

Note: A continuous function f satisfies the functional equation if and only if $f(x) = cx^2$ for some fixed number c and for all x.

23. (B) Let a and b be the solutions of $x^2 + mx + n = 0$; then

$$-m = a + b \qquad -p = a^3 + b^3$$
$$n = ab \qquad q = a^3b^3.$$

Since

$$a^3 + b^3 = (a + b)^3 - 3ab(a + b),$$

we obtain

$$-p = -m^3 + 3mn.$$

24. (D) Since $\frac{1}{n(n+2)} = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$, the desired sum is

$$\frac{1}{2}\left(1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{5} + \ldots + \frac{1}{2N-1} - \frac{1}{2N+1}\right)$$

$$= \frac{1}{2}\left(1 - \frac{1}{2N+1}\right) = \frac{N}{2N+1},$$

with $N = 128$.

OR

Since $\frac{1}{n(n-k)} + \frac{1}{n(n+k)} = \frac{2}{(n-k)(n+k)}$, the desired sum is

$$2\left(\frac{1}{1(5)} + \frac{1}{5(9)} + \ldots + \frac{1}{253(257)}\right) =$$

$$4\left(\frac{1}{1(9)} + \frac{1}{9(17)} + \ldots + \frac{1}{249(257)}\right) = 128\left(\frac{1}{1(257)}\right).$$

OR

The result obtained in the first solution may also be obtained by mathematical induction.

25. (E) If $2^k 3^l 5^m \cdots$ is the factorization of 1005! into powers of distinct primes, then n is the minimum of k and m. Now 201 of the integers between 5 and 1005 are divisible by 5; forty of these 201 integers are divisible by 5^2; eight of these forty integers are divisible by 5^3; and one of these eight integers is divisible by 5^4. Since $k > 502$, $n = m = 201 + 40 + 8 + 1 = 250$.

26. (B) Since A is the sum of the areas of the triangles into which $MNPQ$ is divided by diagonal MP,

$$A = ½ab \sin N + ½cd \sin Q.$$

Similarly,

$$A = ½ad \sin M + ½bc \sin P.$$

Therefore

$$A \leqslant ¼(ab + cd + ad + bc) = \left(\frac{a+c}{2}\right)\left(\frac{b+d}{2}\right)$$

The inequality is an equality if and only if $\sin M = \sin N = \sin P = \sin Q = 1$, i.e., if and only if $MNPQ$ is a rectangle.

27. (C) In a coordinate system having corners of the room as the positive axes, a sphere with radius a tangent to all three coordinate planes has an equation of the form

$$(x - a)^2 + (y - a)^2 + (z - a)^2 = a^2.$$

If the point $(5, 5, 10)$ lies on such a sphere, then

$$2(5 - a)^2 + (10 - a)^2 = a^2$$

$$2a^2 - 40a + 150 = 0$$

$$2(a - 15)(a - 5) = 0.$$

Therefore the radii of the balls are 5 and 15, and the sum of the diameters is 40.

28. (A) Replacing x by x^6 in the equation

$$(x - 1)(x^n + x^{n-1} + \dots + 1) = x^{n+1} - 1$$

yields

$$(x^6 - 1)(x^{6n} + x^{6(n-1)} + \dots + 1) = x^{6(n+1)} - 1.$$

Thus

$$\begin{aligned} g(x^{12}) &= x^{60} + \dots + x^{12} + 1 \\ &= (x^{60} - 1) + \dots + (x^{12} - 1) + 6 \\ &= (x^6 - 1)(x^{54} + \dots) + \dots + (x^6 - 1)(x^6 + 1) + 6 \\ &= g(x)(x - 1)[(x^{54} + \dots) + \dots + (x^6 + 1)] + 6 \end{aligned}$$

$$g(x^{12})/g(x) = (x - 1)[(x^{54} + \dots) + \dots + (x^6 - 1)] + \frac{6}{g(x)},$$

and the remainder is 6.

29. (B) Let $a = x^2$, $b = y^2$ and $c = z^2$. Then

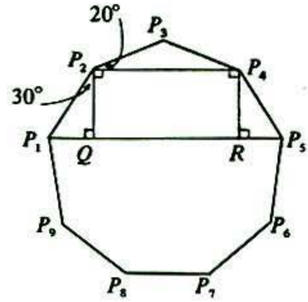
$$0 \leqslant (a - b)^2 + (b - c)^2 + (c - a)^2;$$

$$\frac{ab + bc + ca}{a^2 + b^2 + c^2} \leqslant 1;$$

$$\frac{a^2 + b^2 + c^2 + 2(ab + bc + ca)}{a^2 + b^2 + c^2} \leqslant 3;$$

$$(a + b + c)^2 \leqslant 3(a^2 + b^2 + c^2).$$

Therefore $n \leqslant 3$. Choosing $a = b = c > 0$ shows n is not less than three.



30. (A) In the adjoining figure, $P_1P_2 \cdots P_9$ is a regular nonagon; $P_1P_2 = a$; $P_2P_4 = b$; $P_1P_5 = d$; Q and R lie on P_1P_5; $P_2Q \perp P_1P_5$; $P_4R \perp P_1P_5$. Since $P_2P_3 = P_3P_4$ and the interior angles of a regular nonagon are each

$$\left(\frac{180\,(9-2)}{9}\right)^\circ = 140^\circ;$$

$\measuredangle P_3P_2P_4 = 20^\circ$. Hence $\measuredangle P_1P_2Q = 30^\circ$ and $P_1Q = \frac{a}{2}$. Similarly, $P_5R = \frac{a}{2}$. Thus $d = a + b$.

SOLUTION-ANSWER KEY

TWENTY NINTH ANNUAL H. S.
MATHEMATICS EXAMINATION

1978

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1. **(B)** $1 - \frac{4}{x} + \frac{4}{x^2} = \left(1 - \frac{2}{x}\right)^2 = 0; \frac{2}{x} = 1.$

2. **(C)** If r and A are the radius and area of the circle, respectively, then

$$\begin{aligned} \frac{4}{2\pi r} &= 2r \\ 4 &= 4\pi r^2 \\ 1 &= \pi r^2 = A. \end{aligned}$$

3. **(D)** $\left(x - \frac{1}{x}\right)\left(y + \frac{1}{y}\right) = (x - y)(x + y) = x^2 - y^2.$

4. **(B)** $(a + b + c - d) + (a + b - c + d) + (a - b + c + d) + (-a + b + c + d)$
$= 2(a + b + c + d) = 2222.$

5. **(C)** Let w, x, y and z be the amounts paid by the first, second, third and fourth boy, respectively Then since

$$\begin{aligned} &w + x + y + z = 60 \\ &w = \frac{1}{2}(x + y + z) = \frac{1}{2}(60 - w), \quad w = 20; \\ &x = \frac{1}{3}(w + y + z) = \frac{1}{3}(60 - x), \quad x = 15; \\ &y = \frac{1}{4}(w + x + z) = \frac{1}{4}(60 - y), \quad y = 12; \end{aligned}$$

and $z = 13$.

6. **(E)** If $y \neq 0$, the second equation implies $x = \frac{1}{2}$, and the first equation then implies $y = \pm\frac{1}{2}$. If $y = 0$, the first equation implies $x = 0$ or 1.

7. **(E)** Suppose $A_1, A_2, A_3, A_4, A_5, A_6$ are the vertices of the hexagon listed consecutively, and A_1 and A_3 lie on the parallel sides twelve inches apart. If M is the midpoint of A_1A_3, ΔA_1A_2M is a 30°-60°-90° triangle. Therefore

$$\begin{aligned} \frac{A_1A_2}{6} &= \frac{2}{\sqrt{3}} \\ A_1A_2 &= 4\sqrt{3}. \end{aligned}$$

8. (D) $\frac{a_2 - a_1}{b_2 - b_1} = \frac{(y-x)/3}{(y-x)/4} = \frac{4}{3}.$

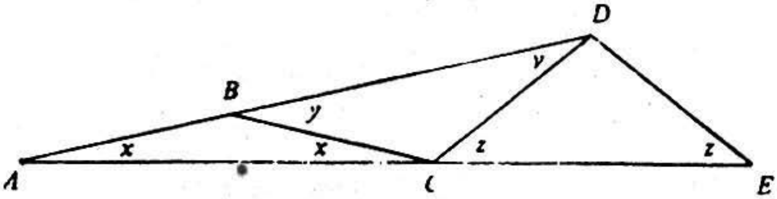
9. (B) $|x - \sqrt{(x-1)^2}| = |x - |x-1|| = |x - (1-x)| = |-1+2x| = 1 - 2x.$

10. (B) For each point A other than P, the point of intersection of circle C with the ray beginning at P and passing through A is the point on circle C closest to A. Therefore the ray beginning at P and passing through B is the set of all points A such that B is the point on circle C which is closest to point A

11. (C) If the line with equation $x + y = r$ is tangent to the circle with equation $x^2 + y^2 = r$, then the distance between the point of tangency $\left(\frac{r}{2}, \frac{r}{2}\right)$ and the origin is $\sqrt{r}$. Therefore, $\left(\frac{r}{2}\right)^2 + \left(\frac{r}{2}\right)^2 = r$ and $r = 2$.

12. (E) Let $x = \measuredangle BAC = \measuredangle BCA$; $y = \measuredangle CBD = \measuredangle CDB$ and $z = \measuredangle DCE = \measuredangle DEC$. Applying the theorem on exterior angles to ΔABC and ΔACD and the theorem on the sum of the interior angles of a triangle to ΔADE yields

$$\begin{aligned} y &= 2x \\ z &= x + y = 3x \\ x + \measuredangle ADE + z &= 180 \\ 140 + 4x &= 180 \\ x &= 10. \end{aligned}$$



13. (B) Since the constant term of a quadratic equation is the product of its roots,

$$b = cd. \quad d = ab$$

Since the coefficient of the x term of a quadratic equation whose x^2 coefficient is one is the negative of the sum of its roots. $-a = c + d$, $c = a + b$, $a + c + d = 0 = a + b + c$, and $b = d$. But the equations $b = cd$ and $d = ab$ imply, since $b = d \neq 0$. that $1 = a = c$ Therefore, $b = d = -2$, and $a + b + c + d = -2$.

14. (C) $b = an - n^2 = (18)_n n - n^2 = (n+8)n - n^2 = (80)_n$.

15. (A) If $\sin x + \cos x = \frac{1}{5}$, then

$$1 - \sin^2 x = \cos^2 x = \left(\frac{1}{5} - \sin x\right)^2$$

$$25 \sin^2 x - 5 \sin x - 12 = 0.$$

Similarly, $\cos x$ satisfies the equation $25t^2 - 5t - 12 = 0$, whose solutions are $\frac{4}{5}$ and $-\frac{3}{5}$. Since $\sin x > 0$, $\sin x = \frac{4}{5}$; and since $\cos x = \frac{1}{5} - \sin x$, $\cos x = -\frac{3}{5}$. Hence, $\tan x = -\frac{4}{3}$.

16. (E) Label the people as $A_1, A_2, \ldots, A_N$ in such a way that A_1 and A_2 were a pair that did not shake hands with each other. Possibly every other pair of people shook hands, so that only A_1 and A_2 did not shake with everyone else. Therefore, at most $N - 2$ people shook hands with everyone else.

17. (D) $\left[f\left(\frac{9+y^2}{y^2}\right)\right]^{\sqrt{\frac{12}{y}}} = \left[\left[f\left(\left(\frac{3}{y}\right)^2 + 1\right)\right]^{\sqrt{\frac{3}{y}}}\right]^2 = k^2.$

18. (C) $\sqrt{n} - \sqrt{n-1} < \frac{1}{100}$;

$$\sqrt{n} + \sqrt{n-1} = \frac{1}{\sqrt{n} - \sqrt{n-1}} > 100;$$

$$\sqrt{2500} + \sqrt{2499} < 100;$$

$$\sqrt{2501} + \sqrt{2500} > 100;$$

$$n = 2501.$$

19. (C) Since $50p + 50(3p) = 1$, $p = .005$; and, since there are seven perfect squares not exceeding 50 and three between 51 and 100, the probability of choosing a perfect square is $7p + 3(3p) = .08$.

20. **(A)** Observe that

$$\frac{a+b-c}{c}+2=\frac{a-b+c}{b}+2=\frac{-a+b+c}{a}+2$$

$$\frac{a+b+c}{c}=\frac{a+b+c}{b}=\frac{a+b+c}{a}.$$

These equalities are satisfied if $a+b+c=0$. If $a+b+c\neq 0$, then dividing each member of the second set of equalities by $a+b+c$ and taking the reciprocals of the results yields $a=b=c$. If $a+b+c=0$, then $x=-1$; if $a=b=c$, then $x=8$.

21. **(A)** If $y=\log_a b$, then $a^y=b$ and $a=b^{1/y}$. Thus

$$\log_b a=\frac{1}{y}=\frac{1}{\log_a b}$$

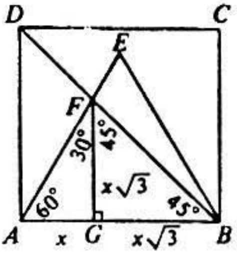
Therefore,

$$\frac{1}{\log_3 x}+\frac{1}{\log_4 x}+\frac{1}{\log_5 x}=\log_x 3+\log_x 4+\log_x 5$$

$$=\log_x 60=\frac{1}{\log_{60} x}.$$

22 **(D)** Since each pair of statements on the card is contradictory, at most one of them is true. The assumption that none of the statements is true implies the fourth statement is true. Hence, there must be exactly one true statement on the card. In fact, it is easy to verify that the third statement is true.

23. **(C)** Let FG be an altitude of ΔAFB, and let x denote the length of AG. From the adjoining figure it may be seen that



$$\sqrt{1+\sqrt{3}}=AB=x(1+\sqrt{3}).$$

$$1+\sqrt{3}=x^2(1+\sqrt{3})^2.$$

$$1=x^2(1+\sqrt{3}).$$

The area of ΔABF is

$$\frac{1}{2}(AB)(FG)=\frac{1}{2}x^2(1+\sqrt{3})\sqrt{3}=\frac{1}{2}\sqrt{3}$$

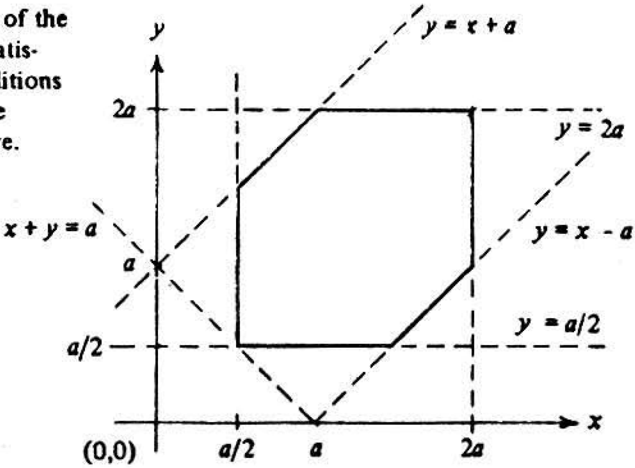
24. (A) Let $a = x(y - z)$ and observe that the identity

$$x(y - z) + y(z - x) + z(x - y) = 0$$

implies

$$a + ar + ar^2 = 0$$
$$1 + r + r^2 = 0.$$

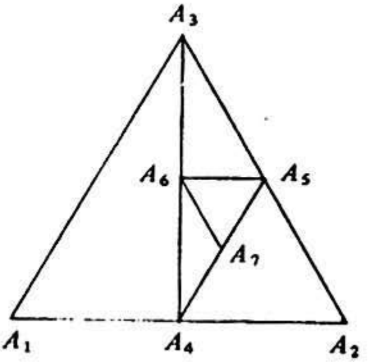
25. (D) The boundary of the set of points satisfying the conditions is shown in the adjoining figure.



26. (B) Let N be the center of circle P. If T is the point of tangency of tangent AB and CH is the altitude to side AB of ΔABC, then $CN + NT > CT > CH$. Since radius CN is minimal, $T = H$ and N is the midpoint of CH; *i.e.*, CH is a diameter of circle P. Since inscribed $\measuredangle QCR$ is 90°, QR is also a diameter. Finally, since $\Delta CBH \sim \Delta ABC$,

$$\frac{CH}{6} = \frac{8}{10}$$
$$QR = CH = 4.8.$$

27. (C) A positive integer has a remainder of 1 when divided by any of the integers from 2 through 11 if and only if the integer is of the form $mt + 1$, where t is a nonnegative integer and $m = 2^3 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 = 27{,}720$ is the least common multiple of 2, 3, . . . , 11. Therefore, consecutive integers with the desired property differ by 27,720.



28. (E) Triangle $A_2A_3A_4$ has vertex angles 60°, 30°, 90°, respectively. Since $\measuredangle A_1A_2A_3 = 60°$, and A_2A_4 and A_2A_5 have the same length, $\Delta A_2A_4A_5$ is equilateral. Therefore, $\Delta A_3A_4A_5$ has vertex angles 30°, 30°, 120°, respectively. Then $\Delta A_4A_5A_6$ has vertex angles 30°, 60°, 90°, respectively. Finally, since $\measuredangle A_4A_5A_6 = 60°$ and A_5A_6 and A_5A_7 have the same length, $\Delta A_5A_6A_7$ is again equilateral. Therefore $\Delta A_nA_{n+1}A_{n+2} \sim \Delta A_{n+4}A_{n+5}A_{n+6}$ with A_n and A_{n+4} as corresponding vertices. Thus $\measuredangle A_{44}A_{45}A_{43} = \measuredangle A_4A_5A_3 = 120°$.

29. (D) Since the length of base AA' of $\Delta AA'B'$ is the same as the length of base AD of ΔABD, and the corresponding altitude of $\Delta AA'B'$ has twice the length of the corresponding altitude of ΔABD,

$$\text{Area } \Delta AA'B' = 2 \text{ Area } \Delta ADB.$$

(Alternately, we could let θ be the measure of $\measuredangle DAB$, and observe

$$\text{Area } \Delta AA'B' = \frac{1}{2}(AD)(2AB)\sin(180° - \theta)$$
$$= 2\frac{1}{2}(AD)(AB)\sin \quad = 2 \text{ Area } \Delta ABD.)$$

Similarly

$$\text{Area } \Delta BB'C' = 2 \text{ Area } \Delta BAC$$
$$\text{Area } \Delta CC'D' = 2 \text{ Area } \Delta CBD$$
$$\text{Area } \Delta DD'A' = 2 \text{ Area } \Delta DCA.$$

Therefore

$$\begin{aligned}\text{Area } A'B'C'D' &= (\text{Area } \Delta AA'B' + \text{Area } \Delta BB'C') \\ &\quad + (\text{Area } \Delta CC'D' + \text{Area } \Delta DD'A') \\ &\quad + \text{Area } ABCD \\ &= 2(\text{Area } \Delta ABD + \text{Area } \Delta BAC) \\ &\quad + 2(\text{Area } \Delta CBD + \text{Area } \Delta DCA) \\ &\quad + \text{Area } ABCD \\ &= 5 \text{ Area } ABCD = 50.\end{aligned}$$

30. **(E)** Let k denote the number of matches in which women defeated men, and let W and M denote the total number of matches won by women and by men, respectively. The total number of matches, the number of matches between a man and a woman, the number of matches between two men and the number of matches between two women are

$$\frac{3n(3n-1)}{2},\ 2n^2,\ \frac{2n(2n-1)}{2},\ \text{and}\ \frac{n(n-1)}{2},$$

respectively. Then, since $k \leqslant 2n^2$, one has

$$\frac{3n(3n-1)}{2k+n(n-1)} = \frac{2(M+W)}{2W} = \frac{M}{W}+1 = \frac{12}{7}$$

$$63n^2 - 21n = 24k + 12n^2 - 12n$$

$$\frac{17n^2-3n}{8} = k < 2n^2$$

$$17n^2 - 3n < 16n^2$$

$$n < 3.$$

Substituting $n = 1, 2, 3$ yields $k = 14/8, 62/8, 144/8$. Since k must be an integer, $n = 3$.

SOLUTION-ANSWER KEY

THIRTIETH ANNUAL H.S.
MATHEMATICS EXAMINATION

1979

30

Sponsored jointly by the

MATHEMATICAL ASSOCIATION OF AMERICA
SOCIETY OF ACTUARIES
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2. Some of the solutions may be intentionally incomplete; crucial steps are shown.
3. The solutions shown here are by no means the only ones possible, nor are they necessarily superior to all alternatives.
4. Even where a "high-powered" method is used, a more elementary procedure is also shown.
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Chairman: Committee on High School Contests
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1. (D) Rectangle *DEFG* has area 18, one quarter of the area of rectangle *ABCD*.

2. (D) $\frac{1}{x}-\frac{1}{y}=\frac{y}{xy}-\frac{x}{xy}=\frac{-xy}{xy}=-1.$

3. (C) Since $\measuredangle DAE = 90^\circ + 60^\circ = 150^\circ$ and $DA = AB = AE$, $\measuredangle AED =$ $\left(\frac{180-150}{2}\right)^\circ = 15^\circ.$

4. (E) $x[x\{x(2-x)-4\}+10]+1 = x[x\{2x-x^2-4\}+10]+1 =$ $x[2x^2-x^3-4x+10]+1 = -x^4+2x^3-4x^2+10x+1.$

5. (D) The numbers whose unit's and hundred's digits are equal are the ones not changed when these digits are interchanged. The largest such even three digit number is 898. The sum of the digits is 25.

6. (A) Add and subtract directly or note that

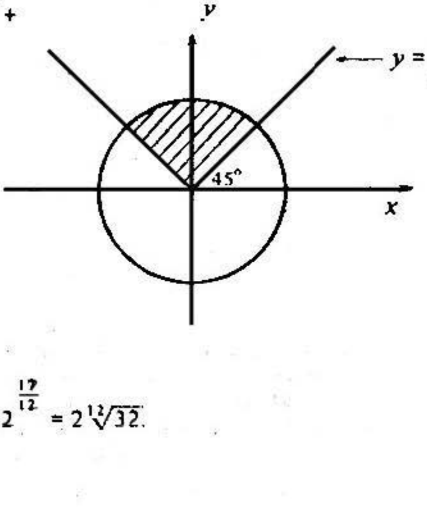
$$\frac{3}{2}+\frac{5}{4}+\cdots+\frac{65}{64}-7=(1+\frac{1}{2})+(1+\frac{1}{4})+\cdots+(1+\frac{1}{64})-7$$

$$=\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{64}-1=-\frac{1}{64}$$

7. (E) Let $x=n^2, n \geqslant 0; (n+1)^2 = n^2+2n+1 = x+2\sqrt{x}+1.$

8. (C) The area of the smallest region bounded by $y=|x|$ and $x^2+y^2=4$ is shown in the adjoining figure. Its area is

$$\frac{1}{4}(\pi\, 2^2)=\pi.$$



9. (E) $\sqrt[3]{4}\ \sqrt[4]{8}=(2^2)^{\frac{1}{3}}(2^3)^{\frac{1}{4}}=2^{\frac{17}{12}}=2\sqrt[12]{32}.$

10. (D) If C is the center of the hexagon, then the area of $Q_1Q_2Q_3Q_4$ is the sum of of the areas of the three equilateral triangles $\Delta Q_1Q_2C, \Delta Q_2Q_3C, \Delta Q_3Q_4C$ each of whose sides have length 2. Therefore, area $Q_1Q_2Q_3Q_4$ =

$$3\left(\frac{2^2\sqrt{3}}{4}\right) = 3\sqrt{3}.$$

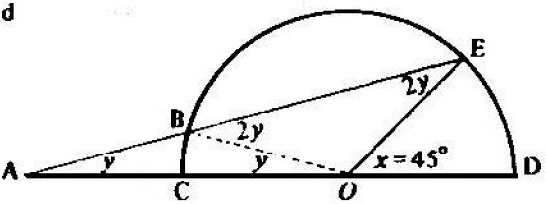
11. (B) Summing the arithmetic progressions yields

$$\frac{115}{116} = \frac{1+3+\cdots+(2n-1)}{2+4+\cdots+2n} = \frac{n^2}{n(n+1)};\ \frac{115}{116} = \frac{n}{n+1};\ n = 115.$$

12. (B) Draw line segment BO, and let x and y denote the measures of $\measuredangle EOD$ and $\measuredangle BAO$, respectively. Observe that $AB = OD = OE = OB$, and apply the theorem on exterior angles of triangles to ΔABO and ΔAEO to obtain $\measuredangle EBO = \measuredangle BEO = 2y$ and

$$x = 3y.$$

Thus $45^\circ = 3y$
$15^\circ = y.$



OR

Since the measure of an angle formed by two secants is half the difference of the intercepted arcs,

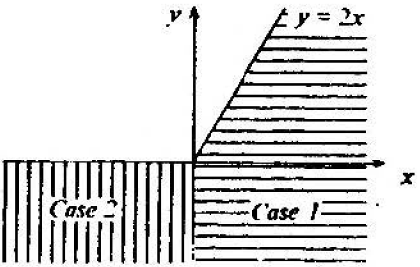
$$y = \frac{1}{2}(x - y),\ y = \frac{x}{3} = \frac{45^\circ}{3} = 15^\circ.$$

13. (A) Consider two cases:

Case 1. If $x \geqslant 0$ then $y - x < \sqrt{x^2}$ if and only if $y - x < x$, or equivalently, $y < 2x$.

Case 2: If $x < 0$ then $y - x < \sqrt{x^2}$ if and only if $y - x < -x$, or equivalently, $y < 0$.

The pairs (x, y) satisfying the conditions described in either Case 1 or Case 2 are exactly the pairs (x, y) such that $y < 0$ or $y < 2x$ (see adjoining diagram for geometric interpretation).



14. (C) If a_n denotes the nth number in the sequence, then

$$a_n = \frac{a_1 a_2 \cdots a_n}{a_1 a_2 \cdots a_{n-1}} = \frac{n^2}{(n-1)^2}$$

for $n \geqslant 2$. Thus, the first five numbers in the sequence are $1, 4, \frac{9}{4}, \frac{16}{9}, \frac{25}{16}$, and the desired sum is $\frac{9}{4} + \frac{25}{16} = \frac{61}{16}$.

15. (E) If each jar contains a total of x liters of solution, then one jar contains $\frac{px}{p+1}$ liters of alcohol and $\frac{x}{p+1}$ liters of water, and the other jar contains $\frac{qx}{q+1}$ liters of alcohol and $\frac{x}{q+1}$ liters of water. The ratio of the volume of alcohol to the volume of water in the mixture is then

$$\frac{\left(\frac{p}{p+1} + \frac{q}{q+1}\right)x}{\left(\frac{1}{p+1} + \frac{1}{q+1}\right)x} = \frac{p(q+1) + q(p+1)}{(q+1) + (p+1)} = \frac{p+q+2pq}{p+q+2}$$

16. (E) Since A_1, A_2, $A_1 + A_2$ are in arithmetic progression,

$$A_2 - A_1 = (A_1 + A_2) - A_2 \quad ; \quad 2A_1 = A_2 .$$

If r is the radius of the smaller circle, then

$$9\pi = A_1 + A_2 = 3A_1 = 3\pi r^2 \quad ; \quad r = \sqrt{3}.$$

17. (C) Since the length of any side of a triangle is less than the sum of the lengths of the other sides,

$x < y - x + z - y = z - x$, which implies $x < \frac{z}{2}$;

$y - x < x + z - y$, which implies $y < x + \frac{z}{2}$;

$z - y < x + y - x$, which implies $y > \frac{z}{2}$.

Therefore, only statements I and II are true.

18. (C) Since $\log_b a = \frac{1}{\log_a b}$, $\log_5 10 = \frac{1}{\log_{10} 5}$

$= \frac{1}{\log_{10} 10 - \log_{10} 2} \approx \frac{1}{.699} \approx \frac{10}{7}$. (The value of $\log_{10} 3$ was not used.)

19. **(A)** Since

$$x^{256} - 256^{32} = x^{(2^8)} - (2^8)^{32} = x^{(2^8)} - 2^{(2^8)}$$
$$= \left(x^{(2^7)} + 2^{(2^7)}\right)\left(x^{(2^6)} + 2^{(2^6)}\right)\cdots(x+2)(x-2),$$

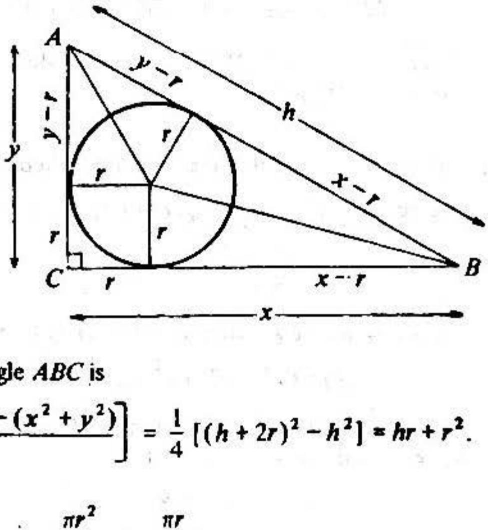
the sum of the squares of the real roots is 8.

20. **(C)** Let $x = \text{Arctan}\, a$ and $y = \text{Arctan}\, b$:

$$(a+1)(b+1) = 2$$
$$(\tan x + 1)(\tan y + 1) = 2$$
$$\tan x + \tan y = 1 - \tan x \tan y$$
$$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = 1$$
$$x + y = \frac{\pi}{4}.$$

21. **(B)** In the adjoining figure, x and y are the lengths of the legs of the triangle, so that

$$h = (y-r) + (x-r)$$
$$= x + y - 2r,$$
$$x + y = h + 2r;$$
$$x^2 + y^2 = h^2.$$



The area of the triangle ABC is

$$\frac{1}{2}xy = \frac{1}{2}\left[\frac{(x+y)^2 - (x^2+y^2)}{2}\right] = \frac{1}{4}[(h+2r)^2 - h^2] = hr + r^2.$$

Thus the desired ratio is $\dfrac{\pi r^2}{hr + r^2} = \dfrac{\pi r}{h + r}$.

22. **(A)** Since $m^3 + 6m^2 + 5m = m(m+1)(m+5)$, this quantity is divisible by 3 for all integers m. To see this, observe that m, $m+5$ or $m+1$ is divisible by 3 if m leaves a remainder of 0, 1 or 2, respectively. However, $27n^3 + 9n^2 + 9n + 1$ always has a remainder of 1 when divided by 3. Therefore, there are no solutions to the given equation.

23. (C) Since the shortest path between a point and a line is the line segment drawn perpendicularly from the point to the line, the desired minimum distance is obtained by symmetry by choosing P and Q to be the midpoints of AB and CD, respectively.

Since PC and PD are altitudes of equilateral triangles ABC and ABD, respectively, $PC = PD = \frac{\sqrt{3}}{2}$. Since Q is the midpoint of side DC, $QC = \frac{1}{2}$. Applying the Pythagorean theorem to ΔCPQ yields

$$PQ = \sqrt{(PC)^2 - (QC)^2} = \sqrt{\frac{3}{4} - \frac{1}{4}} = \frac{\sqrt{2}}{2}.$$

24. (E) Let E be the intersection of lines AB and CD, and let β and θ be the measures of $\measuredangle EBC$ and $\measuredangle ECB$, respectively. Since

$$\cos\beta = -\cos B = \sin C = \sin\theta,$$

$\beta + \theta = 90^\circ$, $\measuredangle BEC$ is a right angle, and

$$BE = BC\sin\theta = 3\ ;\ \ CE = BC\sin\beta = 4.$$

Therefore, $AE = 7$, $DE = 24$ and AD, which is the hypotenuse of right triangle ADE, is 25.

25. (B) Let $a = -\frac{1}{2}$. Applying the remainder theorem yields $r_1 = a^8$, and solving the equality $x^8 = (x-a)q_1(x) + r_1$ for $q_1(x)$ yields

$$q_1(x) = \frac{x^8 - a^8}{x - a} = x^7 + ax^6 + \cdots + a^7$$

[or, by factoring a difference of squares three times,

$$q_1(x) = (x^4 + a^4)(x^2 + a^2)(x + a)].$$

Applying the remainder theorem to determine the remainder when $q_1(x)$ is divided by $x - a$ yields

$$r_2 = q_1(a) = 8a^7 = -\frac{1}{16}.$$

26. (B) Substitute $x = 1$ into the functional equation and solve for the first term on the right side to obtain $f(y + 1) = f(y) + y + 2$. Since $f(1) = 1$, one sees by successively substituting $y = 2, 3, 4, \ldots$ that $f(y) > 0$ for every positive integer. Therefore, for y a positive integer, $f(y + 1) > y + 2 > y + 1$, and $f(n) = n$ has no solutions for integers $n > 1$. Solving the above equation for $f(y)$ yields

$$f(y) = f(y+1) - (y+2).$$

Successively substituting $y = 0, -1, -2, \ldots$ into this equation yields $f(0) = -1, f(-1) = -2, f(-2) = -2, f(-3) = -1, f(-4) = 1$. Now $f(-4) > 0$ and, for $y < -4, -(y+2) > 0$. Thus, for $y < -4, f(y) > 0$. Therefore, $f(n) \neq n$ for $n < -4$; and the solutions $n = 1, -2$ are the only ones.

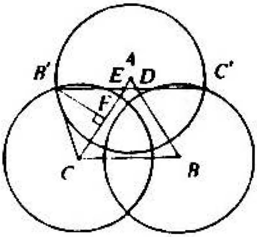
27. **(E)** These six statements are equivalent:

(1) the equation $x^2 + bx + c = 0$ has positive roots;
(2) the equation $x^2 + bx + c = 0$ has real roots, the smaller of which is positive;
(3) $\sqrt{b^2 - 4c}$ is real and $-b - \sqrt{b^2 - 4c} > 0$;
(4) $0 \leqslant b^2 - 4c < b^2$ and $b < 0$;
(5) $0 < c \leqslant \frac{b^2}{4}$ and $b < 0$;
(6) $b = -2$ and $c = 1$; or $b = -3$ and $c = 1$ or 2; or $b = -4$ and $c = 1, 2, 3$ or 4; or $b = -5$ and $c = 1, 2, 3, 4$ or 5.

The roots corresponding to the pairs (b, c) described in (6) will be distinct unless $b^2 = 4c$. Thus, deleting $(b, c) = (-2, 1)$ and $(-4, 4)$ from the list in (6) yields the ten pairs resulting in distinct positive roots.

The desired probability is then $1 - \frac{10}{11^2} = \frac{111}{121}$.

28. **(D)** Let D and E be the points of intersection of $B'C'$ with AB and AC, respectively, and let $B'F$ be the perpendicular drawn from B' to AC. Then $ED \parallel CB$ by symmetry, which implies $\sphericalangle AED$, and hence $\sphericalangle B'EF$, is $60°$.



Applying the Pythagorean theorem to $\Delta B'FC$ yields the equality $B'F = \sqrt{r^2 - 1}$. Now $B'C' = B'E + ED + DC' = 2(B'E) + ED = 2(B'E) + EA$. Therefore,

$$B'C' = 2\left(\frac{2}{\sqrt{3}}\right)B'F + (1 - EF) = 2\left(\frac{2}{\sqrt{3}}\right)\sqrt{r^2 - 1} + \left(1 - \frac{1}{\sqrt{3}}\sqrt{r^2 - 1}\right)$$

$$= 1 + \frac{3}{\sqrt{3}}\sqrt{r^2 - 1} = 1 + \sqrt{3(r^2 - 1)}.$$

29. (E) By observing that $\left[x^3+\frac{1}{x^3}\right]^2 = x^6+2+\frac{1}{x^6}$,

one sees that

$$f(x)=\frac{\left[\left(x+\frac{1}{x}\right)^3\right]^2-\left[x^3+\frac{1}{x^3}\right]^2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)} = \left(x+\frac{1}{x}\right)^3-\left(x^3+\frac{1}{x^3}\right)=3\left(x+\frac{1}{x}\right).$$

Since

$$0 \leqslant \left(\sqrt{x}-\frac{1}{\sqrt{x}}\right)^2 = x+\frac{1}{x}-2. \qquad 2 \leqslant x+\frac{1}{x},$$

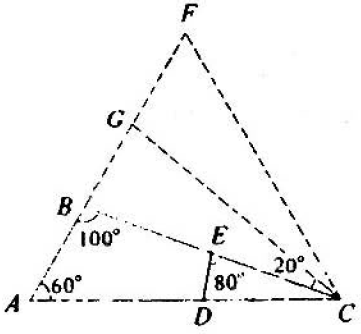
and $f(x)=3\left(x+\frac{1}{x}\right)$ has a minimum value of 6, which is taken on at $x=1$.

30. (B) Let F be the point on the extension of side AB past B for which $AF=1$. Since $AF=AC$ and $\measuredangle FAC=60^\circ$, ΔACF is equilateral. Let G be the point on line segment BF for which $\measuredangle BCG=20^\circ$. Then ΔBCG is similar to ΔDCE and $BC=2(EC)$. Also ΔFGC is congruent to ΔABC. Therefore,

$$\text{Area } \Delta ACF = (\text{Area } \Delta ABC + \text{Area } \Delta GCF) + \text{Area } \Delta BCG$$

$$\frac{\sqrt{3}}{4} = 2 \text{ Area } \Delta ABC + 4 \text{ Area } \Delta CDE$$

$$\frac{\sqrt{3}}{8} = \text{Area } \Delta ABC + 2 \text{ Area } \Delta CDE.$$



SOLUTION-ANSWER PAMPHLET

THIRTY FIRST ANNUAL
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MATHEMATICS EXAMINATION

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1. (C) Since $\frac{100}{7} = 14 + \frac{2}{7}$, the number is 14.

2. (D) The highest powers of x in the factors $(x^2 + 1)^4$ and $(x^3 + 1)^3$ are 8 and 9, respectively. Hence the highest power of x in the product is $8 + 9 = 17$.

3. (E) If $\frac{2x - y}{x + y} = \frac{2}{3}$, then

$$\begin{aligned} 6x - 3y &= 2x + 2y \\ 4x &= 5y \\ \frac{x}{y} &= \frac{5}{4}. \end{aligned}$$

4. (C) The measures of angles $\measuredangle ADC$, $\measuredangle CDE$ and $\measuredangle EDG$ are 90°, 60° and 90°, respectively. Hence the measure of $\measuredangle GDA$ is

$$360^\circ - (90^\circ + 60^\circ + 90^\circ) = 120^\circ.$$

5. (B) Triangle PQC is a 30°-60°-90° right triangle. Since $AQ = CQ$

$$\frac{PQ}{AQ} = \frac{PQ}{CQ} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}.$$

6. (A) Since x is positive the following are equivalent:

$$\begin{aligned} \sqrt{x} &< 2x \\ x &< 4x^2 \\ 1 &< 4x \\ \frac{1}{4} &< x \\ x &> \frac{1}{4}. \end{aligned}$$

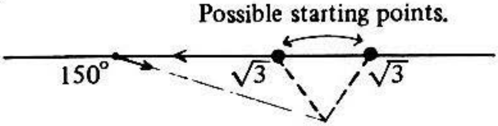
7. (B) By the Pythagorean theorem, diagonal AC has length 5. Since $5^2 + 12^2 = 13^2$, ΔDAC is a right triangle by the converse of the Pythagorean theorem. The area of $ABCD$ is $\left(\frac{1}{2}\right)(3)(4) + \left(\frac{1}{2}\right)(5)(12) = 36$.

8. (A) The given equation is equivalent to each of these equations

$$\frac{a+b}{ab} = \frac{1}{a+b}$$
$$(a+b)^2 = ab$$
$$a^2 + ab + b^2 = 0.$$

Since the last equation is satisfied by the pairs (a, b) such that $a = \frac{-b \pm \sqrt{-3b^2}}{2}$, there are no pairs of real numbers satisfying the original equation.

9. (E) As shown in the adjoining figure, there are two possible starting points; therefore, x is not uniquely determined.



10. (D) Since the teeth are all the same size, equally spaced and are meshed, they all move with the same absolute speed v (v is the distance a point on the circumference moves per unit of time). Let α, β, γ be the angular speeds of A, B, C, respectively. If a, b, c represent the lengths of the circumferences of A, B, C, respectively, then

$$\alpha = \frac{v}{a}, \ \beta = \frac{v}{b}, \ \gamma = \frac{v}{c}.$$

Therefore, $\alpha a = \beta b = \gamma c$ or, equivalently,

$$\frac{\alpha}{\frac{1}{a}} = \frac{\beta}{\frac{1}{b}} = \frac{\gamma}{\frac{1}{c}}.$$

Thus the angular speeds are in the proportion

$$\frac{1}{a} : \frac{1}{b} : \frac{1}{c}.$$

Since a, b, c, are proportional to x, y, z, respectively, the angular speeds are in the proportion

$$\frac{1}{x} : \frac{1}{y} : \frac{1}{z}.$$

Multiplying each term by xyz gives the proportion

$$yz : xz : xy.$$

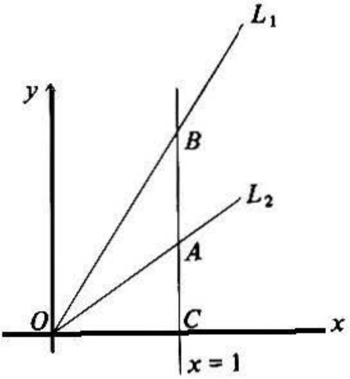
11. (D) The formula for the sum S_n of n terms of an arithmetic progression, whose first term is a and whose common difference is d, is $2S_n = n(2a + (n - 1)d)$. Therefore,

$$\begin{aligned} 200 &= 10(2a + 9d) \\ 20 &= 100(2a + 99d) \\ 2S_{110} &= 110(2a + 109d). \end{aligned}$$

Subtracting the first equation from the second and dividing by 90 yields $2a + 109d = -2$. Hence, $2S_{110} = 110(-2)$, so $S_{110} = -110$.

12. (C) In the adjoining figure, L_1 and L_2 intersect the line $x = 1$ at B and A, respectively; C is the intersection of the line $x = 1$ with the x-axis. Since $OC = 1$, AC is the slope of L_2 and BC is the slope of L_1. Therefore, $AC = n, BC = m$, and $AB = 3n$. Since OA is an angle bisector

$$\frac{OC}{OB} = \frac{AC}{AB}.$$



This yields

$$\frac{1}{OB} = \frac{n}{3n} \text{ and } OB = 3.$$

By the Pythagorean theorem $1 + (4n)^2 = 9$, so $n = \frac{\sqrt{2}}{2}$. Since $m = 4n$, $mn = 4n^2 = 2$.

OR

Let θ_1 and θ_2 be the angles of inclination of lines L_1 and L_2, respectively. Then $m = \tan\theta_1$ and $n = \tan\theta_2$. Since $\theta_1 = 2\theta_2$ and $m = 4n$, $4n = m = \tan\theta_1 = \tan 2\theta_2 = \frac{2\tan\theta_2}{1 - \tan^2\theta_2} = \frac{2n}{1 - n^2}$.

Thus $4n(1 - n^2) = 2n$. Since $n \neq 0$, $2n^2 = 1$, and mn, which equals $4n^2$, is 2.

13. (B) If the bug travels indefinitely, the algebraic sum of the horizontal components of its moves approaches $\frac{4}{5}$, the limit of the geometric series

$$1 - \frac{1}{4} + \frac{1}{16} - \cdots = \frac{1}{1 - \left(-\frac{1}{4}\right)}.$$

Similarly, the algebraic sum of the vertical components of its moves approaches $\frac{2}{5} = \frac{1}{2} - \frac{1}{8} + \frac{1}{32} \cdots$. Therefore, the bug will get arbitrarily close to $\left(\frac{4}{5}, \frac{2}{5}\right)$.

OR

The line segments may be regarded as a complex geometric sequence with $a_1 = 1$ and $r = i/2$. Thus

$$\sum_{i=1}^{\infty} a_i = \frac{a_1}{1-r} = \frac{2}{2-i} = \frac{4+2i}{5}.$$

In coordinate language, the limit is the point $\left(\frac{4}{5}, \frac{2}{5}\right)$.

14. (A) For all $x \neq -\frac{3}{2}$,

$$x = f(f(x)) = \frac{c\left(\frac{cx}{2x+3}\right)}{2\left(\frac{cx}{2x+3}\right) + 3} = \frac{c^2x}{2cx + 6x + 9},$$

which implies $(2c + 6)x + (9 - c^2) = 0$. Therefore, $2c + 6 = 0$ and $9 - c^2 = 0$. Thus, $c = -3$.

15. (B) Let m be the price of the item in cents. Then $(1.04)m = 100n$. Thus $(8)(13)m = (100)^2n$, so $m = (2)(5)^4\frac{n}{13}$. Thus m is an integer if and only if 13 divides n.

16. (B) The edges of the tetrahedron are face diagonals of the cube. Therefore, if s is the length of an edge of the cube, the area of each face of the tetrahedron is

$$\frac{(s\sqrt{2})^2\sqrt{3}}{4} = \frac{s^2\sqrt{3}}{2},$$

and the desired ratio is

$$\frac{6s^2}{4\left(\frac{s^2\sqrt{3}}{2}\right)} = \sqrt{3}.$$

17. (D) Since $i^2 = -1$,

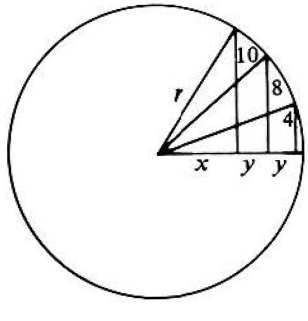
$$(n + i)^4 = n^4 - 6n^2 + 1 + i(4n^3 - 4n).$$

This is real if and only if $4n^3 - 4n = 0$. Since $4n(n^2 - 1) = 0$ if and only if $n = 0, 1, -1$, there are only three values of n for which $(n + i)^4$ is real; $(n + i)^4$ is an integer in all three cases.

18. (D) $\log_b \sin x = a$; $\sin x = b^a$;
$\sin^2 x = b^{2a}$; $\cos x = (1 - b^{2a})^{1/2}$;
$\log_b \cos x = \frac{1}{2} \log_b (1 - b^{2a})$.

19. (D) The adjoining figure is constructed from the given data. We let r be the radius, x the distance from the center of the circle to the closest chord, and y the common distance between the chords. The Pythagorean theorem provides three equations in r, x, and y:

$$r^2 = x^2 + 10^2$$
$$r^2 = (x + y)^2 + 8^2$$
$$r^2 = (x + 2y)^2 + 4^2.$$



Subtracting the first equation from the second yields $0 = 2xy + y^2 - 36$, and subtracting the second equation from the third yields $0 = 2xy + 3y^2 - 48$. Equating the right sides of these last two equations and collecting like terms yields $2y^2 = 12$. Thus, $y = \sqrt{6}$; and by repeated substitutions into the equations above, $r = \frac{5\sqrt{22}}{2}$.

20. (C) The number of ways of choosing 6 coins from 12 is $\binom{12}{6} = 924$. "Having at least 50 cents" will occur if one of the following cases occurs:

(1) Six dimes are drawn.
(2) Five dimes and any other coin are drawn.
(3) Four dimes and two nickels are drawn.

The numbers of ways (1), (2) and (3) can occur are $\binom{6}{6}$, $\binom{6}{5}\binom{6}{1}$ and $\binom{6}{4}\binom{4}{2}$, respectively. The desired probability is, therefore,

$$\frac{\binom{6}{6}+\binom{6}{4}\binom{4}{2}+\binom{6}{5}\binom{6}{1}}{924} = \frac{127}{924}.$$

21. (A) In the adjoining figure the line segment from E to G, the midpoint of DC, is drawn. Then

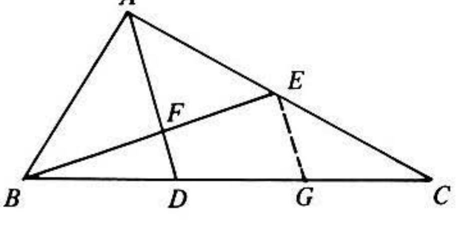
$$\text{area } \Delta EBG = \left(\frac{2}{3}\right)(\text{area } \Delta EBC)$$

$$\text{area } \Delta BDF = \left(\frac{1}{4}\right)(\text{area } \Delta EBG) = \left(\frac{1}{6}\right)(\text{area } \Delta EBC).$$

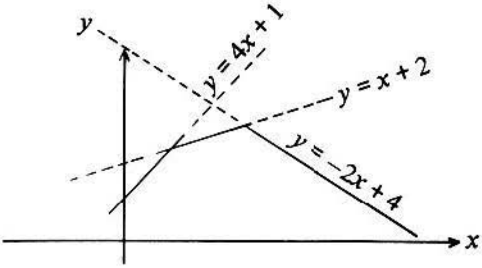
(Note that since EG connects the midpoints of sides AC and DC in ΔACD, EG is parallel to AD). Therefore, area $FDCE = \left(\frac{5}{6}\right)(\text{area } \Delta EBC)$ and

$$\frac{(\text{area } \Delta BDF)}{(\text{area } FDCE)} = \frac{1}{5}.$$

The measure of $\measuredangle CBA$ was not needed.

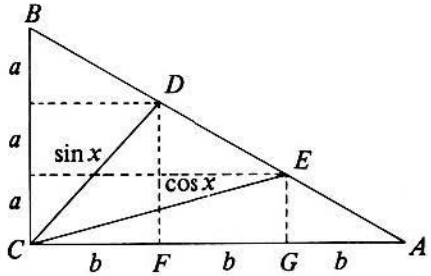


22. (E) In the adjoining figure, the graphs of $y = 4x + 1$, $y = x + 2$ and $y = -2x + 4$ are drawn. The solid line represents the graph of the function f. Its maximum occurs at the intersection of the lines $y = x + 2$ and $y = -2x + 4$. Thus $x = \frac{2}{3}$ and $f\left(\frac{2}{3}\right) = \frac{8}{3}$.



23. (C) Applying the Pythagorean theorem to ΔCDF and ΔCEG in the adjoining figure yields

$$\begin{aligned} 4a^2 + b^2 &= \sin^2 x \\ a^2 + 4b^2 &= \cos^2 x \\ 5(a^2 + b^2) &= 1, \end{aligned}$$



and

$$AB = 3\sqrt{a^2 + b^2} = 3\sqrt{\frac{1}{5}} = 3\,\frac{\sqrt{5}}{5}.$$

24. (D) Since the given polynomial is divisible by $(x - r)^2$ the remainders when it is divided by $(x - r)$ and $(x - r)^2$ must be zero. Using long division the quotient when $8x^3 - 4x^2 - 42x + 45$ is divided by $(x - r)$ is $8x^2 + (8r - 4)x + (8r^2 - 4r - 42)$, the remainder $8r^3 - 4r^2 - 42r + 45$ being zero. Since $(x - r)$ also divides the above quadratic polynomial its remainder must be zero. Thus

$$24r^2 - 8r - 42 = 0.$$

The roots of the last equation are $\frac{3}{2}$ and $-\frac{7}{6}$. Since $r = -\frac{7}{6}$ does not make the first remainder vanish, $r = \frac{3}{2} = 1.5$.

25. (C) Letting $n = 1, \cdots, 5$ in the given equation yields

$$\begin{aligned} 1 &= b\,[\sqrt{1+c}\,] + d \\ 3 &= b\,[\sqrt{2+c}\,] + d \\ 3 &= b\,[\sqrt{3+c}\,] + d \\ 3 &= b\,[\sqrt{4+c}\,] + d \\ 5 &= b\,[\sqrt{5+c}\,] + d \end{aligned}$$

Thus, subtracting the first equation from the second

$$2 = b([\sqrt{2+c}] - [\sqrt{1+c}]).$$

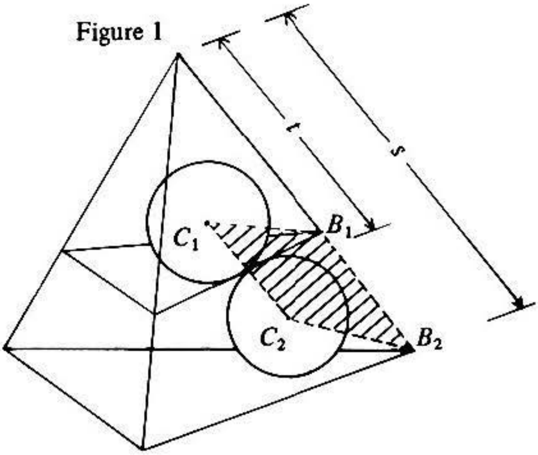
Since the quantity in parentheses is at most one, $b = 2$. Then by subtractions,

$$1 + [\sqrt{1+c}] = [\sqrt{2+c}] = [\sqrt{3+c}] = [\sqrt{4+c}] = [\sqrt{5+c}] - 1.$$

Therefore, for some integers m and k, $2 + c = k^2$ and $5 + c = m^2$. Since the only perfect squares that differ by 3 are 1 and 4, $k = 1$ and $m = 2$. Therefore $c = -1$ and $d = 1$. These are necessary (and as it turns out also, sufficient) conditions.

26. (E) A smaller regular tetrahedron circumscribing just one of the balls may be formed by introducing a plane parallel to the horizontal face as shown in Figure 1. Let the edge of this tetrahedron be t. Next construct the quadrilateral $C_1B_1B_2C_2$, where C_1 and C_2 are centers of two of the balls. By the symmetry of the ball tetrahedron configuration the sides $C_1B_1 = C_2B_2$ are parallel and equal. It follows that

Figure 1



$$B_1B_2 = C_1C_2 = 1 + 1 = 2.$$

Thus, $s = t + 2$, and our problem reduces to that of determining t.

In ΔABC_1, in Figure 2, AC_1 has a length of one radius less than the length b of altitudes DB and AE of the smaller tetrahedron; and AB has length $\frac{2}{3}\left(\frac{\sqrt{3}}{2}t\right)$, since B is the center of a face. Applying the Pythagorean theorem to ΔABC_1, yields

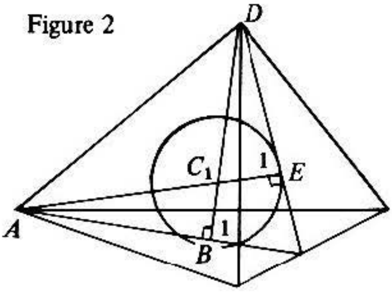
$$(b-1)^2 = 1^2 + \left[\frac{2}{3}\left(\frac{\sqrt{3}}{2}t\right)\right]^2,$$

and applying the Pythagorean theorem to ΔADB yields

$$t^2 = b^2 + \left[\frac{2}{3}\left(\frac{\sqrt{3}}{2}t\right)\right]^2.$$

Solving the second equation for b, substituting this value for b in the first equation, and solving the resulting equation for the positive value of t yields

Figure 2



$t = 2\sqrt{6}$. Thus $s = 2 + t = 2 + 2\sqrt{6}$.

27. (E) Let $a = \sqrt[3]{5 + 2\sqrt{13}}$, $b = \sqrt[3]{5 - 2\sqrt{13}}$ and $x = a + b$. Then

$$\begin{aligned} x^3 &= a^3 + 3a^2b + 3ab^2 + b^3 \\ x^3 &= a^3 + b^3 + 3ab(a + b) \\ x^3 &= 10 + 3\sqrt[3]{-27}\,x. \end{aligned}$$

The last equation is equivalent to $x^3 + 9x - 10 = 0$, or $(x - 1)(x^2 + x + 10) = 0$, whose only real solution is $x = 1$.

28. (C) Let $f(x) = x^2 + x + 1$. Let $g_n(x) = x^{2n} + 1 + (x + 1)^{2n}$, $f(x) = (x - r)(x - r')$, where

$$r = \frac{-1 + \sqrt{-3}}{2} \text{ and } r' = \frac{-1 - \sqrt{-3}}{2}$$

are the roots of $f(x) = 0$. Thus $f(x)$ divides $g_n(x)$ if and only if both $x - r$ and $x - r'$ divide $g_n(x)$. That is, if $g_n(r) = g_n(r') = 0$. Since r and r' are complex conjugates, it suffices to determine those n for which $g_n(r) = 0$.

Note that $g_n(x) = [x^2]^n + [(x + 1)^2]^n + 1$. Also note that r and $r + 1 = \frac{1 + \sqrt{-3}}{2}$ are both 6th roots of 1. (It's easy to see this with Argand diagrams.) Thus r^2 and $(r + 1)^2$ are both cube roots of 1. Thus the value of $g_n(r)$ depends only on the remainder when n is divided by 3. A direct computation (again easiest with Argand diagrams) shows that

$$g_0(r) = 3, g_1(r) = 0, g_2(r) = 0.$$

Thus $f(x)$ does not divide $g_n(x)$ if and only if n is divisible by 3.

29. (A) Adding the given equations and rearranging the terms of the resulting equation yields

$$(x^2 - 2xy + y^2) + (y^2 + 6yz + 9z^2) = 175$$

or

$$(x - y)^2 + (y + 3z)^2 = 175.$$

Since a perfect square can only have 0 or 1 as a remainder when divided by 4, the sum of two perfect squares can only have 0, 1 or 2 as a remainder when divided by 4, and hence the left and right sides of the above equation cannot be equal. There are no solutions.

30. (B) If N is squarish, then there exist single digit positive integers A, B, C, a, b, c such that

$$N = 10^4A^2 + 10^2B^2 + C^2 = (10^2a + 10b + c)^2.$$

The quantity $d = 10b + c$ is less than or equal to 99. Thus $A = a$, for otherwise

$$N = (10^2 a + d)^2 \leqslant 10^4 a^2 + 2(100)99a + (99)^2 < 10^4 a^2 + 2(10)^4 a + 10^4$$
$$= 10^4(a + 1)^2 \leqslant N.$$

Thus

$$8181 \geqslant 10^2B^2 + C^2 = (10^2a + 10b + c)^2 - 10^4a^2$$
$$= 10^3(2ab) + 10^2(b^2 + 2ac) + 10(2bc) + c^2.$$

In particular, $2ab \leqslant 8$. Since no digit of N is zero, $a \geqslant 4$. Thus either $b = 0$ or $a = 4$ and $b = 1$. In the latter case,

$$N = (410 + c)^2 = 168100 + 820c + c^2.$$

Since N has no digit zero, $c \geqslant 1$. But then either the middle two digits or the leftmost two digits are not a square. Thus the latter case is impossible and $b = 0$. Therefore,

$$N = (10^2a + c)^2 = 10^4a^2 + 10^2(2ac) + c^2.$$

Thus $a \geqslant 4, c \geqslant 4$ and $2ac$ is an even two-digit perfect square. It is now easy to check that either $a = 8, c = 4, N = 646416$, or else $a = 4, c = 8$, $N = 166464$.