MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

13th Annual

AMC 10 A

American Mathematics Contest 10 A

Tuesday, February 7, 2012

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom

1. **Answer (D):** Because 20 seconds is $\frac{1}{3}$ of a minute, Cagney can frost $5 \div \frac{1}{3} = 15$ cupcakes in five minutes. Because 30 seconds is $\frac{1}{2}$ of a minute, Lacey can frost $5 \div \frac{1}{2} = 10$ cupcakes in five minutes. Altogether they can frost $15 + 10 = 25$ cupcakes in five minutes.

2. **Answer (E):** The length of each rectangle is equal to the side length of the square. The width of each rectangle is half the side length of the square, so the rectangle's dimensions are 4 by 8.

3. **Answer (E):** The distance from -2 to -6 is $|(-6) - (-2)| = 4$ units. The distance from -6 to 5 is $|5 - (-6)| = 11$ units. Altogether the bug crawls $4 + 11 = 15$ units.

4. **Answer (C):** Ray AB is common to both angles, so the degree measure of $\angle CBD$ is either $24 + 20 = 44$ or $24 - 20 = 4$. The smallest possible degree measure is 4.

5. **Answer (B):** The number of female adult cats was 50, and 25 of those were accompanied by an average of 4 kittens each. Thus the total number of kittens was $25 \cdot 4 = 100$, and the total number of cats and kittens was $100 + 100 = 200$.

6. **Answer (D):** Let $x > 0$ be the first number, and let $y > 0$ be the second number. The first statement implies $xy = 9$. The second statement implies $\frac{1}{x} = \frac{4}{y}$, so $y = 4x$. Substitution yields $x \cdot (4x) = 9$, so $x = \sqrt{\frac{9}{4}} = \frac{3}{2}$. Therefore $x + y = \frac{3}{2} + 4 \cdot \frac{3}{2} = \frac{15}{2}$.

7. **Answer (C):** The ratio of blue marbles to red marbles is $3 : 2$. If the number of red marbles is doubled, the ratio will be $3 : 4$, and the fraction of marbles that are red will be $\frac{4}{3+4} = \frac{4}{7}$.

8. **Answer (D):** Let the three whole numbers be $a < b < c$. The set of sums of pairs of these numbers is $(a + b, a + c, b + c) = (12, 17, 19)$. Thus $2(a + b + c) = (a + b) + (a + c) + (b + c) = 12 + 17 + 19 = 48$, and $a + b + c = 24$. If follows that $(a, b, c) = (24 - 19, 24 - 17, 24 - 12) = (5, 7, 12)$. Therefore the middle number is 7.

9. **Answer (D):** The sum could be 7 only if the even die showed 2 and the odd showed 5, the even showed 4 and the odd showed 3, or the even showed 6 and the odd showed 1. Each of these events can occur in $2 \cdot 2 = 4$ ways. Hence there are 12 ways for a 7 to occur. There are $6 \cdot 6 = 36$ possible outcomes, so the probability that a 7 occurs is $\frac{12}{36} = \frac{1}{3}$.

10. **Answer (C):** Let a be the initial term and d the common difference for the arithmetic sequence. Then the sum of the degree measures of the central angles is

$$a + (a + d) + \cdots + (a + 11d) = 12a + 66d = 360,$$

so $2a + 11d = 60$. Letting $d = 4$ yields the smallest possible positive integer value for a, namely $a = 8$.

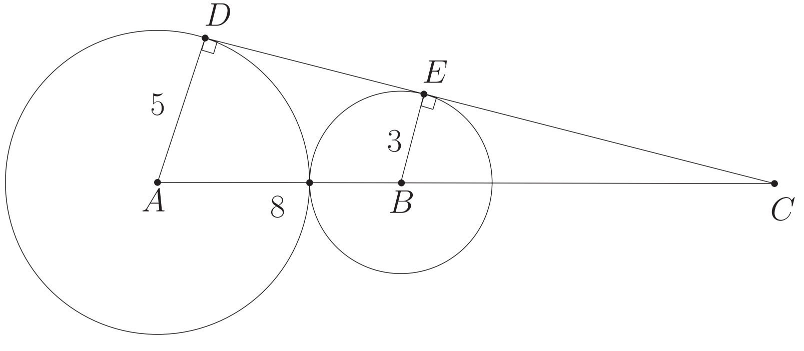
11. **Answer (D):** Let D and E be the points of tangency to circles A and B, respectively, of the common tangent line that intersects ray AB at point C. Then $AD = 5$, $BE = 3$, and $AB = 5+3 = 8$. Because right triangles ADC and BEC are similar, it follows that

$$\frac{BC}{AC} = \frac{BE}{AD},$$

so

$$\frac{BC}{BC + 8} = \frac{3}{5}.$$

Solving gives $BC = 12$.



12. **Answer (A):** There were $200 \cdot 365 = 73000$ non-leap days in the 200-year time period from February 7, 1812 to February 7, 2012. One fourth of those years contained a leap day, except for 1900, so there were $\frac{1}{4} \cdot 200 - 1 = 49$ leap days during that time. Therefore Dickens was born 73049 days before a Tuesday. Because the same day of the week occurs every 7 days and $73049 = 7 \cdot 10435 + 4$, the day of Dickens' birth (February 7, 1812) was 4 days before a Tuesday, which was a Friday.

13. **Answer (C):** If the numbers are arranged in the order a, b, c, d, e, then the iterative average is

$$\frac{\frac{\frac{\frac{a+b}{2}+c}{2}+d}{2}+e}{2} = \frac{a+b+2c+4d+8e}{16}.$$

The largest value is obtained by letting $(a, b, c, d, e) = (1, 2, 3, 4, 5)$ or $(2, 1, 3, 4, 5)$, and the smallest value is obtained by letting $(a, b, c, d, e) = (5, 4, 3, 2, 1)$ or $(4, 5, 3, 2, 1)$. In the former case the iterative average is $65/16$, and in the latter case the iterative average is $31/16$, so the desired difference is

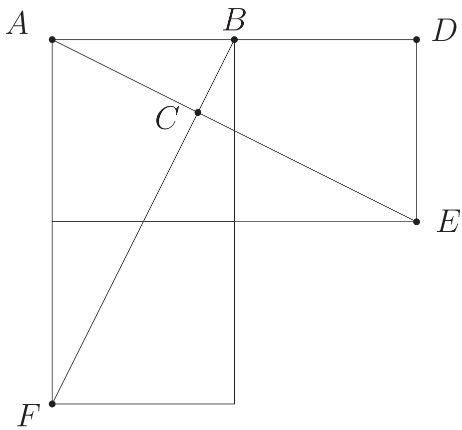
$$\frac{65}{16} - \frac{31}{16} = \frac{34}{16} = \frac{17}{8}.$$

14. **Answer (B):** Separate the modified checkerboard into two parts: the first 30 columns and the last column. The larger section consists of rows, each containing 15 black squares. The last column contains 16 black squares. Thus the total number of black squares is $31 \cdot 15 + 16 = 481$.

OR

There are 16 rows that have 16 black squares and 15 rows that have 15 black squares, so the total number of black squares is $16^2 + 15^2 = 481$.

15. **Answer (B):** Place the figure on the coordinate plane with A at the origin, B on the positive x-axis, and label the other points as shown. Then the equation of line AE is $y = -\frac{1}{2}x$, and the equation of line BF is $y = 2x - 2$. Solving the simultaneous equations shows that $C = (\frac{4}{5}, -\frac{2}{5})$. Therefore $\triangle ABC$ has base $AB = 1$ and altitude $\frac{2}{5}$, so its area is $\frac{1}{5}$.



OR

Congruent right triangles AED and FBA have the property that their corresponding legs are perpendicular; hence their hypotenuses are perpendicular. Therefore $\angle ACB$ is a right angle and $\triangle ACB$ is similar to $\triangle FAB$. Because $AB = 1$ and $BF = \sqrt{5}$, the ratio of the area of $\triangle ACB$ to that of $\triangle FAB$ is 1 to 5. The area of $\triangle FAB$ is 1, so the area of $\triangle ACB$ is $\frac{1}{5}$.

16. **Answer (C):** Label the runners A, B, and C in increasing order of speed. After the start, runner B and runner C will be together again once runner C has run an extra 500 meters. Hence it takes $\frac{500}{5.0-4.8} = 2500$ seconds for runners B and C to be together again. Similarly, it takes $\frac{500}{4.8-4.4} = 1250$ seconds for runner A and runner B to be together again. Runners A and B will also be together at $2 \cdot 1250 = 2500$ seconds, at which time all three runners will be together.

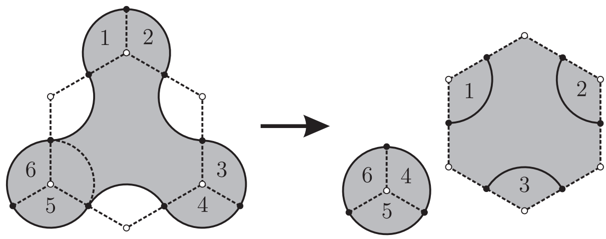
17. **Answer (C):** Note that

$$\frac{a^3 - b^3}{(a-b)^3} = \frac{a^2 + ab + b^2}{a^2 - 2ab + b^2}.$$

Hence the given equation may be written as $3a^2+3ab+3b^2 = 73a^2-146ab+73b^2$. Combining like terms and factoring gives $(10a-7b)(7a-10b) = 0$. Because $a > b$, and a and b are relatively prime, $a = 10$ and $b = 7$. Thus $a - b = 3$.

18. **Answer (E):** The labeled circular sectors in the figure each have the same area because they are all $\frac{2\pi}{3}$-sectors of a circle of radius 1. Therefore the area enclosed by the curve is equal to the area of a circle of radius 1 plus the area of a regular hexagon of side 2. Because the regular hexagon can be partitioned into 6 congruent equilateral triangles of side 2, it follows that the required area is

$$\pi + 6\left(\frac{\sqrt{3}}{4}\cdot 2^2\right) = \pi + 6\sqrt{3}\,.$$



19. **Answer (D):** Let the length of the lunch break be m minutes. Then the three painters each worked $480 - m$ minutes on Monday, the two helpers worked $372 - m$ minutes on Tuesday, and Paula worked $672 - m$ minutes on Wednesday. If Paula paints $p\%$ of the house per minute and her helpers paint a total of $h\%$ of the house per minute, then

$$\begin{aligned}(p+h)(480-m) &= 50,\\ h(372-m) &= 24, \text{and}\\ p(672-m) &= 26.\end{aligned}$$

Adding the last two equations gives $672p + 372h - mp - mh = 50$, and subtracting this equation from the first one gives $108h - 192p = 0$, so $h = \frac{16p}{9}$. Substitution into the first equation then leads to the system

$$\begin{aligned}\frac{25p}{9}(480-m) &= 50,\\ p(672-m) &= 26.\end{aligned}$$

The solution of this system is $p = \frac{1}{24}$ and $m = 48$. Note that $h = \frac{2}{27}$.

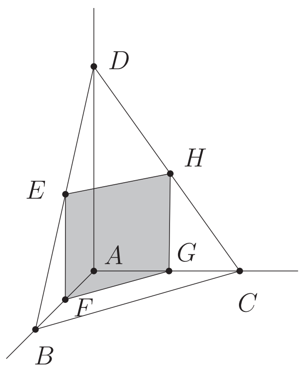
20. **Answer (A):** There are $2^4 = 16$ possible initial colorings for the four corner squares. If their initial coloring is $BBBB$, one of the four cyclic permutations of $BBBW$, or one of the two cyclic permutations of $BWBW$, then all four corner squares are black at the end. If the initial coloring is $WWWW$, one of the four cyclic permutations of $BWWW$, or one of the four cyclic permutations of

$BBWW$, then at least one corner square is white at the end. Hence all four corner squares are black at the end with probability $\frac{7}{16}$. Similarly, all four edge squares are black at the end with probability $\frac{7}{16}$. The center square is black at the end if and only if it was initially black, so it is black at the end with probability $\frac{1}{2}$. The probability that all nine squares are black at the end is $\frac{1}{2} \cdot \left(\frac{7}{16}\right)^2 = \frac{49}{512}$.

21. **Answer (C):** The midpoint formula gives $E = (\frac{1}{2}, 0, \frac{3}{2})$, $F = (\frac{1}{2}, 0, 0)$, $G = (0, 1, 0)$, and $H = (0, 1, \frac{3}{2})$. Note that $EF = GH = \frac{3}{2}$, $\overline{EF} \perp \overline{EH}$, $\overline{GF} \perp \overline{GH}$, and

$$EH = FG = \sqrt{\left(\frac{1}{2}\right)^2 + 1^2} = \frac{\sqrt{5}}{2}.$$

Therefore $EFGH$ is a rectangle with area $\frac{3}{2} \cdot \frac{\sqrt{5}}{2} = \frac{3\sqrt{5}}{4}$.



22. **Answer (A):** The sum of the first k positive integers is $\frac{k(k+1)}{2}$. Therefore the sum of the first k even integers is

$$2 + 4 + 6 + \cdots + 2k = 2(1 + 2 + 3 + \cdots + k) = 2 \cdot \frac{k(k+1)}{2} = k(k+1).$$

The sum of the first k odd integers is

$$(1 + 2 + 3 + \cdots + 2k) - (2 + 4 + 6 + \cdots + 2k) = \frac{2k(2k+1)}{2} - k(k+1) = k^2.$$

The given conditions imply that $m^2 - 212 = n(n+1)$, which may be rewritten as $n^2 + n + (212 - m^2) = 0$. The discriminant for n in this quadratic equation is $1 - 4(212 - m^2) = 4m^2 - 847$, and this must be the square of an odd integer. Let $p^2 = 4m^2 - 847$, and rearrange this equation so that $(2m+p)(2m-p) = 847$.

The only factor pairs for 847 are $847 \cdot 1$, $121 \cdot 7$, and $77 \cdot 11$. Equating these pairs to $2m+p$ and $2m-p$ yields $(m,p) = (212, 423)$, $(32, 57)$, and $(22, 33)$. Note that the corresponding values of n are found using $n = \frac{-1+p}{2}$, which yields 211, 28, and 16, respectively. The sum of the possible values of n is 255.

23. **Answer (B):** This situation can be modeled with a graph having these six people as vertices, in which two vertices are joined by an edge if and only if the corresponding people are internet friends. Let n be the number of friends each person has; then $1 \le n \le 4$. If $n = 1$, then the graph consists of three edges sharing no endpoints. There are 5 choices for Adam's friend and then 3 ways to partition the remaining 4 people into 2 pairs of friends, for a total of $5 \cdot 3 = 15$ possibilities. The case $n = 4$ is complementary, with non-friendship playing the role of friendship, so there are 15 possibilities in that case as well.

For $n = 2$, the graph must consist of cycles, and the only two choices are two triangles (3-cycles) and a hexagon (6-cycle). In the former case, there are $\binom{5}{2} = 10$ ways to choose two friends for Adam and that choice uniquely determines the triangles. In the latter case, every permutation of the six vertices determines a hexagon, but each hexagon is counted $6 \cdot 2 = 12$ times, because the hexagon can start at any vertex and be traversed in either direction. This gives $\frac{6!}{12} = 60$ hexagons, for a total of $10 + 60 = 70$ possibilities. The complementary case $n = 3$ provides 70 more. The total is therefore $15 + 15 + 70 + 70 = 170$.

24. **Answer (E):** Adding the two equations gives

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac = 14,$$

so

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 14.$$

Note that there is a unique way to express 14 as the sum of perfect squares (up to permutations), namely, $14 = 3^2 + 2^2 + 1^2$. Because $a-b$, $b-c$, and $c-a$ are integers with their sum equal to 0 and $a \ge b \ge c$, it follows that $a - c = 3$ and either $a - b = 2$ and $b - c = 1$, or $a - b = 1$ and $b - c = 2$. Therefore either $(a,b,c) = (c+3, c+1, c)$ or $(a,b,c) = (c+3, c+2, c)$. Substituting the relations in the first case into the first given equation yields $2011 = a^2 - c^2 + ab - b^2 = (a-c)(a+c) + (a-b)b = 3(2c+3) + 2(c+1)$. Solving gives $(a,b,c) = (253, 251, 250)$. The second case does not yield an integer solution. Therefore $a = 253$.

25. **Answer (D):** It may be assumed that $x \le y \le z$. Because there are six possible ways of permuting the triple (x, y, z), it follows that the set of all triples (x, y, z) with $0 \le x \le y \le z \le n$ is a region whose volume is $\frac{1}{6}$ of the volume of the cube $[0, n]^3$, that is $\frac{1}{6}n^3$. Let S be the set of triples meeting the required condition. For every $(x, y, z) \in S$ consider the translation $(x, y, z) \mapsto (x', y', z') = (x, y-1, z-2)$. Note that $y' = y - 1 > x = x'$ and $z' = z - 2 > y - 1 = y'$. Thus the image of S under this translation is equal to $\{(x', y', z') : 0 \le x' < y' < z' \le n-2\}$. Again by symmetry of the possible permutations of the triples (x', y', z'), the volume of this set is $\frac{1}{6}(n-2)^3$. Because $\frac{7^3}{9^3} = \frac{343}{729} < \frac{1}{2}$ and $\frac{8^3}{10^3} = \frac{512}{1000} > \frac{1}{2}$, the smallest possible value of n is 10.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steve Dunbar, Michelle Ghrist, Peter Gilchrist, Jerrold Grossman, Dan Kennedy, Joe Kennedy, David Torney, David Wells, LeRoy Wenstrom, and Ron Yannone.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

Academy of Applied Sciences
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Statistical Association
Art of Problem Solving
Awesome Math
Casualty Actuarial Society
D.E. Shaw & Co.
IDEA Math
Jane Street
Mu Alpha Theta
National Council of Teachers of Mathematics
Pi Mu Epsilon
Society of Actuaries
W. H. Freeman and Company

MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

13th Annual

AMC 10 B

American Mathematics Contest 10 B

Wednesday, February 22, 2012

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom

1. **Answer (C):** There are $18-2=16$ more students than rabbits per classroom. Altogether there are $4\cdot 16=64$ more students than rabbits.

2. **Answer (E):** The width of the rectangle is the diameter of the circle, so the width is $2\cdot 5=10$. The length of the rectangle is $2\cdot 10=20$. Therefore the area of the rectangle is $10\cdot 20=200$.

3. **Answer (B):** The given point is 12 units above the horizontal line $y=2000$. The reflected point will be 12 units below the line, and 24 units below the given point. The coordinates of the reflected point are $(1000, 1988)$.

4. **Answer (A):** The 7 marbles left over will fill one more bag of 6 marbles leaving 1 marble left over.

5. **Answer (D):** Tax is 10% and tip is 15%, so her total cost is $100\% + 10\% + 15\% = 125\%$ of her meal. Thus her meal costs $\frac{\$27.50}{1.25} = \$22$.

6. **Answer (A):** Consider x and y as points on the real number line, with x necessarily to the right of y. Then $x-y$ is the distance between x and y. Xiaoli's rounding moved x to the right and moved y to the left. Therefore the distance between them increased, and her estimate is larger than $x-y$.

   To see that the other answer choices are not correct, let $x=2.9$ and $y=2.1$, and round each by 0.1. Then $x-y=0.8$ and Xiaoli's estimated difference is $(2.9+0.1)-(2.1-0.1)=1.0$.

7. **Answer (D):** Let h be the number of holes dug by the chipmunk. Then the chipmunk hid $3h$ acorns, while the squirrel hid $4(h-4)$ acorns. Since they hid the same number of acorns, $3h=4(h-4)$. Solving gives $h=16$. Thus the chipmunk hid $3\cdot 16=48$ acorns.

8. **Answer (B):** If $x-2>0$, then the given inequality is equivalent to $1<x-2<5$, or $3<x<7$. The integer solutions in this case are 4, 5, and 6. If $x-2<0$, then the given inequality is equivalent to $-5<x-2<-1$, or $-3<x<1$. The integer solutions in this case are -2, -1, and 0. The sum of all integer solutions is 12.

OR

The given inequality is equivalent to $1 < |x-2| < 5$. The solution set consists of all numbers whose distance from 2 on the number line is strictly between 1 and 5. Because only integer solutions are sought, this set is $\{-2,-1,0,4,5,6\}$. The required sum is 12.

9. **Answer (A):** The sum of two integers is even if they are both even or both odd. The sum of two integers is odd if one is even and one is odd. Only the middle two integers have an odd sum, namely $41 - 26 = 15$. Hence at least one integer must be even. A list satisfying the given conditions in which there is only one even integer is 1, 25, 1, 14, 1, 15.

10. **Answer (D):** Multiplying the given equation by $6N$ gives $MN = 36$. The divisors of 36 are 1, 2, 3, 4, 6, 9, 12, 18, and 36. Each of these divisors can be paired with a divisor to make a product of 36. Hence there are 9 ordered pairs (M,N).

11. **Answer (A):** There are 3 choices for Saturday (anything except cake) and for the same reason 3 choices for Thursday. Similarly there are 3 choices for Wednesday, Tuesday, Monday, and Sunday (anything except what was to be served the following day). Therefore there are $3^6 = 729$ possible dessert menus.

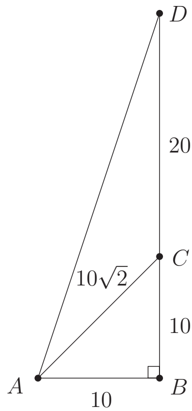
OR

If any dessert could be served on Friday, there would be 4 choices for Sunday and 3 for each of the other six days. There would be a total of $4 \cdot 3^6$ dessert menus for the week, and each dessert would be served on Friday with equal frequency. Because cake is the dessert for Friday, this total is too large by a factor of 4. The actual total is $3^6 = 729$.

12. **Answer (B):** Note that $\angle ABC = 90°$, so $\triangle ABC$ is a $45-45-90°$ triangle. Because hypotenuse $AC = 10\sqrt{2}$, the legs of $\triangle ABC$ have length 10. Therefore $AB = 10$ and $BD = BC + CD = 10 + 20 = 30$. By the Pythagorean Theorem,

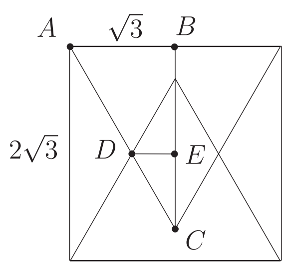
$$AD = \sqrt{10^2 + 30^2} = \sqrt{1000}.$$

Because $31^2 = 961$ and $32^2 = 1024$, it follows that $31 < AD < 32$.



13. **Answer (B):** Let x be Clea's rate of walking and r be the rate of the moving escalator. Because the distance is constant, $24(x+r) = 60x$. Solving for r yields $r = \frac{3}{2}x$. Let t be the time required for Clea to make the escalator trip while just standing on it. Then $rt = 60x$, so $\frac{3}{2}xt = 60x$. Therefore $t = 40$ seconds.

14. **Answer (D):** Construct the altitude for one of the equilateral triangles to its base on the square. Label the vertices of one of the resulting $30-60-90^\circ$ triangles A, B, and C, as shown. Then $AB = \sqrt{3}$ and $BC = 3$. Label one of the intersection points of the two equilateral triangles D and the center of the square E. Then $\triangle CDE$ is a $30-60-90^\circ$ triangle, $CE = 3 - \sqrt{3}$, and $DE = \frac{3-\sqrt{3}}{\sqrt{3}} = \sqrt{3} - 1$. The area of $\triangle CDE$ is $\frac{1}{2} \cdot (3 - \sqrt{3}) \cdot (\sqrt{3} - 1) = 2\sqrt{3} - 3$. Hence the area of the rhombus is $4 \cdot (2\sqrt{3} - 3) = 8\sqrt{3} - 12$.



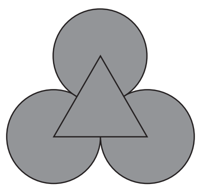
15. **Answer (D):** A total of 15 games are played, so all 6 teams could not be tied for the most wins as this would require $\frac{15}{6} = 2.5$ wins per team. However, it is possible for 5 teams to be tied, each with 3 wins and 2 losses. One such

outcome can be constructed by labeling 5 of the teams A, B, C, D, and E, and placing these labels at distinct points on a circle. If each of these teams beat the 2 labeled teams clockwise from its respective labeled point, as well as the remaining unlabeled team, all 5 would tie with 3 wins and 2 losses.

16. **Answer (A):** Connect the centers of the three circles to form an equilateral triangle with side length 4. Then the requested area is equal to the area of this triangle and a $300°$ sector of each circle. The equilateral triangle has base 4 and altitude $2\sqrt{3}$, so its area is

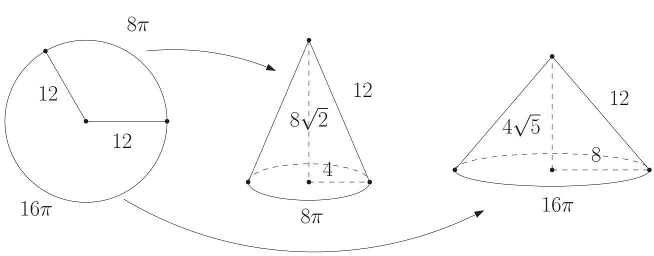
$$\frac{1}{2} \cdot 4 \cdot 2\sqrt{3} = 4\sqrt{3}.$$

The area of each sector is $\frac{300}{360} \cdot \pi \cdot 2^2 = \frac{10\pi}{3}$. Hence the total area is $3 \cdot \frac{10\pi}{3} + 4\sqrt{3} = 10\pi + 4\sqrt{3}$.



17. **Answer (C):** Each sector forms a cone with slant height 12. The circumference of the base of the smaller cone is $\frac{120}{360} \cdot 2 \cdot 12 \cdot \pi = 8\pi$. Hence the radius of the base of the smaller cone is 4 and its height is $\sqrt{12^2 - 4^2} = 8\sqrt{2}$. Similarly, the circumference of the base of the larger cone is 16π. Hence the radius of the base of the larger cone is 8 and its height is $4\sqrt{5}$. The ratio of the volume of the smaller cone to the volume of larger cone is

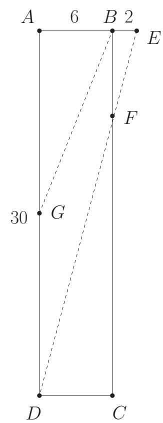
$$\frac{\frac{1}{3}\pi \cdot 4^2 \cdot 8\sqrt{2}}{\frac{1}{3}\pi \cdot 8^2 \cdot 4\sqrt{5}} = \frac{\sqrt{10}}{10}.$$



18. **Answer (C):** On average for every 500 people tested, 1 will test positive because he or she has the disease, while $2\% \cdot 499 \approx 10$ will test positive even though they do not have the disease. In other words, of approximately 11 people who test positive, only 1 has the disease, so the probability is approximately $\frac{1}{11}$.

19. **Answer (C):** Because $\triangle EBF$ is similar to $\triangle EAD$, it follows that $\frac{BF}{AD} = \frac{BE}{AE}$, or $\frac{BF}{30} = \frac{2}{8}$, giving $BF = \frac{15}{2}$. The area of trapezoid $BFDG$ is

$$\frac{1}{2}h(b_1 + b_2) = \frac{1}{2} \cdot AB \cdot (BF + GD) = \frac{1}{2} \cdot 6 \cdot \left(\frac{15}{2} + 15\right) = \frac{135}{2}.$$



20. **Answer (A):** The smallest initial number for which Bernardo wins after one round is the smallest integer solution of $2n + 50 \geq 1000$, which is 475. The smallest initial number for which he wins after two rounds is the smallest integer solution of $2n + 50 \geq 475$, which is 213. Similarly, the smallest initial numbers for which he wins after three and four rounds are 82 and 16, respectively. There is no initial number for which Bernardo wins after more than four rounds. Thus $N = 16$, and the sum of the digits of N is 7.

21. **Answer (A):** Since 4 of the 6 segments have length a, some 3 of the points (call them A, B, and C) must form an equilateral triangle of side length a. The fourth point D must be a distance a from one of A, B, or C, and without loss of generality it can be assumed to be A. Thus D lies on a circle of radius a centered at A. The distance from D to one of the other 2 points (which can be assumed to be B) is $2a$, so $\overline{BD}$ is a diameter of this circle and therefore is the hypotenuse of right triangle DCB with legs of lengths a and b. Thus $b^2 = (2a)^2 - a^2 = 3a^2$, and the ratio of b to a is $\sqrt{3}$.

22. **Answer (B):** If $a_1 = 1$, then the list must be an increasing sequence. Otherwise let $k = a_1$. Then the numbers 1 through $k-1$ must appear in increasing order from right to left, and the numbers from k through 10 must appear in increasing order from left to right. For $2 \le k \le 10$ there are $\binom{9}{k-1}$ ways to choose positions in the list for the numbers from 1 through $k-1$, and the positions of the remaining numbers are then determined. The number of lists is therefore

$$1 + \sum_{k=2}^{10} \binom{9}{k-1} = \sum_{k=0}^{9} \binom{9}{k} = 2^9 = 512.$$

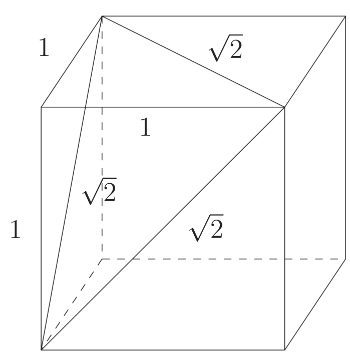
OR

If a_{10} is not 1 or 10, then numbers larger than a_{10} must appear in reverse order in the list, and numbers smaller than a_{10} must appear in order. However, 1 and 10 cannot both appear first in the list, so the placement of either 1 or 10 would violate the given conditions. Hence $a_{10} = 1$ or 10. By similar reasoning, when reading the list from right to left the number that appears next must be the smallest or largest unused integer from 1 to 10. This gives 2 choices for each term until there is one number left. Hence there are $2^9 = 512$ choices.

23. **Answer (D):** The discarded tetrahedron can be viewed as having an isosceles right triangle of side 1 as its base, with an altitude of 1. Therefore its volume is $\frac{1}{6}$. It can also be viewed as having an equilateral triangle of side length $\sqrt{2}$ as its base, in which case its altitude h must satisfy

$$\frac{1}{3} \cdot \frac{\sqrt{3}}{4} \left(\sqrt{2}\right)^2 \cdot h = \frac{1}{6},$$

which implies that $h = \frac{\sqrt{3}}{3}$. The height of the remaining solid is the long diagonal of the cube minus h, which is $\sqrt{3} - \frac{\sqrt{3}}{3} = \frac{2\sqrt{3}}{3}$.



24. **Answer (B):** There are two cases to consider.

Case 1
Each song is liked by two of the girls. Then one of the three pairs of girls likes one of the six possible pairs of songs, one of the remaining pairs of girls likes one of the remaining two songs, and the last pair of girls likes the last song. This case can occur in $3 \cdot 6 \cdot 2 = 36$ ways.

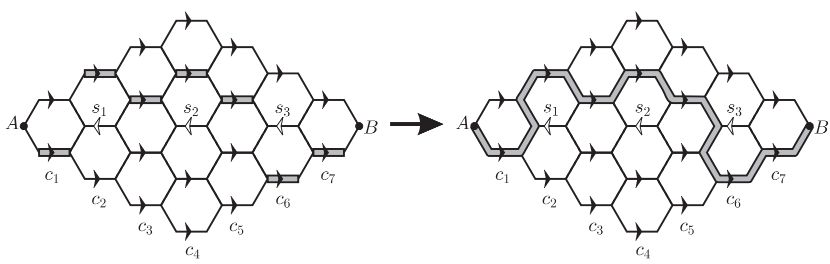
Case 2
Three songs are each liked by a different pair of girls, and the fourth song is liked by at most one girl. There are $4! = 24$ ways to assign the songs to these four categories, and the last song can be liked by Amy, Beth, Jo, or no one. This case can occur in $24 \cdot 4 = 96$ ways.

The total number of possibilities is $96 + 36 = 132$.

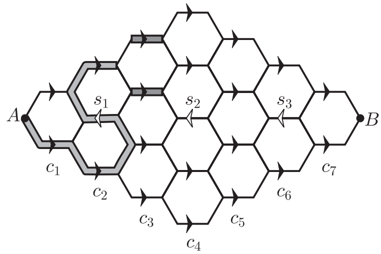
25. **Answer (E):**

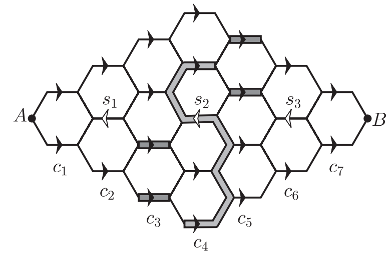
Label the columns having arrows as $c_1, c_2, c_3, \ldots, c_7$ according to the figure. Call those segments that can be traveled only from left to right *forward segments.* Call the segments s_1, s_2, and s_3, in columns c_2, c_4, and c_6, respectively, which can be traveled only from right to left, *back segments*. Denote S as the set of back segments traveled for a path.

First suppose that $S = \emptyset$. Because it is not possible to travel a segment more than once, it follows that the path is uniquely determined by choosing one forward segment in each of the columns c_j. There are 2, 2, 4, 4, 4, 2, and 2 choices for the forward segment in columns c_1, c_2, c_3, c_4, c_5, c_6, and c_7, respectively. This gives a total of 2^{10} total paths in this case.



Next suppose that $S = \{s_1\}$. The two forward segments in c_2, together with s_1, need to be part of the path, and once the forward segment from c_1 is chosen, the order in which the segments of c_2 are traveled is determined. Moreover, there are only 2 choices for possible segments in c_3 depending on the last segment traveled in c_2, either the bottom 2 or the top 2. For the rest of the columns, the path is determined by choosing any forward segment. Thus the total number of paths in this case is $2 \cdot 1 \cdot 2 \cdot 4 \cdot 4 \cdot 2 \cdot 2 = 2^8$, and by symmetry this is also the total for the number of paths when $S = \{s_3\}$. A similar argument gives $2 \cdot 1 \cdot 2 \cdot 4 \cdot 2 \cdot 1 \cdot 2 = 2^6$ trips for the case when $S = \{s_1, s_3\}$.





Suppose $S = \{s_2\}$. Because s_2 is traveled, it follows that 2 forward segments in c_4 need to belong to the path, one of them above s_2 (2 choices) and the other below it (2 choices). Once these are determined, there are 2 possible choices for the order in which these segments are traveled: the bottom forward segment first, then s_2, then the top forward segment, or vice versa. Next, there are only 2 possible forward segments that can be selected in c_3 and also only 2 possible forward segments that can be selected in c_5. The forward segments in c_1, c_2, c_6, and c_7 can be freely selected (2 choices each). This gives a total of $(2^3 \cdot 2 \cdot 2) \cdot 2^4 = 2^9$ paths.

If $S = \{s_1, s_2\}$, then the analysis is similar, except for the last step, where the forward segments of c_1 and c_2 are determined by the previous choices. Thus there are $(2^3 \cdot 2 \cdot 2) \cdot 2^2 = 2^7$ possibilities, and by symmetry the same number when $S = \{s_2, s_3\}$.

Finally, if $S = \{s_1, s_2, s_3\}$, then in the last step, all forward segments of c_1, c_2, c_6, and c_7 are determined by the previous choices and hence there are $2^3 \cdot 2 \cdot 2 = 2^5$ possible paths. Altogether the total number of paths is $2^{10} + 2 \cdot 2^8 + 2^6 + 2^9 + 2 \cdot 2^7 + 2^5 = 2400$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steven Davis, Steve Dunbar, Doug Faires, Sister Josannae Furey, Michelle Ghrist, Peter Gilchrist, Jerrold Grossman, Joe Kennedy, Eugene Veklerov, David Wells, LeRoy Wenstrom, and Ron Yannone.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

Academy of Applied Sciences
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Statistical Association
Art of Problem Solving
Awesome Math
Casualty Actuarial Society
D.E. Shaw & Co.
IDEA Math
Jane Street
Mu Alpha Theta
National Council of Teachers of Mathematics
Pi Mu Epsilon
Society of Actuaries
W. H. Freeman and Company

1. **Answer (C):** A 5-mile taxi ride costs $\$1.50 + 5(\$0.25) = \$2.75$.

2. **Answer (B):** Filling the cup 4 times will give Alice 1 cup of sugar. To get $2\frac{1}{2}$ cups of sugar, she must fill it $4 + 4 + \frac{1}{2} \cdot 4 = 10$ times.

3. **Answer (E):** The legs of $\triangle ABE$ have lengths $AB = 10$ and BE. Therefore $\frac{1}{2} \cdot 10 \cdot BE = 40$, so $BE = 8$.

4. **Answer (C):** The softball team could only have scored twice as many runs as their opponent when they scored an even number of runs. In those games their opponents scored

$$\frac{2}{2} + \frac{4}{2} + \frac{6}{2} + \frac{8}{2} + \frac{10}{2} = 15 \text{ runs.}$$

In the games the softball team lost, their opponents scored

$$(1+1) + (3+1) + (5+1) + (7+1) + (9+1) = 30 \text{ runs.}$$

The total number of runs scored by their opponents was $15 + 30 = 45$ runs.

5. **Answer (B):** The total shared expenses were $105+125+175 = 405$ dollars, so each traveler's fair share was $\frac{1}{3} \cdot 405 = 135$ dollars. Therefore $t = 135 - 105 = 30$ and $d = 135 - 125 = 10$, so $t - d = 30 - 10 = 20$.

OR

Because Dorothy paid 20 dollars more than Tom, Sammy must receive 20 more dollars from Tom than from Dorothy.

6. **Answer (D):** The 5-year-old and the two brothers who went to play baseball account for three of the four brothers who are younger than 10. Because the only age pairs that sum to 16 are 3 and 13, 5 and 11, and 7 and 9, the brothers who went to the movies must be 3 and 13 years old. Hence the 7-year-old and 9-year-old brothers went to play baseball, and Joey is 11.

7. **Answer (C):** Because English is required, the student must choose 3 of the remaining 5 courses. If the student takes both math courses, there are 3 possible choices for the final course. If the student chooses only one of the 2 possible

math courses, then the student must omit one of the 3 remaining courses, for a total of $2 \cdot 3 = 6$ programs. Hence there are $3 + 6 = 9$ programs.

OR

Because English is required, there are 5 remaining courses from which a student must choose 3. Of those $\binom{5}{3}$ possibilities, one does not include a math course. Thus the number of possible programs is $\binom{5}{3} - 1 = 9$.

8. **Answer (C):** Factoring 2^{2012} from each of the terms and simplifying gives

$$\frac{2^{2012}(2^2+1)}{2^{2012}(2^2-1)} = \frac{4+1}{4-1} = \frac{5}{3}\,.$$

9. **Answer (B):** If Shenille attempted x three-point shots and $30 - x$ two-point shots, then she scored a total of $\frac{20}{100} \cdot 3 \cdot x + \frac{30}{100} \cdot 2 \cdot (30 - x) = 18$ points.

   Remark: The given information does not allow the value of x to be determined.

10. **Answer (E):** Because six tenths of the flowers are pink and two thirds of the pink flowers are carnations, $\frac{6}{10} \cdot \frac{2}{3} = \frac{2}{5}$ of the flowers are pink carnations. Because four tenths of the flowers are red and three fourths of the red flowers are carnations, $\frac{4}{10} \cdot \frac{3}{4} = \frac{3}{10}$ of the flowers are red carnations. Therefore $\frac{2}{5} + \frac{3}{10} = \frac{7}{10} = 70\%$ of the flowers are carnations.

11. **Answer (A):** Let n be the number of student council members. Because there are 10 ways of choosing the two-person welcoming committee, it follows that $10 = \binom{n}{2} = \frac{1}{2}n(n-1)$, from which $n = 5$. The number of ways to select the three-person planning committee is $\binom{5}{3} = 10\,$.

12. **Answer (C):** Because $\overline{EF}$ is parallel to $\overline{AB}$, it follows that $\triangle FEC$ is similar to $\triangle ABC$ and $FE = FC$. Thus half of the perimeter of $ADEF$ is $AF + FE = AF + FC = AC = 28$. The entire perimeter is 56.

13. **Answer (B):** Each such three-digit number must have the form aba, where a and b are digits and $a \neq 0$. Such a number will not be divisible by 5 if and only if $a \neq 5$. If a is equal to 1, 2, 3, or 4, then any of the ten choices for b satisfies the requirement. If a is equal to 6, 7, 8, or 9, then there are 8, 6, 4, or 2 choices for b, respectively. This results in $4 \cdot 10 + 8 + 6 + 4 + 2 = 60$ numbers.

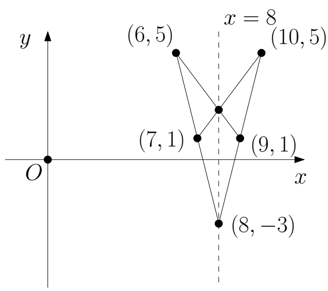
14. **Answer (D):** The large cube has 12 edges, and a portion of each edge remains after the 8 small cubes are removed. All of the 12 edges of each small cube are also edges of the new solid, except for the 3 edges that meet at a vertex of the large cube. Thus the new solid has a total of $12 + 8(12 - 3) = 84$ edges.

15. **Answer (D):** Denote the length of the third side as x, and the altitudes to the sides of lengths 10 and 15 as m and n, respectively. Then twice the area of the triangle is $10m = 15n = \frac{1}{2}x(m+n)$. This implies that $m = \frac{3}{2}n$, so

$$15n = \frac{1}{2}x\left(\frac{3}{2}n + n\right) = \frac{5}{4}xn\,.$$

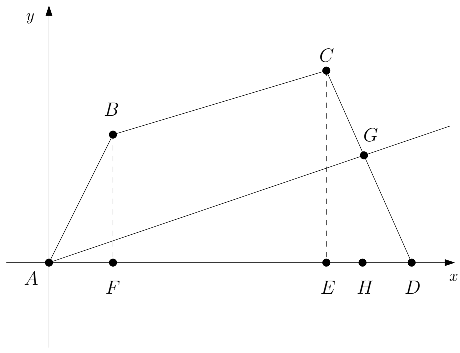
Therefore $15 = \frac{5}{4}x$, and $x = 12$.

16. **Answer (E):** The reflected triangle has vertices $(7,1)$, $(8,-3)$, and $(10,5)$. The point $(9,1)$ is on the line segment from $(10,5)$ to $(8,-3)$. The line segment from $(6,5)$ to $(9,1)$ contains the point $(8,\frac{7}{3})$, which must be on both triangles, and by symmetry the point $(7,1)$ is on the line segment from $(6,5)$ to $(8,-3)$. Therefore the union of the two triangles is also the union of two congruent triangles with disjoint interiors, each having the line segment from $(8,-3)$ to $(8,\frac{7}{3})$ as a base. The altitude of one of the two triangles is the distance from the line $x = 8$ to the point $(10,5)$, which is 2. Hence the union of the triangles has area $2\cdot(\frac{1}{2}\cdot 2\cdot(\frac{7}{3}+3)) = \frac{32}{3}$.



17. **Answer (B):** Alice and Beatrix will visit Daphne together every $3\cdot 4 = 12$ days, so this will happen $\lfloor\frac{365}{12}\rfloor = 30$ times. Likewise Alice and Claire will visit together $\lfloor\frac{365}{3\cdot 5}\rfloor = 24$ times, and Beatrix and Claire will visit together $\lfloor\frac{365}{4\cdot 5}\rfloor = 18$ times. However, each of these counts includes the $\lfloor\frac{365}{3\cdot 4\cdot 5}\rfloor = 6$ times when all three friends visit. The number of days that exactly two friends visit is $(30-6)+(24-6)+(18-6) = 54$.

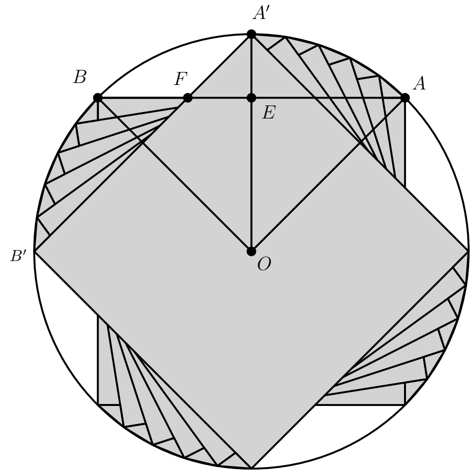
18. **Answer (B):** Let line AG be the required line, with G on $\overline{CD}$. Divide $ABCD$ into triangle ABF, trapezoid $BCEF$, and triangle CDE, as shown. Their areas are 1, 5, and $\frac{3}{2}$, respectively. Hence the area of $ABCD = \frac{15}{2}$, and the area of triangle $ADG = \frac{15}{4}$. Because $AD = 4$, it follows that $GH = \frac{15}{8} = \frac{r}{s}$. The equation of $\overline{CD}$ is $y = -3(x-4)$, so when $y = \frac{15}{8}$, $x = \frac{p}{q} = \frac{27}{8}$. Therefore $p+q+r+s = 58$.



19. **Answer (C):** For the base-b representation of 2013 to end in the digit 3, the base b must exceed 3. Also, b must divide $2013-3 = 2010$, so b must be one of the 16 positive integer factors of $2010 = 2\cdot 3\cdot 5\cdot 67$. Thus there are $16-3 = 13$ bases in which 2013 ends with a 3.

20. **Answer (C):** Let O be the center of unit square $ABCD$, let A and B be rotated to points A' and B', and let $\overline{OA'}$ and $\overline{A'B'}$ intersect $\overline{AB}$ at E and F, respectively. Then one quarter of the region swept out by the interior of the square consists of the $45°$ sector AOA' with radius $\frac{\sqrt{2}}{2}$, isosceles right triangle OEB with leg length $\frac{1}{2}$, and isosceles right triangle $A'EF$ with leg length $\frac{\sqrt{2}-1}{2}$. Thus the area of the region is

$$4\left(\left(\frac{\sqrt{2}}{2}\right)^2\left(\frac{\pi}{8}\right)+\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}-1}{2}\right)^2\right) = 2-\sqrt{2}+\frac{\pi}{4}.$$



21. **Answer (D):** For $1 \leq k \leq 11$, the number of coins remaining in the chest before the k^{th} pirate takes a share is $\frac{12}{12-k}$ times the number remaining afterward. Thus if there are n coins left for the 12^{th} pirate to take, the number of coins originally in the chest is

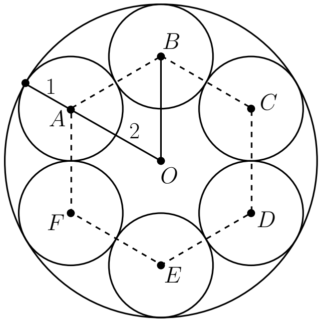
$$\frac{12^{11} \cdot n}{11!} = \frac{2^{22} \cdot 3^{11} \cdot n}{2^8 \cdot 3^4 \cdot 5^2 \cdot 7 \cdot 11} = \frac{2^{14} \cdot 3^7 \cdot n}{5^2 \cdot 7 \cdot 11}.$$

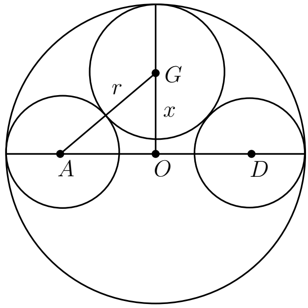
The smallest value of n for which this is a positive integer is $5^2 \cdot 7 \cdot 11 = 1925$. In this case there are

$$2^{14} \cdot 3^7 \cdot \frac{11!}{(12-k)! \cdot 12^{k-1}}$$

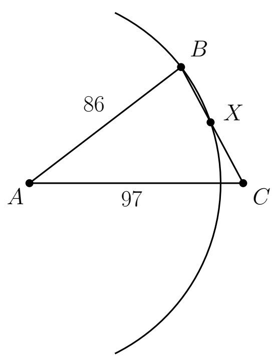
coins left for the k^{th} pirate to take, and note that this amount is an integer for each k. Hence the 12^{th} pirate receives 1925 coins.

22. **Answer (B):** Let the vertices of the regular hexagon be labeled in order A, B, C, D, E, and F. Let O be the center of the hexagon, which is also the center of the largest sphere. Let the eighth sphere have center G and radius r. Because the centers of the six small spheres are each a distance 2 from O and the small spheres have radius 1, the radius of the largest sphere is 3. Because G is equidistant from A and D, the segments $\overline{GO}$ and $\overline{AO}$ are perpendicular. Let x be the distance from G to O. Then $x + r = 3$. The Pythagorean Theorem applied to $\triangle AOG$ gives $(r+1)^2 = 2^2 + x^2 = 4 + (3-r)^2$, which simplifies to $2r + 1 = 13 - 6r$, so $r = \frac{3}{2}$. Note that this shows that the eighth sphere is tangent to $\overline{AD}$ at O.





23. **Answer (D):** By the Power of a Point Theorem, $BC \cdot CX = AC^2 - r^2$ where $r = AB$ is the radius of the circle. Thus $BC \cdot CX = 97^2 - 86^2 = 2013$. Since $BC = BX + CX$ and CX are both integers, they are complementary factors of 2013. Note that $2013 = 3 \cdot 11 \cdot 61$, and $CX < BC < AB + AC = 183$. Thus the only possibility is $CX = 33$ and $BC = 61$.



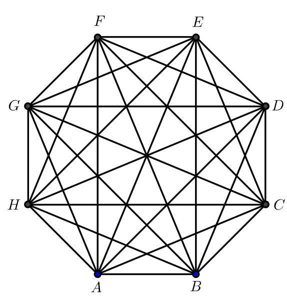
24. **Answer (E):** Call the players from Central A, B, and C, and call the players from Northern X, Y, and Z. Represent the schedule for each Central player by a string of length six consisting of two each of X, Y, and Z. There are $\binom{6}{2}\binom{4}{2} = 90$ possible strings for player A. Assume without loss of generality that the string is $XXYYZZ$. Player B's schedule must be a string with no X's in the first two positions, no Y's in the next two, and no Z's in the last two. If B's string begins with a Y and a Z in either order, the next two letters must be an X and a Z, and the last two must be an X and a Y. Because each pair can be ordered in one of two ways, there are $2^3 = 8$ such strings. If B's string begins with YY or ZZ, it must be $YYZZXX$ or $ZZXXYY$, respectively. Hence there are 10 possible schedules for B for each of the 90 schedules for A, and C's schedule is then determined. The total number of possible schedules is 900.

25. **Answer (A):** Label the octagon $ABCDEFGH$. There are 20 diagonals in all, 5 with endpoints at each vertex. The diagonals are of three types:

- Diagonals that skip over only one vertex, such as $\overline{AC}$ or $\overline{AG}$. These diagonals intersect with each of the five diagonals with endpoints at the skipped vertex.
- Diagonals that skip two vertices, such as $\overline{AD}$ or $\overline{AF}$. These diagonals intersect with four of the five diagonals that have endpoints at each of the two skipped vertices.
- Diagonals that cross to the opposite vertex, such as $\overline{AE}$. These diagonals intersect with three of the five diagonals that have endpoints at each of the three skipped vertices.

Therefore, from any given vertex, the diagonals will intersect other diagonals at $2\cdot 5+2\cdot 8+1\cdot 9 = 35$ points. Counting from all 8 vertices, the total is $8\cdot 35 = 280$ points.

Observe that, by symmetry, all four diagonals that cross to the opposite vertex intersect in the center of the octagon. This single intersection point has been counted 24 times, 3 from each of the 8 vertices. Further observe that at each of the vertices of the smallest internal octagon created by the diagonals, 3 diagonals intersect. For example, $\overline{AD}$ intersects with $\overline{CH}$ on $\overline{BF}$. These 8 intersection points have each been counted 12 times, 2 from each of the 6 affected vertices.The remaining intersection points each involve only two diagonals and each has been counted 4 times, once from each endpoint. These number $\frac{280-24-8\cdot 12}{4} = 40$. There are therefore $1 + 8 + 40 = 49$ distinct intersection points in the interior of the octagon.



The problems and solutions in this contest were proposed by Betsy Bennett, George Brauer, Steve Blasberg, Steve Davis, Marta Eso, Josanne Furey, Michele Ghrist, Jerry Grossman, Peter Gilchrist, Jonathan Kane, Dan Kennedy, Joe Kennedy, Cap Khoury, Roy Roehl, Kevin Wang, Dave Wells, LeRoy Wenstrom, and Woody Wenstrom.

MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

14th Annual

AMC 10 B

American Mathematics Contest 10 B

Wednesday, February 20, 2013

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom

1. **Answer (C):** Simplifying gives

$$\tfrac{2+4+6}{1+3+5} - \tfrac{1+3+5}{2+4+6} = \tfrac{12}{9} - \tfrac{9}{12} = \tfrac{4}{3} - \tfrac{3}{4} = \tfrac{16-9}{12} = \tfrac{7}{12}\,.$$

2. **Answer (A):** The garden is $2 \cdot 15 = 30$ feet wide and $2 \cdot 20 = 40$ feet long. Hence Mr. Green expects $\frac{1}{2} \cdot 30 \cdot 40 = 600$ pounds of potatoes.

3. **Answer (C):** The difference between the high and low temperatures was 16 degrees, so the difference between each of these and the average temperature was 8 degrees. The low temperature was 8 degrees less than the average, so it was $3^\circ - 8^\circ = -5^\circ$.

4. **Answer (D):** The number 201 is the 1^{st} number counted when proceeding backwards from 201 to 3. In turn, 200 is the 2^{nd} number, 199 is the 3^{rd} number, and x is the $(202-x)^{\text{th}}$ number. Therefore 53 is the $(202-53)^{\text{th}}$ number, which is the 149^{th} number.

5. **Answer (B):** Note that $2 \cdot a - a \cdot b = (2-b)a$. This expression is negative when $b > 2$. Hence the product is minimized when a and b are as large as possible. The minimum value is $(2-5) \cdot 5 = -15$.

6. **Answer (C):** The sum of all the ages is $55 \cdot 33 + 33 \cdot 11 = 33 \cdot 66$, so the average of all the ages is

$$\frac{33 \cdot 66}{55 + 33} = \frac{33 \cdot 66}{88} = \frac{33 \cdot 3}{4} = 24.75\,.$$

7. **Answer (B):** The six points divide the circle into six arcs each measuring 60°. By the Inscribed Angle Theorem, the angles of the triangle can only be 30°, 60°, 90°, and 120°. Because the angles of the triangle are pairwise distinct the triangle must be a $30-60-90^\circ$ triangle. Therefore the hypotenuse of the triangle is the diameter of the circle, and the legs have lengths 1 and $\sqrt{3}$. The area of the triangle is $\frac{1}{2} \cdot 1 \cdot \sqrt{3} = \frac{\sqrt{3}}{2}$.

8. **Answer (B):** Let D equal the distance traveled by each car. Then Ray's car uses $\frac{D}{40}$ gallons of gasoline and Tom's car uses $\frac{D}{10}$ gallons of gasoline. The cars

combined miles per gallon of gasoline is

$$\frac{2D}{\left(\frac{D}{40}+\frac{D}{10}\right)}=16.$$

9. **Answer (D):** Note that

$$27{,}000=2^3\cdot 3^3\cdot 5^3.$$

The only three pairwise relatively prime positive integers greater than 1 with a product of 27,000 are 8, 27, and 125. The sum of these numbers is 160.

10. **Answer (C):** Let x denote the number of three-point shots attempted. Then the number of three-point shots made was $0.4x$, resulting in $3(0.4x)=1.2x$ points. The number of two-point shots attempted was $1.5x$, and they were successful on $0.5(1.5x)=0.75x$ of them resulting in $2(0.75x)=1.5x$ points. The number of points scored was $1.2x+1.5x=54$, so $x=20$.

11. **Answer (B):** By completing the square the equation can be rewritten as follows:

$$x^2+y^2=10x-6y-34,$$
$$x^2-10x+25+y^2+6y+9=0,$$
$$(x-5)^2+(y+3)^2=0.$$

Therefore $x=5$ and $y=-3$, so $x+y=2$.

12. **Answer (B):** The five sides of the pentagon are congruent, and the five congruent diagonals are longer than the sides. Once one segment is selected, 4 of the 9 remaining segments have the same length as that segment. Therefore the requested probability is $\frac{4}{9}$.

13. **Answer (E):** Note that Jo starts by saying 1 number, and this is followed by Blair saying 2 numbers, then Jo saying 3 numbers, and so on. After someone completes her turn after saying the number n, then $1+2+3+\cdots+n=\frac{1}{2}n(n+1)$ numbers have been said. If $n=9$ then 45 numbers have been said. Therefore there are $53-45=8$ more numbers that need to be said. The 53^{rd} number said is 8.

14. **Answer (E):** The equation $x♣y = y♣x$ is equivalent to $x^2y - xy^2 = y^2x - yx^2$. This equation is euivalent to gives $2xy(x-y) = 0$. This equation will hold exactly if $x = 0$, $y = 0$, or $x = y$. The solution set consists of three lines: the x-axis, the y-axis, and the line $x = y$.

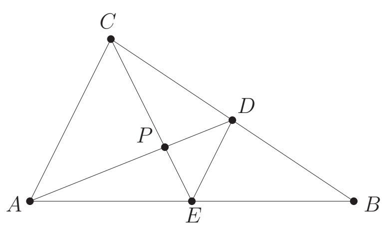
15. **Answer (B):** Let s be the side length of the triangle and h the side length of the hexagon. The hexagon can be subdivided into 6 equilateral triangles by drawing segments from the center of the hexagon to each vertex. Because the areas of the large triangle and hexagon are equal, the triangles in the hexagon each have area $\frac{1}{6}$ of the area of the large triangle. Thus

$$\frac{h}{s} = \sqrt{\frac{1}{6}} \qquad \text{so} \qquad h = \frac{\sqrt{6}}{6}s.$$

The perimeter of the triangle is $a = 3s$ and the perimeter of the hexagon is $b = 6h = \sqrt{6}s$, so

$$\frac{a}{b} = \frac{3s}{\sqrt{6}\,s} = \frac{\sqrt{6}}{2}.$$

16. **Answer (B):** The ratio of PE:PD:DE is 3:4:5 .



Hence by the converse of the Pythagorean Theorem, $\triangle DPE$ is a right triangle. Therefore $\overline{CE}$ is perpendicular to $\overline{AD}$, and the area of $AEDC$ is one-half the product of its diagonals. Because P is the centroid of $\triangle ABC$, it follows that $CE = 3(PE) = 4.5$ and $AD = 3(PD) = 6$. Therefore the area of $AEDC$ is $0.5(4.5)(6) = 13.5$.

OR

From the first solution, triangles CPD, DPE, EPA, and APC are right triangles with right angle at P. The area of trapezoid $AEDC$ is given by the sum of the areas of these four triangles. Because $\overline{DE}$ is parallel to $\overline{AC}$ and D is the midpoint of $\overline{CB}$, triangles BAC and BED are similar with common ratio 2, so $AC = 2 \cdot DE = 5$. Triangles APC and DPE are similar, so $AP = 4$ and $CP = 3$. Thus the area of $AEDC$ is

$$\frac{1}{2}\cdot 4\cdot 1.5+\frac{1}{2}\cdot 3\cdot 4+\frac{1}{2}\cdot 2\cdot 3+\frac{1}{2}\cdot 2\cdot 1.5=13.5\,.$$

17. **Answer (E):** After Alex makes m exchanges at the first booth and n exchanges at the second booth, Alex has $75-(2m-n)$ red tokens, $75-(3n-m)$ blue tokens, and $m+n$ silver tokens. No more exchanges are possible when he has fewer than 2 red tokens and fewer than 3 blue tokens. Therefore no more exchanges are possible if and only if $2m-n\geq 74$ and $3n-m\geq 73$. Equality can be achieved when $(m,n)=(59,44)$, and Alex will have $59+44=103$ silver tokens.

    Note that the following exchanges produce 103 silver tokens:

	Red Tokens	Blue Tokens	Silver Tokens
Exchange 75 blue tokens	100	0	25
Exchange 100 red tokens	0	50	75
Exchange 48 blue tokens	16	2	91
Exchange 16 red tokens	0	10	99
Exchange 9 blue tokens	3	1	102
Exchange 2 red tokens	1	2	103

18. **Answer (D):** First note that the only number between 2000 and 2013 that shares this property is 2002.

    Consider now the numbers in the range 1001 to 1999. There is exactly 1 number, 1001, that shares the property when the units digits is 1. There are exactly 2 numbers, 1102 and 1012, when the units digit is 2; exactly 3 numbers, 1203, 1113, and 1023, when the units digits is 3, and so on. Because the thousands digit is always 1, when the units digit is n, for $1\leq n\leq 9$, the sum of the hundreds and tens digits must be $n-1$. There are exactly n ways for this to occur. Hence there are exactly

$$1+(1+2+\cdots+9)=1+\tfrac{9\cdot 10}{2}=1+45=46$$

    numbers that share this property.

19. **Answer (D):** Let the common difference in the arithmetic sequence be d, so that $a=b+d$ and $c=b-d$. Because the quadratic has exactly one root, $b^2-4ac=0$. Substitution gives $b^2=4(b+d)(b-d)$, and therefore $3b^2=4d^2$. Because $b\geq 0$ and $d\geq 0$, it follows that $\sqrt{3}b=2d$. Thus the real root is

$$\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-b}{2a}=\frac{-b}{2(b+d)}=\frac{-b}{2\left(b+\frac{\sqrt{3}}{2}b\right)}=-2+\sqrt{3}\,.$$

Note that the quadratic equation $x^2 + (4 - 2\sqrt{3})x + 7 - 4\sqrt{3}$ satisfies the given conditions.

20. **Answer (B):** The prime factorization of 2013 is $3 \cdot 11 \cdot 61$. There must be a factor of 61 in the numerator, so $a_1 \geq 61$. Since $a_1!$ will have a factor of 59 and 2013 does not, there must be a factor of 59 in the denominator, and $b_1 \geq 59$. Thus $a_1 + b_1 \geq 120$, and this minimum value can be achieved only if $a_1 = 61$ and $b_1 = 59$. Furthermore, this minimum value is attainable because

$$2013 = \frac{(61!)(11!)(3!)}{(59!)(10!)(5!)}.$$

Thus $|a_1 - b_1| = a_1 - b_1 = 61 - 59 = 2$.

21. **Answer (C):** Let the two sequences be (a_n) and (b_n), and assume without loss of generality that $a_1 < b_1$. The definitions of the sequences imply that $a_7 = 5a_1 + 8a_2 = 5b_1 + 8b_2$, so $5(b_1 - a_1) = 8(a_2 - b_2)$. Because 5 and 8 are relatively prime, 8 divides $b_1 - a_1$ and 5 divides $a_2 - b_2$. It follows that $a_1 \leq b_1 - 8 \leq b_2 - 8 \leq a_2 - 13$. The minimum value of N results from choosing $a_1 = 0$, $b_1 = b_2 = 8$, and $a_2 = 13$, in which case $N = 104$.

22. **Answer (C):** The digit j at J contributes to all four sums, and each of the other digits contributes to exactly one sum. Therefore the sum of all four sums is $3j + (1 + 2 + 3 + \cdots + 9) = 45 + 3j$. Because all four sums are equal, this must be a multiple of 4, so $j = 1$, 5, or 9. For each choice of j, pair up the remaining digits so that each pair has the same sum. For example, for $j = 1$ the pairs are 2 and 9, 3 and 8, 4 and 7, and 5 and 6. Then order the pairs so that they correspond to the vertex pairs (A, E), (B, F), (C, G), (D, H). This results in $2^4 \cdot 4!$ different combinations for each j. Thus the requirements can be met in $2^4 \cdot 4! \cdot 3 = 1152$ ways.

23. **Answer (B):** The Pythagorean Theorem applied to right triangles ABD and ACD gives $AB^2 - BD^2 = AD^2 = AC^2 - CD^2$; that is, $13^2 - BD^2 = 15^2 - (14 - BD)^2$, from which it follows that $BD = 5$, $CD = 9$, and $AD = 12$. Because triangles AED and ADC are similar,

$$\frac{AE}{12} = \frac{DE}{9} = \frac{12}{15},$$

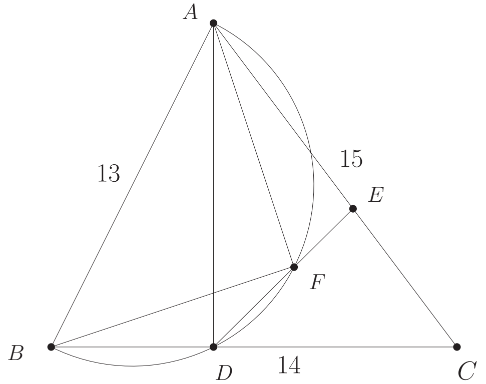
implying that $ED = \frac{36}{5}$ and $AE = \frac{48}{5}$.

Because $\angle AFB = \angle ADB = 90°$, it follows that $ABDF$ is cyclic. Thus $\angle ABD + \angle AFD = 180°$ from which $\angle ABD = \angle AFE$. Therefore right triangles ABD

and AFE are similar. Hence

$$\frac{FE}{5} = \frac{\frac{48}{5}}{12},$$

from which it follows that $FE = 4$. Consequently $DF = DE - FE = \frac{36}{5} - 4 = \frac{16}{5}$.



24. **Answer (A):** Let n denote a nice number from the given set. An integer m has exactly four divisors if and only if $m = p^3$ or $m = pq$, where p and q (with $p > q$) are prime numbers. In the former case, the sum of the four divisor is equal to $1 + p + p^2 + p^3$. Note that $1 + 11 + 11^2 + 11^3 < 2010 \leq n$ and $1 + 13 + 13^2 + 13^3 > 2019 \geq n$. Therefore we must have $m = pq$ and $n = 1+q+p+pq = (1+q)(1+p)$. Because p is odd, n must be an even number. If $q = 2$, then n must be divisible by 3. In the given set only $2010 = (1+2)(1+669)$ and $2016 = (1 + 2)(1 + 671)$ satisfy these requirements. However neither 669 nor 671 are prime. If q is odd, then n must be divisible by 4. In the given set, only 2012 and 2016 are divisible by 4. None of the pairs of factors of 2012, namely $1 \cdot 2012$, $2 \cdot 1006$, $4 \cdot 503$, gives rise to primes p and q. This leaves $2016 = (1 + 3)(1 + 503)$, which is the only nice number in the given set.

   Remark: Note that 2016 is nice in five ways. The other four ways are $(1+7)(1+251)$, $(1 + 11)(1 + 167)$, $(1 + 23)(1 + 83)$, and $(1 + 41)(1 + 47)$.

25. **Answer (E):** Expand the set of three-digit positive integers to include integers N, $0 \leq N \leq 99$, with leading zeros appended. Because $\text{lcm}(5^2, 6^2, 10^2) = 900$, such an integer N meets the required condition if and only if $N + 900$ does. Therefore N can be considered to be chosen from the set of integers between 000 and 899, inclusive. Suppose that the last two digits in order of the base-5 representation of N are a_1 and a_0. Similarly, suppose that the last two digits of the base-6 representation of N are b_1 and b_0. By assumption, $2N \equiv a_0 + b_0$

(mod 10), but $N \equiv a_0 \pmod 5$ and so

$$a_0 + b_0 \equiv 2N \equiv 2a_0 \pmod{10}.$$

Thus $a_0 \equiv b_0 \pmod{10}$ and because $0 \le a_0 \le 4$ and $0 \le b_0 \le 5$, it follows that $a_0 = b_0$. Because $N \equiv a_0 \pmod 5$, it follows that there is an integer N_1 such that $N = 5N_1 + a_0$. Also, $N \equiv a_0 \pmod 6$ implies that $5N_1 + a_0 \equiv a_0 \pmod 6$ and so $N_1 \equiv 0 \pmod 6$. It follows that $N_1 = 6N_2$ for some integer N_2 and so $N = 30N_2 + a_0$. Similarly, $N \equiv 5a_1 + a_0 \pmod{25}$ implies that $30N_2+a_0 \equiv 5a_1+a_0 \pmod{25}$ and then $N_2 \equiv 6N_2 \equiv a_1 \pmod 5$. It follows that $N_2 = 5N_3 + a_1$ for some integer N_3 and so $N = 150N_3 + 30a_1 + a_0$. Once more, $N \equiv 6b_1+a_0 \pmod{36}$ implies that $6N_3-6a_1+a_0 \equiv 150N_3+30a_1+a_0 \equiv 6b_1+a_0 \pmod{36}$ and then $N_3 \equiv a_1 + b_1 \pmod 6$. It follows that $N_3 = 6N_4 + a_1 + b_1$ for some integer N_4 and so $N = 900N_4 + 180a_1 + 150b_1 + a_0$. Finally, $2N \equiv 10(a_1 + b_1) + 2a_0 \pmod{100}$ implies that

$$60a_1 + 2a_0 \equiv 360a_1 + 300b_1 + 2a_0 \equiv 10a_1 + 10b_1 + 2a_0 \pmod{100}.$$

Therefore $5a_1 \equiv b_1 \pmod{10}$, equivalently, $b_1 \equiv 0 \pmod 5$ and $a_1 \equiv b_1 \pmod 2$. Conversely, if $N = 900N_4+180a_1+150b_1+a_0$, $a_0 = b_0$, and $5a_1 \equiv b_1 \pmod{10}$, then $2N \equiv 60a_1 + 2a_0 = 10(a_1 + 5a_1) + a_0 + b_0 \equiv 10(a_1 + b_1) + (a_0 + b_0) \pmod{100}$. Because $0 \le a_1 \le 4$ and $0 \le b_1 \le 5$, it follows that there are exactly 5 different pairs (a_1, b_1), namely $(0,0)$, $(2,0)$, $(4,0)$, $(1,5)$, and $(3,5)$. Each of these can be combined with 5 different values of a_0 $(0 \le a_0 \le 4)$, to determine exactly 25 different numbers N with the required property.

The problems and solutions in this contest were proposed by Betsy Bennett, Steve Blasberg, Tom Butts, Steve Davis, Doug Faires, Zuming Feng, Michelle Ghrist, Jerry Grossman, Elgin Johnston, Jonathan Kane, Joe Kennedy, Cap Khoury, Dave Wells, and LeRoy Wenstrom.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

Academy of Applied Sciences
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Statistical Association
Art of Problem Solving
Awesome Math
Casualty Actuarial Society
D.E. Shaw & Co.
Delta Airlines
Jane Street Capital
Math for America
Mu Alpha Theta
National Council of Teachers of Mathematics
Pi Mu Epsilon
Society for Industrial and Applied Mathematics (SIAM)

MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

15th Annual

AMC 10 A

American Mathematics Contest 10 A

Tuesday, February 4, 2014

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom

1. **Answer (C):** Note that

$$10 \cdot \left(\frac{1}{2} + \frac{1}{5} + \frac{1}{10}\right)^{-1} = 10 \cdot \left(\frac{8}{10}\right)^{-1} = \frac{25}{2}.$$

2. **Answer (C):** Because Roy's cat eats $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$ of a can of cat food each day, the cat eats 7 cans of cat food in 12 days. Therefore the cat eats $7 - \frac{7}{12} = 6\frac{5}{12}$ cans in 11 days and $6\frac{5}{12} - \frac{7}{12} = 5\frac{5}{6}$ cans in 10 days. The cat finishes the cat food in the box on the 11th day, which is Thursday.

3. **Answer (E):** In the morning, Bridget sells half of her loaves of bread for $\frac{1}{2} \cdot 48 \cdot \$2.50 = \$60$. In the afternoon, she sells $\frac{2}{3} \cdot 24 = 16$ loaves of bread for $16 \cdot \frac{1}{2} \cdot \$2.50 = \$20$. Finally, she sells the remaining 8 loaves of bread for \$8. Her total cost is $48 \cdot \$0.75 = \$36$. Her profit is $60 + 20 + 8 - 36 = 52$ dollars.

4. **Answer (B):** If Ralph passed the orange house first, then because the blue and yellow houses are not neighbors, the house color ordering must be orange, blue, red, yellow. If Ralph passed the blue house first, then there are 2 possible placements for the yellow house, and each choice determines the placement of the orange and red houses. These 2 house color orderings are blue, orange, yellow, red, and blue, orange, red, yellow. There are 3 possible orderings for the colored houses.

5. **Answer (C):** Because over 50% of the students scored 90 or lower, and over 50% of the students scored 90 or higher, the median score is 90. The mean score is

$$\frac{10}{100} \cdot 70 + \frac{35}{100} \cdot 80 + \frac{30}{100} \cdot 90 + \frac{25}{100} \cdot 100 = 87,$$

for a difference of $90 - 87 = 3$.

6. **Answer (A):** One cow gives $\frac{b}{a}$ gallons in c days, so one cow gives $\frac{b}{ac}$ gallons in 1 day. Thus d cows will give $\frac{bd}{ac}$ gallons in 1 day. In e days d cows will give $\frac{bde}{ac}$ gallons of milk.

7. **Answer (B):** Note that $x + y < a + y < a + b$, so inequality I is true. If $x = -2$, $y = -2$, $a = -1$, and $b = -1$, then none of the other three inequalities is true.

8. **Answer (D):** Note that $\frac{n!(n+1)!}{2} = (n!)^2 \cdot \frac{n+1}{2}$, which is a perfect square if and only if $\frac{n+1}{2}$ is a perfect square. Only choice D satisfies this condition.

9. **Answer (C):** The area of the triangle is $\frac{1}{2} \cdot 2\sqrt{3} \cdot 6 = 6\sqrt{3}$. By the Pythagorean Theorem, the hypotenuse has length $4\sqrt{3}$. The desired altitude has length $\frac{6\sqrt{3}}{\frac{1}{2} \cdot 4\sqrt{3}} = 3$.

10. **Answer (B):** The five consecutive integers starting with a are a, $a+1$, $a+2$, $a+3$, and $a+4$. Their average is $a+2=b$. The average of five consecutive integers starting with b is $b+2=a+4$.

11. **Answer (C):** Let $P > 100$ be the listed price. Then the price reductions in dollars are as follows:

    Coupon 1: $\frac{P}{10}$
    Coupon 2: 20
    Coupon 3: $\frac{18}{100}(P-100)$

    Coupon 1 gives a greater price reduction than coupon 2 when $\frac{P}{10} > 20$, that is, $P > 200$. Coupon 1 gives a greater price reduction than coupon 3 when $\frac{P}{10} > \frac{18}{100}(P-100)$, that is, $P < 225$. The only choice that satisfies these inequalities is \$219.95.

12. **Answer (C):** Each of the 6 sectors has radius 3 and central angle $120°$. Their combined area is $6 \cdot \frac{1}{3} \cdot \pi \cdot 3^2 = 18\pi$. The hexagon can be partitioned into 6 equilateral triangles each having side length 6, so the hexagon has area $6 \cdot \frac{\sqrt{3}}{4} \cdot 6^2 = 54\sqrt{3}$. The shaded region has area $54\sqrt{3} - 18\pi$.

13. **Answer (C):** The three squares each have area 1, and $\triangle ABC$ has area $\frac{\sqrt{3}}{4}$. Note that $\angle EAF = 360° - 60° - 2 \cdot 90° = 120°$. Thus the altitude from A in isosceles $\triangle EAF$ partitions the triangle into two $30-60-90°$ right triangles, each with hypotenuse 1. It follows that $\triangle EAF$ has base $EF = \sqrt{3}$ and altitude $\frac{1}{2}$, so its area is $\frac{\sqrt{3}}{4}$. Similarly, triangles GCH and DBI each have area $\frac{\sqrt{3}}{4}$. Therefore the area of hexagon $DEFGHI$ is $3 \cdot \frac{\sqrt{3}}{4} + 3 \cdot 1 + \frac{\sqrt{3}}{4} = 3 + \sqrt{3}$.

14. **Answer (D):** Let the y-intercepts of lines PA and QA be $\pm b$. Then their slopes are $\frac{8 \pm b}{6}$. Setting the product of the slopes to -1 and solving yields $b = \pm 10$. Therefore $\triangle APQ$ has base 20 and altitude 6, for an area of 60.

15. **Answer (C):** Let d be the remaining distance after one hour of driving, and let t be the remaining time until his flight. Then $d = 35(t+1)$, and $d = 50(t-0.5)$. Solving gives $t = 4$ and $d = 175$. The total distance from home to the airport is $175 + 35 = 210$ miles.

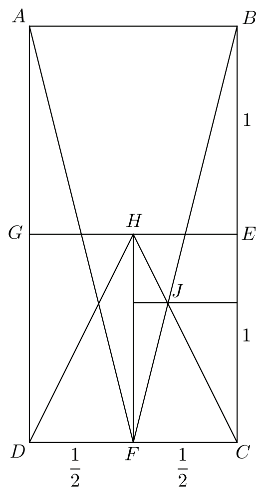
OR

Let d be the distance between David's home and the airport. The time required to drive the entire distance at 35 MPH is $\frac{d}{35}$ hours. The time required to drive at 35 MPH for the first 35 miles and 50 MPH for the remaining $d - 35$ miles is $1 + \frac{d-35}{50}$. The second trip is 1.5 hours quicker than the first, so

$$\frac{d}{35} - \left(1 + \frac{d-35}{50}\right) = 1.5.$$

Solving yields $d = 210$ miles.

16. **Answer (E):** Let J be the intersection point of $\overline{BF}$ and $\overline{HC}$. Then $\triangle JHF$ is similar to $\triangle JCB$ with ratio $1 : 2$. The length of the altitude of $\triangle JHF$ to $\overline{HF}$ plus the length of the altitude of $\triangle JCB$ to $\overline{CB}$ is $FC = \frac{1}{2}$. Thus $\triangle JHF$ has altitude $\frac{1}{6}$ and base 1, and its area is $\frac{1}{12}$. The shaded area is twice the area of $\triangle JHF$, or $\frac{1}{6}$.



OR

Place the figure on the coordinate plane with H at the origin. Then the equation of line DH is $y = 2x$, and the equation of line AF is $y = -4x - 1$. Solving the equations simultaneously shows that the leftmost point of the shaded region has x-coordinate $-\frac{1}{6}$. The kite therefore has diagonals $\frac{1}{3}$ and 1, so its area is $\frac{1}{2} \cdot \frac{1}{3} \cdot 1 = \frac{1}{6}$.

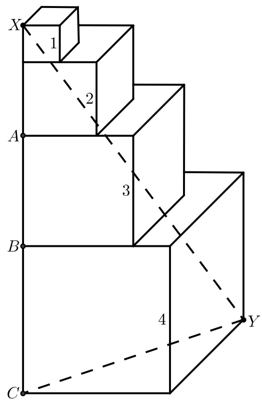
17. **Answer (D):** Each roll of the three dice can be recorded as an ordered triple (a, b, c) of the three values appearing on the dice. There are 6^3 equally likely triples possible. For the sum of two of the values in the triple to equal the third value, the triple must be a permutation of one of the triples $(1,1,2)$, $(1,2,3)$, $(1,3,4)$, $(1,4,5)$, $(1,5,6)$, $(2,2,4)$, $(2,3,5)$, $(2,4,6)$, or $(3,3,6)$. There are $3! = 6$ permutations of the values (a, b, c) when a, b, and c are distinct, and 3 permutations of the values when two of the values are equal. Thus there are $6 \cdot 6 + 3 \cdot 3 = 45$ triples where the sum of two of the values equals the third. The requested probability is $\frac{45}{6^3} = \frac{5}{24}$.

OR

There are 36 outcomes when a pair of dice are rolled, and the probability of rolling a total of $2, 3, 4, 5$, or 6 is $\frac{1}{36}, \frac{2}{36}, \frac{3}{36}, \frac{4}{36}$, and $\frac{5}{36}$, respectively. The probability that another die matches this total is $\frac{1}{6}$, and there are 3 ways to choose the die that matches the total of the other two. Thus the requested probability is $3(\frac{1}{36} \cdot \frac{1}{6} + \frac{2}{36} \cdot \frac{1}{6} + \frac{3}{36} \cdot \frac{1}{6} + \frac{4}{36} \cdot \frac{1}{6} + \frac{5}{36} \cdot \frac{1}{6}) = 3 \cdot \frac{15}{36} \cdot \frac{1}{6} = \frac{5}{24}$.

18. **Answer (B):** Let the square have vertices A, B, C, D in counterclockwise order. Without loss of generality assume that $A = (0, 0)$ and $B = (x, 1)$ for some $x > 0$. Because D is the image of B under a $90°$ counterclockwise rotation about A, the coordinates of D are $(-1, x)$, so $x = 4$. Therefore the area of the square is $(AB)^2 = 4^2 + 1^2 = 17$. Note that $C = (3, 5)$, and $ABCD$ is indeed a square.

19. **Answer (A):** Label vertices A, B, and C as shown. Note that $XC = 10$ and $CY = \sqrt{4^2 + 4^2} = 4\sqrt{2}$. Because $\triangle XYC$ is a right triangle, $XY = \sqrt{10^2 + (4\sqrt{2})^2} = 2\sqrt{33}$. The ratio of BX to CX is $\frac{3}{5}$, so in the top face of the bottom cube the distance from B to $\overline{XY}$ is $4\sqrt{2} \cdot \frac{3}{5} = \frac{12\sqrt{2}}{5}$. This distance is less than $3\sqrt{2}$, so $\overline{XY}$ pierces the top and bottom faces of the cube with side length 3. The ratio of AB to XC is $\frac{3}{10}$, so the length of $\overline{XY}$ that is inside the cube with side length 3 is $\frac{3}{10} \cdot 2\sqrt{33} = \frac{3\sqrt{33}}{5}$.



OR

Place the figure in a 3-dimensional coordinate system with the lower left front corner at $(0,0,0)$, $X = (0,0,10)$, and $Y = (4,4,0)$. Then line XY consists of all points of the form $(4t, 4t, 10 - 10t)$. This line intersects the bottom face of the cube with side length 3 when $10 - 10t = 4$, or $t = \frac{3}{5}$; this is the point $(\frac{12}{5}, \frac{12}{5}, 4)$, and because $\frac{12}{5} < 3$, the point indeed lies on that face. Similarly, line XY intersects the top face of the cube with side length 3 when $10 - 10t = 7$, or $t = \frac{3}{10}$; this is the point $(\frac{6}{5}, \frac{6}{5}, 7)$. Therefore the desired length is

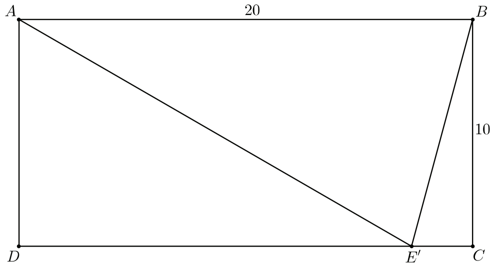
$$\sqrt{\left(\frac{12}{5} - \frac{6}{5}\right)^2 + \left(\frac{12}{5} - \frac{6}{5}\right)^2 + (4-7)^2} = \frac{3}{5}\sqrt{33}.$$

20. **Answer (D):** By direct multiplication, $8 \cdot 888\ldots 8 = 7111\ldots 104$, where the product has 2 fewer ones than the number of digits in $888\ldots 8$. Because $7 + 4 = 11$, the product must have $1000 - 11 = 989$ ones, so $k - 2 = 989$ and $k = 991$.

21. **Answer (E):** Setting $y = 0$ in both equations and solving for x gives $x = -\frac{5}{a} = -\frac{b}{3}$, so $ab = 15$. Only four pairs of positive integers (a, b) have product 15, namely $(1, 15)$, $(15, 1)$, $(3, 5)$, and $(5, 3)$. Therefore the four possible points on the x-axis have coordinates -5, $-\frac{1}{3}$, $-\frac{5}{3}$, and -1, the sum of which is -8.

22. **Answer (E):** Let E' be the point on $\overline{CD}$ such that $AE' = AB = 2AD$. Then $\triangle ADE'$ is a $30-60-90^\circ$ triangle, so $\angle DAE' = 60^\circ$. Hence $\angle BAE' = 30^\circ$. Also,

$AE' = AB$ implies that $\angle E'BA = \angle BE'A = 75°$, and then $\angle CBE' = 15°$. Thus it follows that E' and E are the same point. Therefore, $AE = AE' = AB = 20$.



23. **Answer (C):** Without loss of generality, assume that the rectangle has dimensions 3 by $\sqrt{3}$. Then the fold has length 2, and the overlapping areas are equilateral triangles each with area $\frac{\sqrt{3}}{4} \cdot 2^2$. The new shape has area $3\sqrt{3} - \frac{\sqrt{3}}{4} \cdot 2^2 = 2\sqrt{3}$, and the desired ratio is $2\sqrt{3} : 3\sqrt{3} = 2 : 3$.

24. **Answer (A):** After the nth iteration there will be $4 + 5 + 6 + \cdots + (n + 3) = \frac{(n+3)(n+4)}{2} - 6 = \frac{n(n+7)}{2}$ numbers listed, and $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$ numbers skipped. The first number to be listed on the $(n + 1)$st iteration will be one more than the sum of these, or $n^2 + 4n + 1$.

    It is necessary to find the greatest integer value of n such that $\frac{n(n+7)}{2} < 500{,}000$. This implies that $n(n + 7) < 1{,}000{,}000$. Note that, for $n = 993$, this product becomes $993 \cdot 1000 = 993{,}000$. Next observe that, in general, $(a + k)(b + k) = ab + (a + b)k + k^2$ so $(993 + k)(1000 + k) = 993{,}000 + 1993k + k^2$. By inspection, the largest integer value of k that will satisfy the above inequality is 3 and the n needed is 996. After the 996th iteration, there will be $\frac{993{,}000+1993\cdot3+9}{2} = \frac{998{,}988}{2} = 499{,}494$ numbers in the sequence. The 997th iteration will begin with the number $996^2 + 4 \cdot 996 + 1 = 996 \cdot 1000 + 1 = 996{,}001$.

    The 506th number in the 997th iteration will be the 500,000th number in the sequence. This is $996{,}001 + 505 = 996{,}506$.

25. **Answer (B):** Because $2^2 < 5$ and $2^3 > 5$, there are either two or three integer powers of 2 strictly between any two consecutive integer powers of 5. Thus for

each n there is at most one m satisfying the given inequalities, and the question asks for the number of cases in which there are three powers rather than two. Let d (respectively, t) be the number of nonnegative integers n less than 867 such that there are exactly two (respectively, three) powers of 2 strictly between 5^n and 5^{n+1}. Because $2^{2013} < 5^{867} < 2^{2014}$, it follows that $d + t = 867$ and $2d + 3t = 2013$. Solving the system yields $t = 279$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Steve Blasberg, Tom Butts, Steven Davis, Peter Gilchrist, Jerry Grossman, Jon Kane, Joe Kennedy, Gerald Kraus, Roger Waggoner, Kevin Wang, David Wells, LeRoy Wenstrom, and Ronald Yannone.

The American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

Academy of Applied Sciences
American Institute of Mathematics
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Statistical Association
Art of Problem Solving
Association of Symbolic Logic
Awesome Math
Casualty Actuarial Society
Conference Board of the Mathematical Sciences
The D.E. Shaw Group
IDEA Math
Jane Street Capital
Math For America
Math Training Center
Mu Alpha Theta
Pi Mu Epsilon
Society for Industrial and Applied Math – SIAM
W.H. Freeman

MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

15th Annual

AMC 10 B

American Mathematics Contest 10 A

Wednesday, February 19, 2014

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom

1. **Answer (C):** Leah has 7 pennies and 6 nickels, which are worth 37 cents.

2. **Answer (E):** Note that

$$\frac{2^3+2^3}{2^{-3}+2^{-3}} = \frac{2\cdot 2^3}{2\cdot 2^{-3}} = 2^6 = 64.$$

3. **Answer (E):** The fraction of Randy's trip driven on pavement was $1-\frac{1}{3}-\frac{1}{5}=\frac{7}{15}$. Therefore the entire trip was $20 \div \frac{7}{15} = \frac{300}{7}$ miles.

4. **Answer (B):** Let a muffin cost m dollars and a banana cost b dollars. Then $2(4m+3b) = 2m+16b$, and simplifying gives $m = \frac{5}{3}b$.

5. **Answer (A):** Denote the height of a pane by $5x$ and the width by $2x$. Then the square window has height $2\cdot 5x+6$ inches and width $4\cdot 2x+10$ inches. Solving $2\cdot 5x+6 = 4\cdot 2x+10$ gives $x=2$. The side length of the square window is 26 inches.

6. **Answer (C):** The special allows Orvin to purchase balloons at $\frac{1+\frac{2}{3}}{2} = \frac{5}{6}$ times the regular price. Because Orvin had just enough money to purchase 30 balloons at the regular price, he may now purchase $30\cdot\frac{6}{5} = 36$ balloons.

7. **Answer (A):** The fraction by which A is greater than B is simply the positive difference $A-B$ divided by B. The percent difference is 100 times this, or $100\left(\frac{A-B}{B}\right)$.

8. **Answer (E):** The truck travels for $3\cdot 60 = 180$ seconds, at a rate of $\frac{b}{6t}\cdot\frac{1}{3}$ yards per second. Hence the truck travels $180\cdot\frac{b}{6t}\cdot\frac{1}{3} = \frac{10b}{t}$ yards.

9. **Answer (A):** Note that

$$2014 = \frac{\frac{1}{w}+\frac{1}{z}}{\frac{1}{w}-\frac{1}{z}} = \frac{\frac{w+z}{wz}}{\frac{z-w}{wz}} = \frac{w+z}{z-w}.$$

Because $\frac{w+z}{z-w} = -\frac{w+z}{w-z}$, the requested value is -2014.

10. **Answer (C):** As indicated by the leftmost column $A+B \leq 9$. Then both the second and fourth columns show that $C = 0$. Because A, B, and C are distinct digits, D must be at least 3. The following values for (A, B, C, D) show that D may be any of the 7 digits that are at least 3: $(1, 2, 0, 3)$, $(1, 3, 0, 4)$, $(2, 3, 0, 5)$, $(2, 4, 0, 6)$, $(2, 5, 0, 7)$, $(2, 6, 0, 8)$, $(2, 7, 0, 9)$.

11. **Answer (C):** If P is the price paid for an item, then the discounted prices with the three given discounts are given by the following calculations:

    (1) $(0.85)^2 P = 0.7225P$ for a discount of 27.75%

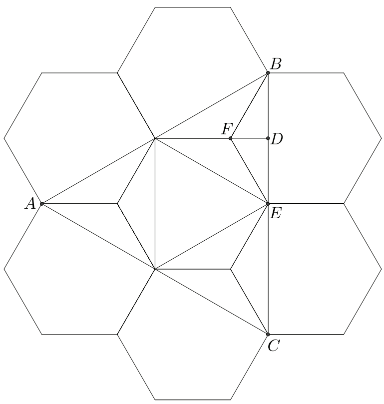
    (2) $(0.9)^3 P = 0.729P$ for a discount of 27.1%

    (3) $(0.75) \cdot (0.95)P = 0.7125P$ for a discount of 28.75%

    The smallest integer greater than 27.75, 27.1, and 28.75 is 29.

12. **Answer (C):** By inspection, the five smallest positive divisors of 2,014,000,000 are 1, 2, 4, 5, and 8. Therefore the fifth largest divisor is $\frac{2{,}014{,}000{,}000}{8} =$ 251,750,000.

13. **Answer (B):** Label points E and F as shown in the figure, and let D be the midpoint of $\overline{BE}$. Because $\triangle BFD$ is a $30-60-90^\circ$ triangle with hypotenuse 1, the length of $\overline{BD}$ is $\frac{\sqrt{3}}{2}$, and therefore $BC = 2\sqrt{3}$. It follows that the area of $\triangle ABC$ is $\frac{\sqrt{3}}{4} \cdot (2\sqrt{3})^2 = 3\sqrt{3}$.



OR

Notice that $AE = 3$ since AE is composed of a hexagon side (length 1) and the longest diagonal of a hexagon (length 2). Triangle ABE is $30-60-90°$, so $BE = \frac{3}{\sqrt{3}} = \sqrt{3}$. The area of $\triangle ABC$ is $AE \cdot BE = 3\sqrt{3}$.

14. **Answer (D):** Let m be the total mileage of the trip. Then m must be a multiple of 55. Also, because $m = cba - abc = 99(c-a)$, it is a multiple of 9. Therefore m is a multiple of 495. Because m is at most a 3-digit number and a is not equal to 0, $m = 495$. Therefore $c - a = 5$. Because $a+b+c \le 7$, the only possible abc is 106, so $a^2 + b^2 + c^2 = 1 + 0 + 36 = 37$.

OR

Let m be the total mileage of the trip. Then m must be a multiple of 55. Also, because $m = cba - abc = 99(c-a)$, $c - a$ is a multiple of 5. Because $a \ge 1$ and $a+b+c \le 7$, it follows that $c = 6$ and $a = 1$. Therefore $b = 0$, so $a^2+b^2+c^2 = 37$.

15. **Answer (A):** Let $AD = \sqrt{3}$. Because $\angle ADE = 30°$, it follows that $AE = 1$ and $DE = 2$. Now $\angle EDF = 30°$ and $\angle DEF = 120°$, so $\triangle DEF$ is isosceles and $EF = 2$. Thus the area of $\triangle DEF$ (with $\overline{EF}$ viewed as the base) is $\frac{1}{2} \cdot 2 \cdot \sqrt{3} = \sqrt{3}$, and the desired ratio is $\frac{\sqrt{3}}{\sqrt{3} \cdot 2\sqrt{3}} = \frac{\sqrt{3}}{6}$.

16. **Answer (B):** If exactly three of the four dice show the same number, then there are 6 possible choices for the repeated value and 5 possible choices for the non-repeated value. The non-repeated value may appear on any one of the 4 dice, so there are $6 \cdot 5 \cdot 4 = 120$ possible ways for such a result to occur. There are 6 ways for all four dice to show the same value. There are 6^4 total possible outcomes for the four dice. The probability of the desired result is $\frac{120+6}{6^4} = \frac{7}{72}$.

17. **Answer (D):** Note that

$$\begin{aligned}
10^{1002} - 4^{501} &= 2^{1002} \cdot 5^{1002} - 2^{1002} \\
&= 2^{1002}(5^{1002} - 1) \\
&= 2^{1002}(5^{501} - 1)(5^{501} + 1) \\
&= 2^{1002}(5-1)(5^{500} + 5^{499} + \cdots + 5 + 1)(5+1)(5^{500} - 5^{499} + \cdots \\
&\qquad - 5 + 1) \\
&= 2^{1005}(3)(5^{500} + 5^{499} + \cdots + 5 + 1)(5^{500} - 5^{499} + \cdots - 5 + 1).
\end{aligned}$$

Because each of the last two factors is a sum of an odd number of odd terms, they are both odd. The greatest power of 2 is 2^{1005}.

18. **Answer (E):** The numbers in the list have a sum of $11 \cdot 10 = 110$. The value of the 11th number is maximized when the sum of the first ten numbers is minimized subject to the following conditions.

- If the numbers are arranged in nondecreasing order, the sixth number is 9.
- The number 8 occurs either 2, 3, 4, or 5 times, and all other numbers occur fewer times.

If 8 occurs 5 times, the smallest possible sum of the first 10 numbers is

$$8+8+8+8+8+9+9+9+9+10=86.$$

If 8 occurs 4 times, the smallest possible sum of the first 10 numbers is

$$1+8+8+8+8+9+9+9+10+10=80.$$

If 8 occurs 3 times, the smallest possible sum of the first 10 numbers is

$$1+1+8+8+8+9+9+10+10+11=75.$$

If 8 occurs 2 times, the smallest possible sum of the first 10 numbers is

$$1+2+3+8+8+9+10+11+12+13=77.$$

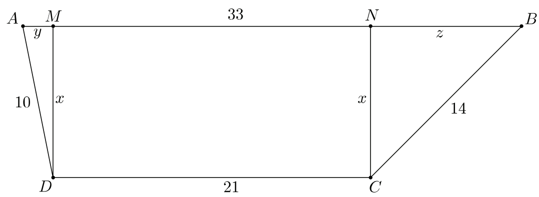
Thus the largest possible value of the 11th number is $110-75=35$.

19. **Answer (D):** Let A be the first point chosen on the outer circle, let chords $\overline{AB}$ and $\overline{AC}$ on the outer circle be tangent to the inner circle at D and E, respectively, and let O be the common center of the two circles. Triangle ADO has a right angle at D, $OA=2$, and $OD=1$, so $\angle OAD=30^\circ$. Similarly, $\angle OAE=30^\circ$, so $\angle BAC=\angle DAE=60^\circ$, and minor arc $BC=120^\circ$. If X is the second point chosen on the outer circle, then chord $\overline{AX}$ intersects the inner circle if and only if X is on minor arc BC. Therefore the requested probability is $\frac{120^\circ}{360^\circ}=\frac{1}{3}$.

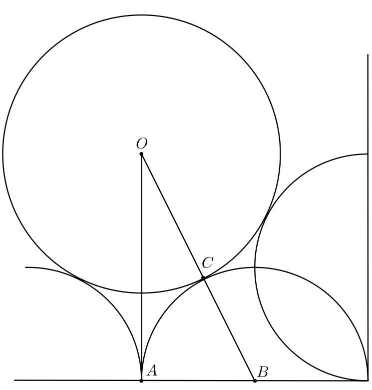
20. **Answer (C):** Note that $x^4-51x^2+50=(x^2-50)(x^2-1)$, so the roots of the polynomial are ± 1 and $\pm\sqrt{50}$. Arranged from least to greatest, these roots are approximately $-7.1,-1,1,7.1$. The polynomial takes negative values on the intervals $(-7.1,-1)$ and $(1,7.1)$, which include 12 integers: -7, -6, -5, -4, -3, -2, 2, 3, 4, 5, 6, 7.

21. **Answer (B):** Assume without loss of generality that $DA=10$ and $BC=14$. Let M and N be the feet of the perpendicular segments to $\overline{AB}$ from D and

C, respectively. The four points A, M, N, B appear on $\overline{AB}$ in that order. Let $x = DM = CN$, $y = AM$, and $z = NB$. Then $x^2 + y^2 = 10^2 = 100$, $x^2 + z^2 = 14^2 = 196$, and $y + 21 + z = 33$. Therefore $z = 12 - y$, and it follows that $\sqrt{196 - x^2} = 12 - \sqrt{100 - x^2}$. Squaring and simplifying gives $24\sqrt{100 - x^2} = 48$, so $x^2 = 96$ and $y = \sqrt{100 - 96} = 2$. The square of the length of the shorter diagonal, $\overline{AC}$, is $(y + 21)^2 + x^2 = 23^2 + 96 = 625$, so $AC = 25$.



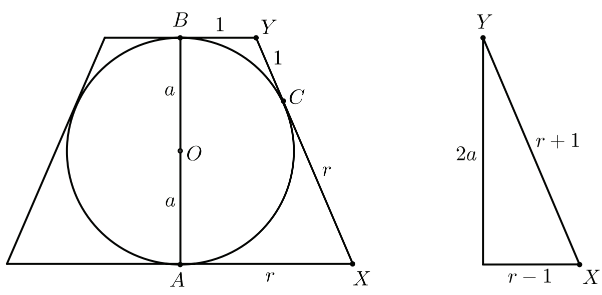
22. **Answer (B):** Let O be the center of the circle and choose one of the semicircles to have center point B. Label the point of tangency C and point A as in the figure. In $\triangle OAB$, $AB = \frac{1}{2}$ and $OA = 1$, so $OB = \frac{\sqrt{5}}{2}$. Because $BC = \frac{1}{2}$, $OC = \frac{\sqrt{5}}{2} - \frac{1}{2} = \frac{\sqrt{5}-1}{2}$.



23. **Answer (E):** Assume without loss of generality that the radius of the top base of the truncated cone (frustum) is 1. Denote the radius of the bottom base by r and the radius of the sphere by a. The figure on the left is a side view of the frustum. Applying the Pythagorean Theorem to the triangle on the right yields $r = a^2$. The volume of the frustum is

$$\frac{1}{3}\pi(r^2 + r \cdot 1 + 1^2) \cdot 2a = \frac{1}{3}\pi(a^4 + a^2 + 1) \cdot 2a.$$

Setting this equal to twice the volume of the sphere, $\frac{4}{3}\pi a^3$, and simplifying gives $a^4 - 3a^2 + 1 = 0$, or $r^2 - 3r + 1 = 0$. Therefore $r = \frac{3+\sqrt{5}}{2}$.



24. **Answer (B):** The circular arrangement 14352 is bad because the sum 6 cannot be achieved with consecutive numbers, and the circular arrangement 23154 is bad because the sum 7 cannot be so achieved. It remains to show that these are the only bad arrangements. Given a circular arrangement, sums 1 through 5 can be achieved with a single number, and if the sum n can be achieved, then the sum $15 - n$ can be achieved using the complementary subset. Therefore an arrangement is not bad as long as sums 6 and 7 can be achieved. Suppose 6 cannot be achieved. Then 1 and 5 cannot be adjacent, so by a suitable rotation and/or reflection, the arrangement is $1bc5e$. Furthermore, $\{b, c\}$ cannot equal $\{2, 3\}$ because $1 + 2 + 3 = 6$; similarly $\{b, c\}$ cannot equal $\{2, 4\}$. It follows that $e = 2$, which then forces the arrangement to be 14352 in order to avoid consecutive 213. This arrangement is bad. Next suppose that 7 cannot be achieved. Then 2 and 5 cannot be adjacent, so again without loss of generality

the arrangement is $2bc5e$. Reasoning as before, $\{b,c\}$ cannot equal $\{3,4\}$ or $\{1,4\}$, so $e=4$, and then $b=3$ and $c=1$, to avoid consecutive 421; therefore the arrangement is 23154, which is also bad. Thus there are only two bad arrangements up to rotation and reflection.

25. **Answer (C):** First note that once the frog is on pad 5, it has probability $\frac{1}{2}$ of eventually being eaten by the snake, and a probability $\frac{1}{2}$ of eventually exiting the pond without being eaten. It is therefore necessary only to determine the probability that the frog on pad 1 will reach pad 5 before being eaten.

Consider the frog's jumps in pairs. The frog on pad 1 will advance to pad 3 with probability $\frac{9}{10}\cdot\frac{8}{10}=\frac{72}{100}$, will be back at pad 1 with probability $\frac{9}{10}\cdot\frac{2}{10}=\frac{18}{100}$, and will retreat to pad 0 and be eaten with probability $\frac{1}{10}$. Because the frog will eventually make it to pad 3 or make it to pad 0, the probability that it ultimately makes it to pad 3 is $\frac{72}{100}\div\left(\frac{72}{100}+\frac{10}{100}\right)=\frac{36}{41}$, and the probability that it ultimately makes it to pad 0 is $\frac{10}{100}\div\left(\frac{72}{100}+\frac{10}{100}\right)=\frac{5}{41}$.

Similarly, in a pair of jumps the frog will advance from pad 3 to pad 5 with probability $\frac{7}{10}\cdot\frac{6}{10}=\frac{42}{100}$, will be back at pad 3 with probability $\frac{7}{10}\cdot\frac{4}{10}+\frac{3}{10}\cdot\frac{8}{10}=\frac{52}{100}$, and will retreat to pad 1 with probability $\frac{3}{10}\cdot\frac{2}{10}=\frac{6}{100}$. Because the frog will ultimately make it to pad 5 or pad 1 from pad 3, the probability that it ultimately makes it to pad 5 is $\frac{42}{100}\div\left(\frac{42}{100}+\frac{6}{100}\right)=\frac{7}{8}$, and the probability that it ultimately makes it to pad 1 is $\frac{6}{100}\div\left(\frac{42}{100}+\frac{6}{100}\right)=\frac{1}{8}$.

The sequences of pairs of moves by which the frog will advance to pad 5 without being eaten are

$$1\to3\to5,1\to3\to1\to3\to5,1\to3\to1\to3\to1\to3\to5,$$

and so on. The sum of the respective probabilities of reaching pad 5 is then

$$\begin{aligned}&\frac{36}{41}\cdot\frac{7}{8}+\frac{36}{41}\cdot\frac{1}{8}\cdot\frac{36}{41}\cdot\frac{7}{8}+\frac{36}{41}\cdot\frac{1}{8}\cdot\frac{36}{41}\cdot\frac{1}{8}\cdot\frac{36}{41}\cdot\frac{7}{8}+\cdots\\&=\frac{63}{82}\left(1+\frac{9}{82}+\left(\frac{9}{82}\right)^2+\cdots\right)\\&=\frac{63}{82}\div\left(1-\frac{9}{82}\right)\\&=\frac{63}{73}.\end{aligned}$$

Therefore the requested probability is $\frac{1}{2}\cdot\frac{63}{73}=\frac{63}{146}$.

OR

For $1\le j\le 5$, let p_j be the probability that the frog eventually reaches pad 10 starting at pad j. By symmetry $p_5=\frac{1}{2}$. For the frog to reach pad 10 starting from pad 4, the frog goes either to pad 3 with probability $\frac{2}{5}$ or to pad 5 with probability $\frac{3}{5}$, and then continues on a successful sequence from either of these pads. Thus $p_4 = \frac{2}{5}p_3 + \frac{3}{5}p_5 = \frac{2}{5}p_3 + \frac{3}{10}$. Similarly, to reach pad 10 starting from pad 3, the frog goes either to pad 2 with probability $\frac{3}{10}$ or to pad 4 with probability $\frac{7}{10}$. Thus $p_3 = \frac{3}{10}p_2 + \frac{7}{10}p_4$, and substituting from the previous equation for p_4 gives $p_3 = \frac{5}{12}p_2 + \frac{7}{24}$. In the same way, $p_2 = \frac{1}{5}p_1 + \frac{4}{5}p_3$ and after substituting for p_3 gives $p_2 = \frac{3}{10}p_1 + \frac{7}{20}$. Lastly, for the frog to escape starting from pad 1, it is necessary for it to get to pad 2 with probability $\frac{9}{10}$, and then escape starting from pad 2. Thus $p_1 = \frac{9}{10}p_2 = \frac{9}{10}(\frac{3}{10}p_1 + \frac{7}{20})$, and solving the equation gives $p_1 = \frac{63}{146}$.

Note: This type of random process is called a Markov process.

The problems and solutions in this contest were proposed by Steve Blasberg, Tom Butts, Peter Gilchrist, Jerry Grossman, Jon Kane, Joe Kennedy, Cap Khoury, Stuart Sidney, Kevin Wang, David Wells, LeRoy Wenstrom, and Ronald Yannone.

The American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors
Academy of Applied Sciences
American Institute of Mathematics
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Statistical Association
Art of Problem Solving
Association of Symbolic Logic
Awesome Math
Casualty Actuarial Society
Conference Board of the Mathematical Sciences
The D.E. Shaw Group
IDEA Math
Jane Street Capital
Math For America
Math Training Center
Mu Alpha Theta
Pi Mu Epsilon
Society for Industrial and Applied Math – SIAM
W.H. Freeman

MATHEMATICAL ASSOCIATION OF AMERICA
CELEBRATING A CENTURY OF ADVANCING MATHEMATICS

Solutions Pamphlet

American Mathematics Competitions

16th Annual

AMC 10 A

American Mathematics Contest 10A

Tuesday, February 3, 2015

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs conceptual, elementary vs advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
Phone 800.527.3690 | Fax 240.396.5647 | amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Silvia Fernandez

1. **Answer (C):**

$$(1-1+25+0)^{-1}\times 5=\frac{1}{25}\times 5=\frac{1}{5}$$

2. **Answer (D):** Counting 3 edges per tile gives a total of $3\cdot 25=75$ edges, and exactly 1 edge per square tile is missing. So there are exactly $84-75=9$ square tiles.

OR

Let x be the number of square tiles in the box. Then there are $25-x$ triangular tiles and $4x+3(25-x)=84$ edges. Solving for x gives $x=9$ square tiles.

3. **Answer (D):** Five vertical and five horizontal toothpicks must be added to complete the fourth step. Six vertical and six horizontal toothpicks must be added to complete the fifth step. This is a total of 22 toothpicks added.

4. **Answer (B):** Let m be the number of eggs that Mia has. Then Sofia has $2m$ eggs and Pablo has $6m$ eggs. If the total of $9m$ eggs is to be divided equally, each person will have $3m$ eggs. Therefore Pablo should give $2m$ eggs to Mia and m eggs to Sofia. The fraction of his eggs he should give to Sofia is $\frac{m}{6m}=\frac{1}{6}$.

5. **Answer (E):** The sum of the 14 test scores was $14\cdot 80=1120$. The sum of all 15 test scores was $15\cdot 81=1215$. Therefore Payton's score was $1215-1120=95$.

OR

To bring the average up to 81, the total must include 1 more point for each of the 14 students, in addition to 81 points for Payton. Therefore Payton's score was $81+14=95$.

6. **Answer (B):** Let x and y be the two positive numbers, with $x>y$. Then $x+y=5(x-y)$. Thus $4x=6y$, so $\frac{x}{y}=\frac{3}{2}$.

7. **Answer (B):** The difference between consecutive terms is 3, and the difference between the first and last terms is $73-13=60=20\cdot 3$. Therefore the number of terms is $20+1=21$.

Note: The kth term in the sequence is $3k+10$.

8. **Answer (B):** Let p be Pete's present age, and let c be Claire's age. Then $p-2=3(c-2)$ and $p-4=4(c-4)$. Solving these equations gives $p=20$ and $c=8$. Thus Pete is 12 years older than Claire, so the ratio of their ages will be $2:1$ when Claire is 12 years old. That will occur $12-8=4$ years from now.

9. **Answer (D):** Let r, h, R, H be the radii and heights of the first and second cylinders, respectively. The volumes are equal, so $\pi r^2 h = \pi R^2 H$. Also $R = r + 0.1r = 1.1r$. Thus $\pi r^2 h = \pi(1.1r)^2 H = \pi(1.21r^2)H$. Dividing by πr^2 yields $h = 1.21H = H + 0.21H$. Thus the first height is 21% more than the second height.

10. **Answer (C):** In the alphabet the letter b is adjacent to both a and c. So in any rearrangement, b can only be adjacent to d, and thus b must be the first or last letter in the rearrangement. Similarly, the letter c can only be adjacent to a, so c must be the first or last letter in the rearrangement. Thus the only two acceptable rearrangements are $bdac$ and $cadb$.

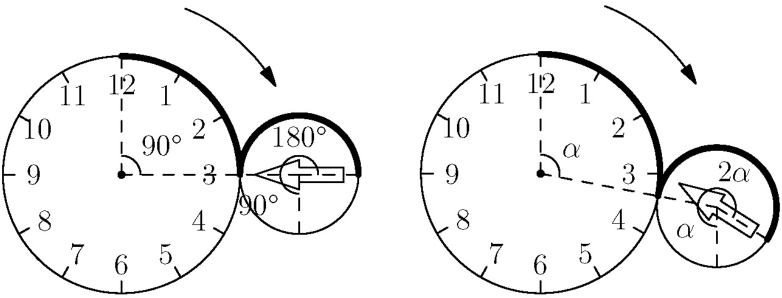
11. **Answer (C):** Let the sides of the rectangle have lengths $3a$ and $4a$. By the Pythagorean Theorem, the diagonal has length $5a$. Because $5a = d$, the side lengths are $\frac{3}{5}d$ and $\frac{4}{5}d$. Therefore the area is $\frac{3}{5}d \cdot \frac{4}{5}d = \frac{12}{25}d^2$, so $k = \frac{12}{25}$.

12. **Answer (C):** The equation is equivalent to $1 = y^2 - 2x^2y + x^4 = (y-x^2)^2$, or $y - x^2 = \pm 1$. The graph consists of two parabolas, $y = x^2 + 1$ and $y = x^2 - 1$. Thus a and b are $\pi + 1$ and $\pi - 1$, and their difference is 2. Indeed, the answer would still be 2 if $\sqrt{\pi}$ were replaced by any real number.

13. **Answer (C):** If Claudia only has 10-cent coins, then she can make 12 different values. Otherwise, suppose that the number of 10-cent coins is d and thus the number of 5-cent coins is $12-d$. Then she can make any value that is a multiple of 5 from 5 to $10d + 5(12-d) = 5(d+12)$. Therefore $d + 12 = 17$, and $d = 5$.

14. **Answer (C):** The circumference of the disk is half the circumference of the clock face. As the disk rolls $\frac{1}{4}$ of the way around the circumference of the clock face (from 12 o'clock to 3 o'clock), the disk rolls through $\frac{1}{2}$ of its own circumference. At that point, the arrow of the disk is pointing at the point of tangency, so the arrow on the disk will have turned $\frac{3}{4}$ of one revolution. In general, as the disk rolls through an angle α around the clock face, the arrow on the disk turns through an angle 3α on the disk. The arrow will again be

pointing in the upward vertical direction when the disk has turned through 1 complete revolution, and the angle traversed on the clock face is $\frac{1}{3}$ of the way around the face. The point of tangency will be at 4 o'clock.

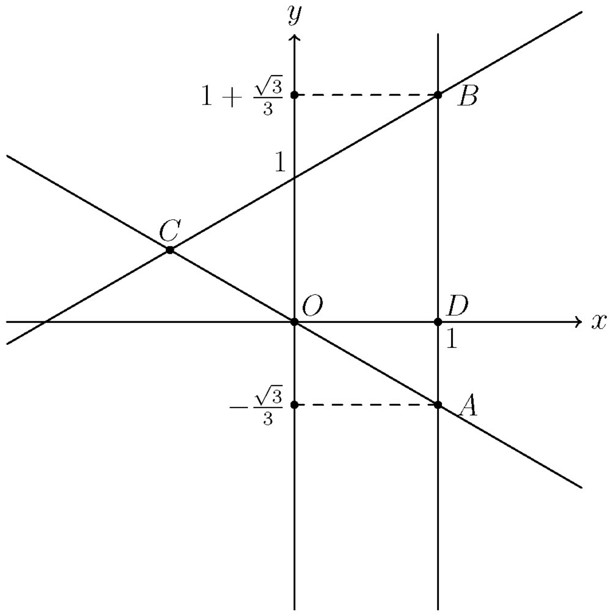


15. **Answer (B):** Because $\frac{x+1}{y+1} = \frac{11}{10} \cdot \frac{x}{y}$, it follows that $10y - 11x - xy = 0$ and so $(10 - x)(11 + y) = 110 = 2 \cdot 5 \cdot 11$. The only possible values of $10 - x$ are 5, 2, and 1 because x and y are positive integers. Thus the possible values of x are 5, 8, and 9. Of the resulting fractions $\frac{5}{11}$, $\frac{8}{44}$, and $\frac{9}{99}$, only the first is in simplest terms.

16. **Answer (B):** Expanding the binomials and subtracting the equations yields $x^2 - y^2 = 3(x - y)$. Because $x - y \neq 0$, it follows that $x + y = 3$. Adding the equations gives $x^2 + y^2 = 5(x + y) = 5 \cdot 3 = 15$.

    Note: The two solutions are $(x, y) = (\frac{3}{2} + \frac{\sqrt{21}}{2}, \frac{3}{2} - \frac{\sqrt{21}}{2})$ and $(\frac{3}{2} - \frac{\sqrt{21}}{2}, \frac{3}{2} + \frac{\sqrt{21}}{2})$.

17. **Answer (D):** Label the vertices of the equilateral triangle A, B, and C so that A is on the line $x = 1$ and B is on both lines $x = 1$ and $y = 1 + \frac{\sqrt{3}}{3}x$. Then $B = (1, 1 + \frac{\sqrt{3}}{3})$. Let O be the origin and $D = (1, 0)$. Because $\triangle ABC$ is equilateral, $\angle CAB = 60°$, and $\triangle OAD$ is a $30-60-90°$ triangle. Because $OD = 1$, $AD = \frac{\sqrt{3}}{3}$ and $AB = AD + DB = \frac{\sqrt{3}}{3} + (1 + \frac{\sqrt{3}}{3}) = 1 + \frac{2\sqrt{3}}{3}$. The perimeter of $\triangle ABC$ is $3 \cdot AB = 3 + 2\sqrt{3}$. Indeed, $\triangle ABC$ is equilateral with $C = (-\frac{\sqrt{3}}{2}, \frac{1}{2})$.



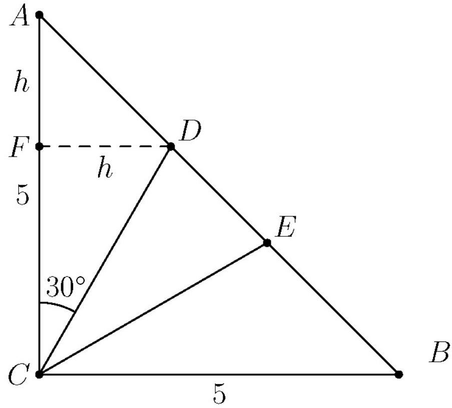
18. **Answer (E):** Because $1000 = 3 \cdot 16^2 + 14 \cdot 16 + 8$, the largest number less than 1000 whose hexadecimal representation contains only numeric digits is $3 \cdot 16^2 + 9 \cdot 16 + 9$. Thus the number of such positive integers is $n = 4 \cdot 10 \cdot 10 - 1 = 399$ ($0 \cdot 16^2 + 0 \cdot 16 + 0 = 0$ is excluded), and the sum of the digits of n is 21.

19. **Answer (D):** Because the area is 12.5, it follows that $AC = BC = 5$. Label D and E so that D is closer to A than to B. Let F be the foot of the perpendicular to $\overline{AC}$ passing through D. Let $h = FD$. Then $AF = h$ because $\triangle ADF$ is an isosceles right triangle, and $CF = h\sqrt{3}$ because $\triangle CDF$ is a $30-60-90^\circ$ triangle. So $h + h\sqrt{3} = AC = 5$ and

$$h = \frac{5}{1+\sqrt{3}} = \frac{5\sqrt{3}-5}{2}.$$

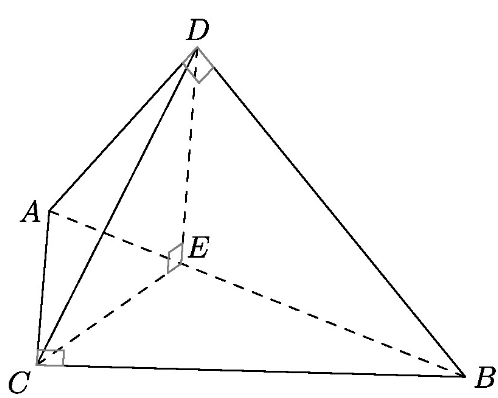
Thus the area of $\triangle CDE$ is

$$\frac{25}{2} - 2 \cdot \frac{1}{2} \cdot 5 \cdot \frac{5\sqrt{3}-5}{2} = \frac{50-25\sqrt{3}}{2}.$$



20. **Answer (B):** Let x and y be the lengths of the sides of the rectangle. Then $A + P = xy + 2x + 2y = (x + 2)(y + 2) - 4$, so $A + P + 4$ must be the product of two factors, each of which is greater than 2. Because the only factorization of $102 + 4 = 106$ into two factors greater than 1 is $2 \cdot 53$, $A + P$ cannot equal 102. Because $100 + 4 = 104 = 4 \cdot 26$, $104 + 4 = 108 = 3 \cdot 36$, $106 + 4 = 110 = 5 \cdot 22$, and $108 + 4 = 112 = 4 \cdot 28$, the other choices equal $A + P$ for rectangles with dimensions 2×24, 1×34, 3×20, and 2×26, respectively.

21. **Answer (C):** Triangles ABC and ABD are $3-4-5$ right triangles with area 6. Let $\overline{CE}$ be the altitude of $\triangle ABC$. Then $CE = \frac{12}{5}$. Likewise in $\triangle ABD$, $DE = \frac{12}{5}$. Triangle CDE has sides $\frac{12}{5}$, $\frac{12}{5}$, and $\frac{12}{5}\sqrt{2}$, so it is an isosceles right triangle with right angle CED. Therefore $\overline{DE}$ is the altitude of the tetrahedron to base ABC. The tetrahedron's volume is $\frac{1}{3} \cdot 6 \cdot \frac{12}{5} = \frac{24}{5}$.



22. **Answer (A):** There are $2^8 = 256$ equally likely outcomes of the coin tosses. Classify the possible arrangements around the table according to the number of heads flipped. There is 1 possibility with no heads, and there are 8 possibilities with exactly one head. There are $\binom{8}{2} = 28$ possibilities with exactly two heads, 8 of which have two adjacent heads. There are $\binom{8}{3} = 56$ possibilities with exactly three heads, of which 8 have three adjacent heads and $8 \cdot 4$ have exactly two adjacent heads (8 possibilities to place the two adjacent heads and 4 possibilities to place the third head). Finally, there are 2 possibilities using exactly four heads where no two of them are adjacent (heads and tails must alternate). There cannot be more than four heads without two of them being adjacent. Therefore there are $1 + 8 + (28 - 8) + (56 - 8 - 32) + 2 = 47$ possibilities with no adjacent heads, and the probability is $\frac{47}{256}$.

23. **Answer (C):** The zeros of f are integers and their sum is a, so a is an integer. If r is an integer zero, then $r^2 - ar + 2a = 0$ or

$$a = \frac{r^2}{r-2} = r + 2 + \frac{4}{r-2}.$$

So $\frac{4}{r-2} = a - r - 2$ must be an integer, and the possible values of r are 6, 4, 3, 1, 0, and -2. The possible values of a are 9, 8, 0, and -1, all of which yield integer zeros of f, and their sum is 16.

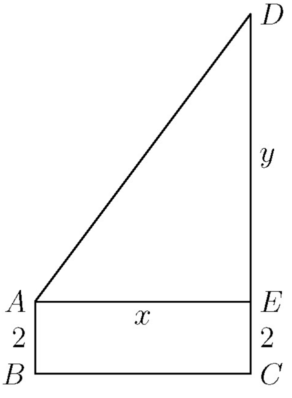
OR

As above, a must be an integer. The function f has zeros at

$$x = \frac{a \pm \sqrt{a^2 - 8a}}{2}.$$

These values are integers only if $a^2 - 8a = w^2$ for some integer w. Solving for a in terms of w gives $a = 4 \pm \sqrt{16 + w^2}$, so $16 + w^2$ must be a perfect square. The only integer solutions for w are 0 and ± 3, from which it follows that the values of a are 0, 8, 9, and -1, all of which yield integer values of x. The requested sum is 16.

24. **Answer (B):** In every such quadrilateral, $CD \geq AB$. Let E be the foot of the perpendicular from A to $\overline{CD}$; then $CE = 2$ and $AE = BC$. Let $x = AE$ and $y = DE$; then $AD = 2 + y$. By the Pythagorean Theorem, $x^2 + y^2 = (2+y)^2$, or $x^2 = 4 + 4y$. Therefore x is even, say $x = 2z$, and $z^2 = 1 + y$. The perimeter of the quadrilateral is $x + 2y + 6 = 2z^2 + 2z + 4$. Increasing positive integer values of z give the required quadrilaterals, with increasing perimeter. For $z = 31$ the perimeter is 1988, and for $z = 32$ the perimeter is 2116. Therefore there are 31 such quadrilaterals.

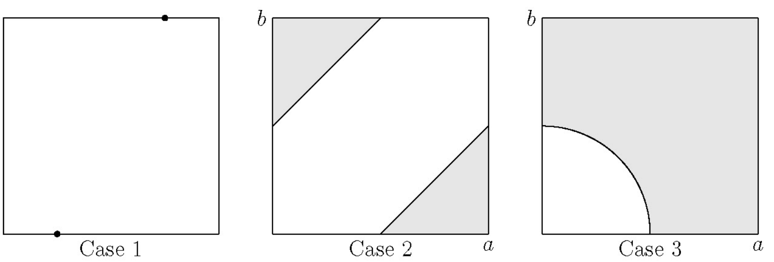


25. **Answer (A):** Let the square have vertices $(0,0)$, $(1,0)$, $(1,1)$, and $(0,1)$, and consider three cases.

Case 1: The chosen points are on opposite sides of the square. In this case the distance between the points is at least $\frac{1}{2}$ with probability 1.

Case 2: The chosen points are on the same side of the square. It may be assumed that the points are $(a,0)$ and $(b,0)$. The pairs of points in the ab-plane that meet the requirement are those within the square $0 \le a \le 1$, $0 \le b \le 1$ that satisfy either $b \ge a + \frac{1}{2}$ or $b \le a - \frac{1}{2}$. These inequalities describe the union of two isosceles right triangles with leg length $\frac{1}{2}$, together with their interiors. The area of the region is $\frac{1}{4}$, and the area of the square is 1, so the probability that the pair of points meets the requirement in this case is $\frac{1}{4}$.

Case 3: The chosen points are on adjacent sides of the square. It may be assumed that the points are $(a,0)$ and $(0,b)$. The pairs of points in the ab-plane that meet the requirement are those within the square $0 \le a \le 1$, $0 \le b \le 1$ that satisfy $\sqrt{a^2+b^2} \ge \frac{1}{2}$. These inequalities describe the region inside the square and outside a quarter-circle of radius $\frac{1}{2}$. The area of this region is $1-\frac{1}{4}\pi(\frac{1}{2})^2 = 1-\frac{\pi}{16}$, which is also the probability that the pair of points meets the requirement in this case.



Case 1 Case 2 Case 3

Cases 1 and 2 each occur with probability $\frac{1}{4}$, and Case 3 occurs with probability $\frac{1}{2}$. The requested probability is

$$\frac{1}{4}\cdot 1+\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\left(1-\frac{\pi}{16}\right)=\frac{26-\pi}{32},$$

and $a+b+c=59$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Steve Blasberg, Tom Butts, Steven Davis, Steve Dunbar, Silvia Fernandez, Charles Garner, Peter Gilchrist, Jerry Grossman, Jon Kane, Dan Kennedy, Joe Kennedy, Michael Khoury, Roger Waggoner, Dave Wells, Ronald Yannone, and Carl Yerger.

The

MAA American Mathematics Competitions

are supported by

Academy of Applied Science
Akamai Foundation
American Mathematical Society
American Statistical Association
Art of Problem Solving, Inc.
Casualty Actuarial Society
Conference Board of the Mathematical Sciences
The D.E. Shaw Group
Google
IDEA MATH, LLC
Jane Street Capital
Math for America Inc.
Mu Alpha Theta
National Council of Teachers of Mathematics
Simons Foundation
Society for Industrial & Applied Mathematics

MATHEMATICAL ASSOCIATION OF AMERICA
CELEBRATING A CENTURY OF ADVANCING MATHEMATICS

Solutions Pamphlet

American Mathematics Competitions

16th Annual

AMC 10 B

American Mathematics Contest 10B

Wednesday, February 25, 2015

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs conceptual, elementary vs advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
Phone 800.527.3690 | Fax 240.396.5647 | amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Silvia Fernandez

1. **Answer (C):**

$$2-(-2)^{-2}=2-\frac{1}{(-2)^2}=2-\frac{1}{4}=\frac{7}{4}$$

2. **Answer (B):** The first two tasks together took 100 minutes—from 1:00 to 2:40. Therefore each task took 50 minutes. Marie began the third task at 2:40 and finished 50 minutes later, at 3:30 PM.

3. **Answer (A):** Let x be the integer Isaac wrote two times, and let y be the integer Isaac wrote three times. Then $2x+3y=100$. If $x=28$, then $3y=100-2\cdot 28=44$, and y cannot be an integer. Therefore $y=28$ and $2x=100-3\cdot 28=16$, so $x=8$.

4. **Answer (C):** After the first three siblings ate, there was $1-\frac{1}{5}-\frac{1}{3}-\frac{1}{4}=\frac{13}{60}$ of the pizza left for Dan to eat, so Dan ate more than $\frac{1}{5}=\frac{12}{60}$ but less than $\frac{1}{4}=\frac{15}{60}$ of the pizza. Because $\frac{1}{3}>\frac{1}{4}>\frac{13}{60}>\frac{1}{5}$, the order is Beth, Cyril, Dan, Alex.

5. **Answer (B):** Marta finished 6th, so Jack finished 5th. Therefore Todd finished 3rd and Rand finished 2nd. Because Hikmet was 6 places behind Rand, it was Hikmet who finished 8th. (David finished 10th.)

6. **Answer (E):** Marley plays basketball on Monday and golf on Wednesday. Because she cannot run three of the four consecutive days between Thursday and Sunday, she must run on Tuesday. From Thursday to Sunday she runs, swims, and plays tennis, but she cannot play tennis the day after running or swimming. So she must play tennis on Thursday. She must swim on Saturday, and run on Friday and Sunday, so that she does not run on consecutive days.

7. **Answer (A):**

$$\begin{aligned}((1\diamond 2)\diamond 3)-(1\diamond(2\diamond 3)) &= \left(\left(1-\frac{1}{2}\right)-\frac{1}{3}\right)-\left(1-\left(2-\frac{1}{3}\right)^{-1}\right)\\ &= \frac{1}{6}-\left(1-\frac{3}{5}\right)=\frac{1}{6}-\frac{2}{5}=-\frac{7}{30}\end{aligned}$$

8. **Answer (E):** The first rotation results in Figure 1, the reflection in Figure 2, and the half turn in Figure 3.

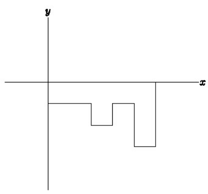


Figure 1

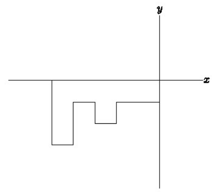


Figure 2

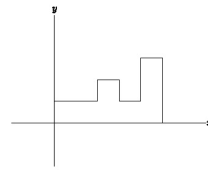


Figure 3

9. **Answer (B):** The shaded area is obtained by subtracting the area of the semicircle from the area of the quarter circle:

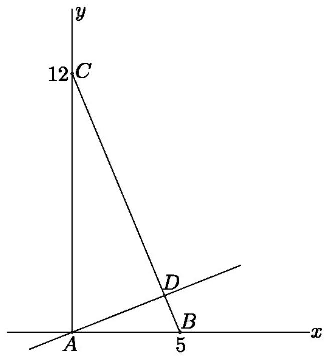
$$\frac{1}{4}\pi \cdot 3^2 - \frac{1}{2}\pi \left(\frac{3}{2}\right)^2 = \frac{9\pi}{4} - \frac{9\pi}{8} = \frac{9\pi}{8}.$$

10. **Answer (C):** There are 2014 negative integers strictly greater than -2015, and exactly half of them, or 1007, are odd. The product of an odd number of negative numbers is negative. Furthermore, because all factors are odd and some of them are multiples of 5, this product is an odd multiple of 5 and therefore has units digit 5.

11. **Answer (B):** There are four one-digit primes (2, 3, 5, and 7), which can be used to form $4^2 = 16$ two-digit numbers with prime digits. Of these two-digit numbers, only 23, 37, 53, and 73 are prime. So there are $4 + 16 = 20$ numbers less than 100 whose digits are prime, and $4 + 4 = 8$ of them are prime. The probability is $\frac{8}{20} = \frac{2}{5}$.

12. **Answer (A):** The circle intersects the line $y = -x$ at the points $A = (-5, 5)$ and $B = (5, -5)$. Segment $\overline{AB}$ is a chord of the circle and contains 11 points with integer coordinates.

13. **Answer (E):** Label the vertices of the triangle $A = (0,0)$, $B = (5,0)$, and $C = (0,12)$. By the Pythagorean Theorem $BC = 13$. Two altitudes are 5 and 12. Let $\overline{AD}$ be the third altitude. The area of this triangle is 30, so $\frac{1}{2} \cdot AD \cdot BC = 30$. Therefore $AD = \frac{2 \cdot 30}{BC} = \frac{60}{13}$. The sum of the lengths of the altitudes is $5 + 12 + \frac{60}{13} = \frac{281}{13}$.



14. **Answer (D):** If $(x-a)(x-b)+(x-b)(x-c) = 0$, then $(x-b)(2x-(a+c)) = 0$, so the two roots are b and $\frac{a+c}{2}$. The maximum value of their sum is $9+\frac{8+7}{2} = 16.5$.

15. **Answer (B):** Let h be the number of horses and c be the number of cows. There are then $3h$ people, $9h$ ducks, and $4c$ sheep in Hamlet. The total population of Hamlet is $13h + 5c$, where h and c are whole numbers. A number N can be the population only if there exists a whole number value for h such that $N - 13h$ is a whole number multiple of 5. This is possible for all the provided numbers except 47, as follows: $41-13\cdot 2 = 5\cdot 3$, $59-13\cdot 3 = 5\cdot 4$, $61-13\cdot 2 = 5\cdot 7$, and $66 - 13 \cdot 2 = 5 \cdot 8$

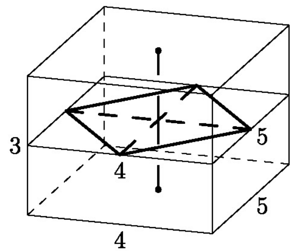
None of 47, $47 - 13 = 34$, $47 - 13 \cdot 2 = 21$, and $47 - 13 \cdot 3 = 8$ is a multiple of 5. Therefore 47 cannot be the population of Hamlet.

Note: In fact, 47 is the largest number that cannot be the population.

16. **Answer (C):** There are 9 assignments satisfying the condition: $(4,2,1)$, $(6,2,1)$, $(8,2,1)$, $(10,2,1)$, $(6,3,1)$, $(9,3,1)$, $(8,4,1)$, $(10,5,1)$, and $(8,4,2)$. There are $10 \cdot 9 \cdot 8 = 720$ possible assignments, so the probability is $\frac{9}{720} = \frac{1}{80}$.

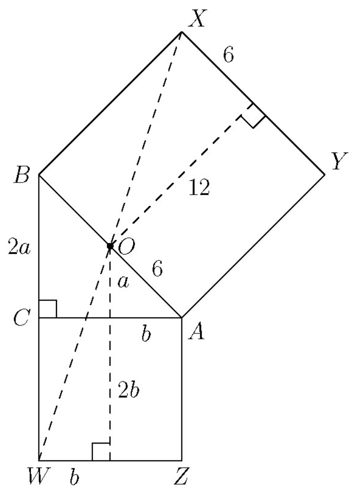
17. **Answer (B):** Consider the octahedron to be two pyramids whose base is a rhombus in the middle horizontal plane, as shown below. One pyramid points

up, the other down. The area of the base is the area of 4 right triangles with legs 2 and $\frac{5}{2}$, or 10. The altitude of each pyramid is half that of the prism or $\frac{3}{2}$. The volume of the octahedron is $2 \cdot \frac{1}{3} \cdot 10 \cdot \frac{3}{2} = 10$.

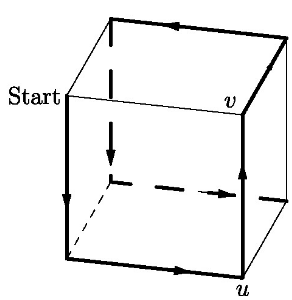


18. **Answer (D):** A coin can be tossed once, twice, or three times. View the problem as tossing each coin three times. If all three tosses are tails then the coin ends on a tail; however, if any of the three tosses is a head then the coin ends on a head (the subsequent tosses can be ignored). Thus each coin has a 7 out of 8 chance of landing on heads. Therefore the expected number of heads is $\frac{7}{8} \cdot 64 = 56$.

19. **Answer (C):** Let O be the center of the circle on which X, Y, Z, and W lie. Then O lies on the perpendicular bisectors of segments $\overline{XY}$ and $\overline{ZW}$, and $OX = OW$. Note that segments $\overline{XY}$ and $\overline{AB}$ have the same perpendicular bisector and segments $\overline{ZW}$ and $\overline{AC}$ have the same perpendicular bisector, from which it follows that O lies on the perpendicular bisectors of segments $\overline{AB}$ and $\overline{AC}$; that is, O is the circumcenter of $\triangle ABC$. Because $\angle C = 90°$, O is the midpoint of hypotenuse $\overline{AB}$. Let $a = \frac{1}{2}BC$ and $b = \frac{1}{2}CA$. Then $a^2 + b^2 = 6^2$ and $12^2 + 6^2 = OX^2 = OW^2 = b^2 + (a + 2b)^2$. Solving these two equations simultaneously gives $a = b = 3\sqrt{2}$. Thus the perimeter of $\triangle ABC$ is $12+2a+2b = 12 + 12\sqrt{2}$.



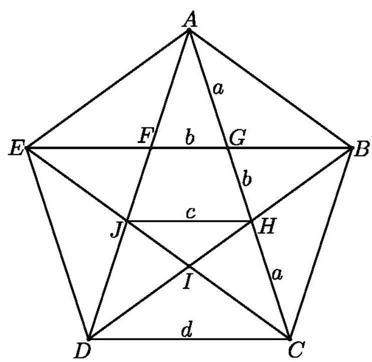
20. **Answer (A):** The first two edges of Erin's crawl can be chosen in $3 \cdot 2 = 6$ ways. These edges share a unique face of the cube, called the initial face. At this point, Erin is standing at a vertex u and there is only one unvisited vertex v of the initial face. If v is not visited right after u, then Erin visits all vertices adjacent to v before v. This means that once Erin reaches v, she cannot continue her crawl to any unvisited vertex, and v cannot be her last visited vertex because v is adjacent to her starting point. Thus v must be visited right after u. There are only two ways to visit the remaining four vertices (clockwise or counterclockwise around the face opposite to the initial face) and exactly one of them cannot be followed by a return to the starting vertex. Therefore there are exactly 6 paths in all.



21. **Answer (D):** Assume that there are t steps in this staircase and it took Dash $d+1$ jumps. Then the possible values of t are $5d+1, 5d+2, 5d+3, 5d+4, 5d+5$. On the other hand, it took Cozy $d+20$ jumps, and $t = 2d+39$ or $t = 2d+40$.

There are 10 possible combinations but only 3 of them lead to integer values of d: $t = 5d + 3 = 2d + 39$, or $t = 5d + 1 = 2d + 40$, or $t = 5d + 4 = 2d + 40$. The possible values of t are 63, 66, and 64, and $s = 63 + 66 + 64 = 193$. The answer is $1 + 9 + 3 = 13$.

22. **Answer (D):** Triangles AGB and CHJ are isosceles and congruent, so $AG = HC = HJ = 1$. Triangles AFG and BGH are congruent, so $FG = GH$. Triangles AGF, AHJ, and ACD are similar, so $\frac{a}{b} = \frac{a+b}{c} = \frac{2a+b}{d}$.
Because $a = c = 1$, the first equation becomes $\frac{1}{b} = \frac{1+b}{1}$ or $b^2 + b - 1 = 0$, so $b = \frac{-1+\sqrt{5}}{2}$. Substituting this in the second equation gives $d = \frac{1+\sqrt{5}}{2}$, so $b + c + d = 1 + \sqrt{5}$.



23. **Answer (B):** Because there are ample factors of 2, it is enough to count the number of factors of 5. Let $f(n)$ be the number of factors of 5 in positive integers less than or equal to n. For n from 5 to 9, $f(n) = 1$. In order for $f(2n)$ to equal 3, $2n$ must be between 15 and 19, inclusive. Therefore $n = 8$ or $n = 9$. For n from 10 to 14, $f(n) = 2$. In order for $f(2n)$ to equal 6, $2n$ must be between 25 and 29, inclusive. Hence, $n = 13$ or $n = 14$. Thus the four smallest integers n that satisfy the specified condition are 8, 9, 13, and 14. Their sum is 44 and the sum of the digits of 44 is 8.

OR

In fact there are only 4 possible values of n. By Legendre's Theorem, if $n!$ ends in k zeros and $(2n)!$ ends in k' zeros, then

$$k = \left\lfloor \frac{n}{5} \right\rfloor + \left\lfloor \frac{n}{5^2} \right\rfloor + \left\lfloor \frac{n}{5^3} \right\rfloor + \cdots + \left\lfloor \frac{n}{5^j} \right\rfloor,$$

$$k' = \left\lfloor \frac{2n}{5} \right\rfloor + \left\lfloor \frac{2n}{5^2} \right\rfloor + \left\lfloor \frac{2n}{5^3} \right\rfloor + \cdots + \left\lfloor \frac{2n}{5^j} \right\rfloor + \left\lfloor \frac{2n}{5^{j+1}} \right\rfloor,$$

where j is the highest power of 5 not exceeding n, and thus the highest power of 5 not exceeding $2n$ is at most $j+1$. If x is a real number, then $\lfloor 2x \rfloor \leq 2\lfloor x \rfloor + 1$. So $\left\lfloor \frac{2n}{5^i} \right\rfloor \leq 2\left\lfloor \frac{n}{5^i} \right\rfloor + 1$ for each $1 \leq i \leq j+1$. Adding these inequalities yields $k' \leq 2k + j + 1$. If $n \geq 15$, then $k > 2 + j - 1 = j + 1$ and $k' < 3k$. For $n = 13$ and $n = 14$, $k = 2$ and $k' = 5 + 1 = 6 = 3k$. For $n \leq 12$, $k = \left\lfloor \frac{n}{5} \right\rfloor$ and $k' = \left\lfloor \frac{2n}{5} \right\rfloor$; in this case $k' = 3k$ only for $n = 8$ and $n = 9$. So $s = 8 + 9 + 13 + 14 = 44$ and the answer is $4 + 4 = 8$.

24. **Answer (D):** Note that for any natural number k, when Aaron reaches point $(k, -k)$, he will have just completed visiting all of the grid points within the square with vertices at $(k, -k)$, (k, k), $(-k, k)$, and $(-k, -k)$. Thus the point $(k, -k)$ is equal to $p_{(2k+1)^2-1}$. It follows that $p_{2024} = p_{(2\cdot 22+1)^2-1} = (22, -22)$. Because $2024 - 2015 = 9$, the point $p_{2015} = (22 - 9, -22) = (13, -22)$.

25. **Answer (B):** Because the volume and surface area are numerically equal, $abc = 2(ab+ac+bc)$. Rewriting the equation as $ab(c-6)+ac(b-6)+bc(a-6) = 0$ shows that $a \leq 6$. The original equation can also be written as $(a-2)bc - 2ab - 2ac = 0$. Note that if $a = 2$, this becomes $b + c = 0$, and there are no solutions. Otherwise, multiplying both sides by $a - 2$ and adding $4a^2$ to both sides gives $[(a-2)b - 2a][(a-2)c - 2a] = 4a^2$. Consider the possible values of a.

$a = 1$: $(b+2)(c+2) = 4$
There are no solutions in positive integers.

$a = 3$: $(b-6)(c-6) = 36$
The 5 solutions for (b, c) are $(7, 42)$, $(8, 24)$, $(9, 18)$, $(10, 15)$, and $(12, 12)$.

$a = 4$: $(b-4)(c-4) = 16$
The 3 solutions for (b, c) are $(5, 20)$, $(6, 12)$, and $(8, 8)$.

$a = 5$: $(3b-10)(3c-10) = 100$
Each factor must be congruent to 2 modulo 3, so the possible pairs of factors are $(2, 50)$ and $(5, 20)$. The solutions for (b, c) are $(4, 20)$ and $(5, 10)$, but only $(5, 10)$ has $a \leq b$.

$a = 6$: $(b-3)(c-3) = 9$
The solutions for (b, c) are $(4, 12)$ and $(6, 6)$, but only $(6, 6)$ has $a \leq b$.

Thus in all there are 10 ordered triples (a, b, c): $(3, 7, 42)$, $(3, 8, 24)$, $(3, 9, 18)$, $(3, 10, 15)$, $(3, 12, 12)$, $(4, 5, 20)$, $(4, 6, 12)$, $(4, 8, 8)$, $(5, 5, 10)$, and $(6, 6, 6)$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Tom Butts, Steve Dunbar, Marta Eso, Zuming Feng, Silvia Fernandez, Charles Garner, Peter Gilchrist, Jerry Grossman, Joe Kennedy, Dan Kennedy, Cap Khoury, Steve Miller, David Wells, and Carl Yerger.

The

MAA American Mathematics Competitions

are supported by

Academy of Applied Science
Akamai Foundation
American Mathematical Society
American Statistical Association
Art of Problem Solving, Inc.
Casualty Actuarial Society
Conference Board of the Mathematical Sciences
The D.E. Shaw Group
Google
IDEA MATH, LLC
Jane Street Capital
Math for America Inc.
Mu Alpha Theta
National Council of Teachers of Mathematics
Simons Foundation
Society for Industrial & Applied Mathematics
Star League, Inc.

MATHEMATICAL ASSOCIATION OF AMERICA

CELEBRATING A CENTURY OF ADVANCING MATHEMATICS

Solutions Pamphlet

MAA American Mathematics Competitions

17th Annual

AMC 10A

American Mathematics Contest 10A

Tuesday, February 2, 2016

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
Phone 800.527.3690 | Fax 240.396.5647 | amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Silvia Fernandez.

1. **Answer (B):**

$$\frac{11! - 10!}{9!} = \frac{10! \cdot (11-1)}{9!} = \frac{10 \cdot 9! \cdot 10}{9!} = 100$$

2. **Answer (C):** The equation can be written $10^x \cdot (10^2)^{2x} = (10^3)^5$ or $10^x \cdot 10^{4x} = 10^{15}$. Thus $10^{5x} = 10^{15}$, so $5x = 15$ and $x = 3$.

3. **Answer (C):** Because $\$12.50 = 50 \cdot \$0.25$, Ben spent $\$50$. David spent $\$50 - \$12.50 = \$37.50$, and the two together paid $\$87.50$.

4. **Answer (B):**

$$\frac{3}{8} - \left(-\frac{2}{5}\right)\left\lfloor \frac{\frac{3}{8}}{-\frac{2}{5}} \right\rfloor = \frac{3}{8} + \frac{2}{5}\left\lfloor -\frac{15}{16} \right\rfloor = \frac{3}{8} + \frac{2}{5}(-1) = -\frac{1}{40}$$

5. **Answer (D):** Let the dimensions of the box be x, $3x$, and $4x$. Then the volume of the box is $12x^3$. Therefore the volume must be 12 times the cube of an integer. Among the choices, only $48 = 4 \cdot 12$, $96 = 8 \cdot 12$, and $144 = 12 \cdot 12$ are multiples of 12, and only for 96 is the other factor a perfect cube.

6. **Answer (D):** Each time Emilio replaces a 2 in the ones position by 1, Ximena's sum is decreased by 1. When Emilio replaces a 2 in the tens position by 1, Ximena's sum is decreased by 10. Ximena wrote 3 twos in the ones position $(2, 12, 22)$ and 10 twos in the tens position $(20, 21, 22, \ldots, 29)$. Thus Ximena's sum is greater than Emilio's sum by $3 \cdot 1 + 10 \cdot 10 = 103$.

7. **Answer (D):** The mean of the data values is

$$\frac{60 + 100 + x + 40 + 50 + 200 + 90}{7} = \frac{x + 540}{7} = x.$$

Solving this equation for x gives $x = 90$. Thus the data in nondecreasing order are $40, 50, 60, 90, 90, 100, 200$, so the median is 90 and the mode is 90, as required.

8. **Answer (C):** Working backwards, Fox must have approached the bridge for the third time with 20 coins in order to have no coins left after paying the

toll. In the second crossing he must have started with 30 coins in order to have $20 + 40 = 60$ before paying the toll. So he must have started with 35 coins in order to have $30 + 40 = 70$ before paying the toll for the first crossing.

OR

Let c be the number of coins Fox had at the beginning. After three crossings he had $2(2(2c - 40) - 40) - 40 = 8c - 280$ coins. Setting this equal to 0 and solving gives $c = 35$.

9. **Answer (D):** There are

$$1 + 2 + \cdots + N = \frac{N(N+1)}{2}$$

coins in the array. Therefore $N(N+1) = 2 \cdot 2016 = 4032$. Because $N(N+1) \approx N^2$, it follows that $N \approx \sqrt{4032} \approx \sqrt{2^{12}} = 2^6 = 64$. Indeed, $63 \cdot 64 = 4032$, so $N = 63$ and the sum of the digits of N is 9.

10. **Answer (B):** Let the inner rectangle's length be x feet; then its area is x square feet. The middle region has area $3(x + 2) - x = 2x + 6$, so the difference in the arithmetic sequence is equal to $(2x+6)-x = x+6$. The outer region has area $5(x + 4) - 3(x + 2) = 2x + 14$, so the difference in the arithmetic sequence is also equal to $(2x + 14) - (2x + 6) = 8$. From $x + 6 = 8$, it follows that $x = 2$. The regions have areas 2, 10, and 18.

11. **Answer (D):** The diagonal of the rectangle from upper left to lower right divides the shaded region into four triangles. Two of them have a 1-unit horizontal base and altitude $\frac{1}{2} \cdot 5 = 2\frac{1}{2}$, and the other two have a 1-unit vertical base and altitude $\frac{1}{2} \cdot 8 = 4$. Therefore the total area is $2 \cdot \frac{1}{2} \cdot 1 \cdot 2\frac{1}{2} + 2 \cdot \frac{1}{2} \cdot 1 \cdot 4 = 6\frac{1}{2}$.

12. **Answer (A):** The product of three integers is odd if and only if all three integers are odd. There are 1008 odd integers among the 2016 integers in the given range. The probability that all the selected integers are odd is

$$p = \frac{1008}{2016} \cdot \frac{1007}{2015} \cdot \frac{1006}{2014}.$$

The first factor is $\frac{1}{2}$ and each of the other factors is less than $\frac{1}{2}$, so $p < \frac{1}{8}$.

13. **Answer (B):** The total number of seats moved to the right among the five friends must equal the total number of seats moved to the left. One of Dee and

Edie moved some number of seats to the right, and the other moved the same number of seats to the left. Because Bea moved two seats to the right and Ceci moved one seat to the left, Ada must also move one seat to the left upon her return. Because her new seat is an end seat and its number cannot be 5, it must be seat 1. Therefore Ada occupied seat 2 before she got up. The order before moving was Bea-Ada-Ceci-Dee-Edie (or Bea-Ada-Ceci-Edie-Dee), and the order after moving was Ada-Ceci-Bea-Edie-Dee (or Ada-Ceci-Bea-Dee-Edie).

14. **Answer (C):** If the sum uses n twos and m threes, then $2n + 3m = 2016$. Therefore $n = \frac{2016-3m}{2}$. Both m and n will be nonnegative integers if and only if m is an even integer from 0 to 672. Thus there are $\frac{672}{2} + 1 = 337$ ways to form the sum.

15. **Answer (A):** The circle of dough has radius 3 inches. The area of the remaining dough is $3^2 \cdot \pi - 7\pi = 2\pi$ in^2. Let r be the radius in inches of the scrap cookie; then $2\pi = \pi r^2$. Therefore $r = \sqrt{2}$ inches.

16. **Answer (D):** After reflection about the x-axis, the coordinates of the image are $A'(0,-2)$, $B'(-3,-2)$, and $C'(-3,0)$. The counterclockwise 90°-rotation around the origin maps this triangle to the triangle with vertices $A''(2,0)$, $B''(2,-3)$, and $C''(0,-3)$. Notice that the final image can be mapped to the original triangle by interchanging the x- and y-coordinates, which corresponds to a reflection about the line $y = x$.

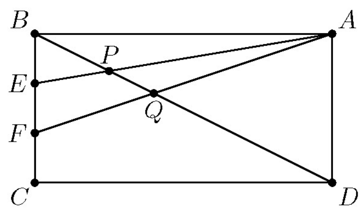
17. **Answer (A):** Let $N = 5k$, where k is a positive integer. There are $5k + 1$ equally likely possible positions for the red ball in the line of balls. Number these $0, 1, 2, 3, \ldots, 5k - 1, 5k$ from one end. The red ball will *not* divide the green balls so that at least $\frac{3}{5}$ of them are on the same side if it is in position $2k + 1, 2k + 2, \ldots, 3k - 1$. This includes $(3k - 1) - 2k = k - 1$ positions. The probability that $\frac{3}{5}$ or more of the green balls will be on the same side is therefore $1 - \frac{k-1}{5k+1} = \frac{4k+2}{5k+1}$.

    Solving the inequality $\frac{4k+2}{5k+1} < \frac{321}{400}$ for k yields $k > \frac{479}{5} = 95\frac{4}{5}$. The value of k corresponding to the required least value of N is therefore 96, so $N = 480$. The sum of the digits of N is 12.

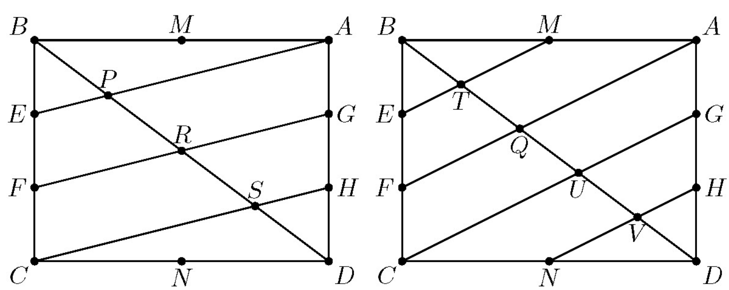
18. **Answer (C):** The sum of the four numbers on the vertices of each face must be $\frac{1}{6} \cdot 3 \cdot (1 + 2 + \cdots + 8) = 18$. The only sets of four of the numbers that include 1 and have a sum of 18 are $\{1, 2, 7, 8\}$, $\{1, 3, 6, 8\}$, $\{1, 4, 5, 8\}$, and $\{1, 4, 6, 7\}$. Three of these sets contain both 1 and 8. Because two specific vertices can

belong to at most two faces, the vertices of one face must be labeled with the numbers $1, 4, 6, 7$, and two of the faces must include vertices labeled 1 and 8. Thus 1 and 8 must mark two adjacent vertices. The cube can be rotated so that the vertex labeled 1 is at the lower left front, and the vertex labeled 8 is at the lower right front. The numbers 4, 6, and 7 must label vertices on the left face. There are $3! = 6$ ways to assign these three labels to the three remaining vertices of the left face. Then the numbers 5, 3, and 2 must label the vertices of the right face adjacent to the vertices labeled 4, 6, and 7, respectively. Hence there are 6 possible arrangements.

19. **Answer (E):** Triangles APD and EPB are similar and $BE : DA = 1 : 3$, so $BP = \frac{1}{4}BD$. Triangles AQD and FQB are similar and $BF : DA = 2 : 3$, so $BQ = \frac{2}{5}BD$ and $QD = \frac{3}{5}BD$. Then $PQ = BQ - BP = (\frac{2}{5} - \frac{1}{4})BD = \frac{3}{20}BD$. Thus $BP : PQ : QD = \frac{1}{4} : \frac{3}{20} : \frac{3}{5} = 5 : 3 : 12$, and $r + s + t = 5 + 3 + 12 = 20$.



Note: The answer is independent of the dimensions of the original rectangle. Consider the figures below, showing the rectangle $ABCD$ with points E and F trisecting side $\overline{BC}$. Let G and H trisect $\overline{AD}$, and let M and N be the midpoints of $\overline{AB}$ and $\overline{CD}$. Then the segments $\overline{AE}$, $\overline{GF}$, and $\overline{HC}$ are equally spaced, implying that $BP = PR = RS = SD$ and showing that $BP : PD : BD = 1 : 3 : 4 = 5 : 15 : 20$. The segments $\overline{ME}$, $\overline{AF}$, $\overline{GC}$, and $\overline{HN}$ are also equally spaced, implying that $BT = TQ = QU = UV = VD$ and showing that $BQ : QD : BD = 2 : 3 : 5 = 8 : 12 : 20$. It then follows that $BP : PQ : QD = 5 : (15 - 12) : 12 = 5 : 3 : 12$.



20. **Answer (B):** If a term contains all four variables a, b, c, and d, then it has the form $a^{i+1}b^{j+1}c^{k+1}d^{l+1}1^m$ for some nonnegative integers i, j, k, l, and m such that $(i+1)+(j+1)+(k+1)+(l+1)+m=N$ or $i+j+k+l+m=N-4$. The number of terms can be counted using the stars and bars technique. The number of linear arrangements of $N-4$ stars and 4 bars corresponds to the number of possible values of i, j, k, l, and m. Namely, in each arrangement the bars separate the stars into five groups (some of them can be empty) whose sizes are the values of i, j, k, l, and m. There are

$$\binom{N-4+4}{4}=\binom{N}{4}=\frac{N(N-1)(N-2)(N-3)}{4\cdot 3\cdot 2\cdot 1}=1001=7\cdot 11\cdot 13$$

such arrangements. So $N(N-1)(N-2)(N-3)=4\cdot 3\cdot 2\cdot 7\cdot 11\cdot 13=14\cdot 13\cdot 12\cdot 11$. Thus the answer is $N=14$.

21. **Answer (D):** Let X be the foot of the perpendicular from P to $\overline{QQ'}$, and let Y be the foot of the perpendicular from Q to $\overline{RR'}$. By the Pythagorean Theorem,

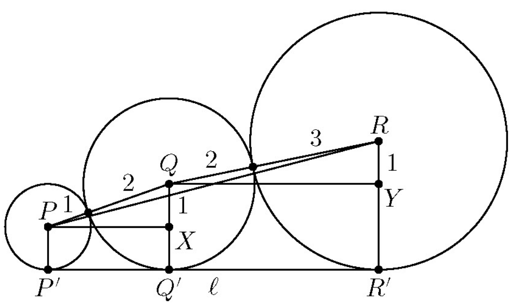
$$P'Q'=PX=\sqrt{(2+1)^2-(2-1)^2}=\sqrt{8}$$

and

$$Q'R'=QY=\sqrt{(3+2)^2-(3-2)^2}=\sqrt{24}.$$

The required area can be computed as the sum of the areas of the two smaller trapezoids, $PQQ'P'$ and $QRR'Q'$, minus the area of the large trapezoid, $PRR'P'$:

$$\frac{1+2}{2}\sqrt{8}+\frac{2+3}{2}\sqrt{24}-\frac{1+3}{2}\left(\sqrt{8}+\sqrt{24}\right)=\sqrt{6}-\sqrt{2}.$$



22. **Answer (D):** Let $110n^3=p_1^{r_1}p_2^{r_2}\cdots p_k^{r_k}$, where the p_j are distinct primes and the r_j are positive integers. Then $\tau(110n^3)$, the number of positive integer divisors of $110n^3$, is given by

$$\tau(110n^3)=(r_1+1)(r_2+1)\cdots(r_k+1)=110.$$

Because $110 = 2 \cdot 5 \cdot 11$, it follows that $k = 3$, $\{p_1, p_2, p_3\} = \{2, 5, 11\}$, and, without loss of generality, $r_1 = 1$, $r_2 = 4$, and $r_3 = 10$. Therefore

$$n^3 = \frac{p_1 \cdot p_2^4 \cdot p_3^{10}}{110} = p_2^3 \cdot p_3^9, \quad \text{so} \quad n = p_2 \cdot p_3^3.$$

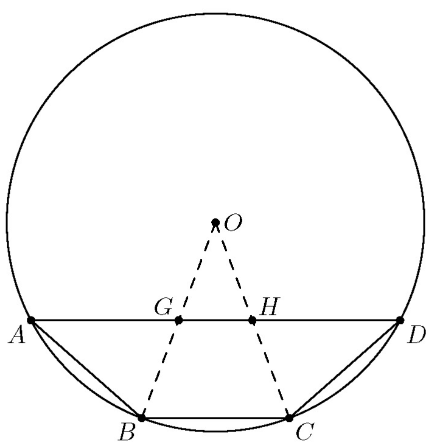
It follows that $81n^4 = 3^4 \cdot p_2^4 \cdot p_3^{12}$, and because 3, p_2, and p_3 are distinct primes, $\tau(81n^4) = 5 \cdot 5 \cdot 13 = 325$.

23. **Answer (A):** From the given properties, $a \diamondsuit 1 = a \diamondsuit (a \diamondsuit a) = (a \diamondsuit a) \cdot a = 1 \cdot a = a$ for all nonzero a. Then for nonzero a and b, $a = a \diamondsuit 1 = a \diamondsuit (b \diamondsuit b) = (a \diamondsuit b) \cdot b$. It follows that $a \diamondsuit b = \frac{a}{b}$. Thus

$$100 = 2016 \diamondsuit (6 \diamondsuit x) = 2016 \diamondsuit \frac{6}{x} = \frac{2016}{\frac{6}{x}} = 336x,$$

so $x = \frac{100}{336} = \frac{25}{84}$. The requested sum is $25 + 84 = 109$.

24. **Answer (E):** Let $ABCD$ be the given quadrilateral inscribed in the circle centered at O, with $AB = BC = CD = 200$, as shown in the figure. Because the chords $\overline{AB}$, $\overline{BC}$, and $\overline{CD}$ are shorter than the radius, each of $\angle AOB$, $\angle BOC$, and $\angle COD$ is less than 60°, so O is outside the quadrilateral $ABCD$. Let G and H be the intersections of $\overline{AD}$ with $\overline{OB}$ and $\overline{OC}$, respectively. Because $\overline{AD}$ and $\overline{BC}$ are parallel, and $\triangle OAB$ and $\triangle OBC$ are congruent and isosceles, it follows that $\angle ABO = \angle OBC = \angle OGH = \angle AGB$. Thus $\triangle ABG$, $\triangle OGH$, and $\triangle OBC$ are similar and isosceles with $\frac{AB}{BG} = \frac{OG}{GH} = \frac{OB}{BC} = \frac{200\sqrt{2}}{200} = \sqrt{2}$. Then $AG = AB = 200$, $BG = \frac{AB}{\sqrt{2}} = \frac{200}{\sqrt{2}} = 100\sqrt{2}$, and $GH = \frac{OG}{\sqrt{2}} = \frac{BO-BG}{\sqrt{2}} = \frac{200\sqrt{2}-100\sqrt{2}}{\sqrt{2}} = 100$. Therefore $AD = AG + GH + HD = 200 + 100 + 200 = 500$.



25. **Answer (A):** Because $\operatorname{lcm}(x, y) = 2^3 \cdot 3^2$ and $\operatorname{lcm}(x, z) = 2^3 \cdot 3 \cdot 5^2$, it follows that 5^2 divides z, but neither x nor y is divisible by 5. Furthermore, y is divisible by 3^2, and neither x nor z is divisible by 3^2, but at least one of x or z is divisible by 3. Finally, because $\operatorname{lcm}(y, z) = 2^2 \cdot 3^2 \cdot 5^2$, at least one of y or z is divisible by 2^2, but neither is divisible by 2^3. However, x must be divisible by 2^3. Thus $x = 2^3 \cdot 3^j$, $y = 2^k \cdot 3^2$, and $z = 2^m \cdot 3^n \cdot 5^2$, where $\max(j, n) = 1$ and $\max(k, m) = 2$. There are 3 choices for (j, n) and 5 choices for (k, m), so there are 15 possible ordered triples (x, y, z).

Problems and solutions were contributed by Sam Baethge, Tom Butts, Barb Currier, Steve Dunbar, Marta Eso, Jacek Fabrykowski, Silvia Fernandez, Chuck Garner, Peter Gilchrist, Jerry Grossman, Elgin Johnston, Joe Kennedy, Krassimir Penev, and David Wells.

The

MAA American Mathematics Competitions

are supported by

The Akamai Foundation
American Mathematical Society
American Statistical Association
Art of Problem Solving
Casualty Actuarial Society
Collaborator's Circle
Conference Board of the Mathematical Sciences
The D.E. Shaw Group
Expii
IDEA MATH
Jane Street Capital
MathWorks
Mu Alpha Theta
National Council of Teachers of Mathematics
Simons Foundation
Society for Industrial and Applied Mathematics
Star League
Tudor Investment Corporation
Two Sigma

MAA 100

MATHEMATICAL ASSOCIATION OF AMERICA

CELEBRATING A CENTURY OF ADVANCING MATHEMATICS

Solutions Pamphlet

MAA American Mathematics Competitions

17th Annual

AMC 10B

American Mathematics Contest 10B

Wednesday, February 17, 2016

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
Phone 800.527.3690 | Fax 240.396.5647 | amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Silvia Fernandez.

1. **Answer (D):**

$$\frac{2(\frac{1}{2})^{-1}+\frac{(\frac{1}{2})^{-1}}{2}}{\frac{1}{2}}=\left(2\cdot 2+\frac{2}{2}\right)\cdot 2=10$$

2. **Answer (B):**

$$\frac{2\heartsuit 4}{4\heartsuit 2}=\frac{2^3\cdot 4^2}{4^3\cdot 2^2}=\frac{2}{4}=\frac{1}{2}$$

3. **Answer (D):**

$$\begin{aligned}&\Bigg|\Big||-2016|-(-2016)\Big|-|-2016|\Bigg|-(-2016)\\&\qquad=\Bigg|\Big|2016+2016\Big|-2016\Bigg|+2016=2016+2016=4032\end{aligned}$$

4. **Answer (B):** It took Zoey $1+2+3+\cdots+15=\frac{15\cdot 16}{2}=120$ days to read the 15 books. Because $120=7\cdot 17+1$, it follows that Zoey finished the 15th book on the same day of the week as the first, a Monday.

5. **Answer (D):** Because the mean is 8, it follows that the sum of the ages of all Amanda's cousins is $8\cdot 4=32$. Because the median age is 5, the sum of the two middle ages is $5\cdot 2=10$. Then the sum of the ages of Amanda's youngest and oldest cousins is $32-10=22$.

6. **Answer (B):** Because S has to be greater than 300, the digit sum has to be at least 4, and an example like $197+203=400$ shows that 4 is indeed the smallest possible value.

7. **Answer (C):** Let α and β be the measures of the angles, with $\alpha<\beta$. Then $\frac{\beta}{\alpha}=\frac{5}{4}$. Because $\alpha<\beta$, it follows that $90^\circ-\beta<90^\circ-\alpha$, so $90^\circ-\alpha=2(90^\circ-\beta)$. This leads to the system of linear equations $4\beta-5\alpha=0$ and $2\beta-\alpha=90^\circ$. Solving the system gives $\alpha=60^\circ$, $\beta=75^\circ$. The requested sum is $\alpha+\beta=135^\circ$.

8. **Answer (A):** Positive even powers of numbers ending in 5 end in 25. The tens digit of the difference is the tens digit of $25-17=08$, or 0.

9. **Answer (C):** Let the vertex of the triangle that lies in the first quadrant be (x, x^2). Then the base of the triangle is $2x$ and the height is x^2, so $\frac{1}{2} \cdot 2x \cdot x^2 = 64$. Thus $x^3 = 64$, $x = 4$, and $BC = 2x = 8$.

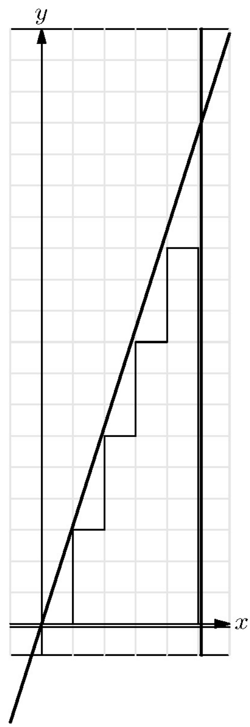
10. **Answer (D):** The weight of an object of uniform density is proportional to its volume. The volume of the triangular piece of wood of uniform thickness is proportional to the area of the triangle. The side length of the second piece is $\frac{5}{3}$ times the side length of the first piece, so the area of the second piece is $\left(\frac{5}{3}\right)^2$ times the area of the first piece. Therefore the weight is $12 \cdot \left(\frac{5}{3}\right)^2 = \frac{100}{3} \approx 33.3$ ounces.

11. **Answer (B):** Let x be the number of posts along the shorter side; then there are $2x$ posts along the longer side. When counting the number of posts on all the sides of the garden, each corner post is counted twice, so $2x + 2(2x) = 20 + 4$. Solving this equation gives $x = 4$. Thus the dimensions of the rectangle are $(4-1) \cdot 4 = 12$ yards by $(8-1) \cdot 4 = 28$ yards. The requested area is given by the product of these dimensions, $12 \cdot 28 = 336$ square yards.

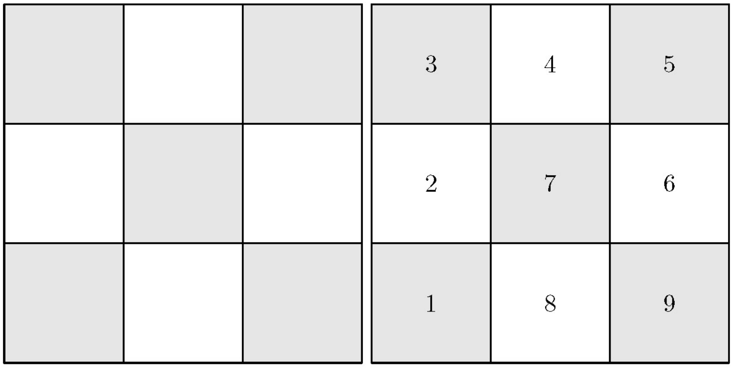
12. **Answer (D):** The product of two integers is odd if and only if both integers are odd. Thus the probability that the product is odd is $\frac{3}{5} \cdot \frac{2}{4} = 0.3$, and the probability that the product is even is $1 - 0.3 = 0.7$.

13. **Answer (D):** Let x denote the number of sets of quadruplets. Then $1000 = 4 \cdot x + 3 \cdot (4x) + 2 \cdot (3 \cdot 4x) = 40x$. Thus $x = 25$, and the number of babies in sets of quadruplets is $4 \cdot 25 = 100$.

14. **Answer (D):** Note that $3 < \pi < 4$, $6 < 2\pi < 7$, $9 < 3\pi < 10$, and $12 < 4\pi < 13$. Therefore there are 3 1-by-1 squares of the desired type in the strip $1 \le x \le 2$, 6 1-by-1 squares in the strip $2 \le x \le 3$, 9 1-by-1 squares in the strip $3 \le x \le 4$, and 12 1-by-1 squares in the strip $4 \le x \le 5$. Furthermore there are 2 2-by-2 squares in the strip $1 \le x \le 3$, 5 2-by-2 squares in the strip $2 \le x \le 4$, and 8 2-by-2 squares in the strip $3 \le x \le 5$. There is 1 3-by-3 square in the strip $1 \le x \le 4$, and there are 4 3-by-3 squares in the strip $2 \le x \le 5$. There are no 4-by-4 or larger squares. Thus in all there are $3 + 6 + 9 + 12 + 2 + 5 + 8 + 1 + 4 = 50$ squares of the desired type within the given region.



15. **Answer (C):** Shade the squares in a checkerboard pattern as shown in the first figure. Because consecutive numbers must be in adjacent squares, the shaded squares will contain either five odd numbers or five even numbers. Because there are only four even numbers available, the shaded squares contain the five odd numbers. Thus the sum of the numbers in all five shaded squares is $1+3+5+7+9=25$. Because all but the center add up to $18=25-7$, the center number must be 7. The situation described is actually possible, as the second figure demonstrates.



16. **Answer (E):** Let r be the common ratio of the geometric series; then

$$S = \frac{1}{r} + 1 + r + r^2 + \cdots = \frac{\frac{1}{r}}{1-r} = \frac{1}{r-r^2}.$$

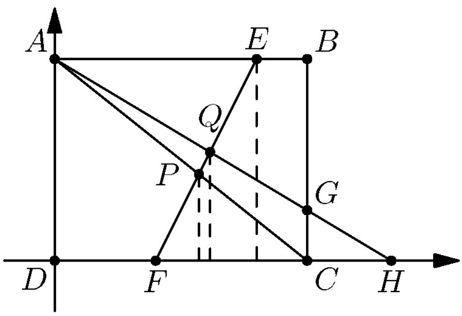
Because $S > 0$, the smallest value of S occurs when the value of $r - r^2$ is maximized. The graph of $f(r) = r - r^2$ is a downward-opening parabola with vertex $(\frac{1}{2}, \frac{1}{4})$, so the smallest possible value of S is $\frac{1}{(\frac{1}{4})} = 4$. The optimal series is $2, 1, \frac{1}{2}, \frac{1}{4}, \ldots$.

17. **Answer (D):** Suppose that one pair of opposite faces of the cube are assigned the numbers a and b, a second pair of opposite faces are assigned the numbers c and d, and the remaining pair of opposite faces are assigned the numbers e and f. Then the needed sum of products is $ace + acf + ade + adf + bce + bcf + bde + bdf = (a+b)(c+d)(e+f)$. The sum of these three factors is $2+3+4+5+6+7 = 27$. A product of positive numbers whose sum is fixed is maximized when the factors are all equal. Thus the greatest possible value occurs when $a+b = c+d = e+f = 9$, as in $(a,b,c,d,e,f) = (2,7,3,6,4,5)$. This results in the value $9^3 = 729$.

18. **Answer (E):** A sum of consecutive integers is equal to the number of integers in the sum multiplied by their median. Note that $345 = 3 \cdot 5 \cdot 23$. If there are an odd number of integers in the sum, then the median and the number of integers must be complementary factors of 345. The only possibilities are 3 integers with median $5 \cdot 23 = 115$, 5 integers with median $3 \cdot 23 = 69$, $3 \cdot 5 = 15$ integers with median 23, and 23 integers with median $3 \cdot 5 = 15$. Having more integers in the sum would force some of the integers to be negative. If there are an even number of integers in the sum, say $2k$, then the median will be $\frac{j}{2}$,

where k and j are complementary factors of 345. The possibilities are 2 integers with median $\frac{345}{2}$, 6 integers with median $\frac{115}{2}$, and 10 integers with median $\frac{69}{2}$. Again, having more integers in the sum would force some of the integers to be negative. This gives a total of 7 solutions.

19. **Answer (D):** Triangles AEP and CFP are similar and $FP : EP = CF : AE = 3 : 4$, so $FP = \frac{3}{7}EF$. Extend $\overline{AG}$ and $\overline{FC}$ to meet at point H; then $\triangle AEQ$ and $\triangle HFQ$ are similar. Note that $\triangle HCG$ and $\triangle ABG$ are similar with sides in a ratio of $1 : 3$, so $CH = \frac{1}{3} \cdot 5$ and $FH = 3 + \frac{5}{3} = \frac{14}{3}$. Then $FQ : EQ = \frac{14}{3} : 4 = 7 : 6$, so $FQ = \frac{7}{13}FE$. Thus $PQ = FQ - FP = \left(\frac{7}{13} - \frac{3}{7}\right)FE = \frac{10}{91}FE$ and $\frac{PQ}{FE} = \frac{10}{91}$.



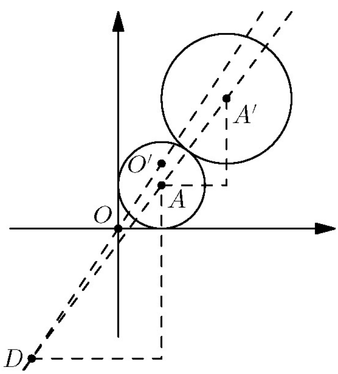
OR

Place the figure in the coordinate plane with D at the origin, A at $(0, 4)$, and C at $(5, 0)$. Then the equations of lines AC, AG, and EF are $y = -\frac{4}{5}x + 4$, $y = -\frac{3}{5}x + 4$, and $y = 2x - 4$, respectively. The intersections can be found by solving simultaneous linear equations: $P\left(\frac{20}{7}, \frac{12}{7}\right)$ and $Q\left(\frac{40}{13}, \frac{28}{13}\right)$. Because F, P, Q, and E are aligned, ratios of distances between these points are the same as ratios of the corresponding distances between their coordinates. Then

$$\frac{PQ}{FE} = \frac{\frac{40}{13} - \frac{20}{7}}{4 - 2} = \frac{10}{91}.$$

20. **Answer (C):** The scale factor for this transformation is $\frac{3}{2}$. The center of the dilation, D, must lie along ray $A'A$ (with A between A' and D), and its distance from A must be $\frac{2}{3}$ of its distance from A'. Because A is 3 units to the left of and 4 units below A', the center of the dilation must be 6 units to the left of and 8 units below A, placing it at $D(-4, -6)$. The origin is $\sqrt{(-4)^2 + (-6)^2} = 2\sqrt{13}$ units

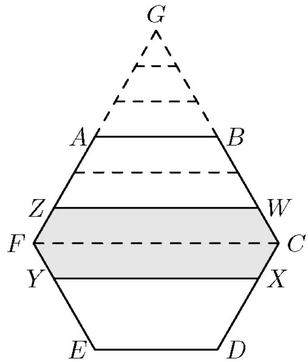
from D, so the dilation must move it half that far, or $\sqrt{13}$ units. Alternatively, note that the origin is 4 units to the right of and 6 units above D, so its image must be 6 units to the right of and 9 units above D; therefore it is located at $(2, 3)$, a distance $\sqrt{2^2 + 3^2} = \sqrt{13}$ from the origin.



21. **Answer (B):** The graph of the equation is symmetric about both axes. In the first quadrant, the equation is equivalent to $x^2 + y^2 - x - y = 0$. Completing the square gives $(x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 = \frac{1}{2}$, so the graph in the first quadrant is an arc of the circle that is centered at $C(\frac{1}{2}, \frac{1}{2})$ and contains the points $A(1, 0)$ and $B(0, 1)$. Because C is the midpoint of $\overline{AB}$, the arc is a semicircle. The region enclosed by the graph in the first quadrant is the union of isosceles right triangle AOB, where $O(0, 0)$ is the origin, and a semicircle with diameter $\overline{AB}$. The triangle and the semicircle have areas $\frac{1}{2}$ and $\frac{1}{2} \cdot \pi \left(\frac{\sqrt{2}}{2}\right)^2 = \frac{\pi}{4}$, respectively, so the area of the region enclosed by the graph in all quadrants is $4(\frac{1}{2} + \frac{\pi}{4}) = \pi + 2$.

22. **Answer (A):** There must have been $10 + 10 + 1 = 21$ teams, and therefore there were $\binom{21}{3} = \frac{21 \cdot 20 \cdot 19}{6} = 1330$ subsets $\{A, B, C\}$ of three teams. If such a subset does not satisfy the stated condition, then it consists of a team that beat both of the others. To count such subsets, note that there are 21 choices for the winning team and $\binom{10}{2} = 45$ choices for the other two teams in the subset. This gives $21 \cdot 45 = 945$ such subsets. The required answer is $1330 - 945 = 385$. To see that such a scenario is possible, arrange the teams in a circle, and let each team beat the 10 teams that follow it in clockwise order around the circle.

23. **Answer (C):** Extend sides $\overline{CB}$ and $\overline{FA}$ to meet at G. Note that $FC = 2AB$ and $ZW = \frac{5}{3}AB$. Then the areas of $\triangle BAG$, $\triangle WZG$, and $\triangle CFG$ are in the ratio $1^2 : (\frac{5}{3})^2 : 2^2 = 9 : 25 : 36$. Thus $\frac{[ZWCF]}{[ABCF]} = \frac{36-25}{36-9} = \frac{11}{27}$, and by symmetry, $\frac{[WCXYFZ]}{[ABCDEF]} = \frac{11}{27}$ also.



OR

Suppose that $AB = 1$; then $FZ = \frac{1}{3}$ and $FC = 2$. Trapezoid $WCFZ$, which is the upper half of hexagon $WCXYFZ$, can be tiled by 11 equilateral triangles of side length $\frac{1}{3}$, and the lower half similarly, making 22 such triangles. Hexagon $ABCDEF$ can be tiled by 6 equilateral triangles of side length 1, and each of these can be tiled by 9 equilateral triangles of side length $\frac{1}{3}$, making a total of $6 \cdot 9 = 54$ small triangles. The required ratio is $\frac{22}{54} = \frac{11}{27}$.

24. **Answer (D):** Let k be the common difference for the arithmetic sequence. If $b = c$ or $c = d$, then $k = bc - ab = cd - bc$ must be a multiple of 10, so $b = c = d$. However, the two-digit integers bc and cd are then equal, a contradiction. Therefore either (b, c, d) or $(b, c, d+10)$ is an increasing arithmetic sequence.

Case 1: (b, c, d) is an increasing arithmetic sequence. In this case the additions of k to ab and bc do not involve any carries, so (a, b, c) also forms an increasing arithmetic sequence, as does (a, b, c, d). Let $n = b - a$. If $n = 1$, the possible values of a are 1, 2, 3, 4, 5, and 6. If $n = 2$, the possible values of a are 1, 2, and 3. There are no possibilities with $n \geq 3$. Thus in this case there are 9 integers that have the required property: 1234, 2345, 3456, 4567, 5678, 6789, 1357, 2468, and 3579.

Case 2: $(b, c, d + 10)$ is an increasing arithmetic sequence. In this case the addition of k to bc involves a carry, so $(a, b, c - 1)$ forms a nondecreasing arithmetic sequence, as does $(b, c-1, (d+10)-2) = (b, c-1, d+8)$. Hence $(a, b, c - 1, d + 8)$ is a nondecreasing arithmetic sequence. Again letting $n = b - a$, note that $0 \leq c = d + (9 - n) \leq 9$ and $1 \leq a = d + (8 - 3n) \leq 9$. The only integers with the required properties are 8890 with $n = 0$; 5680 and 6791 with $n = 1$; 2470, 3581, and 4692 with $n = 2$; and 1482 and 2593 with $n = 3$. Thus in this case there are 8 integers that have the required property.

The total number of integers with the required property is $9 + 8 = 17$.

25. **Answer (A):** Note that for any x, $f(x+1) = \sum_{k=2}^{10}(\lfloor kx + k \rfloor - k\lfloor x + 1 \rfloor) = \sum_{k=2}^{10}(\lfloor kx \rfloor + k - k\lfloor x \rfloor - k) = f(x)$. This implies that $f(x)$ is periodic with period 1. Thus the number of distinct values that $f(x)$ assumes is the same as the number of distinct values that $f(x)$ assumes for $0 \le x < 1$. For these x, $\lfloor x \rfloor = 0$, so $f(x) = \sum_{k=2}^{10}\lfloor kx \rfloor$, which is a nondecreasing function of x. This function increases at exactly those values of x expressible as a fraction of positive integers with denominator between 2 and 10. There are 31 such values between 0 and 1. They are $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{1}{6}, \frac{5}{6}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}, \frac{1}{8}, \frac{3}{8}, \frac{5}{8}, \frac{7}{8}, \frac{1}{9}, \frac{2}{9}, \frac{4}{9}, \frac{5}{9}, \frac{7}{9}, \frac{8}{9}, \frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}$. Thus $f(0) = 0$ and $f(x)$ increases 31 times for x between 0 and 1, showing that $f(x)$ assumes 32 distinct values.

Problems and solutions were contributed by Sam Baethge, Tom Butts, Barb Currier, Marta Eso, Silvia Fernandez, Chuck Garner, Jerry Grossman, Michael Khoury, Hugh Montgomery, Harold Reiter, and David Wells.

The

MAA American Mathematics Competitions

are supported by

The Akamai Foundation
American Mathematical Society
American Statistical Association
Art of Problem Solving
Casualty Actuarial Society
Collaborator's Circle
Conference Board of the Mathematical Sciences
The D.E. Shaw Group
Expii
IDEA MATH
Jane Street Capital
MathWorks
Mu Alpha Theta
National Council of Teachers of Mathematics
Simons Foundation
Society for Industrial and Applied Mathematics
Star League
Tudor Investment Corporation
Two Sigma

MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

18th Annual

AMC 10A

American Mathematics Competition 10A

Tuesday, February 7, 2017

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
Phone 800.527.3690 | Fax 240.396.5647 | amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by
MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the
co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (C):**

$$\begin{aligned}(2(2(2(2(2(2+1)+1)+1)+1)+1)+1)\\ &= (2(2(2(2(2(3)+1)+1)+1)+1)+1)\\ &= (2(2(2(2(7)+1)+1)+1)+1)\\ &= (2(2(2(15)+1)+1)+1)\\ &= (2(2(31)+1)+1)\\ &= (2(63)+1)\\ &= 127\end{aligned}$$

   Observe that each intermediate result is 1 less than a power of 2.

2. **Answer (D):** The cheapest popsicles cost $\$3.00 \div 5 = \$0.60$ each. Because $14 \cdot \$0.60 = \$8.40$ and Pablo has just \$8, he could not pay for 14 popsicles even if he were allowed to buy partial boxes. The best he can hope for is 13 popsicles, and he can achieve that by buying two 5-popsicle boxes (for \$6) and one 3-popsicle box (for \$2).

   OR

   If Pablo buys two single popsicles for \$1 each, he could have bought a 3-popsicle box for the same amount of money. Similarly, if Pablo buys three single popsicles or both one 3-popsicle box and one single popsicle, he could have bought a 5-popsicle box for the same amount of money. If Pablo buys two 3-popsicle boxes, he could have bought a 5-popsicle box and a single popsicle for the same amount of money. The previous statements imply that a maximum number of popsicles for a given amount of money can be obtained by buying either at most one single popsicle and the rest 5-popsicle boxes, or a single 3-popsicle box and the rest 5-popsicle boxes. When Pablo has \$8, he can obtain the maximum number of popsicles by buying two 5-popsicle boxes and one 3-popsicle box. This gives a total of $2 \cdot 5 + 1 \cdot 3 = 13$ popsicles.

3. **Answer (B):** The area of the garden is $15 \cdot 10 = 150$ square feet, and the combined area of the six flower beds is $6 \cdot 6 \cdot 2 = 72$ square feet. Therefore the area of the walkways is $150 - 72 = 78$ square feet.

4. **Answer (B):** After exactly half a minute there will be 3 toys in the box and 27 toys outside the box. During the next half-minute, Mia takes 2 toys out and her mom puts 3 toys into the box. This

means that during this half-minute the number of toys in the box was increased by 1. The same argument applies to each of the following half-minutes until all the toys are in the box for the first time. Therefore it takes $1 + 27 \cdot 1 = 28$ half-minutes, which is 14 minutes, to complete the task.

5. **Answer (C):** Let the two numbers be x and y. Then $x + y = 4xy$. Dividing this equation by xy gives $\frac{1}{y} + \frac{1}{x} = 4$. One such pair of numbers is $x = \frac{1}{3}$, $y = 1$.

6. **Answer (B):** The given statement is logically equivalent to its contrapositive: If a student did not receive an A on the exam, then the student did not get all the multiple choice questions right, which means that he got at least one of them wrong. None of the other statements follows logically from the given implication; the teacher made no promises concerning students who did not get all the multiple choice questions right. In particular, a statement does not imply its inverse or its converse; and the negation of the statement that Lewis got all the questions right is not the statement that he got all the questions wrong.

7. **Answer (A):** If the square had side length x, then Jerry's path had length $2x$, and Silvia's path along the diagonal, by the Pythagorean Theorem, had length $\sqrt{2}x$. Therefore Silvia's trip was shorter by $2x - \sqrt{2}x$, and the required percentage is
$$\frac{2x - \sqrt{2}x}{2x} = 1 - \frac{\sqrt{2}}{2} \approx 1 - 0.707 = 0.293 = 29.3\%.$$
The closest of the answer choices is 30%.

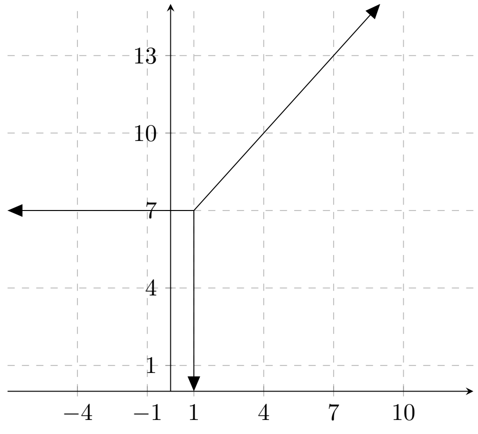
8. **Answer (B):** Each of the 20 people who know each other shakes hands with 10 people. Each of the 10 people who know no one shakes hands with 29 people. Because each handshake involves two people, the number of handshakes is $\frac{1}{2}(20 \cdot 10 + 10 \cdot 29) = 245$.

9. **Answer (C):** Note that Penny is going downhill on the segment on which Minnie is going uphill, and vice versa. Minnie needs $\frac{10}{5}$ hours to go from A to B, $\frac{15}{30}$ hours to go from B to C, and $\frac{20}{20}$ hours to go from C to A, a total of $3\frac{1}{2}$ hours. Penny's time is $\frac{20}{30} + \frac{15}{10} + \frac{10}{40} = 2\frac{5}{12}$ hours. It takes Minnie $3\frac{1}{2} - 2\frac{5}{12} = 1\frac{1}{12}$ hours, which is 65 minutes, longer.

10. **Answer (B):** Four rods can form a quadrilateral with positive area if and only if the length of the longest rod is less than the sum of the lengths of the other three. Therefore if the fourth rod has length n cm, then n must satisfy the inequalities $15 < 3+7+n$ and $n < 3+7+15$, that is, $5 < n < 25$. Because n is an integer, it must be one of the 19 integers from 6 to 24, inclusive. However, the rods of lengths 7 cm and 15 cm have already been chosen, so the number of rods that Joy can choose is $19 - 2 = 17$.

11. **Answer (D):** Let $h = AB$. The region consists of a solid circular cylinder of radius 3 and height h, together with two solid hemispheres of radius 3 centered at A and B. The volume of the cylinder is $\pi \cdot 3^2 \cdot h = 9\pi h$, and the two hemispheres have a combined volume of $\frac{4}{3}\pi \cdot 3^3 = 36\pi$. Therefore $9\pi h + 36\pi = 216\pi$, and $h = 20$.

12. **Answer (E):** Suppose that the two larger quantities are the first and the second. Then $3 = x + 2 \geq y - 4$. This is equivalent to $x = 1$ and $y \leq 7$, and its graph is the downward-pointing ray with endpoint $(1, 7)$. Similarly, if the two larger quantities are the first and third, then $3 = y - 4 \geq x + 2$. This is equivalent to $y = 7$ and $x \leq 1$, and its graph is the leftward-pointing ray with endpoint $(1, 7)$. Finally, if the two larger quantities are the second and third, then $x + 2 = y - 4 \geq 3$. This is equivalent to $y = x + 6$ and $x \geq 1$, and its graph is the ray with endpoint $(1, 7)$ that points upward and to the right. Thus the graph consists of three rays with common endpoint $(1, 7)$.



Note: This problem is related to a relatively new area of mathematics called tropical geometry.

13. **Answer (D):** The sequence starts $0, 1, 1, 2, 0, 2, 2, 1, 0, 1, 1, 2, \ldots$. Notice that the pattern repeats and the period is 8. Thus no matter which 8 consecutive numbers are added, the answer will be $0 + 1 + 1 + 2 + 0 + 2 + 2 + 1 = 9$.

14. **Answer (D):** Let M be the cost of Roger's movie ticket, and let S be the cost of Roger's soda. Then $M = 0.20(A-S)$ and $S = 0.05(A-M)$. Thus $5M + S = A$ and $M + 20S = A$. Solving the system for M and S in terms of A gives $M = \frac{19}{99}A$ and $S = \frac{4}{99}A$. The total cost of the movie ticket and soda as a fraction of A is $\frac{23}{99} = 0.2323\ldots \approx 23\%$.

15. **Answer (C):** Half of the time Laurent will pick a number between 2017 and 4034, in which case the probability that his number will be greater than Chloé's number is 1. The other half of the time, he will pick a number between 0 and 2017, and by symmetry his number will be the larger one in half of those cases. Therefore the requested probability is $\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot \frac{1}{2} = \frac{3}{4}$.

OR

The choices of numbers can be represented in the coordinate plane by points in the rectangle with vertices at $(0, 0)$, $(2017, 0)$, $(2017, 4034)$, and $(0, 4034)$. The portion of the rectangle representing the event that Laurent's number is greater than Chloé's number is the portion above the line segment with endpoints $(0, 0)$ and $(2017, 2017)$. This area is $\frac{3}{4}$ of the area of the entire rectangle, so the requested probability is $\frac{3}{4}$.

16. **Answer (B):** Horse k will again be at the starting point after t minutes if and only if k is a divisor of t. Let $I(t)$ be the number of integers k with $1 \le k \le 10$ that divide t. Then $I(1) = 1$, $I(2) = 2$, $I(3) = 2$, $I(4) = 3$, $I(5) = 2$, $I(6) = 4$, $I(7) = 2$, $I(8) = 4$, $I(9) = 3$, $I(10) = 4$, $I(11) = 1$, and $I(12) = 5$. Thus $T = 12$ and the requested sum of digits is $1 + 2 = 3$.

17. **Answer (D):** The ratio $\frac{PQ}{RS}$ has its greatest value when PQ is as large as possible and RS is as small as possible. Points P, Q, R, and S have coordinates among $(\pm 5, 0)$, $(\pm 4, \pm 3)$, $(\pm 4, \mp 3)$, $(\pm 3, \pm 4)$, $(\pm 3, \mp 4)$, and $(0, \pm 5)$. In order for the distance between two of these points to be irrational, the two points must not form a diameter, and they must not have the same x-coordinate or y-coordinate. If

$R = (a, b)$ and $S = (a', b')$, then $|a - a'| \geq 1$ and $|b - b'| \geq 1$. Because $(3, 4)$ and $(4, 3)$ achieve this, they are as close as two points can be, $\sqrt{2}$ units apart. If $P = (a, b)$ and $Q = (a', b')$, then PQ is maximized when the distance from (a', b') to $(-a, -b)$ is minimized. Because $|a + a'| \geq 1$ and $|b + b'| \geq 1$, the points $(3, -4)$ and $(-4, 3)$ are as far apart as possible, $\sqrt{98}$ units. Therefore the greatest possible ratio is $\frac{\sqrt{98}}{\sqrt{2}} = \sqrt{49} = 7$.

18. **Answer (D):** Let x be the probability that Amelia wins. Then $x = \frac{1}{3} + \left(1 - \frac{1}{3}\right)\left(1 - \frac{2}{5}\right)x$, because either Amelia wins on the first toss, or, if she and Blaine both get tails, then the chance of her winning from that point onward is also x. Solving this equation gives $x = \frac{5}{9}$. The requested difference is $9 - 5 = 4$.

OR

The probability that Amelia wins on the first toss is $\frac{1}{3}$, the probability that Amelia wins on the second toss is $\frac{2}{3} \cdot \frac{3}{5} \cdot \frac{1}{3}$, and so on. Therefore the probability that Amelia wins is

$$\begin{aligned}\frac{1}{3} + \frac{2}{3} \cdot \frac{3}{5} \cdot \frac{1}{3} + \left(\frac{2}{3}\right)^2 \cdot \left(\frac{3}{5}\right)^2 \cdot \frac{1}{3} + \cdots &= \frac{1}{3} \cdot \left(1 + \frac{2}{5} + \left(\frac{2}{5}\right)^2 + \cdots\right) \\ &= \frac{1}{3} \cdot \frac{1}{1 - \frac{2}{5}} \\ &= \frac{5}{9}.\end{aligned}$$

OR

Let $n \geq 0$ be the greatest integer such that Amelia and Blaine each toss tails n times in a row. Then the game will end in the next round of tosses, either because Amelia tosses a head, which will occur with probability $\frac{1}{3}$, or because Amelia tosses a tail and Blaine tosses a head, which will occur with probability $\frac{2}{3} \cdot \frac{2}{5} = \frac{4}{15}$. The probability that it is Amelia who wins is therefore

$$\frac{\frac{1}{3}}{\frac{1}{3} + \frac{4}{15}} = \frac{5}{9}.$$

19. **Answer (C):** Let X be the set of ways to seat the five people in which Alice sits next to Bob. Let Y be the set of ways to seat the

five people in which Alice sits next to Carla. Let Z be the set of ways to seat the five people in which Derek sits next to Eric. The required answer is $5! - |X \cup Y \cup Z|$. The Inclusion–Exclusion Principle gives

$$|X\cup Y\cup Z| = \big(|X|+|Y|+|Z|\big)-\big(|X\cap Y|+|X\cap Z|+|Y\cap Z|\big)+|X\cap Y\cap Z|.$$

Viewing Alice and Bob as a unit in which either can sit on the other's left side shows that there are $2 \cdot 4! = 48$ elements of X. Similarly there are 48 elements of Y and 48 elements of Z. Viewing Alice, Bob, and Carla as a unit with Alice in the middle shows that $|X \cap Y| = 2 \cdot 3! = 12$. Viewing Alice and Bob as a unit and Derek and Eric as a unit shows that $|X \cap Z| = 2 \cdot 2 \cdot 3! = 24$. Similarly $|Y \cap Z| = 24$. Finally, there are $2 \cdot 2 \cdot 2! = 8$ elements of $X \cap Y \cap Z$. Therefore $|X \cup Y \cup Z| = (48+48+48)-(12+24+24)+8 = 92$, and the answer is $120 - 92 = 28$.

OR

There are three cases based on where Alice is seated.

• If Alice takes the first or last chair, then Derek or Eric must be seated next to her, Bob or Carla must then take the middle chair, and either of the remaining two individuals can be seated in either of the other two chairs. This gives a total of $2^4 = 16$ arrangements.

• If Alice is seated in the second or fourth chair, then Derek and Eric will take the seats on her two sides, and this can be done in two ways. Bob and Carla can be seated in the two remaining chairs in two ways, which yields a total of $2^3 = 8$ arrangements.

• If Alice sits in the middle chair, then Derek and Eric will be seated on her two sides, with Bob and Carla seated in the first and last chairs. This results in $2^2 = 4$ arrangements.

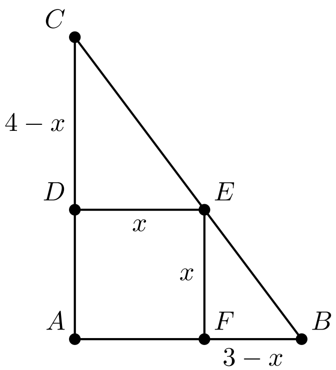
Thus there are $16 + 8 + 4 = 28$ possible arrangements in total.

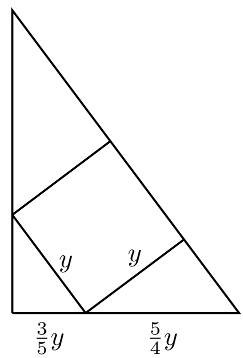
20. **Answer (D):** Note that $S(n+1) = S(n) + 1$ unless the numeral for n ends with a 9. Moreover, if the numeral for n ends with exactly k 9s, then $S(n+1) = S(n) + 1 - 9k$. Thus the possible values of $S(n+1)$ when $S(n) = 1274$ are all of the form $1275 - 9k$, where $k \in \{0, 1, 2, 3, \ldots, 141\}$. Of the choices, only 1239 can be formed in this manner, and $S(n+1)$ will equal 1239 if, for example, n consists of 4 consecutive 9s preceded by 1238 1s.

OR

The value of a positive integer is congruent to the sum of its digits modulo 9. Therefore $n \equiv S(n) = 1274 \equiv 5 \pmod 9$, so $S(n+1) \equiv n+1 \equiv 6 \pmod 9$. Of the given choices, only 1239 meets this requirement.

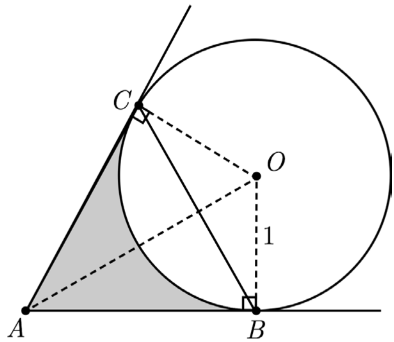
21. **Answer (D):** In the first figure $\triangle FEB \sim \triangle DCE$, so $\frac{x}{3-x} = \frac{4-x}{x}$ and $x = \frac{12}{7}$. In the second figure, the small triangles are similar to the large one, so the lengths of the portions of the side of length 3 are as shown. Solving $\frac{3}{5}y + \frac{5}{4}y = 3$ yields $y = \frac{60}{37}$. Thus $\frac{x}{y} = \frac{12}{7} \cdot \frac{37}{60} = \frac{37}{35}$.





22. **Answer (E):** Let O be the center of the circle, and without loss of generality, assume that radius $OB = 1$. Because $\triangle ABO$ is a $30-60-90^\circ$ right triangle, $AO = 2$ and $AB = BC = \sqrt{3}$. Kite $ABOC$ has diagonals of lengths 2 and $\sqrt{3}$, so its area is $\sqrt{3}$. Because $\angle BOC = 120^\circ$, the area of the sector cut off by $\angle BOC$ is $\frac{1}{3}\pi$. The area of the portion of $\triangle ABC$ lying outside the circle (shaded in the figure) is therefore $\sqrt{3} - \frac{1}{3}\pi$. The area of $\triangle ABC$ is $\frac{1}{4}\sqrt{3}\left(\sqrt{3}\right)^2 = \frac{3}{4}\sqrt{3}$, so the requested fraction is

$$\frac{\sqrt{3} - \frac{1}{3}\pi}{\frac{3}{4}\sqrt{3}} = \frac{4}{3} - \frac{4\sqrt{3}\pi}{27}.$$



23. **Answer (B):** There are $\binom{25}{3} = \frac{25\cdot24\cdot23}{6} = 2300$ ways to choose three vertices, but in some cases they will fall on a line. There are $5\cdot\binom{5}{3} = 50$ that fall on a horizontal line, another 50 that fall on a vertical line, $\binom{5}{3}+2\binom{4}{3}+2\binom{3}{3} = 20$ that fall on a line with slope 1, another 20 that fall on a line with slope -1, and 3 each that fall on lines with slopes 2, -2, $\frac{1}{2}$, and $-\frac{1}{2}$. Therefore the answer is $2300-50-50-20-20-12 = 2148$.

24. **Answer (C):** Let q be the additional root of $f(x)$. Then

$$\begin{aligned} f(x) &= (x-q)(x^3+ax^2+x+10) \\ &= x^4+(a-q)x^3+(1-qa)x^2+(10-q)x-10q. \end{aligned}$$

Thus $100 = 10-q$, so $q = -90$ and $c = -10q = 900$. Also $1 = a-q = a+90$, so $a = -89$. It follows, using the factored form of f shown above, that $f(1) = \big(1-(-90)\big)\cdot(1-89+1+10) = 91\cdot(-77) = -7007$.

25. **Answer (A):** Recall the divisibility test for 11: A three-digit number $\underline{a}\,\underline{b}\,\underline{c}$ is divisible by 11 if and only if $a-b+c$ is divisible by 11. The smallest and largest three-digit multiples of 11 are, respectively, $110 = 10\cdot11$ and $990 = 90\cdot11$, so the number of three-digit multiples of 11 is $90-10+1 = 81$. They may be grouped as follows:

- There are 9 multiples of 11 that have the form $\underline{a}\,\underline{a}\,\underline{0}$ for $1 \le a \le 9$. They can each be permuted to form a total of 2 three-digit integers. In each case $\underline{a}\,\underline{a}\,\underline{0}$ is a multiple of 11 and $\underline{a}\,\underline{0}\,\underline{a}$ is not, so these 9 multiples of 11 give 18 integers with the required property.

- There are 8 multiples of 11 that have the form $\underline{a}\,\underline{b}\,\underline{a}$, namely 121, 242, 363, 484, 616, 737, 858, and 979. They can each be permuted to form a total of 3 three-digit integers. In each case $\underline{a}\,\underline{b}\,\underline{a}$ is a multiple of 11, but neither $\underline{a}\,\underline{a}\,\underline{b}$ nor $\underline{b}\,\underline{a}\,\underline{a}$ is, so these 8 multiples of 11 give 24 integers with the required property.

- If a three-digit multiple of 11 has distinct digits and one digit is 0, it must have the form $\underline{a}\,\underline{0}\,\underline{c}$ with $a+c=11$. There are 8 such integers, namely 209, 308, 407, $\ldots$, 902. They can each be permuted to form a total of 4 three-digit integers, but these 8 multiples of 11 give only 4 distinct sets of permutations, leading to $4 \cdot 4 = 16$ integers with the required property.

- The remaining $81-(9+8+8)=56$ three-digit multiples of 11 all have the form $\underline{a}\,\underline{b}\,\underline{c}$, where a, b, and c are distinct nonzero digits. They can each be permuted to form a total of 6 three-digit integers, and in each case both $\underline{a}\,\underline{b}\,\underline{c}$ and $\underline{c}\,\underline{b}\,\underline{a}$—and only these—are multiples of 11. Therefore these 56 multiples of 11 give only 28 distinct sets of permutations, leading to $28 \cdot 6 = 168$ integers with the required property.

The total number of integers with the required property is $18+24+16+168=226$.

Problems and solutions were contributed by Stephen Adams, Paul Baird-Smith, Steven Davis, Marta Eso, Silvia Fernandez, Peter Gilchrist, Jerrold Grossman, Jonathan Kane, Joe Kennedy, Michael Khoury, Steven Miller, Hugh Montgomery, Yasick Nemenov, Mark Saul, Walter Stromquist, Roger Waggoner, Dave Wells, and Carl Yerger.

The

MAA American Mathematics Competitions

are supported by

Academy of Applied Science
Akamai Foundation
American Mathematical Society
American Statistical Association
Ansatz Capital
Army Educational Outreach Program
Art of Problem Solving
Casualty Actuarial Society
Conference Board of the Mathematical Sciences
The DE Shaw Group
Dropbox
Expii, Inc
IDEA MATH, LLC
Jane Street Capital
MathWorks
Mu Alpha Theta
National Council of Teachers of Mathematics
Simons Foundation
Society for Industrial and Applied Mathematics
Star League
Susquehanna International Group
Tudor Investment Corp
Two Sigma

MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

18th Annual

AMC 10B

American Mathematics Competition 10B

Wednesday, February 15, 2017

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
Phone 800.527.3690 | Fax 240.396.5647 | amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (B):** Working backwards, switching the digits of the numbers 71, 72, 73, 74, and 75 and subtracting 11 gives, respectively, 6, 16, 26, 36, and 46. Only 6 and 36 are divisible by 3, and only $36 \div 3 = 12$ is a two-digit number.

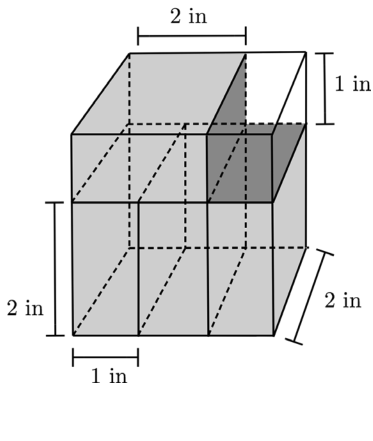
2. **Answer (C):** Each lap took Sofia $\frac{100 \text{ m}}{4 \text{ m/s}} + \frac{300 \text{ m}}{5 \text{ m/s}} = 85$ seconds, so 5 laps took her $5 \cdot 85 = 425$ seconds, which is 7 minutes and 5 seconds.

3. **Answer (E):** Adding the inequalities $y > -1$ and $z > 1$ yields $y + z > 0$. The other four choices give negative values if, for example, $x = \frac{1}{8}$, $y = -\frac{1}{4}$, and $z = \frac{3}{2}$.

4. **Answer (D):** The given equation implies that $3x + y = -2(x - 3y)$, which is equivalent to $x = y$. Therefore
$$\frac{x + 3y}{3x - y} = \frac{4y}{2y} = 2.$$

5. **Answer (D):** Suppose Camilla originally had b blueberry jelly beans and c cherry jelly beans. After eating 10 pieces of each kind, she now has $b - 10$ blueberry jelly beans and $c - 10$ cherry jelly beans. The conditions of the problem are equivalent to the equations $b = 2c$ and $b - 10 = 3(c - 10)$. Then $2c - 10 = 3c - 30$, which means that $c = 20$ and $b = 2 \cdot 20 = 40$.

6. **Answer (B):** A possible arrangement of 4 blocks is shown by the figure.



Four blocks do not completely fill the box because the combined volume of the blocks is only $4(2 \cdot 2 \cdot 1) = 16$ cubic inches, whereas the volume of the box is $3 \cdot 2 \cdot 3 = 18$ cubic inches. Because the unused space, $18 - 16 = 2$ cubic inches, is less than the volume of a block, 4 cubic inches, no more than 4 blocks can fit in the box.

7. **Answer (C):** Let $2d$ be the distance in kilometers to the friend's house. Then Samia bicycled distance d at rate 17 and walked distance d at rate 5, for a total time of

$$\frac{d}{17} + \frac{d}{5} = \frac{44}{60}$$

hours. Solving this equation yields $d = \frac{17}{6} = 2.833\ldots$. Therefore Samia walked about 2.8 kilometers.

8. **Answer (C):** The altitude $\overline{AD}$ lies on a line of symmetry for the isosceles triangle. Under reflection about this line, B will be sent to C. Because B is obtained from D by adding 3 to the x-coordinate and subtracting 6 from the y-coordinate, C is obtained from D by subtracting 3 from the x-coordinate and adding 6 to the y-coordinate. Thus the third vertex C has coordinates $(-1 - 3, 3 + 6) = (-4, 9)$.

OR

To find the coordinates of $C(x, y)$, note that D is the midpoint of $\overline{BC}$. Therefore

$$\frac{x+2}{2} = -1 \qquad \text{and} \qquad \frac{y-3}{2} = 3.$$

Solving these equations gives $x = -4$ and $y = 9$, so $C = (-4, 9)$.

9. **Answer (D):** The probability of getting all 3 questions right is $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$. Because there are 3 ways to get 2 of the questions right and 1 wrong, the probability of getting exactly 2 right is $3\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = \frac{6}{27}$. Therefore the probability of winning is $\frac{1}{27} + \frac{6}{27} = \frac{7}{27}$.

10. **Answer (E):** Because the lines are perpendicular, their slopes, $\frac{a}{2}$ and $-\frac{2}{b}$, are negative reciprocals, so $a = b$. Substituting b for a and using the point $(1, -5)$ yields the equations $b + 10 = c$ and $2 - 5b = -c$. Adding the two equations yields $12 - 4b = 0$, so $b = 3$. Thus $c = 3 + 10 = 13$.

11. **Answer (D):** The students who like dancing but say they dislike it constitute $60\% \cdot (100\% - 80\%) = 12\%$ of the students. Similarly, the students who dislike dancing and say they dislike it constitute $(100\% - 60\%) \cdot 90\% = 36\%$ of the students. Therefore the requested fraction is $\frac{12}{12+36} = \frac{1}{4} = 25\%$.

12. **Answer (A):** For Elmer's old car, let M be the fuel efficiency in kilometers per liter, and let C be the cost of fuel in dollars per liter. Then for his new car, the fuel efficiency is $1.5M$, and the cost of fuel is $1.2C$. The cost in dollars per kilometer for the old car is $\frac{C}{M}$, and for the new car it is $\frac{1.2C}{1.5M} = 0.8\frac{C}{M}$. Therefore, fuel for the long trip will cost 20% less in Elmer's new car.

13. **Answer (C):** Let x, y, and z be the number of people taking exactly one, two, and three classes, respectively. The condition that each student in the program takes at least one class is equivalent to the equation $x + y + z = 20$. The condition that there are 9 students taking at least two classes is equivalent to the equation $y + z = 9$. The sum $10 + 13 + 9 = 32$ counts once the students taking one class, twice the students taking two classes, and three times the students taking three classes. Then $x + 2y + 3z = 32$, which is equivalent to $z = 32 - (x + y + z) - (y + z) = 32 - 20 - 9 = 3$.

OR

Let Y, B, and P be the sets of students taking yoga, bridge, and painting, respectively. By the Inclusion–Exclusion Principle,

$$|Y \cup B \cup P| = |Y| + |B| + |P| - (|Y \cap B| + |Y \cap P| + |B \cap P|) + |Y \cap B \cap P|.$$

Furthermore, $|Y \cap B| + |Y \cap P| + |B \cap P| = 9 + 2|Y \cap B \cap P|$, because in tabulating the students taking at least two classes by considering the pairs of classes one by one, the students taking all three classes are counted three times rather than just once. Thus

$$20 = 10 + 13 + 9 - (9 + 2|Y \cap B \cap P|) + |Y \cap B \cap P| = 23 - |Y \cap B \cap P|,$$

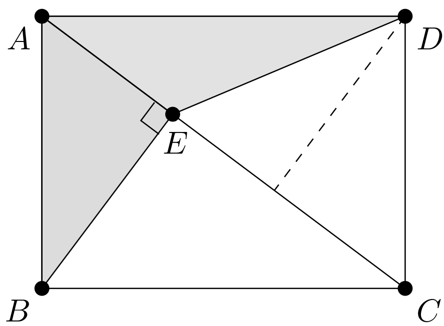
so the number of students taking all three classes is $|Y \cap B \cap P| = 3$.

14. **Answer (D):** An integer will have a remainder of 1 when divided by 5 if and only if the units digit is either 1 or 6. The randomly selected positive integer will itself have a units digit of each of the numbers

from 0 through 9 with equal probability. This digit of N alone will determine the units digit of N^{16}. Computing the 16th power of each of these 10 digits by squaring the units digit four times yields one 0, one 5, four 1s, and four 6s. The probability is therefore $\frac{8}{10} = \frac{4}{5}$.

Note: This result also follows from Fermat's Little Theorem.

15. **Answer (E):** Triangles ADE and ABE have the same area because they share the base $\overline{AE}$ and, by symmetry, they have the same height. By the Pythagorean Theorem, $AC = 5$. Because $\triangle ABE \sim \triangle ACB$, the ratio of their areas is the square of the ratio of their corresponding sides. Their hypotenuses have lengths 3 and 5, respectively, so their areas are in the ratio 9 to 25. The area of $\triangle ACB$ is half that of the rectangle, so the area of $\triangle ABE$ is $\frac{9}{25} \cdot 6 = \frac{54}{25}$. Thus the area of $\triangle ADE$ is also $\frac{54}{25}$.

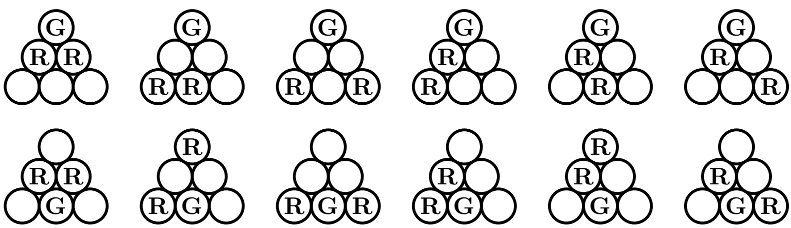


16. **Answer (A):** It will be easier to count the complementary set. There are 9 one-digit numerals that do not contain the digit 0, $9 \cdot 9 = 81$ two-digit numerals that do not contain the digit 0, $9 \cdot 9 \cdot 9 = 729$ three-digit numerals that do not contain the digit 0, and $1 \cdot 9 \cdot 9 \cdot 9 = 729$ four-digit numerals starting with 1 that do not contain the digit 0, a total of 1548. All four-digit numerals between 2000 and 2017, inclusive, contain the digit 0. Therefore $2017 - 1548 = 469$ numerals in the required range do contain the digit 0.

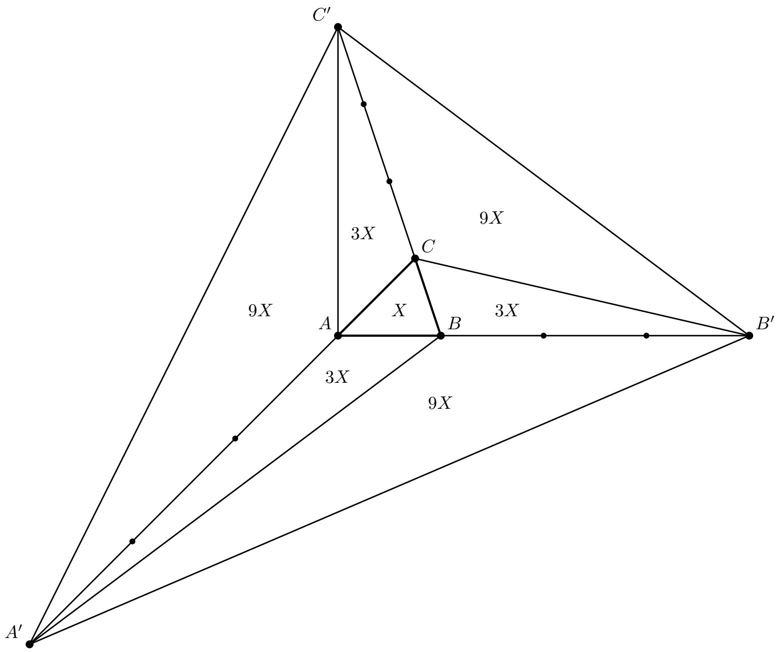
17. **Answer (B):** The monotonous positive integers with one digit or increasing digits can be put into a one-to-one correspondence with the nonempty subsets of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. The number of such subsets is $2^9 - 1 = 511$. The monotonous positive integers with one digit or decreasing digits can be put into a one-to-one correspondence

with the subsets of $\{0,1,2,3,4,5,6,7,8,9\}$ other than $\emptyset$ and $\{0\}$. The number of these is $2^{10}-2=1022$. The single-digit numbers are included in both sets, so there are $511+1022-9=1524$ monotonous positive integers.

18. **Answer (D):** By symmetry, there are just two cases for the position of the green disk: corner or non-corner. If a corner disk is painted green, then there is 1 case in which both red disks are adjacent to the green disk, there are 2 cases in which neither red disk is adjacent to the green disk, and there are 3 cases in which exactly one of the red disks is adjacent to the green disk. Similarly, if a non-corner disk is painted green, then there is 1 case in which neither red disk is in a corner, there are 2 cases in which both red disks are in a corner, and there are 3 cases in which exactly one of the red disks is in a corner. The total number of paintings is $1+2+3+1+2+3=12$.

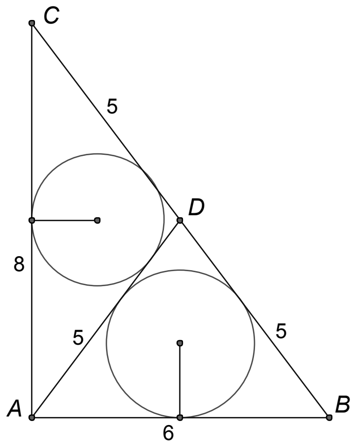


19. **Answer (E):** Draw segments $\overline{CB'}$, $\overline{AC'}$, and $\overline{BA'}$. Let X be the area of $\triangle ABC$. Because $\triangle BB'C$ has a base 3 times as long and the same altitude, its area is $3X$. Similarly, the areas of $\triangle AA'B$ and $\triangle CC'A$ are also $3X$. Furthermore, $\triangle AA'C'$ has 3 times the base and the same height as $\triangle ACC'$, so its area is $9X$. The areas of $\triangle CC'B'$ and $\triangle BB'A'$ are also $9X$ by the same reasoning. Therefore the area of $\triangle A'B'C'$ is $X+3(3X)+3(9X)=37X$, and the requested ratio is $37:1$. Note that nothing in this argument requires $\triangle ABC$ to be equilateral.



20. **Answer (B):** There are $\left\lfloor \frac{21}{2} \right\rfloor + \left\lfloor \frac{21}{4} \right\rfloor + \left\lfloor \frac{21}{8} \right\rfloor + \left\lfloor \frac{21}{16} \right\rfloor = 10+5+2+1 = 18$ powers of 2 in the prime factorization of 21!. Thus $21! = 2^{18}k$, where k is odd. A divisor of 21! must be of the form $2^i b$ where $0 \le i \le 18$ and b is a divisor of k. For each choice of b, there is one odd divisor of 21! and 18 even divisors. Therefore the probability that a randomly chosen divisor is odd is $\frac{1}{19}$. In fact, $21! = 2^{18} \cdot 3^9 \cdot 5^4 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 19$, so it has $19 \cdot 10 \cdot 5 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 60{,}800$ positive integer divisors, of which $10 \cdot 5 \cdot 4 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 3{,}200$ are odd.

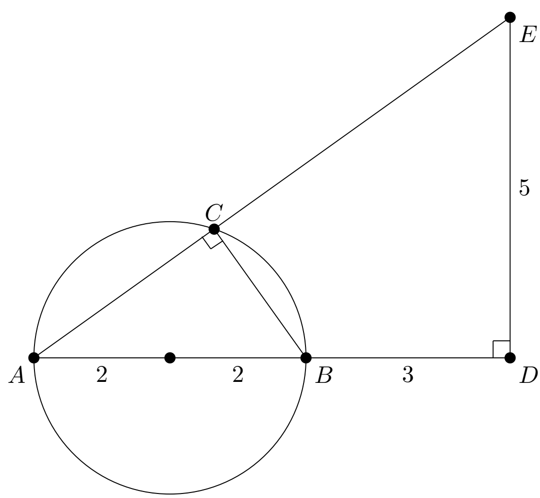
21. **Answer (D):** By the converse of the Pythagorean Theorem, $\angle BAC$ is a right angle, so $BD = CD = AD = 5$, and the area of each of the small triangles is 12 (half the area of $\triangle ABC$). The area of $\triangle ABD$ is equal to its semiperimeter, $\frac{1}{2} \cdot (5+5+6) = 8$, multiplied by the radius of the inscribed circle, so the radius is $\frac{12}{8} = \frac{3}{2}$. Similarly, the radius of the inscribed circle of $\triangle ACD$ is $\frac{4}{3}$. The requested sum is $\frac{3}{2} + \frac{4}{3} = \frac{17}{6}$.



22. **Answer (D):** Because $\angle ACB$ is inscribed in a semicircle, it is a right angle. Therefore $\triangle ABC$ is similar to $\triangle AED$, so their areas are related as AB^2 is to AE^2. Because $AB^2 = 4^2 = 16$ and, by the Pythagorean Theorem,

$$AE^2 = (4+3)^2 + 5^2 = 74,$$

this ratio is $\frac{16}{74} = \frac{8}{37}$. The area of $\triangle AED$ is $\frac{35}{2}$, so the area of $\triangle ABC$ is $\frac{35}{2} \cdot \frac{8}{37} = \frac{140}{37}$.



23. **Answer (C):** The remainder when N is divided by 5 is clearly 4. A positive integer is divisible by 9 if and only if the sum of its digits is divisible by 9. The sum of the digits of N is $4(0+1+2+\cdots+9)+10\cdot 1+10\cdot 2+10\cdot 3+(4+0)+(4+1)+(4+2)+(4+3)+(4+4)=270$, so N must be a multiple of 9. Then $N-9$ must also be a multiple of 9, and the last digit of $N-9$ is 5, so it is also a multiple of 5. Thus $N-9$ is a multiple of 45, and N leaves a remainder of 9 when divided by 45.

24. **Answer (C):** Assume without loss of generality that two of the vertices of the triangle are on the branch of the hyperbola in the first quadrant. This forces the centroid of the triangle to be the vertex $(1,1)$ of the hyperbola. Because the vertices of the triangle are equidistant from the centroid, the first-quadrant vertices must be $\left(a,\frac{1}{a}\right)$ and $\left(\frac{1}{a},a\right)$ for some positive number a. By symmetry, the third vertex must be $(-1,-1)$. The distance between the vertex $(-1,-1)$ and the centroid $(1,1)$ is $2\sqrt{2}$, so the altitude of the triangle must be $\frac{3}{2}\cdot 2\sqrt{2}=3\sqrt{2}$, which makes the side length of the triangle $s=\frac{2}{\sqrt{3}}\cdot 3\sqrt{2}=2\sqrt{6}$. The required area is $\frac{\sqrt{3}}{4}s^2=6\sqrt{3}$. The requested value is $(6\sqrt{3})^2=108$. In fact, the vertices of the equilateral triangle are $(-1,-1)$, $(2+\sqrt{3},2-\sqrt{3})$, and $(2-\sqrt{3},2+\sqrt{3})$.

25. **Answer (E):** Let S be the sum of Isabella's 7 scores. Then S is a multiple of 7, and

$$658=91+92+93+\cdots+97\le S\le 94+95+96+\cdots+100=679,$$

so S is one of 658, 665, 672, or 679. Because $S-95$ is a multiple of 6, it follows that $S=665$. Thus the sum of Isabella's first 6 scores was $665-95=570$, which is a multiple of 5, and the sum of her first 5 scores was also a multiple of 5. Therefore her sixth score must have been a multiple of 5. Because her seventh score was 95 and her scores were all different, her sixth score was 100. One possible sequence of scores is $91,93,92,96,98,100,95$.

Problems and solutions were contributed by Stephen Adams, Thomas Butts, Steven Davis, Steven Dunbar, Marta Eso, Silvia Fernandez, Zachary Franco, Jesse Freeman, Devin Gardella, Peter Gilchrist, Jerrold Grossman, Jonathan Kane, Joe Kennedy, Michael Khoury, Pamela Mishkin, Hugh Montgomery, Mark Saul, Roger Waggoner, Dave Wells, and Carl Yerger.

The

MAA American Mathematics Competitions

are supported by

Academy of Applied Science
Akamai Foundation
American Mathematical Society
American Statistical Association
Ansatz Capital
Army Educational Outreach Program
Art of Problem Solving
Casualty Actuarial Society
Conference Board of the Mathematical Sciences
The DE Shaw Group
Dropbox
Expii, Inc
IDEA MATH, LLC
Jane Street Capital
MathWorks
Mu Alpha Theta
National Council of Teachers of Mathematics
Simons Foundation
Society for Industrial and Applied Mathematics
Star League
Susquehanna International Group
Tudor Investment Corp
Two Sigma

MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

19th Annual

AMC 10A

American Mathematics Competition 10A

Wednesday, February 7, 2018

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
Phone 800.527.3690 | Fax 240.396.5647 | amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by
MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the
co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (B):** Computing inside to outside yields:

$$\begin{aligned}\left(\left((2+1)^{-1}+1\right)^{-1}+1\right)^{-1}+1 &= \left(\left(\frac{4}{3}\right)^{-1}+1\right)^{-1}+1\\ &= \left(\frac{7}{4}\right)^{-1}+1\\ &= \frac{11}{7}.\end{aligned}$$

   Note: The successive denominators and numerators of numbers obtained from this pattern are the Lucas numbers.

2. **Answer (A):** Let L, J, and A be the amounts of soda that Liliane, Jacqueline, and Alice have, respectively. The given information implies that $L = 1.50J = \frac{3}{2}J$ and $A = 1.25J = \frac{5}{4}J$, and hence $J = \frac{4}{5}A$. Then

$$L = \frac{3}{2}\cdot\frac{4}{5}A = \frac{6}{5}A = 1.20A,$$

   so Liliane has 20% more soda than Alice.

3. **Answer (E):** Converting 10! seconds to days gives

$$\frac{10!}{60\cdot 60\cdot 24} = \frac{10\cdot 9\cdot 8\cdot 7\cdot 6\cdot 120}{60\cdot 120\cdot 12} = \frac{9\cdot 8\cdot 7}{12} = 42.$$

   Because 30 days after January 1 is January 31, 42 days after January 1 is February 12.

4. **Answer (E):** There are 4 choices for the periods in which the mathematics courses can be taken: periods 1, 3, 5; periods 1, 3, 6; periods 1, 4, 6; and periods 2, 4, 6. Each choice of periods allows $3! = 6$ ways to order the 3 mathematics courses. Therefore there are $4\cdot 6 = 24$ ways of arranging a schedule.

5. **Answer (D):** Because the statements of Alice, Bob, and Charlie are all incorrect, the actual distance d satisfies $d < 6$, $d > 5$, and $d > 4$. Hence the actual distance lies in the interval $(5, 6)$.

6. **Answer (B):** Let N be the number of votes cast. Then $0.65N$ of them were like votes, and $0.35N$ of them were dislike votes. The current score for Sangho's video is then $0.65N - 0.35N = 0.3N = 90$. Thus $N = 90 \div (0.3) = 300$.

7. **Answer (E):** Because $4000 = 2^5 \cdot 5^3$,

$$4000 \cdot \left(\frac{2}{5}\right)^n = 2^{5+n} \cdot 5^{3-n}.$$

This product will be an integer if and only if both of the factors 2^{5+n} and 5^{3-n} are integers, which happens if and only if both exponents are nonnegative. Therefore the given expression is an integer if and only if $5+n \geq 0$ and $3-n \geq 0$. The solutions are exactly the integers satisfying $-5 \leq n \leq 3$. There are $3 - (-5) + 1 = 9$ such values.

8. **Answer (C):** Let n be the number of 5-cent coins Joe has, and let x be the requested value—the number of 25-cent coins Joe has minus the number of 5-cent coins he has. Then Joe has $(n + 3)$ 10-cent coins and $(n + x)$ 25-cent coins. The given information leads to the equations

$$\begin{aligned} n + (n+3) + (n+x) &= 23 \\ 5n + 10(n+3) + 25(n+x) &= 320. \end{aligned}$$

These equations simplify to $3n + x = 20$ and $8n + 5x = 58$. Solving these equations simultaneously yields $n = 6$ and $x = 2$. Joe has 2 more 25-cent coins than 5-cent coins. Indeed, Joe has 6 5-cent coins, 9 10-cent coins, and 8 25-cent coins.

9. **Answer (E):** The length of the base $\overline{DE}$ of $\triangle ADE$ is 4 times the length of the base of a small triangle, so the area of $\triangle ADE$ is $4^2 \cdot 1 = 16$. Therefore the area of $DBCE$ is the area of $\triangle ABC$ minus the area of $\triangle ADE$, which is $40 - 16 = 24$.

10. **Answer (A):** Let

$$a = \sqrt{49 - x^2} - \sqrt{25 - x^2} \quad \text{and} \quad b = \sqrt{49 - x^2} + \sqrt{25 - x^2}.$$

Then $ab = (49 - x^2) - (25 - x^2) = 24$, so $b = \frac{24}{a} = \frac{24}{3} = 8$.

OR

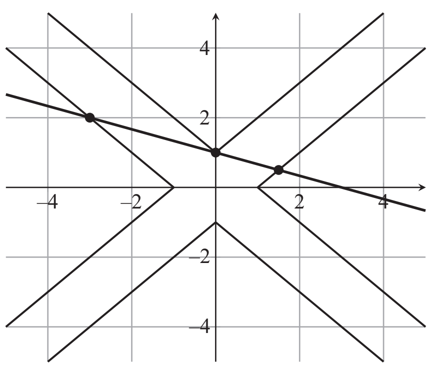
The given equation can be solved directly. Adding $\sqrt{25-x^2}$ to both sides of the equation and squaring leads to $15 = 6\sqrt{25-x^2}$. Solving for x^2 gives $x^2 = \frac{75}{4}$. Substituting this value into $\sqrt{49-x^2}+\sqrt{25-x^2}$ gives the value 8.

11. **Answer (E):** The only ways to achieve a sum of 10 by adding 7 unordered integers between 1 and 6 inclusive are (i) six 1s and one 4; (ii) five 1s, one 2, and one 3; or (iii) four 1s and three 2s. The number of ways to order the outcomes among the 7 dice are 7 in case (i), $7 \cdot 6 = 42$ in case (ii), and $\binom{7}{3} = 35$ in case (iii). There are 6^7 possible outcomes. Therefore $n = 7 + 42 + 35 = 84$.

OR

The number of ways to achieve a sum of 10 by adding 7 ordered integers between 1 and 6, inclusive, is the same as the number of ways to insert 6 bars in the spaces between stars in a row of 10 stars (with no more than one bar per space). For example, the sum $1+1+2+1+3+1+1$ corresponds to $*|*|**|*|***|*|*$. The number of ways of inserting 6 bars in the 9 spaces in a row of 10 stars is $\binom{9}{6} = 84$. (This approach is referred to as "stars and bars".)

12. **Answer (C):** The graph of the system is shown below.



The graph of the first equation is a line with x-intercept $(3, 0)$ and y-intercept $(0, 1)$. To draw the graph of the second equation, consider the equation quadrant by quadrant. In the first quadrant $x > 0$ and $y > 0$, and thus the second equation is equivalent to $|x-y| = 1$, which in turn is equivalent to $y = x \pm 1$. Its graph consists of the rays with endpoints $(0,1)$ and $(1,0)$, as shown. In the second quadrant $x<0$ and $y>0$. The corresponding graph is the reflection of the first quadrant graph across the y-axis. The rest of the graph can be sketched by further reflections of the first-quadrant graph across the coordinate axes, resulting in the figure shown. There are 3 intersection points: $(-3,2)$, $(0,1)$, and $\left(\frac{3}{2},\frac{1}{2}\right)$, as shown.

OR

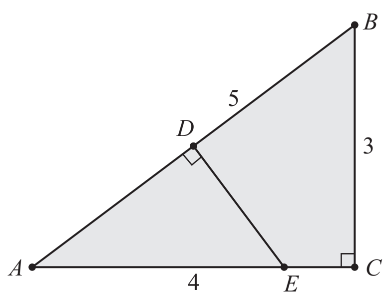
The second equation implies that $x=y\pm 1$ or $x=-y\pm 1$. There are four cases:

- If $x=y+1$, then $(y+1)+3y=3$, so $(x,y)=\left(\frac{3}{2},\frac{1}{2}\right)$.
- If $x=y-1$, then $(y-1)+3y=3$, so $(x,y)=(0,1)$.
- If $x=-y+1$, then $(-y+1)+3y=3$, so again $(x,y)=(0,1)$.
- If $x=-y-1$, then $(-y-1)+3y=3$, so $(x,y)=(-3,2)$.

It may be checked that each of these ordered pairs actually satisfies the given equations, so the total number of solutions is 3.

13. **Answer (D):** The paper's long edge $\overline{AB}$ is the hypotenuse of right triangle ACB, and the crease lies along the perpendicular bisector of $\overline{AB}$. Because $AC>BC$, the crease hits $\overline{AC}$ rather than $\overline{BC}$. Let D be the midpoint of $\overline{AB}$, and let E be the intersection of $\overline{AC}$ and the line through D perpendicular to $\overline{AB}$. Then the crease in the paper is $\overline{DE}$. Because $\triangle ADE\sim\triangle ACB$, it follows that $\frac{DE}{AD}=\frac{CB}{AC}=\frac{3}{4}$. Thus

$$DE=AD\cdot\frac{CB}{AC}=\frac{5}{2}\cdot\frac{3}{4}=\frac{15}{8}.$$



14. **Answer (A):** Because the powers-of-3 terms greatly dominate the powers-of-2 terms, the given fraction should be close to

$$\frac{3^{100}}{3^{96}}=3^4=81.$$

Note that

$$\left(3^{100}+2^{100}\right)-81\left(3^{96}+2^{96}\right)=2^{100}-81\cdot 2^{96}=(16-81)\cdot 2^{96}<0,$$

so the given fraction is less than 81. On the other hand

$$\left(3^{100}+2^{100}\right)-80\left(3^{96}+2^{96}\right)=3^{96}(81-80)-2^{96}(80-16)=3^{96}-2^{102}.$$

Because $3^2>2^3$,

$$3^{96}=\left(3^2\right)^{48}>\left(2^3\right)^{48}=2^{144}>2^{102},$$

it follows that

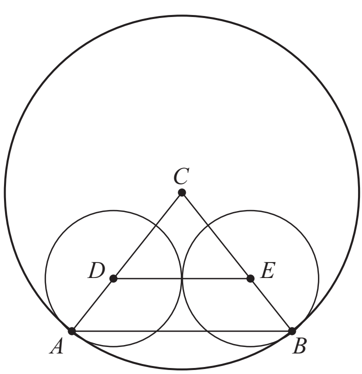
$$\left(3^{100}+2^{100}\right)-80\left(3^{96}+2^{96}\right)>0,$$

and the given fraction is greater than 80. Therefore the greatest integer less than or equal to the given fraction is 80.

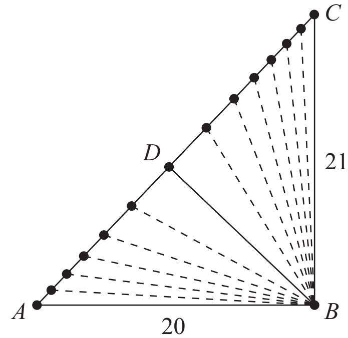
15. **Answer (D):** Let C be the center of the larger circle, and let D and E be the centers of the two smaller circles, as shown. Points C, D, and A are collinear because the radii are perpendicular to the common tangent at the point of tangency, and so are C, E, and B. These points form two isosceles triangles that share a vertex angle. Thus $\triangle CAB \sim \triangle CDE$, and therefore $\frac{AB}{DE}=\frac{CA}{CD}$, so

$$AB=\frac{DE\cdot CA}{CD}=\frac{(5+5)\cdot 13}{13-5}=\frac{65}{4},$$

and the requested sum is $65+4=69$.



16. **Answer (D):** The area of $\triangle ABC$ is 210. Let D be the foot of the altitude from B to $\overline{AC}$. By the Pythagorean Theorem, $AC = \sqrt{20^2 + 21^2} = 29$, so $210 = \frac{1}{2} \cdot 29 \cdot BD$, and $BD = 14\frac{14}{29}$. Two segments of every length from 15 through 19 can be constructed from B to $\overline{AC}$. In addition to these 10 segments and the 2 legs, there is a segment of length 20 from B to a point on $\overline{AC}$ near C, for a total of 13 segments with integer length.



17. **Answer (C):** If $1 \in S$, then S can have only 1 element, not 6 elements. If 2 is the least element of S, then 2, 3, 5, 7, 9, and 11 are available to be in S, but 3 and 9 cannot both be in S, so the largest possible size of S is 5. If 3 is the least element, then 3, 4, 5, 7, 8, 10, and 11 are available, but at most one of 4 and 8 can be in S and at most one of 5 and 10 can be in S, so again S has size at most 5. The set $S = \{4, 6, 7, 9, 10, 11\}$ has the required property, so 4 is the least possible element of S.

OR

At most one integer can be selected for S from each of the following 6 groups: $\{1, 2, 4, 8\}$, $\{3, 6, 12\}$, $\{5, 10\}$, $\{7\}$, $\{9\}$, and $\{11\}$. Because S consists of 6 distinct integers, exactly one integer must be selected from each of these 6 groups. Therefore 7, 9, and 11 must be in S. Because 9 is in S, 3 must not be in S. This implies that either 6 or 12 must be selected from the second group, so neither 1 nor 2 can be selected from the first group. If 4 is selected from the first group, the collection of integers $\{4, 5, 6, 7, 9, 11\}$ is one possibility for the set S. Therefore 4 is the least possible element of S.

Note: The two collections given in the solutions are the only ones with least element 4 that have the given property. This problem is a special case of the following result of Paul Erdős from the 1930s:

Given n integers $a_1, a_2, \ldots, a_n$, no one of them dividing any other, with $a_1 < a_2 < \cdots < a_n \le 2n$, let the integer k be determined by the inequalities $3^k < 2n < 3^{k+1}$. Then $a_1 \ge 2^k$, and this bound is sharp.

18. **Answer (D):** Let S be the set of integers, both negative and nonnegative, having the given form. Increasing the value of a_i by 1 for $0 \le i \le 7$ creates a one-to-one correspondence between S and the ternary (base 3) representation of the integers from 0 through $3^8 - 1$, so S contains $3^8 = 6561$ elements. One of those is 0, and by symmetry, half of the others are positive, so S contains $1 + \frac{1}{2} \cdot (6561 - 1) = 3281$ elements.

OR

First note that if an integer N can be written in this form, then $N - 1$ can also be written in this form as long as not all the a_i in the representation of N are equal to -1. A procedure to alter the representation of N so that it will represent $N - 1$ instead is to find the least value of i such that $a_i \ne -1$, reduce the value of that a_i by 1, and set $a_i = 1$ for all lower values of i. By the formula for the sum of a finite geometric series, the greatest integer that can be written in the given form is

$$\frac{3^8 - 1}{3 - 1} = 3280.$$

Therefore, 3281 nonnegative integers can be written in this form, namely all the integers from 0 through 3280, inclusive. (The negative integers from -3280 through -1 can also be written in this way.)

OR

Think of the indicated sum as an expansion in base 3 using "digits" -1, 0, and 1. Note that the leftmost digit a_k of any positive integer that can be written in this form cannot be negative and therefore must be 1. Then there are 3 choices for each of the remaining k digits to the right of a_k, resulting in 3^k positive integers that can be written in the indicated form. Thus there are

$$\sum_{k=0}^{7} 3^k = \frac{3^8 - 1}{3 - 1} = 3280$$

positive numbers of the indicated form. Because 0 can also be written in this form, the number of nonnegative integers that can be written in the indicated form is 3281.

19. **Answer (E):** For $m \in \{11, 13, 15, 17, 19\}$, let $p(m)$ denote the probability that m^n has units digit 1, where n is chosen at random from the set $S = \{1999, 2000, 2001, \ldots, 2018\}$. Then the desired probability is equal to $\frac{1}{5}\big(p(11) + p(13) + p(15) + p(17) + p(19)\big)$. Because any positive integral power of 11 always has units digit 1, $p(11) = 1$, and because any positive integral power of 15 always has units digit 5, $p(15) = 0$. Note that S has 20 elements, exactly 5 of which are congruent to j mod 4 for each of $j = 0, 1, 2, 3$. The units digits of powers of 13 and 17 cycle in groups of 4. More precisely,

$$\left(13^k \bmod 10\right)_{k=1999}^{2018} = (7, 1, 3, 9, 7, 1, \ldots, 3, 9)$$

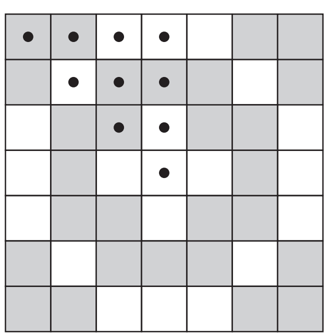
and

$$\left(17^k \bmod 10\right)_{k=1999}^{2018} = (3, 1, 7, 9, 3, 1, \ldots, 7, 9).$$

Thus $p(13) = p(17) = \frac{5}{20} = \frac{1}{4}$. Finally, note that the units digit of 19^k is 1 or 9, according to whether k is even or odd, respectively. Thus $p(19) = \frac{1}{2}$. Hence the requested probability is

$$\frac{1}{5}\left(1 + \frac{1}{4} + 0 + \frac{1}{4} + \frac{1}{2}\right) = \frac{2}{5}.$$

20. **Answer (B):** None of the squares that are marked with dots in the sample scanning code shown below can be mapped to any other marked square by reflections or non-identity rotations. Therefore these 10 squares can be arbitrarily colored black or white in a symmetric scanning code, with the exception of “all black” and “all white”. On the other hand, reflections or rotations will map these squares to all the other squares in the scanning code, so once these 10 colors are specified, the symmetric scanning code is completely determined. Thus there are $2^{10} - 2 = 1022$ symmetric scanning codes.



OR

The diagram below shows the orbits of each square under rotations and reflections. Because the scanning code must look the same under these transformations, all squares in the same orbit must get the same color, but one is free to choose the color for each orbit, except for the choice of "all black" and "all white". Because there are 10 orbits, there are $2^{10} - 2 = 1022$ symmetric scanning codes.

A	B	C	D	C	B	A
B	E	F	G	F	E	B
C	F	H	I	H	F	C
D	G	I	J	I	G	D
C	F	H	I	H	F	C
B	E	F	G	F	E	B
A	B	C	D	C	B	A

21. **Answer (E):** Solving the second equation for x^2 gives $x^2 = y + a$, and substituting into the first equation gives $y^2 + y + (a - a^2) = 0$. The polynomial in y can be factored as $(y + (1 - a))(y + a)$, so the solutions are $y = a - 1$ and $y = -a$. (Alternatively, the solutions can be obtained using the quadratic formula.) The corresponding equations for x are $x^2 = 2a - 1$ and $x^2 = 0$. The second equation always has the solution $x = 0$, corresponding to the point of tangency at the vertex of the parabola $y = x^2 - a$. The first equation has 2 solutions if and only if $a > \frac{1}{2}$, corresponding to the 2 symmetric intersection points of the parabola with the circle. Thus the two curves intersect at 3 points if and only if $a > \frac{1}{2}$.

OR

Substituting the value for y from the second equation into the first equation yields

$$x^2 + \left(x^2 - a\right)^2 = a^2,$$

which is equivalent to

$$x^2\left(x^2 - (2a - 1)\right) = 0.$$

The first factor gives the solution $x = 0$, and the second factor gives 2 other solutions if $a > \frac{1}{2}$ and no other solutions if $a \leq \frac{1}{2}$. Thus there are 3 solutions if and only if $a > \frac{1}{2}$.

22. **Answer (D):** Because $\gcd(a, b) = 24 = 2^3 \cdot 3$ and $\gcd(b, c) = 36 = 2^2 \cdot 3^2$, it follows that a is divisible by 2 and 3 but not by 3^2. Similarly, because $\gcd(b, c) = 2^2 \cdot 3^2$ and $\gcd(c, d) = 54 = 2 \cdot 3^3$, it follows that d is divisible by 2 and 3 but not by 2^2. Therefore $\gcd(d, a) = 2 \cdot 3 \cdot n$, where n is a product of primes that do not include 2 or 3. Because $70 < \gcd(d, a) < 100$ and n is an integer, it must be that $12 \leq n \leq 16$, so $n = 13$, and 13 must also be a divisor of a. The conditions are satisfied if $a = 2^3 \cdot 3 \cdot 13 = 312$, $b = 2^3 \cdot 3^2 = 72$, $c = 2^2 \cdot 3^3 = 108$, and $d = 2 \cdot 3^3 \cdot 13 = 702$.

23. **Answer (D):** Let the triangle's vertices in the coordinate plane be $(4, 0)$, $(0, 3)$, and $(0, 0)$, with $[0, s] \times [0, s]$ representing the unplanted portion of the field. The equation of the hypotenuse is $3x+4y-12 = 0$, so the distance from (s, s), the corner of S closest to the hypotenuse, to this line is given by

$$\frac{|3s + 4s - 12|}{\sqrt{3^2 + 4^2}}.$$

Setting this equal to 2 and solving for s gives $s = \frac{22}{7}$ and $s = \frac{2}{7}$, and the former is rejected because the square must lie within the triangle. The unplanted area is thus $\left(\frac{2}{7}\right)^2 = \frac{4}{49}$, and the requested fraction is

$$1 - \frac{\frac{4}{49}}{\frac{1}{2} \cdot 4 \cdot 3} = \frac{145}{147}.$$

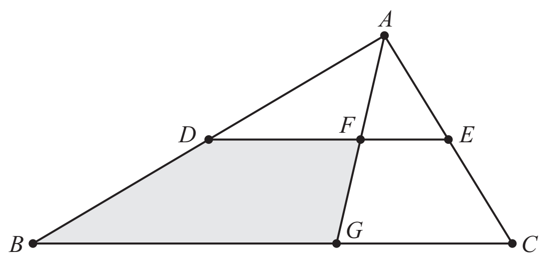
OR

Let the given triangle be described as $\triangle ABC$ with the right angle at B and $AB = 3$. Let D be the vertex of the square that is in the interior of the triangle, and let s be the edge length of the square. Then two sides of the square along with line segments $\overline{AD}$ and $\overline{CD}$ decompose $\triangle ABC$ into four regions. These regions are a triangle with base 5 and height 2, the unplanted square with side s, a right triangle with legs s and $3 - s$, and a right triangle with legs s and $4 - s$. The sum of the areas of these four regions is

$$\frac{1}{2} \cdot 5 \cdot 2 + s^2 + \frac{1}{2}s(3 - s) + \frac{1}{2}s(4 - s) = 5 + \frac{7}{2}s,$$

and the area of $\triangle ABC$ is 6. Solving $5 + \frac{7}{2}s = 6$ for s gives $s = \frac{2}{7}$, and the solution concludes as above.

24. **Answer (D):** Because AB is $\frac{5}{6}$ of $AB + AC$, it follows from the Angle Bisector Theorem that DF is $\frac{5}{6}$ of DE, and BG is $\frac{5}{6}$ of BC. Because trapezoids $FDBG$ and $EDBC$ have the same height, the area of $FDBG$ is $\frac{5}{6}$ of the area of $EDBC$. Furthermore, the area of $\triangle ADE$ is $\frac{1}{4}$ of the area of $\triangle ABC$, so its area is 30, and the area of trapezoid $EDBC$ is $120 - 30 = 90$. Therefore the area of quadrilateral $FDBG$ is $\frac{5}{6} \cdot 90 = 75$.



Note: The figure (not drawn to scale) shows the situation in which $\angle ACB$ is acute. In this case $BC \approx 59.0$ and $\angle BAC \approx 151°$. It is also possible for $\angle ACB$ to be obtuse, with $BC \approx 41.5$ and $\angle BAC \approx 29°$. These values can be calculated using the Law of Cosines and the sine formula for area.

25. **Answer (D):** The equation $C_n - B_n = A_n^2$ is equivalent to

$$c \cdot \frac{10^{2n} - 1}{9} - b \cdot \frac{10^n - 1}{9} = a^2 \left(\frac{10^n - 1}{9}\right)^2.$$

Dividing by $10^n - 1$ and clearing fractions yields

$$(9c - a^2) \cdot 10^n = 9b - 9c - a^2.$$

As this must hold for two different values n_1 and n_2, there are two such equations, and subtracting them gives

$$\left(9c - a^2\right)\left(10^{n_1} - 10^{n_2}\right) = 0.$$

The second factor is non-zero, so $9c - a^2 = 0$ and thus $9b - 9c - a^2 = 0$. From this it follows that $c = \left(\frac{a}{3}\right)^2$ and $b = 2c$. Hence digit a must be 3, 6, or 9, with corresponding values 1, 4, or 9 for c, and 2, 8, or

18 for b. The case $b = 18$ is invalid, so there are just two triples of possible values for a, b, and c, namely $(3, 2, 1)$ and $(6, 8, 4)$. In fact, in these cases, $C_n - B_n = A_n^2$ for *all* positive integers n; for example, $4444 - 88 = 4356 = 66^2$. The second triple has the greater coordinate sum, $6 + 8 + 4 = 18$.

Problems and solutions were contributed by David Altizio, Risto Atanasov, George Bauer, Ivan Borsenco, Thomas Butts, Barbara Currier, Steve Dunbar, Marta Eso, Zachary Franco, Peter Gilchrist, Jerrold Grossman, Chris Jeuell, Jonathan Kane, Joe Kennedy, Michael Khoury, Hugh Montgomery, Mohamed Omar, Albert Otto, Joachim Rebholz, Michael Tang, David Wells, and Carl Yerger.

The

MAA American Mathematics Competitions

are supported by

Innovator's Circle

The D. E. Shaw Group
Susquehanna International Group
Tudor Investment Corporation
Two Sigma

Winner's Circle

MathWorks

Achiever's Circle

Art of Problem Solving
Jane Street Capital

Sustainer's Circle

American Mathematical Society
Ansatz Capital
Army Educational Outreach Program

Collaborator's Circle

American Statistical Association
Casualty Actuarial Society
Conference Board of the Mathematical Sciences
Mu Alpha Theta
Society for Industrial and Applied Mathematics

MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

MAA American Mathematics Competitions

19th Annual

AMC 10B

American Mathematics Competition 10B

Thursday, February 15, 2018

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions for this contest during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet, or media of any type is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
Phone 800.527.3690 | Fax 240.396.5647 | amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by MAA's Subcommittee on the AMC10/AMC12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (A):** The total area of cornbread is $20 \cdot 18 = 360$ in^2. Because each piece of cornbread has area $2 \cdot 2 = 4$ in^2, the pan contains $360 \div 4 = 90$ pieces of cornbread.

OR

When cut, there are $20 \div 2 = 10$ pieces of cornbread along a long side of the pan and $18 \div 2 = 9$ pieces along a short side, so there are $10 \cdot 9 = 90$ pieces.

2. **Answer (D):** Sam covered $\frac{1}{2} \cdot 60 = 30$ miles during the first 30 minutes and $\frac{1}{2} \cdot 65 = 32.5$ miles during the second 30 minutes, so he needed to cover $96 - 30 - 32.5 = 33.5$ miles during the last 30 minutes. Thus his average speed during the last 30 minutes was
$$\frac{33.5 \text{ miles}}{\frac{1}{2} \text{ hour}} = 67 \text{ mph}.$$

3. **Answer (B):** Both the multiplications and the addition can be performed in either order, so each possible value can be obtained by putting the 1 in the first position and one of the other three numbers in the second position. Therefore the only possible values are
$$\begin{aligned} &(1 \times 2) + (3 \times 4) = 14, \\ &(1 \times 3) + (2 \times 4) = 11, \\ \text{and} \quad &(1 \times 4) + (2 \times 3) = 10, \end{aligned}$$
so just 3 different values can be obtained.

4. **Answer (B):** Without loss of generality, assume that $X \le Y \le Z$. Then the geometric description of the problem can be translated into the system of equations, $XY = 24$, $XZ = 48$, and $YZ = 72$. Dividing the second equation by the first yields $\frac{Z}{Y} = 2$, so $Z = 2Y$. Then $72 = YZ = 2Y^2$, so $Y^2 = 36$. Because Y is positive, $Y = 6$. It follows that $X = 24 \div 6 = 4$ and $Z = 72 \div 6 = 12$, so $X + Y + Z = 22$.

OR

With X, Y, and Z as above, multiply the three equations to give
$$X^2Y^2Z^2 = 24 \cdot 48 \cdot 72 = 24 \cdot 24 \cdot 2 \cdot 24 \cdot 3 = 24^2 \cdot 144 = (24 \cdot 12)^2.$$
Therefore $XYZ = 24 \cdot 12$, and dividing successively by the three equations gives $Z = 12$, $Y = 6$, and $X = 4$, so $X + Y + Z = 22$.

5. **Answer (D):** The number of qualifying subsets equals the difference between the total number of subsets of $\{2,3,4,5,6,7,8,9\}$ and the number of such subsets containing no prime numbers, which is the number of subsets of $\{4,6,8,9\}$. A set with n elements has 2^n subsets, so the requested number is $2^8 - 2^4 = 256 - 16 = 240$.

OR

A subset meeting the condition must be the union of a nonempty subset of $\{2,3,5,7\}$ and a subset of $\{4,6,8,9\}$. There are $2^4 - 1 = 15$ of the former and $2^4 = 16$ of the latter, which gives $15 \cdot 16 = 240$ choices in all.

6. **Answer (D):** Three draws will be required if and only if the first two chips drawn have a sum of 4 or less. The draws $(1,2)$, $(2,1)$, $(1,3)$, and $(3,1)$ are the only draws meeting this condition. There are $5 \cdot 4 = 20$ possible two-chip draws, so the requested probability is $\frac{4}{20} = \frac{1}{5}$. (Note that all 20 possible two-chip draws are considered in determining the denominator, even though some draws will end after the first chip is drawn.)

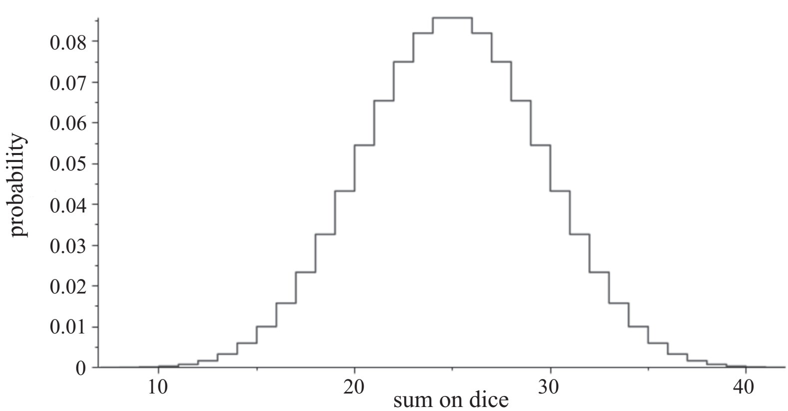
7. **Answer (D):** Suppose without loss of generality that each small semicircle has radius 1; then the large semicircle has radius N. The area of each small semicircle is $\frac{\pi}{2}$, and the area of the large semicircle is $N^2 \cdot \frac{\pi}{2}$. The combined area A of the N small semicircles is $N \cdot \frac{\pi}{2}$, and the area B inside the large semicircle but outside the small semicircles is
$$N^2 \cdot \frac{\pi}{2} - N \cdot \frac{\pi}{2} = \left(N^2 - N\right) \cdot \frac{\pi}{2}.$$
Thus the ratio $A : B$ of the areas is $N : \left(N^2 - N\right)$, which is $1 : (N-1)$. Because this ratio is given to be $1 : 18$, it follows that $N - 1 = 18$ and $N = 19$.

8. **Answer (C):** In the staircase with n steps, the number of vertical toothpicks is
$$1 + 2 + 3 + \cdots + n + n = \frac{n(n+1)}{2} + n.$$
There are an equal number of horizontal toothpicks, for a total of $n(n+1) + 2n$ toothpicks. Solving $n(n+1) + 2n = 180$ with $n > 0$ yields $n = 12$.

OR

By inspection, the number of toothpicks for staircases consisting of 1, 2, and 3 steps are 4, 10, and 18, respectively. The n-step staircase is obtained from the $(n-1)$-step staircase by adding $n+1$ horizontal toothpicks and $n+1$ vertical toothpicks. With this observation, the pattern can be continued so that 28, 40, 54, 70, 88, 108, 130, 154, and 180 are the numbers of toothpicks used to construct staircases consisting of 4 through 12 steps, respectively. Therefore 180 toothpicks are needed for the 12-step staircase.

9. **Answer (D):** Without loss of generality, one can assume that the numbers on opposite faces of each die add up to 7. In other words, the 1 is opposite the 6, the 2 is opposite the 5, and the 3 is opposite the 4. (In fact, standard dice are numbered in this way.) The top faces give a sum of 10 if and only if the bottom faces give a sum of $7 \cdot 7 - 10 = 39$. By symmetry, the probability that the top faces give a sum of 39 is also p. The distribution of the outcomes of the dice rolls has the bell-shaped graph shown below, so no other outcome has the same probability as 10 and 39.



10. **Answer (E):** The volume of the rectangular pyramid with base $BCHE$ and apex M equals the volume of the given rectangular parallelepiped, which is 6, minus the combined volume of triangular prism $AEHDCB$, tetrahedron $BEFM$, and tetrahedron $CGHM$. Tetrahedra $BEFM$ and $CGHM$ each have three right angles at F and G, respectively, and the edges of the tetrahedra emanating from F and G have lengths 2, 3, and $\frac{1}{2}$, so the volume of each of these tetrahedra

is $\frac{1}{6} \cdot (2 \cdot 3 \cdot \frac{1}{2}) = \frac{1}{2}$. The volume of the triangluar prism $AEHDCB$ is 3 because it is half the volume of the rectangular parallelepiped. Therefore the requested volume is $6 - 3 - \frac{1}{2} - \frac{1}{2} = 2$.

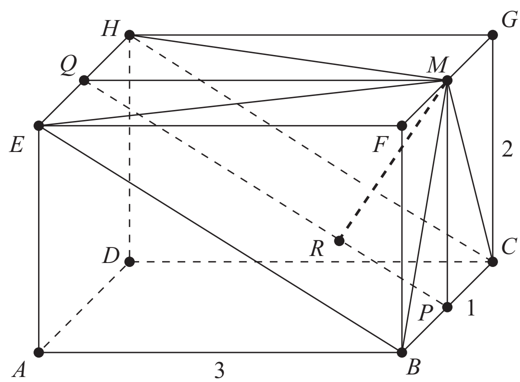
OR

Let P and Q be the midpoints of $\overline{BC}$ and $\overline{EH}$, respectively. By the Pythagorean Theorem $PQ = \sqrt{13}$. Let R be the foot of the perpendicular from M to $\overline{PQ}$. Then $\triangle PMQ \sim \triangle PRM$, so

$$\frac{3}{\sqrt{13}} = \frac{MQ}{PQ} = \frac{RM}{PM} = \frac{RM}{2} \qquad \text{and} \qquad RM = \frac{6}{\sqrt{13}}.$$

The requested volume of the pyramid is $\frac{1}{3}$ times the area of the base times the height, which is

$$\frac{1}{3} \cdot \left(\sqrt{13} \cdot 1\right) \cdot \frac{6}{\sqrt{13}} = 2.$$



11. **Answer (C):** If $p = 3$, then $p^2 + 26 = 35 = 5 \cdot 7$. If p is a prime number other than 3, then $p = 3k \pm 1$ for some positive integer k. In that case

$$p^2 + 26 = (3k \pm 1)^2 + 26 = 9k^2 \pm 6k + 27 = 3(3k^2 \pm 2k + 9)$$

is a multiple of 3 and is not prime. The smallest counterexamples for the other choices are $5^2 + 16 = 41$, $7^2 + 24 = 73$, $5^2 + 46 = 71$, and $19^2 + 96 = 457$.

12. **Answer (C):** Let O be the center of the circle. Triangle ABC is a right triangle, and O is the midpoint of the hypotenuse $\overline{AB}$. Then

$\overline{OC}$ is a radius, and it is also one of the medians of the triangle. The centroid is located one third of the way along the median from O to C, so the centroid traces out a circle with center O and radius $\frac{1}{3} \cdot 12 = 4$ (except for the two missing points corresponding to $C = A$ or $C = B$). The area of this smaller circle is then $\pi \cdot 4^2 = 16\pi \approx 16 \cdot 3.14 \approx 50$.

13. **Answer (C):** The numbers in the given sequence are of the form $10^n + 1$ for $n = 2, 3, \ldots, 2019$. If n is even, say $n = 2k$ for some positive integer k, then $10^n + 1 = 100^k + 1 \equiv (-1)^k + 1 \pmod{101}$. Thus $10^n + 1$ is divisible by 101 if and only if k is odd, which means $n = 2, 6, 10, \ldots, 2018$. There are $\frac{1}{4}(2018 - 2) + 1 = 505$ such values. On the other hand, if n is odd, say $n = 2k + 1$ for some positive integer k, then

$$10^n + 1 = 10 \cdot 10^{n-1} + 1 = 10 \cdot 100^k + 1 \equiv 10 \cdot (-1)^k + 1 \pmod{101},$$

which is congruent to 9 or 11, and $10^n + 1$ is not divisible by 101 in this case.

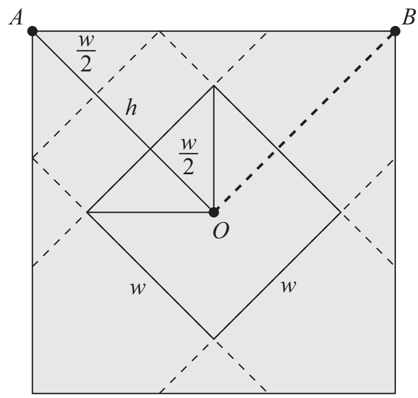
14. **Answer (D):** The list has $2018 - 10 = 2008$ entries that are not equal to the mode. Because the mode is unique, each of these 2008 entries can occur at most 9 times. There must be at least $\left\lceil \frac{2008}{9} \right\rceil = 224$ distinct values in the list that are different from the mode, because if there were fewer than this many such values, then the size of the list would be at most $9 \cdot (\left\lceil \frac{2008}{9} \right\rceil - 1) + 10 = 2017 < 2018$. (The ceiling function notation $\lceil x \rceil$ represents the least integer greater than or equal to x.) Therefore the least possible number of distinct values that can occur in the list is 225. One list satisfying the conditions of the problem contains 9 instances of each of the numbers 1 through 223, 10 instances of the number 224, and one instance of 225.

15. **Answer (A):** The figure shows that the distance AO from a corner of the wrapping paper to the center is

$$\frac{w}{2} + h + \frac{w}{2} = w + h.$$

The side of the wrapping paper, $\overline{AB}$ in the figure, is the hypotenuse of a $45-45-90°$ right triangle, so its length is $\sqrt{2} \cdot AO = \sqrt{2}(w + h)$. Therefore the area of the wrapping paper is

$$\left(\sqrt{2}(w + h)\right)^2 = 2(w + h)^2.$$



OR

The area of the wrapping paper, excluding the four small triangles indicated by the dashed lines, is equal to the surface area of the box, which is $2w^2 + 4wh$. The four triangles are isosceles right triangles with leg length h, so their combined area is $4 \cdot \frac{1}{2}h^2 = 2h^2$. Thus the total area of the wrapping paper is $2w^2 + 4wh + 2h^2 = 2(w+h)^2$.

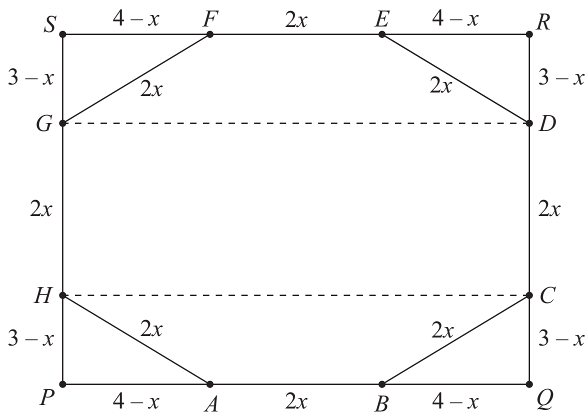
16. **Answer (E):** Let n be an integer. Because $n^3 - n = (n-1)n(n+1)$, it follows that $n^3 - n$ has at least one prime factor of 2 and one prime factor of 3 and therefore is divisible by 6. Thus $n^3 \equiv n \pmod 6$. Then

$$a_1^3 + a_2^3 + \cdots + a_{2018}^3 \equiv a_1 + a_2 + \cdots + a_{2018} \equiv 2018^{2018} \pmod 6.$$

Because $2018 \equiv 2 \pmod 6$, the powers of 2018 modulo 6 are alternately $2, 4, 2, 4, \ldots$, so $2018^{2018} \equiv 4 \pmod 6$. Therefore the remainder when $a_1^3 + a_2^3 + \cdots + a_{2018}^3$ is divided by 6 is 4.

17. **Answer (B):** Because $AP < 4 = \frac{1}{2}PQ$, it follows that A is closer to P than it is to Q and that A is between points P and B. Because $AP = BQ$, $AH = BC$, and angles APH and BQC are right angles, $\triangle APH \cong \triangle BQC$. Thus $PH = QC$, and $PQCH$ is a rectangle. Because $CD = HG$, it follows that $HCDG$ is also a rectangle. Thus $GDRS$ is a rectangle and $DR = GS$, and it follows that $\triangle ERD \cong \triangle FSG$. Therefore segment $\overline{EF}$ is centered in $\overline{RS}$ just as congruent segment $\overline{AB}$ is centered in $\overline{PQ}$. Therefore $\triangle ERD \cong \triangle BQC$, and $\overline{CD}$ is also centered in $\overline{QR}$. Let $2x$ be the side length $AB = BC =$

$CD = DE = EF = FG = GH = HA$ of the regular octagon; then $AP = BQ = 4 - x$ and $QC = RD = 3 - x$. Applying the Pythagorean Theorem to $\triangle BQC$ yields $(4-x)^2 + (3-x)^2 = (2x)^2$, which simplifies to $2x^2 + 14x - 25 = 0$. Thus $x = \frac{1}{2} \cdot (-7 \pm 3\sqrt{11})$, and because $x > 0$, it follows that $2x = -7 + 3\sqrt{11}$. Hence $k + m + n = -7 + 3 + 11 = 7$.



18. **Answer (D):** Let X, Y, and Z denote the three different families in some order. Then the only possible arrangements are to have the second row be members of XYZ and the third row be members of ZXY, or to have the second row be members of XYZ and the third row be members of YZX. Note that these are not the same, because in the first case one sibling pair occupy the right-most seat in the second row and the left-most seat in the third row, whereas in the second case this does not happen. (Having members of XYX in the second row does not work because then the third row must be members of ZYZ to avoid consecutive members of Z; but in this case one of the Y siblings would be seated directly in front of the other Y sibling.) In each of these 2 cases there are $3! = 6$ ways to assign the families to the letters and $2^3 = 8$ ways to position the boy and girl within the seats assigned to the families. Therefore the total number of seating arrangements is $2 \cdot 6 \cdot 8 = 96$.

19. **Answer (E):** Let Chloe be n years old today, so she is $n - 1$ years older than Zoe. For integers $y \geq 0$, Chloe's age will be a multiple of Zoe's age y years from now if and only if

$$\frac{n+y}{1+y} = 1 + \frac{n-1}{1+y}$$

is an integer, that is, $1+y$ is a divisor of $n-1$. Thus $n-1$ has exactly 9 positive integer divisors, so the prime factorization of $n-1$ has one of the two forms p^2q^2 or p^8. There are no two-digit integers of the form p^8, and the only one of the form p^2q^2 is $2^2 \cdot 3^2 = 36$. Therefore Chloe is 37 years old today, and Joey is 38. His age will be a multiple of Zoe's age in y years if and only if $1+y$ is a divisor of $38-1=37$. The nonnegative integer solutions for y are 0 and 36, so the only other time Joey's age will be a multiple of Zoe's age will be when he is $38+36=74$ years old. The requested sum is $7+4=11$.

20. **Answer (B):** Applying the recursion for several steps leads to the conjecture that

$$f(n) = \begin{cases} n+2 & \text{if } n \equiv 0 \pmod 6, \\ n & \text{if } n \equiv 1 \pmod 6, \\ n-1 & \text{if } n \equiv 2 \pmod 6, \\ n & \text{if } n \equiv 3 \pmod 6, \\ n+2 & \text{if } n \equiv 4 \pmod 6, \\ n+3 & \text{if } n \equiv 5 \pmod 6. \end{cases}$$

The conjecture can be verified using the strong form of mathematical induction with two base cases and six inductive steps. For example, if $n \equiv 2 \pmod 6$, then $n = 6k+2$ for some nonnegative integer k and

$$\begin{aligned} f(n) &= f(6k+2) \\ &= f(6k+1) - f(6k) + 6k + 2 \\ &= (6k+1) - (6k+2) + 6k + 2 \\ &= 6k+1 \\ &= n-1. \end{aligned}$$

Therefore $f(2018) = f(6 \cdot 336 + 2) = 2018 - 1 = 2017$.

OR

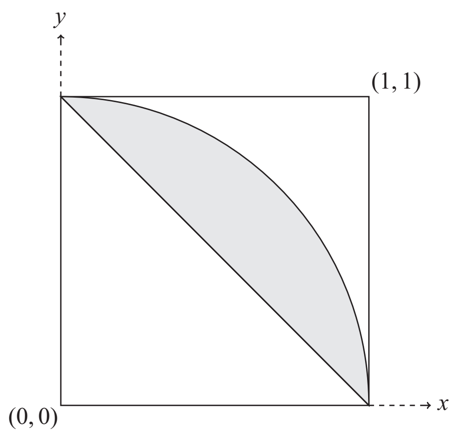
Note that

$$\begin{aligned} f(n) &= f(n-1) - f(n-2) + n \\ &= [f(n-2) - f(n-3) + (n-1)] - f(n-2) + n \\ &= -[f(n-4) - f(n-5) + (n-3)] + 2n - 1 \\ &= -[f(n-5) - f(n-6) + (n-4)] + f(n-5) + n + 2 \\ &= f(n-6) + 6. \end{aligned}$$

It follows that $f(2018) = f(2) + 2016 = 2017$.

21. **Answer (C):** Let d be the next divisor of n after 323. Then $\gcd(d, 323) \neq 1$, because otherwise $n \geq 323d > 323^2 > 100^2 = 10000$, contrary to the fact that n is a 4-digit number. Therefore $d - 323 \geq \gcd(d, 323) > 1$. The prime factorization of 323 is $17 \cdot 19$. Thus the next divisor of n is at least $323 + 17 = 340 = 17 \cdot 20$. Indeed, 340 will be the next number in Mary's list when $n = 17 \cdot 19 \cdot 20 = 6460$.

22. **Answer (C):** The set of all possible ordered pairs (x, y) occupies the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$ in the Cartesian plane. The numbers x, y, and 1 are the side lengths of a triangle if and only if $x + y > 1$, which means that (x, y) lies above the line $y = 1 - x$. By a generalization of the Pythagorean Theorem, the triangle is obtuse if and only if, in addition, $x^2 + y^2 < 1^2$, which means that (x, y) lies inside the circle of radius 1 centered at the origin. Within the unit square, the region inside the circle of radius 1 centered at the origin has area $\frac{\pi}{4}$, and the region below the line $y = 1 - x$ has area $\frac{1}{2}$. Therefore the ordered pairs that meet the required conditions occupy a region with area $\frac{\pi}{4} - \frac{1}{2} = \frac{\pi - 2}{4}$. The area of the unit square is 1, so the required probability is also $\frac{\pi-2}{4} \approx \frac{1.14}{4} = 0.285$, which is closest to 0.29.



23. **Answer (B):** Recall that $a \cdot b = \gcd(a, b) \cdot \operatorname{lcm}(a, b)$. Let $x = \operatorname{lcm}(a, b)$ and $y = \gcd(a, b)$. The given equation is then $xy + 63 = 20x + 12y$, which can be rewritten as

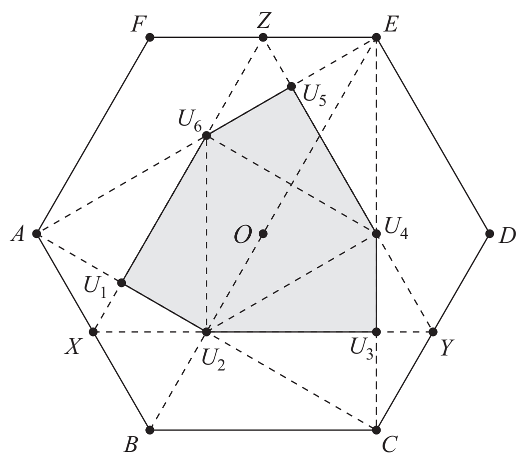
$$(x - 12)(y - 20) = 240 - 63 = 177 = 3 \cdot 59 = 1 \cdot 177.$$

Because x and y are integers, one of the following must be true:

- $x - 12 = 1$ and $y - 20 = 177$,
- $x - 12 = 177$ and $y - 20 = 1$,
- $x - 12 = 3$ and $y - 20 = 59$,
- $x - 12 = 59$ and $y - 20 = 3$.

Therefore (x, y) must be $(13, 197)$, $(189, 21)$, $(15, 79)$, or $(71, 23)$. Because x must be a multiple of y, only $(x, y) = (189, 21)$ is possible. Therefore $\gcd(a, b) = 21 = 7 \cdot 3$, and $\operatorname{lcm}(a, b) = 189 = 7 \cdot 3^3$. Both a and b are divisible by 7 but not by 7^2; one of a and b is divisible by 3 but not 3^2, and the other is divisible by 3^3 but not 3^4; and neither is divisible by any other prime. Therefore one of them is $7 \cdot 3 = 21$ and the other is $7 \cdot 3^3 = 189$. There are 2 ordered pairs, $(a, b) = (21, 189)$ and $(a, b) = (189, 21)$.

24. **Answer (C):** Let O be the center of the regular hexagon. Points B, O, E are collinear and $BE = BO + OE = 2$. Trapezoid $FABE$ is isosceles, and $\overline{XZ}$ is its midline. Hence $XZ = \frac{3}{2}$ and analogously $XY = ZY = \frac{3}{2}$.



Denote by U_1 the intersection of $\overline{AC}$ and $\overline{XZ}$ and by U_2 the intersection of $\overline{AC}$ and $\overline{XY}$. It is easy to see that $\triangle AXU_1$ and $\triangle U_2XU_1$ are congruent $30-60-90°$ right triangles.

By symmetry the area of the convex hexagon enclosed by the intersection of $\triangle ACE$ and $\triangle XYZ$, shaded in the figure, is equal to the area of $\triangle XYZ$ minus 3 times the area of $\triangle U_2XU_1$. The hypotenuse of $\triangle U_2XU_1$ is $XU_2 = AX = \frac{1}{2}$, so the area of $\triangle U_2XU_1$ is

$$\frac{1}{2} \cdot \frac{\sqrt{3}}{4} \cdot \left(\frac{1}{2}\right)^2 = \frac{1}{32}\sqrt{3}.$$

The area of the equilateral triangle XYZ with side length $\frac{3}{2}$ is equal to $\frac{1}{4}\sqrt{3} \cdot \left(\frac{3}{2}\right)^2 = \frac{9}{16}\sqrt{3}$. Hence the area of the shaded hexagon is

$$\frac{9}{16}\sqrt{3} - 3 \cdot \frac{1}{32}\sqrt{3} = 3\sqrt{3}\left(\frac{3}{16} - \frac{1}{32}\right) = \frac{15}{32}\sqrt{3}.$$

OR

Let U_1 and U_2 be as above, and continue labeling the vertices of the shaded hexagon counterclockwise with U_3, U_4, U_5, and U_6 as shown. The area of $\triangle ACE$ is half the area of hexagon $ABCDEF$. Triangle $U_2U_4U_6$ is the midpoint triangle of $\triangle ACE$, so its area is $\frac{1}{4}$ of the area of $\triangle ACE$, and thus $\frac{1}{8}$ of the area of $ABCDEF$. Each of $\triangle U_2U_3U_4$, $\triangle U_4U_5U_6$, and $\triangle U_6U_1U_2$ is congruent to half of $\triangle U_2U_4U_6$, so the total shaded area is $\frac{5}{2}$ times the area of $\triangle U_2U_4U_6$ and therefore $\frac{5}{2} \cdot \frac{1}{8} = \frac{5}{16}$ of the area of $ABCDEF$. The area of $ABCDEF$ is $6 \cdot \frac{\sqrt{3}}{4} \cdot 1^2$, so the requested area is $\frac{15}{32}\sqrt{3}$.

25. **Answer (C):** Let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x. Then $0 \le \{x\} < 1$. The given equation is equivalent to $x^2 = 10{,}000\{x\}$, that is,

$$\frac{x^2}{10{,}000} = \{x\}.$$

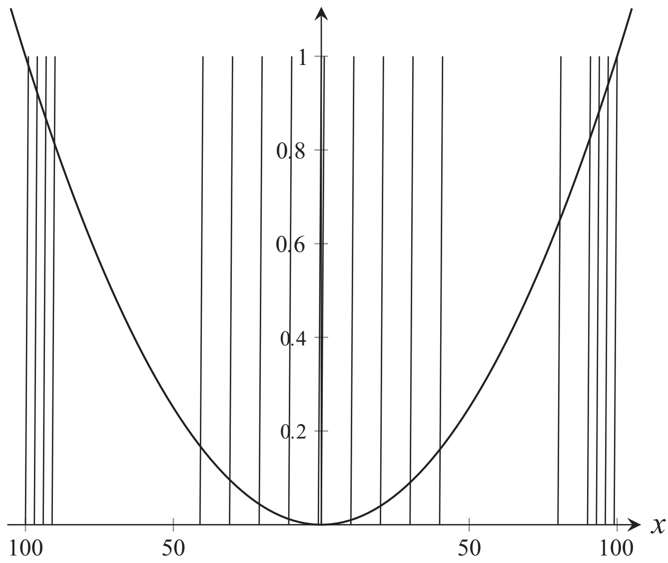
Therefore if x satisfies the equation, then

$$0 \le \frac{x^2}{10{,}000} < 1.$$

This implies that $x^2 < 10{,}000$, so $-100 < x < 100$. The figure shows a sketch of the graphs of

$$f(x) = \frac{x^2}{10{,}000} \qquad \text{and} \qquad g(x) = \{x\}$$

for $-100 < x < 100$ on the same coordinate axes. The graph of g consists of the 200 half-open line segments with slope 1 connecting the points $(k, 0)$ and $(k + 1, 1)$ for $k = -100, -99, \ldots, 98, 99$. (The endpoints of these intervals that lie on the x-axis are part of the graph, but the endpoints with y-coordinate 1 are not.) It is clear that there is one intersection point for x lying in each of the intervals $[-100, -99)$, $[-99, -98)$, $[-98, -97)$, $\ldots$, $[-1, 0)$, $[0, 1)$, $[1, 2)$, $\ldots$, $[97, 98)$, $[98, 99)$ but no others. Thus the equation has 199 solutions.



OR

The solutions to the equation correspond to points of intersection of the graphs $y = 10000\lfloor x \rfloor$ and $y = 10000x - x^2$. There will be a point of intersection any time the parabola intersects the half-open horizontal segment from the point $(a, 10000a)$ to the point $(a+1, 10000a)$, where a is an integer. This occurs for every integer value of a for which

$$10000a - a^2 \leq 10000a < 10000(a+1) - (a+1)^2.$$

This is equivalent to $(a+1)^2 < 10000$, which occurs if and only if $-101 < a < 99$. Thus points of intersection occur on the intervals $[a, a+1)$ for $a = -100, -99, -98, \ldots, -1, 0, 1, \ldots, 97, 98$, resulting in 199 points of intersection.

Problems and solutions were contributed by Risto Atanasov, Chris Bolognese, Ivan Borsenco, Thomas Butts, Barbara Currier, Steven Davis, Steve Dunbar, Marta Eso, Zuming Feng, Jerrold Grossman, Chris Jeuell, Jonathan Kane, Joe Kennedy, Michael Khoury, Norbert Kuenzi, Hugh Montgomery, Mohamed Omar, Albert Otto, Joachim Rebholz, David Wells, and Carl Yerger.

The

MAA American Mathematics Competitions

are supported by

Innovator's Circle

The D. E. Shaw Group
Susquehanna International Group
Tudor Investment Corporation
Two Sigma

Winner's Circle

MathWorks

Achiever's Circle

Art of Problem Solving
Jane Street Capital

Sustainer's Circle

American Mathematical Society
Ansatz Capital
Army Educational Outreach Program

Collaborator's Circle

American Statistical Association
Casualty Actuarial Society
Conference Board of the Mathematical Sciences
Mu Alpha Theta
Society for Industrial and Applied Mathematics

2018 AMC 10B Answer Key

Problem 1

- **Answer:** A
- **Solution 1:** The area of the pan is $20 \cdot 18$ = 360. Since the area of each piece is 4, there are $\frac{360}{4} = 90$ pieces. Thus, the answer is $\boxed{A}$.
- **Solution 2:** By dividing the each of the dimensions by 2, we get a 10×9 grid which makes 90 pieces. Thus, the answer is $\boxed{A}$.

Problem 2

- **Answer:** D
- **Solution:** Let Sam drive at exactly 60 mph in the first half hour, 65 mph in the second half hour, and x mph in the third half hour. Due to $rt = d$, and that 30 min is half an hour, he covered $60 \cdot \frac{1}{2} = 30$ miles in the first 30 mins. SImilarly, he covered $\frac{65}{2}$ miles in the 2nd half hour period. The problem states that Sam drove 96 miles in 90 min, so that means that he must have covered $96 - \left(30 + \frac{65}{2}\right) = 33\frac{1}{2}$ miles in the third half hour period. $rt = d$, so $x \cdot \frac{1}{2} = 33\frac{1}{2}$. Therefore, Sam was driving $\boxed{\textbf{(D) } 67}$ miles per hour in the third half hour.

Problem 3

- **Answer:** B
- **Solution 1:** We have $\binom{4}{2}$ ways to choose the pairs, and we have 2 ways for the values to be switched so $\frac{6}{2} = \boxed{3.}$ (harry1234)
- **Solution 2:** We have four available numbers $(1, 2, 3, 4)$. Because different permutations do not matter because they are all addition and multiplication, if we put 1 on the first space, it is obvious there are $\boxed{3}$ possible outcomes $(2, 3, 4)$.

Problem 4

- Answer: B
- **Solution 1:** Let X be the length of the shortest dimension and Z be the length of the longest dimension. Thus, $XY = 24$, $YZ = 72$, and $XZ = 48$. Divide the first to equations to get $\frac{Z}{X} = 3$. Then, multiply by the last equation to get $Z^2 = 144$, giving $Z = 12$. Following, $X = 4$ and $Y = 6$. The final answer is $4 + 6 + 12 = 22$. $\boxed{B}$

- **Solution 2:** Simply use guess and check to find that the dimensions are 4 by 6 by 12. Therefore, the answer is $4+6+12=22$. $\boxed{B}$

Problem 5:

- **Answer:** D
- **Solution 1:** Consider finding the number of subsets that do not contain any primes. There are four primes in the set: 2, 3, 5, and 7. This means that the number of subsets without any primes is the number of subsets of $\{4,6,8,9\}$, which is just $2^4=16$. The number of subsets with at least one prime is the number of subsets minus the number of subsets without any primes. The number of subsets is $2^8=256$. Thus, the answer is $256-16=240$. $\boxed{D}$
- **Solution 2 (Using Answer Choices):** Well, there are 4 composite numbers, and you can list them in a 1 number format, a 2 number, 3 number, and a 4 number format. Now, we can use permutations $\binom{4}{1}+\binom{4}{2}+\binom{4}{3}+\binom{4}{4}=15$. Using the answer choices, the only multiple of 15 is $\boxed{\textbf{(D) }240}$

Problem 6:

- **Answer:** D
- **Solution 1:** Notice that the only two ways such that more than 2 draws are required are $1,2$; $1,3$; $2,1$; and $3,1$. Notice that each of those cases has a $\frac{1}{5}\cdot\frac{1}{4}$ chance, so the answer is $\frac{1}{5}\cdot\frac{1}{4}\cdot 4=\frac{1}{5}$, or $\boxed{D}$.
- **Solution 2:** Notice that only the first two draws are important, it doesn' t matter what number we get third because no matter what combination of 3 numbers is picked, the sum will always be greater than 5. Also, note that it is necessary to draw a 1 in order to have 3 draws, otherwise 5 will be attainable in two or less draws. So the probability of getting a 1 is $\frac{1}{5}$. It is necessary to pull either a 2 or 3 on the next draw and the probability of that is $\frac{1}{2}$. But, the order of the draws can be switched so we get: $\frac{1}{5}\cdot\frac{1}{2}\cdot 2=\frac{1}{5}$, or $\boxed{D}$ By: Soccer_JAMS

Problem 7:

- **Answer:** D
- **Solution 1 (Work using Answer Choices) :** Use the answer choices and calculate them. The one that works is D
- **Solution 2 (More Algebraic Approach):** Let the number of semicircles be n and let the radius of each semicircle to be r. To find the total area of all of the small semicircles, we have $n\cdot\frac{\pi\cdot r^2}{2}$. Next, we have to find the area of the larger semicircle. The radius of the large semicircle can be deduced to be $n\cdot r$. So, the area of the larger semicircle is $\frac{\pi\cdot n^2\cdot r^2}{2}$. Now that we have found the area of both A and B, we can find the ratio. $\frac{A}{B}=\frac{1}{18}$, so part-to-whole ratio is $1:19$.When we divide the area of the small semicircles combined by the

area of the larger semicircles, we get $\frac{1}{n}$. This is equal to $\frac{1}{19}$. By setting them equal, we find that $n = 19$. This is our answer, which corresponds to choice $\boxed{\text{(D)}19}$. Solution by: Archimedes15; edited correct by kevinmathz

Problem 8:

- **Answer:** C
- **Solution 1:** A staircase with n steps contains $4 + 6 + 8 + ... + 2n + 2$ toothpicks. This can be rewritten as $(n+1)(n+2) - 2$.So, $(n+1)(n+2) - 2 = 180$ So, $(n+1)(n+2) = 182.$ Inspection could tell us that $13 * 14 = 182$, so the answer is $13 - 1 = \boxed{(C) 12}$
- **Solution 2:** Layer 1: 4 stepsLayer $1, 2$: 10 stepsLayer $1, 2, 3$: 18 stepsLayer $1, 2, 3, 4$: 28 stepsFrom inspection, we can see that with each increase in layer the difference in toothpicks between the current layer and the previous increases by 2. Using this pattern:
  $4, 10, 18, 28, 40, 54, 70, 88, 108, 130, 154, 180$

  From this we see that the solution is indeed $\boxed{(C) 12}$

  By: Soccer_JAMS

Problem 9:

- **Answer:** D
- **Solution 1:** It can be seen that the probability of rolling the smallest number possible is the same as the probability of rolling the largest number possible, the probability of rolling the second smallest number possible is the same as the probability of rolling the second largest number possible, and so on. This is because the number of ways to add a certain number of ones to an assortment of 7 ones is the same as the number of ways to take away a certain number of ones from an assortment of 7 6s. So, we can match up the values to find the sum with the same probability as 10. We can start by noticing that 7 is the smallest possible roll and 42 is the largest possible role. The pairs with the same probability are as follows: (7, 42), (8, 41), (9, 40), (10, 39), (11, 38)... However, we need to find the number that matches up with 10. So, we can stop at (10, 39) and deduce that the sum with equal probability as 10 is 39. So, the correct answer is $\textbf{(D)}\ 39$, and we are done. Written By: Archimedes15
- **Solution 2:** Let' s call the unknown value x. By symmetry, we realize that the difference between 10 and the minimum value of the rolls is equal to the difference between the maximum and x. So,$10 - 7 = 42 - x$, $x = 39$ and our answer is $\textbf{(D)}$ By: Soccer_JAMS
- **Solution 3 (Simple Logic):** For the sums to have equal probability, the average sum of both sets of 7 dies has to be (6+1) x 7 = 49. Since having 10 is similar to not having 10, you just subtract 10 from the expected total sum. 49 – 10 = 39 so the answer is $\textbf{(D)}$ By: epicmonster
- **Solution 4:** The expected value of the sums of the die rolls is $3.5 * 7 = 24.5$, and since the probabilities should be distributed symmetrically on both sides of 24.5, the answer is $24.5 + 24.5 - 10 = 39$, which is $\textbf{(D)}$. By: dajeff

Problem 10:

- **Answer:** E

- **Solution 1**: Consider the cross-sectional plane and label its area b. Note that the volume of the triangular prism that encloses the pyramid is $bh/2 = 3$, and we want the rectangular pyramid that shares the base and height with the triangular prism. The volume of the pyramid is $bh/3$, so the answer is $\boxed{2}$. (AOPS12142015)
- **Solution 2:** We can start by finding the total volume of the parallelepiped. It is $2 \cdot 3 \cdot 1 = 6$, because a rectangular parallelepiped is a rectangular prism. Next, we can consider the wedge-shaped section made when the plane $BCHE$ cuts the figure. We can find the volume of the triangular pyramid with base EFB and apex M. The area of EFB is $\frac{1}{2} \cdot 2 \cdot 3 = 3$. Since BC is given to be 1, we have that FM is $\frac{1}{2}$. Using the formula for the volume of a triangular pyramid, we have $V = \frac{1}{3} \cdot \frac{1}{2} \cdot 3 = \frac{1}{2}$. Also, since the triangular pyramid with base HGC and apex M has the exact same dimensions, it has volume $\frac{1}{2}$ as well. The original wedge we considered in the last step has volume 3, because it is half of the volume of the parallelepiped. We can subtract out the parts we found to have $3 - \frac{1}{2} \cdot 2 = 2$. Thus, the volume of the figure we are trying to find is 2. This means that the correct answer choice is $\boxed{E}$. Written by: Archimedes15NOTE: For those who think that it isn't a rectangular prism, please read the problem. It says "rectangular parallelepiped." If a parallelepiped is such that all of the faces are rectangles, it is a rectangular prism.

Problem 11:

- **Answer:** C
- **Solution 1:** Because squares of a non-multiple of 3 is always $1 \mod 3$, the only expression is always a multiple of 3 is $\boxed{\textbf{(C)}\ p^2+26}$. This is excluding when $p = 0 \mod 3$, which only occurs when $p = 3$, then $p^2 + 26 = 35$ which is still composite.
- **Solution 2 (Bad Solution):** We proceed with guess and check: $5^2 + 16 = 41 \qquad 7^2 + 24 = 73 \qquad 5^2 + 46 = 71 \qquad 19^2 + 96 = 457$. Clearly only $\boxed{\textbf{(C)}}$ is our only option left. (franchester)
- **Solution 3:** From Fermat's Little Theorom, $p^2 \equiv 1 \pmod 3$ if p is coprime with 3. So for any $n \equiv 2 \pmod 3$, $p^2 + n \equiv 0 \pmod 3$ – divisible by 3, so not a prime. The only choice $\equiv 2 \pmod 3$ is $\boxed{\textbf{(C)}}$
- **Solution 4:** Primes can only be 1 or $-1 \mod 6$. Therefore, the square of a prime can only be $1 \mod 6$. $p^2 + 26$ then must be $3 \mod 6$, so it is always divisible by 3. Therefore, the answer is $\boxed{\text{(C)}}$.

Problem 12:

- **Answer:** C
- **Solution 1:** Let $A = (-12, 0), B = (12, 0)$. Therefore, C lies on the circle with equation $x^2 + y^2 = 144$. Let it have coordinates (x, y). Since we know the centroid of a triangle with vertices with coordinates of $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is $\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$, the centroid of $\triangle ABC$ is $\left(\frac{x}{3}, \frac{y}{3}\right)$. Because $x^2 + y^2 = 144$, we know that $\left(\frac{x}{3}\right)^2 + \left(\frac{y}{3}\right)^2 = 16$, so the curve is a circle centered at the origin. Therefore, its area is $16\pi \approx \boxed{\text{(C) } 50}$. -tdeng

- **Solution 2 (no coordinates):** We know the centroid of a triangle splits the medians into segments of ratio 2:1, and the median of the triangle that goes to the center of the circle is the radius (length 12), so the length from the centroid of the triangle to the center of the circle is always $\frac{1}{3} \cdot 12 = 4$. The area of a circle with radius 4 is 16π, or around $\boxed{\textbf{(C) } 50}$. -That_Crazy_Book_Nerd

Problem 13

- **Answer:** C
- **Solution 1:** Note that $10^{2k}+1$ for some odd k will suffice $\mod 101$. Each $2k \in \{2,4,6,\dots,2018\}$, so the answer is $\boxed{\textbf{(C) } 505}$ (AOPS12142015)
- **Solution 2 :** If we divide each number by 101, we see a pattern occuring in every 4 numbers. $101, 1000001, 10000000001, \dots$. We divide 2018 by 4 to get 504 with 2 left over. One divisible number will be in the 2 left over, so out answer is $\boxed{\textbf{(C) } 505}$.
- **Solution 3:** Note that 909 is divisible by 101, and thus 9999 is too. We know that 101 is divisible and 1001 isn' t so let us start from 10001. We subtract 9999 to get 2. Likewise from 100001 we subtract, but we instead subtract 9999 times 10 or 99990 to get 11. We do it again and multiply the 9' s by 10 to get 101. Following the same knowledge, we can use mod 101 to finish the problem. The sequence will just be subtracting 1, multiplying by 10, then adding 1. Thus the sequence is $0, -9, -99(2), 11, 0, \dots$ Thus it repeats every four. Consider the sequence after the 1st term and we have 2017 numbers. Divide 2017 by four to get 504 remainder 1. Thus the answer is 504 plus the 1st term or $\boxed{\textbf{(C) } 505}$. -googleghosh

Problem 14

- **Answer:** D
- **Solution:** To minimize the number of values, we want to maximize the number of times they appear. So, we could have 223 numbers appear 9 times, 1 number appear once, and the mode appear 10 times, giving us a total of $223+1+1$ = $\boxed{\textbf{(D) } 225}$

Problem 15

- Answer: A
- **Solution:** Consider one-quarter of the image (the wrapping paper is divided up into 4 congruent squares). The length of each dotted line is h. The area of the rectangle that is w by h is wh. The combined figure of the two triangles with base h is a square with h as its diagonal. Using the Pythagorean Theorem, each side of this square is $\sqrt{\frac{h^2}{2}}$. Thus, the area is the side length squared which is $\frac{h^2}{2}$. Similarly, the combined figure of the two triangles with base w is a square with area $\frac{w^2}{2}$ Adding all of these together, we get $\frac{w^2}{2}+\frac{h^2}{2}+wh$. Since we have four of these areas in the entire wrapping paper, we multiply this by 4, getting $4(\frac{w^2}{2}+\frac{h^2}{2}+wh) = 2(w^2+h^2+2wh) = \boxed{\textbf{(A) } 2(w+h)^2}$.

Problem 16

- Answer：E
- **Solution 1：** By Euler's Totient Theorem, $n^3 \equiv n \pmod 6$ Alternatively, one could simply list out all the residues to the third power $\mod 6$ Therefore the answer is congruent to $2018^{2018} \equiv 2^{2018} \pmod 6 = \boxed{(E) 4}$
- **Solution 2 (not very good one)：** Note that $(a_1+a_2+\cdots+a_{2018})^3 = a_1^3 + a_2^3 + \cdots + a_{2018}^3 + 3a_1^2(a_1+a_2+\cdots+a_{2018}-a_1) + 3a_2^2(a_1+a_2+\cdots+a_{2018}-a_2) + \cdots + 3a_{2018}^2(a_1+a_2+\cdots+a_{2018}-a_{2018}) + 6\prod_{i\neq j\neq k}^{2018} a_ia_ja_k$ Note that $a_1^3+a_2^3+\cdots+a_{2018}^3+3a_1^2(a_1+a_2+\cdots+a_{2018}-a_1)+3a_2^2(a_1+a_2+\cdots+a_{2018}-a_2)+\cdots+3a_{2018}^2(a_1+a_2+\cdots+a_{2018}-a_{2018})+6\prod_{i\neq j\neq k}^{2018} a_ia_ja_k \equiv a_1^3+a_2^3+\cdots+a_{2018}^3+3a_1^2(2018-a_1)+3a_2^2(2018-a_2)+\cdots+3a_{2018}^2(2018-a_{2018}) \equiv -2(a_1^3+a_2^3+\cdots+a_{2018}^3) \pmod 6$ Therefore, $-2(a_1^3+a_2^3+\cdots+a_{2018}^3) \equiv (2018^{2018})^3 \equiv (2^{2018})^3 \equiv 4^3 \equiv 4 \pmod 6$. Thus, $a_1^3+a_2^3+\cdots+a_{2018}^3 \equiv 1 \pmod 3$. However, since cubing preserves parity, and the sum of the individual terms is even, the some of the cubes is also even, and our answer is $\boxed{\text{(E) } 4}$
- **Solution 3：** We first note that $1^3+2^3+\ldots=(1+2+\ldots)^2$. So what we are trying to find is what $(2018^{2018})^2=(2018^{4036})$ mod 6. We start by noting that 2018 is congruent to 2 mod 6. So we are trying to find (2^{4036}) mod 6. Instead of trying to do this with some number theory skills, we could just look for a pattern. We start with small powers of 2 and see that 2^1 is 2 mod 6, 2^2 is 4 mod 6, 2^3 is 2 mod 6, 2^4 is 4 mod 6, and so on... So we see that since (2^{4036}) has an even power, it must be congruent to 4 mod 6, thus giving our answer $\boxed{\text{(E) } 4}$. You can prove this pattern using mods. But I thought this was easier. -TheMagician
- **Solution 4 (Lazy solution)：** Assume $a_1, a_2, \ldots a_{2017}$ are multiples of 6 and find $2018^{2018} mod 6$ (which happens to be 4). Then ${a_1}^3+\ldots+{a_{2018}}^3$ is congruent to $64 mod 6$ or just 4. -Patrick4Presiden

Problem 17

- Answer：B
- **Solution 1：** Let $AP=BQ=x$. Then $AB=8-2x$. Now notice that since $CD=8-2x$ we have $QC=DR=x-1$. Thus by the Pythagorean Theorem we have $x^2+(x-1)^2=(8-2x)^2$ which becomes $2x^2-30x+63=0 \implies x=\frac{15-3\sqrt{11}}{2}$. Our answer is $8-(15-3\sqrt{11})=3\sqrt{11}-7 \implies \boxed{\text{(B) } 7}$. (Mudkipswims42)
- **Solution 2：** Denote the length of the equilateral octagon as x. The length of $\overline{BQ}$ can be expressed as $\frac{8-x}{2}$. By Pythagoras, we find that:

$$\left(\frac{8-x}{2}\right)^2+\overline{CQ}^2=x^2 \implies \overline{CQ}=\sqrt{x^2-\left(\frac{8-x}{2}\right)^2}$$

Since $\overline{CQ}=\overline{DR}$, we can say that $x+2\sqrt{x^2-\left(\frac{8-x}{2}\right)^2}=6 \implies x=-7\pm3\sqrt{11}$. We can discard the negative solution, so $k+m+n=-7+3+11=\boxed{\textbf{(B) } 7}$ ~ blitzkrieg21

Problem 18:

- Answer: D
- **Solution 1 (Casework):** We can begin to put this into cases. Let' s call the pairs a, b and c, and assume that a member of pair a is sitting in the leftmost seat of the second row. We can have the following cases then.Case 1: Second Row: a b c Third Row: b c aCase 2: Second Row: a c b Third Row: c b aCase 3: Second Row: a b c Third Row: c a b
  Case 4: Second Row: a c b Third Row: b a c

  For each of the four cases, we can flip the siblings, as they are distinct. So, each of the cases has $2 \cdot 2 \cdot 2 = 8$ possibilities. Since there are four cases, when pair a has someone in the leftmost seat of the second row, there are 32 ways to rearrange it. However, someone from either pair a, b, or c could be sitting in the leftmost seat of the second row. So, we have to multiply it by 3 to get our answer of $32 \cdot 3 = 96$. So, the correct answer is $\boxed{D}$.

  Written By: Archimedes15

- **Solution 2:** Call the siblings A_1, A_2, B_1, B_2, C_1, and C_2.There are 6 choices for the child in the first seat, and it doesn' t matter which one takes it, so suppose Without loss of generality (https://www.linstitute.net/goto/https://artofproblemsolving.com/wiki/index.php?title=Without_loss_of_generality) that A_1 takes it (a ○ is an empty seat):A_1 ○ ○
  ○ ○ ○ Then there are 4 choices for the second seat (B_1, B_2, C_1, or C_2). Again, it doesn' t matter who takes the seat, so WLOG suppose it is B_1:
  $A_1 B_1$ ○
  ○ ○ ○

  The last seat in the first row cannot be A_2 because it would be impossible to create a second row that satisfies the conditions. Therefore, it must be C_1 or C_2. Suppose WLOG that it is C_1. There are two ways to create a second row:

  $A_1 B_1 C_1$
  $B_2 C_2 A_2$

  $A_1 B_1 C_1$
  $C_2 A_2 B_2$

  Therefore, there are $6 \cdot 4 \cdot 2 \cdot 2 = \boxed{96}$ possible seating arrangements.

  Written by: R1ceming

- **Solution 3 (Using the Answers):** Notice how given an arrangement of the children that works (the answers tell us there is at least one) we can swap each pair of the siblings in one of 2 ways for 2^3 = 8 arrangements, and each of the 3 pairs can take each others' spaces in 3! = 6 ways. This means that the answer must be divisible by 48.

  Written by: Kevin Schmidt

Problem 19

- **Answer:** E

- **Solution 1:** Let Joey's age be j, Chloe's age be c, and we know that Zoe's age is 1. We know that there must be 9 values $k \in \mathbb{Z}$ such that $c+k=a(1+k)$ where a is an integer. Therefore, $c-1+(1+k)=a(1+k)$ and $c-1=(1+k)(a-1)$. Therefore, we know that, as there are 9 solutions for k, there must be 9 solutions for $c-1$. We know that this must be a perfect square. Testing perfect squares, we see that $c-1=36$, so $c=37$. Therefore, $j=38$. Now, since $j-1=37$, by similar logic, $37=(1+k)(a-1)$, so $k=36$ and Joey will be $38+36=74$ and the sum of the digits is $\boxed{\text{(E) } 11}$
- **Solution 2:** Here's a different way of saying your solution. If a number is a multiple of both Chloe's age and Zoe's age, then it is a multiple of their difference. Since the difference between their ages does not change, then that means the difference between their ages has 9 factors. Therefore, the difference between Chloe and Zoe's age is 36, so Chloe is 37, and Joey is 38. The common factor that will divide both of their ages is 37, so Joey will be 74. 7 + 4 = $\boxed{\text{(E) } 11}$
- **Solution 3:** Similar approach to above, just explained less concisely and more in terms of the problem (less algebra-y)Let $C+n$ denote Chloe's age, $J+n$ denote Joey's age, and $Z+n$ denote Zoe's age, where n is the number of years from now. We are told that $C+n$ is a multiple of $Z+n$ exactly nine times. Because $Z+n$ is 1 at $n=0$ and will increase until greater than $C-Z$, it will hit every natural number less than $C-Z$, including every factor of $C-Z$. For $C+n$ to be an integral multiple of $Z+n$, the difference $C-Z$ must also be a multiple of Z, which happens if Z is a factor of $C-Z$. Therefore, $C-Z$ has nine factors. The smallest number that has nine positive factors is $2^2 3^2=36$ (we want it to be small so that Joey will not have reached three digits of age before his age is a multiple of Zoe's). We also know $Z=1$ and $J=C+1$. Thus,$C-Z=36J-Z=37$
  By our above logic, the next time $J-Z$ is a multiple of $Z+n$ will occur when $Z+n$ is a factor of $J-Z$. Because 37 is prime, the next time this happens is at $Z+n=37$, when $J+n=74$. $7+4=\boxed{\textbf{(E)} 11}$

Problem 20

- **Answer:** B
- **Solution 1:** $f(n)=f(n-1)-f(n-2)+n$
  $=(f(n-2)-f(n-3)+n-1)-f(n-2)+n= 2n-1-f(n-3)$
  $=2n-1-(2(n-3)-1-f(n-6))$
  $=f(n-6)+6$

  Thus, $f(2018)=2016+f(2)=2017.$ $\boxed{B}$

- **Solution 2:** Start out by listing some terms of the sequence.$f(1)=1f(2)=1f(3)=3$
  $f(4)=6$

  $f(5)=8$

  $f(6)=8$

  $f(7)=7$

  $f(8)=7$

  $f(9)=9$

  $f(10)=12$

  $f(11)=14$

  $f(12)=14$

$f(13) = 13$

$f(14) = 13$

$f(15) = 15 \cdots\cdots$ Notice that $f(n) = n$ whenever n is an odd multiple of 3, and the pattern of numbers that follow will always be +3, +2, +0, -1, +0. The closest odd multiple of 3 to 2018 is 2013, so we have

$f(2013) = 2013$

$f(2014) = 2016$

$f(2015) = 2018$

$f(2016) = 2018$

$f(2017) = 2017$

$f(2018) = \boxed{(B) 2017}$.

- **Solution 3 (Bashy Pattern Finding):** Writing out the first few values, we get: $1, 1, 3, 6, 8, 8, 7, 7, 9, 12, 14, 14, 13, 13, 15, 18, 20, 20, 19, 19...$. Examining, we see that every number x where $x \equiv 1 \pmod 6$ has $f(x) = x$, $f(x+1) = f(x) = x$, and $f(x-1) = f(x-2) = x+1$. The greatest number that' s 1 (mod 6) and less 2018 is 2017, so we have $f(2017) = f(2018) = 2017.$ $\boxed{B}$

Problem 21

- Answer: C
- **Solution 1:** Prime factorizing 323 gives you $17 \cdot 19$. Looking at the answer choices, $\boxed{\text{(C) } 340}$ is the smallest number divisible by 17 or 19.
- **Solution 2:** Let the next largest divisor be k. Suppose $\gcd(k, 323) = 1$. Then, as $323|n$, $k|n$, therefore, $323 \cdot k|n.$ However, because $k > 323$, $323k > 323 \cdot 324 > 9999$. Therefore, $\gcd(k, 323) > 1$. Note that $323 = 17 \cdot 19$. Therefore, the smallest the gcd can be is 17 and our answer is $323 + 17 = \boxed{\text{(C) } 340}$.

Problem 22

- Answer: C
- **Solution**: Note that the obtuse angle in the triangle has to be opposite the side that is always length 1. This is because the largest angle is always opposite the largest side, and if 2 sides of the triangle were 1, the last side would have to be greater than 1 to make an obtuse triangle. Using this observation, we can set up a law of cosines where the angle is opposite 1:$1^2 = x^2 + y^2 - 2xy\cos(\theta)$where x and y are the sides that go from $[0, 1]$ and θ is the angle opposite the side of length 1.By isolating $\cos(\theta)$, we get:
$$\frac{1 - x^2 - y^2}{-2xy} = \cos(\theta)$$
For θ to be obtuse, $\cos(\theta)$ must be negative. Therefore, $\dfrac{1 - x^2 - y^2}{-2xy}$ is negative. Since x and y must be positive, $-2xy$ must be negative, so we must make $1 - x^2 - y^2$ positive. From here, we can set up the inequality $x^2 + y^2 < 1$ Additionally, to satisfy the definition of a triangle, we need: $x + y > 1$ The solution

should be the overlap between the two equations in the 1st quadrant.

By observing that $x^2 + y^2 < 1$ is the equation for a circle, the amount that is in the 1st quadrant is $\frac{\pi}{4}$. The line can also be seen as a chord that goes from $(0, 1)$ to $(1, 0)$. By cutting off the triangle of area $\frac{1}{2}$ that is not part of the overlap, we get $\frac{\pi}{4} - \frac{1}{2} \approx \boxed{0.29}$.

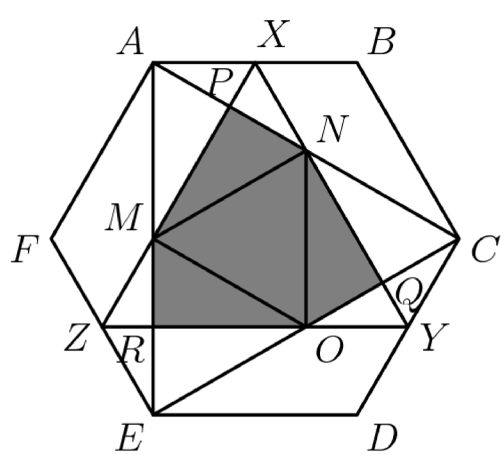
-allenle873

Problem 23

- **Answer:** B
- **Solution :** Let $x = lcm(a, b)$, and $y = gcd(a, b)$. Therefore, $a \cdot b = lcm(a, b) \cdot gcd(a, b) = x \cdot y$. Thus, the equation becomes$x \cdot y + 63 = 20x + 12y$ $x \cdot y - 20x - 12y + 63 = 0$Using Simon's Favorite Factoring Trick, we rewrite this equation as$(x - 12)(y - 20) - 240 + 63 = 0$
$(x - 12)(y - 20) = 177$
Since $177 = 3 \cdot 59$ and $x > y$, we have $x - 12 = 59$ and $y - 20 = 3$, or $x - 12 = 177$ and $y - 20 = 1$. This gives us the solutions $(71, 23)$ and $(189, 21)$. Obviously, the first pair does not work. The second pair can be ordered in two ways. Thus, the answer is $\boxed{2}$. (awesomeag)

Problem 24

- Answer: C
- **Solution 1:**



The desired area (hexagon $MPNQOR$) consists of an equilateral triangle ($\triangle MNO$) and three right triangles ($\triangle MPN$, $\triangle NQO$, and $\triangle ORM$).

Notice that $\overline{AD}$ (not shown) and $\overline{BC}$ are parallel. $\overline{XY}$ divides transversals $\overline{AB}$ and $\overline{CD}$ into a $1:1$ ratio. Thus, it must also divide transversal $\overline{AC}$ and transversal $\overline{CO}$ into a $1:1$ ratio. By symmetry, the same applies for $\overline{CE}$ and $\overline{EA}$ as well as $\overline{EM}$ and $\overline{AN}$

In $\triangle ACE$, we see that $\frac{[MNO]}{[ACE]} = \frac{1}{4}$ and $\frac{[MPN]}{[ACE]} = \frac{1}{8}$. Our desired area becomes

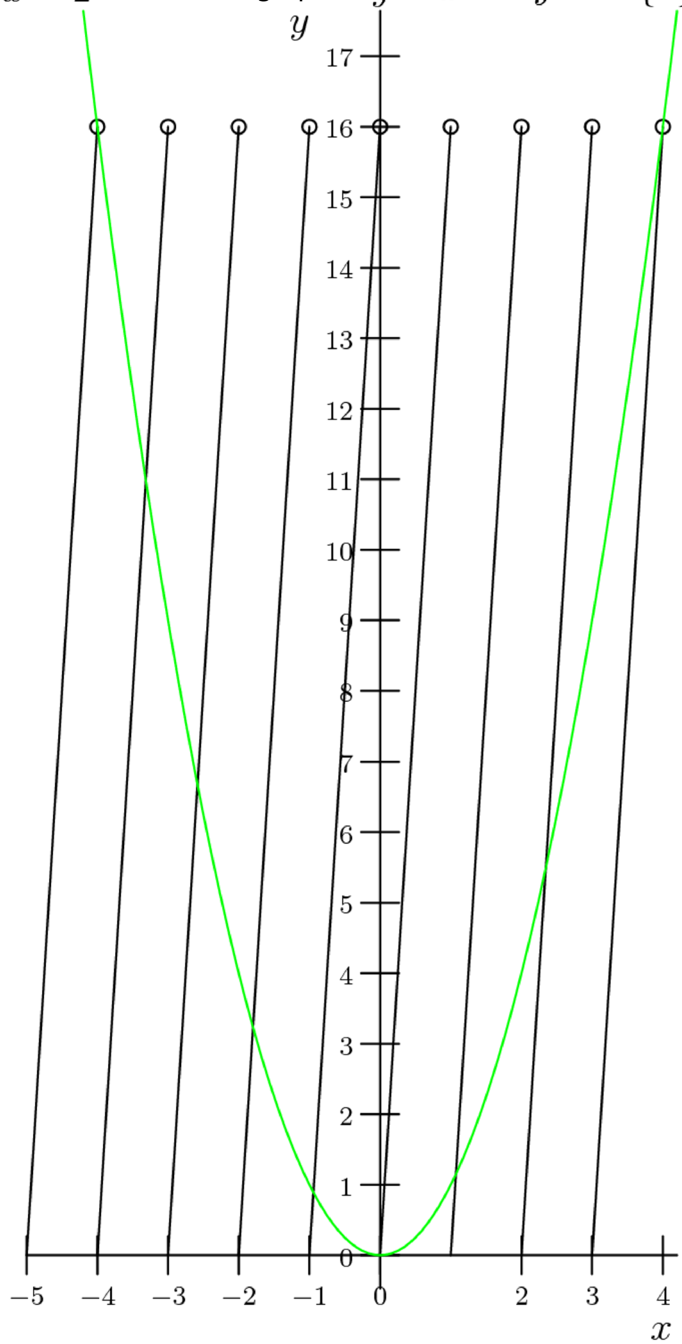
$$(\frac{1}{4}+3\cdot\frac{1}{8})\cdot\frac{(\sqrt{3})^2\cdot\sqrt{3}}{4}=\frac{15}{32}\sqrt{3}=\boxed{C}$$

- **Solution 2:** Now, if we look at the figure, we can see that the complement of the hexagon we are trying to find is composed of 3 isosceles trapezoids (AXFZ, XBCY, and ZYED), and 3 right triangles (With one vertice on each of X, Y, and Z). We know that one base of each trapezoid is just the side length of the hexagon which is 1, and the other base is 3/2 (It is halfway in between the side and the longest diagonal) with a height of $\sqrt{3}/4$ (by using the Pythagorean theorem and the fact that it is an isosceles trapezoid) to give each trapezoid having an area of $5\sqrt{3}/16$ for a total area of $15\sqrt{3}/16$ (Alternatively, we could have calculated the area of hexagon ABCDEF and subtracted the area of triangle XYZ, which, as we showed before, had a side length of 3/2). Now, we need to find the area of each of the small triangles, which, if we look at the triangle that has a vertice on X, is similar to the triangle with a base of YC = 1/2. Using similar triangles we calculate the base to be 1/4 and the height to be $\sqrt{3}/4$ giving us an area of $\sqrt{3}/32$ per triangle, and a total area of $3\sqrt{3}/32$. Adding the two areas together, we get $15\sqrt{3}/16+3\sqrt{3}/32=33*sqrt3/32$. Finding the total area, we get $6*1^2*\sqrt{3}/4=3\sqrt{3}/2$. Taking the complement, we get $3\sqrt{3}/2-33\sqrt{3}/32=15\sqrt{3}/32=\boxed{C}$
- **Solution 3 (Trig) :** Notice, the area of the convex hexagon formed through the intersection of the 2 triangles can be found by finding the area of the triangle formed by the midpoints of the sides and subtracting the smaller triangles that are formed by the region inside this triangle but outside the other triangle. First, let’ s find the area of the area of the triangle formed by the midpoint of the sides. Notice, this is an equilateral triangle, thus all we need is to find the length of its side. To do this, we look at the isosceles trapezoid outside this triangle but inside the outer hexagon. Since the interior angle of a regular hexagon is 120° and the trapezoid is isosceles, we know that the angle opposite is 60°, and thus the side length of this triangle is $1+2(\frac{1}{2}\cos(60^{\circ})=1+\frac{1}{2}=\frac{3}{2}$. So the area of this triangle is $\frac{\sqrt{3}}{4}s^2=\frac{9\sqrt{3}}{16}$ Now let’ s find the area of the smaller triangles. Notice, triangle ACE cuts off smaller isosceles triangles from the outer hexagon. The base of these isosceles triangles is perpendicular to the base of the isosceles trapezoid mentioned before, thus we can use trigonometric ratios to find the base and height of these smaller triangles, which are all congruent due to the rotational symmetry of a regular hexagon. The area is then $\frac{1}{2}(\frac{1}{2})\cos(60^{\circ}))(\frac{1}{2}\sin(60^{\circ}))=\frac{\sqrt{3}}{32}$ and the sum of the areas is $3\cdot\frac{\sqrt{3}}{32}=\frac{3\sqrt{3}}{32}$ Therefore, the area of the convex hexagon is
$$\frac{9\sqrt{3}}{16}-\frac{3\sqrt{3}}{32}=\frac{18\sqrt{3}}{32}-\frac{3\sqrt{3}}{32}=\boxed{\frac{15\sqrt{3}}{32}}\implies\boxed{C}$$

Problem 25

- Answer：C
- **Solution 1:** This rewrites itself to $x^2=10,000\{x\}$.Graphing $y=10,000\{x\}$ and $y=x^2$ we see that the former is a set of line segments with slope $10,000$ from 0 to 1 with a hole at $x=1$, then 1 to 2 with a hole at

$x = 2$ etc.Here is a graph of $y = x^2$ and $y = 16\{x\}$ for visualization.



Now notice that when $x = \pm 100$ then graph has a hole at $(\pm 100, 10,000)$ which the equation $y = x^2$ passes through and then continues upwards. Thus our set of possible solutions is bounded by $(-100, 100)$. We can see that $y = x^2$ intersects each of the lines once and there are $99 - (-99) + 1 = 199$ lines for an answer of $\boxed{\text{(C) } 199}$.

- **Solution 2 (Alternative, Bashy Solution) :** Same as the first solution, $x^2 = 10,000\{x\}$.We can write x as $\lfloor x \rfloor + \{x\}$. Expanding everything, we get a quadratic in x in terms of $\lfloor x \rfloor$: $\{x\}^2 + (2\lfloor x \rfloor - 10,000)\{x\} + \lfloor x \rfloor^2 = 0$We use the quadratic formula to solve for {x}: $\{x\} = \frac{-2\lfloor x \rfloor + 10,000 \pm \sqrt{(-2\lfloor x \rfloor + 10,000^2 - 4\lfloor x \rfloor^2)}}{2}$Since $0 \le \{x\} < 1$, we get an inequality which we can then solve. After simplifying a lot, we get that $\lfloor x \rfloor^2 + 2\lfloor x \rfloor - 9999 < 0$. Solving over the integers, $-101 < \lfloor x \rfloor < 99$, and since $\lfloor x \rfloor$ is an integer, there are $\boxed{\text{(C) } 199}$ solutions. Each value of $\lfloor x \rfloor$ should correspond to one value of x, so we are done.

- **Solution 3:** Let $x = a + k$ where a is the integer portion of x and k is the decimal portion. We can then rewrite the problem below:$(a+k)^2 + 10000a = 10000(a+k)$From here, we get
$(a+k)^2 + 10000a = 10000a + 10000k$
Solving for $a+k$...

  $$(a+k)^2 = 10000k$$

  $$a+k = \pm 100\sqrt{k}$$

  Because $0 \leq k < 1$, we know that $a+k$ cannot be less than or equal to -100 nor greater than or equal to 100. Therefore:

  $$-99 \leq a+k = x \leq 99$$

  There are 199 elements in this range, so the answer is $\boxed{\textbf{C } 199}$

- **Solution 4:** As in the first solution, we can write $x^2 = 10,000\{x\}$.Now obviously, $0 < \{x\} < 1$.So, x can get be $x \rightarrow [-99, 99]$.Hence, the answer is $\boxed{199}$.
-Pi_3.14_Squared

MAA AMC

American Mathematics Competitions

Solutions Pamphlet

MAA American Mathematics Competitions

20th Annual

AMC 10A

American Mathematics Competition 10A

Thursday, February 7, 2019

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *Copies of the problem booklet and solution pamphlet may be shared with your students for educational purposes. However, the publication, reproduction, or communication of the problems or solutions for this competition with anyone outside of the classroom is a violation of the competition rules. This includes dissemination via copier, telephone, email, internet, or media of any type.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
Phone 800.527.3690 | Fax 240.396.5647 | amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by MAA's Subcommittee on the AMC 10/AMC 12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (C):**

$$2^{\left(0^{\left(1^9\right)}\right)} + \left(\left(2^0\right)^1\right)^9 = 2^{\left(0^1\right)} + \left((1)^1\right)^9 = 2^0 + 1^9 = 1 + 1 = 2$$

2. **Answer (A):** Both 20! and 15! have at least 3 factors of 2 and at least 3 factors of 5, so both are multiples of $10^3 = 1000$ and therefore have 0s as their last three digits. The digit in the hundreds place of the difference is therefore $0 - 0 = 0$.

3. **Answer (D):** If B denotes Bonita's age last year, then Ana's age last year was $5B$. This year Bonita's age is $B+1$, and Ana's age is $5B+1$. So the given condition is $5B+1 = (B+1)^2$. This quadratic equation simplifies to $B^2 = 3B$. Because the two girls were born in different years, $B \neq 0$, so $B = 3$. Last year Bonita was 3 and Ana was $5 \cdot 3 = 15$, so they were born $15 - 3 = 12$ years apart. As a check, note that Ana's age this year, 16, is indeed the square of Bonita's age this year, 4.

4. **Answer (B):** The greatest number of balls that can be drawn without getting 15 of one color is obtained if and only if 14 red, 14 green, 14 yellow, 13 blue, 11 white, and 9 black balls are drawn, a total of 75 balls. When another ball is drawn, it will be the 15th ball of one of the colors red, green, or yellow. Thus the requested minimum is $75 + 1 = 76$.

5. **Answer (D):** The sum $(-44) + (-43) + \cdots + 43 + 44 + 45 = 45$. This sum has 90 consecutive integers. There is no longer list because for the sum of consecutive integers to be positive, there must be more positive integers than negative integers. Further, if there are more than 90 consecutive integers as part of a list that sums to a positive number, then there must be a positive integer greater than 45 that is not cancelled out by its additive inverse.

OR

Suppose that the consecutive integers are $a, a+1, \ldots, a+n-1$; their sum then equals $na + \frac{n(n-1)}{2}$. Therefore

$$na + \frac{n(n-1)}{2} = 45,$$

so $n(2a + n - 1) = 90$, which implies that $n \leq 90$. A sequence of 90 consecutive integers with sum equal to 45 indeed exists, as observed in the first solution.

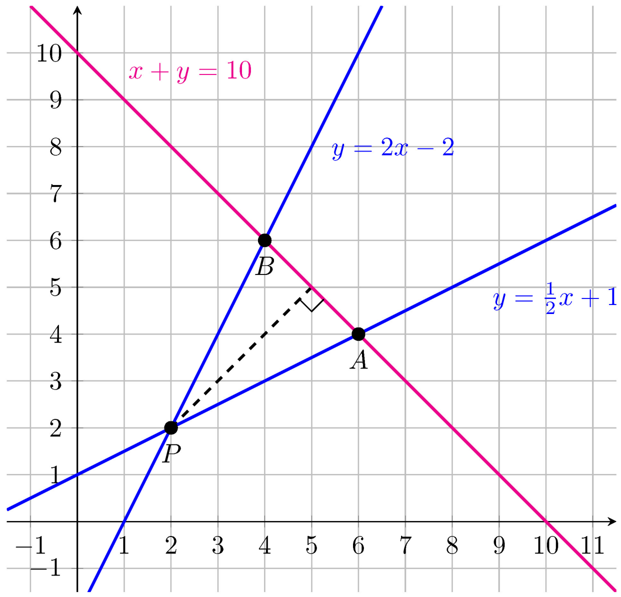
6. **Answer (C):** In a square or non-square rectangle, the diagonals are congruent and bisect each other, and their point of intersection is equidistant from all four vertices. This point also lies on the perpendicular bisectors of all four sides. In a non-square rhombus or a parallelogram that is not a rectangle or rhombus, the perpendicular bisectors of parallel sides do not meet, so no point could be equidistant from all four vertices. Finally, in an isosceles trapezoid that is not a parallelogram, the perpendicular bisectors of the parallel sides are the same line, and the perpendicular bisectors of the nonparallel sides meet at a point on this line; that point is equidistant from all four vertices. In summary, 3 of the types of given quadrilaterals—the first, second, and fifth in the list—have the required property.

OR

The required point exists if and only if the quadrilateral can be inscribed in a circle, in which case the point is the center of the circle. A square, rectangle, and isosceles trapezoid can each be inscribed in a circle, but a non-square rhombus and a non-rectangular parallelogram cannot. Therefore the required point exists for 3 of the listed types of quadrilaterals.

7. **Answer (C):** Let $P(2,2)$ be the intersection point. The two lines have equations $y = \frac{1}{2}x+1$ and $y = 2x-2$. They intersect $x+y = 10$ at $A(6,4)$ and $B(4,6)$. Consider $\overline{AB}$ to be the base of the triangle; then the altitude of the triangle is the segment joining $(2,2)$ and $(5,5)$. By the Distance Formula, the area of $\triangle PAB$ is

$$\frac{1}{2} \cdot \sqrt{(6-4)^2 + (4-6)^2} \cdot \sqrt{(5-2)^2 + (5-2)^2} = \frac{1}{2} \cdot 2\sqrt{2} \cdot 3\sqrt{2} = 6.$$

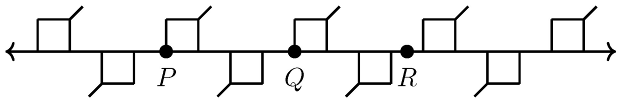


Note: The area of the triangle with vertices $(2, 2)$, $(6, 4)$, and $(4, 6)$ can be calculated in a number of other ways, such as by enclosing it in a 4×4 square with sides parallel to the coordinate axes and subtracting the areas of three right triangles; by splitting it into two triangles with the line $y = 4$; by the shoelace formula:

$$\frac{1}{2} \cdot \Big|(2 \cdot 4 + 6 \cdot 6 + 4 \cdot 2) - (6 \cdot 2 + 4 \cdot 4 + 2 \cdot 6)\Big| = 6;$$

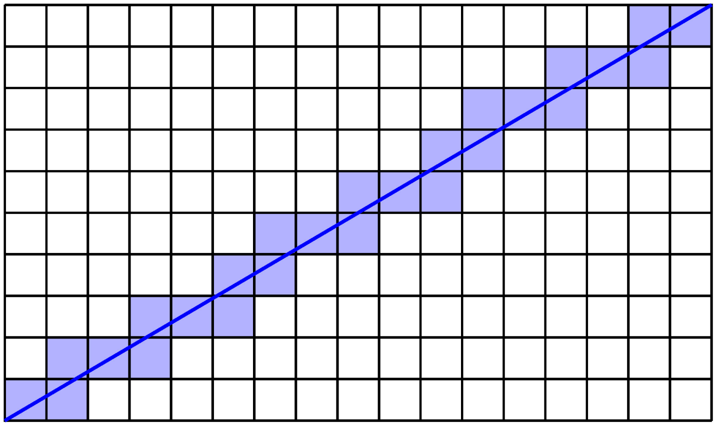
or by observing that there are 4 lattice points in the interior of the triangle and 6 lattice points on the boundary, and using Pick's Formula: $4 + \frac{6}{2} - 1 = 6$.

8. **Answer (C):** A translation in the direction parallel to line ℓ by an amount equal to the distance between the left sides of successive squares above the line (or any integer multiple thereof), will take the figure to itself. The translation vector could be $\overrightarrow{PQ}$ in the figure below. In addition, a rotation of 180° around any point on line ℓ that is halfway between the bases on ℓ of a square above the line and a nearest square below the line, such as point R in the figure, will also take the figure to itself. Either of the given reflections, however, will result in a figure in which the "tails" attached to the squares above the line are on the left side of the squares instead of the right side. Therefore 2 of the listed non-identity transformations will transform this figure into itself.



9. **Answer (B):** The sum of the first n positive integers is $\frac{n(n+1)}{2}$, and the product of the first n positive integers is $n! = n \cdot (n-1) \cdots 2 \cdot 1$. If $n > 1$ is odd, then $n \cdot \frac{n+1}{2}$ divides $n \cdot (n-1)!$ because $\frac{n+1}{2}$ is an integer between 1 and n. If n is even, then $\frac{n}{2} \cdot (n+1)$ does not divide $n!$ if and only if $n+1$ is prime. Because 997 is the greatest three-digit prime number, the greatest three-digit positive integer n for which the sum of the first n positive integers is not a divisor of the product of the first n positive integers is $997 - 1 = 996$.

10. **Answer (C):** Because 10 and 17 are relatively prime, the diagonal does not cross the boundaries between tiles at any corner point of the tiles. In order for the bug to move from one vertex of the rectangle to the opposite vertex, the bug must cross 9 edges in one direction and 16 edges in the other direction, a total of 25 edges. Each time the bug crosses an edge, it enters a new tile. Counting the tile it started on as well, the bug visits a total of $1 + 25 = 26$ tiles.

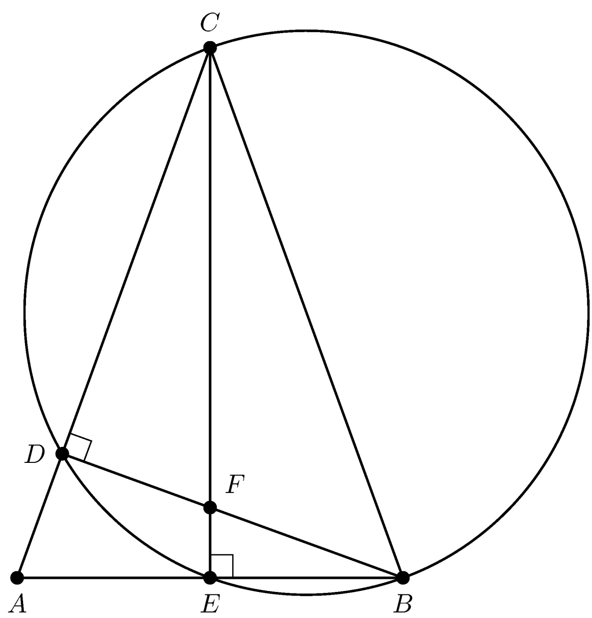


11. **Answer (C):** Because $201^9 = 3^9 \cdot 67^9$, a square divisor has the form $3^a \cdot 67^b$ where $a, b \in \{0, 2, 4, 6, 8\}$, and a cubic divisor has the form $3^a \cdot 67^b$ where $a, b \in \{0, 3, 6, 9\}$. A number is both a square and a cube if and only if it is a sixth power, so it has the form $3^a \cdot 67^b$ where $a, b \in \{0, 6\}$. Thus there are $5 \cdot 5 = 25$ square divisors, $4 \cdot 4 = 16$ cubic divisors, and $2 \cdot 2 = 4$ divisors that are sixth powers. Therefore the number of divisors that are squares and/or cubes is $25 + 16 - 4 = 37$.

12. **Answer (E):** Each of the values 1 through 28 is a mode, so $d = \frac{14+15}{2} = 14.5$. There are $15 \cdot 12 = 180 < \frac{365}{2}$ data values less than or equal to 15, and there are $16 \cdot 12 = 192 > \frac{365}{2}$ values less than or equal to 16. Therefore more than half of the values are greater than or equal to 16 and more than half of the values are less than or equal to 16, so $M = 16$. To see the relationship between μ and 16, note that if every month had 31 days, then there would be 12 of each value from 1 to 31, and the mean would be 16; because the actual data are missing some of the larger values, $\mu < 16$. To see the relationship between μ and 14.5, note that if every month had 28 days, then there would be 12 of each value from 1 to 28, and the mean would be 14.5; because the actual data consist of all of these values together with some larger values, $\mu > 14.5$. Therefore $d = 14.5 < \mu < 16 = M$.

13. **Answer (D):** Because $BC = AC$ and $\angle ACB = 40^\circ$, it follows that $\angle BAC = \angle ABC = 70^\circ$. Because $\angle BAC = \frac{1}{2}\left(\widehat{BC} - \widehat{DE}\right)$ and $\widehat{BC} = 180^\circ$, it follows that $\widehat{DE} = 40^\circ$. Then

$$\angle BFC = \frac{1}{2}\left(\widehat{BC} + \widehat{DE}\right) = \frac{1}{2}\left(180^\circ + 40^\circ\right) = 110^\circ.$$



OR

Because D and E lie on the circle with diameter $\overline{BC}$, both $\angle BDC$ and $\angle BEC$ are right angles, so $\angle ADF$ and $\angle AEF$ are also right angles. Therefore in quadrilateral $AEFD$

$$\angle DFE = 180^\circ - \angle DAE = 180^\circ - \frac{1}{2}(180^\circ - \angle ACB) = 110^\circ,$$

and $\angle BFC$ has the same measure.

14. **Answer (D):** There are several cases to consider.

If all four lines are concurrent, then there is 1 intersection point.

If three of the lines are concurrent and the fourth line is parallel to one of those three, then there are 3 intersection points. If three of the lines are concurrent and the fourth line is parallel to none of those three, then there are 4 intersection points.

In the remaining cases no three lines are concurrent. If they are all parallel, then there are 0 intersection points.

If only three of them are parallel, then there are again 3 intersection points.

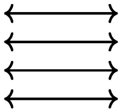
If two of them are parallel but no three are mutually parallel, then there are either again 4 intersection points, if the other two lines are parallel to each other; or 5 intersection points, if the other two lines intersect.

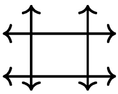
In the final case, every line intersects every other line, giving 6 points of intersection.

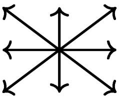
These are all the cases, so the requested sum is $1+3+4+0+5+6 = 19$.

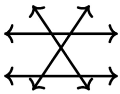
OR

The problem deals with the possible number of intersection points when four distinct lines are drawn in the plane. The maximum number of intersection points is $\binom{4}{2} = 6$. The diagram shows that the number of intersection points can be 0, 1, 3, 4, 5, and 6.

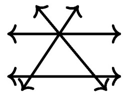

0


4

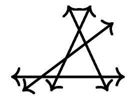

1



5



3



6

To see that there cannot be 2 points of intersection, suppose to the contrary that A and B are the only points of intersection among the four lines. If A and B lie on one of the lines, then the other three lines, ℓ, m, and n, must all pass through A or B, with at least one line passing through each. Without loss of generality, suppose ℓ contains A, and m and n contain B. At most one of m and n can be parallel to ℓ, and thus there will be a third intersection point. Otherwise it must be the case that A is the intersection of two of the lines, and B is the intersection of the other two lines; say A is the intersection of k and ℓ, and B is the intersection of m and n. Again, m and n cannot both be parallel to ℓ, and thus there must be another point of intersection.

As in the first solution, the requested sum is $0+1+3+4+5+6 = 19$.

Note: Students who wish to pursue further interesting questions about arrangements of points and lines, such as the dual problem to this one (how many lines n points can determine), might want to start by looking up the Sylvester–Gallai Theorem.

15. **Answer (E):** The sequence begins $1, \frac{3}{7}, \frac{3}{11}, \frac{3}{15}, \frac{3}{19}, \ldots$. This pattern leads to the conjecture that $a_n = \frac{3}{4n-1}$. Checking the initial conditions $n = 1$ and $n = 2$, and observing that for $n \geq 3$,

$$\begin{aligned}
\frac{\frac{3}{4(n-2)-1} \cdot \frac{3}{4(n-1)-1}}{2 \cdot \frac{3}{4(n-2)-1} - \frac{3}{4(n-1)-1}} &= \frac{\frac{3}{4n-9} \cdot \frac{3}{4n-5}}{\frac{6}{4n-9} - \frac{3}{4n-5}} \\
&= \frac{9}{6(4n-5) - 3(4n-9)} \\
&= \frac{9}{12n-3} = \frac{3}{4n-1},
\end{aligned}$$

confirms the conjecture. Therefore $a_{2019} = \frac{3}{4 \cdot 2019 - 1} = \frac{3}{8075}$, and the requested sum is $3 + 8075 = 8078$.

OR

Taking the reciprocal of both sides of the recurrence gives

$$\frac{1}{a_n} = \frac{2a_{n-2} - a_{n-1}}{a_{n-2} \cdot a_{n-1}} = \frac{2}{a_{n-1}} - \frac{1}{a_{n-2}},$$

which is equivalent to

$$\frac{1}{a_n} - \frac{1}{a_{n-1}} = \frac{1}{a_{n-1}} - \frac{1}{a_{n-2}}.$$

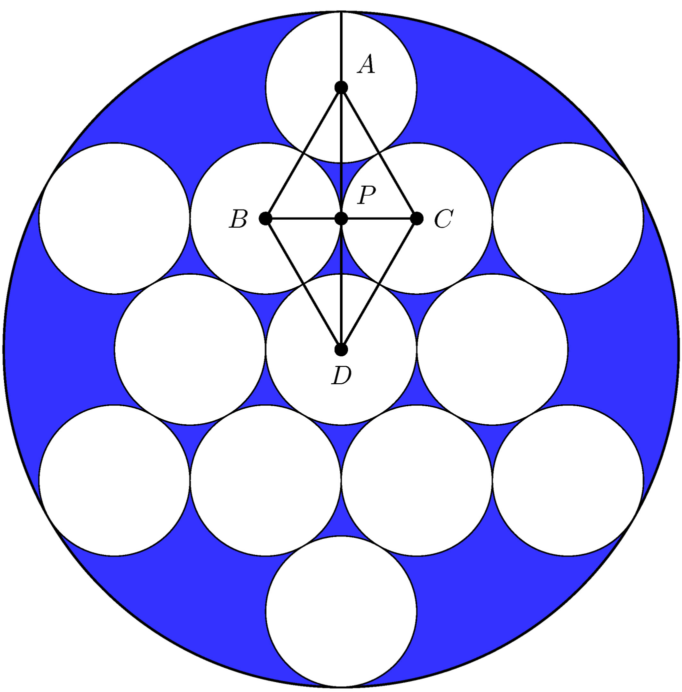
Thus the sequence of reciprocals is an arithmetic sequence. Its first term is 1, and its common difference is $\frac{7}{3} - 1 = \frac{4}{3}$. Its 2019th term is

$$\frac{1}{a_{2019}} = 1 + 2018 \cdot \frac{4}{3} = \frac{8075}{3},$$

so $a_{2019} = \frac{3}{8075}$, and the requested sum is $3 + 8075 = 8078$.

16. **Answer (A):** Let A, B, C, and D be the centers of four of the circles as shown below, and let P be the intersection of the diagonals of rhombus $ABDC$. Then $PC = 1$ and $AC = 2$, so $AP = \sqrt{3}$; similarly $PD = \sqrt{3}$. The radius of the large circle is therefore $1 + 2\sqrt{3}$. The requested area is

$$\pi\left(1 + 2\sqrt{3}\right)^2 - 13\pi = 4\pi\sqrt{3}.$$



Note: This problem is related to the question of how densely the plane can be packed with congruent circles—how much wasted space there is with the most efficient packing. It has been proved that the best arrangement is the one shown in this problem, with each circle surrounded by six others. The fraction of the plane covered by the circles is $\frac{\pi}{6}\sqrt{3} \approx 0.9069$.

17. **Answer (D):** Of the 9 cubes available, 1 cube will not be used. Because there are three different kinds of cubes and limited numbers of each kind, there are three different possibilities for the set of cubes that are used. One possibility is 1 red cube, 3 blue cubes, and 4 green cubes; the second possibility is 2 red cubes, 2 blue cubes, and 4 green cubes; and the third possibility is 2 red cubes, 3 blue cubes, and 3 green cubes. Cubes of the same color are indistinguishable. Hence the number of different towers is

$$\frac{8!}{1! \cdot 3! \cdot 4!} + \frac{8!}{2! \cdot 2! \cdot 4!} + \frac{8!}{2! \cdot 3! \cdot 3!} = 1{,}260.$$

OR

There is a one-to-one correspondence between towers of height 9 and towers of height 8 by viewing the top cube in a tower of height 9 as the cube that is not used in a tower of height 8. The number of different towers of height 9 is given by

$$\frac{9!}{2! \cdot 3! \cdot 4!} = 1{,}260.$$

Note: A generalization of the two solutions put together yields the following extension of Pascal's Identity:

$$\frac{(a+b+c-1)!}{(a-1)! \cdot b! \cdot c!} + \frac{(a+b+c-1)!}{a! \cdot (b-1)! \cdot c!} + \frac{(a+b+c-1)!}{a! \cdot b! \cdot (c-1)!} = \frac{(a+b+c)!}{a! \cdot b! \cdot c!}.$$

18. **Answer (D):** The number $0.\overline{23}_k$ is the sum of an infinite geometric series with first term $\frac{2}{k} + \frac{3}{k^2}$ and common ratio $\frac{1}{k^2}$. Therefore the sum is

$$\frac{\frac{2}{k} + \frac{3}{k^2}}{1 - \frac{1}{k^2}} = \frac{2k+3}{k^2-1} = \frac{7}{51}.$$

Then $0 = 7k^2 - 102k - 160 = (k-16)(7k+10)$, and therefore $k = 16$.

OR

Let $x = 0.\overline{23}_k$. Then $(k^2-1)x = 23.\overline{0}_k = 2k+3$, so $\frac{2k+3}{k^2-1} = \frac{7}{51}$ and the solution proceeds as above.

Note: If $0 < a < q$, $\gcd(a,q) = 1$, and $\gcd(k,q) = 1$, then the base-k representation of the fraction $\frac{a}{q}$ has least period equal to the order of k modulo q. In the case at hand, $k = 16$ and $q = 51 = 3 \cdot 17$. Then $\phi(q) = 2 \cdot 16 = 32$ and $k^2 = 256 = 5 \cdot 51 + 1$, so k has order 2 modulo 51.

19. **Answer (B):** Observe that

$$\begin{aligned}
&(x+1)(x+4)(x+2)(x+3) + 2019 \\
&\qquad = \left(x^2+5x+4\right)\left(x^2+5x+6\right) + 2019 \\
&\qquad = \left[\left(x^2+5x+5\right)-1\right]\left[\left(x^2+5x+5\right)+1\right] + 2019 \\
&\qquad = \left(x^2+5x+5\right)^2 - 1 + 2019 \\
&\qquad = \left(x^2+5x+5\right)^2 + 2018.
\end{aligned}$$

Because $(x^2+5x+5)^2 \ge 0$ for all x and equals 0 for $x = \frac{-5\pm\sqrt{5}}{2}$, it follows that the requested minimum value is 2018.

OR

Let $r = x + \frac{5}{2}$. Then

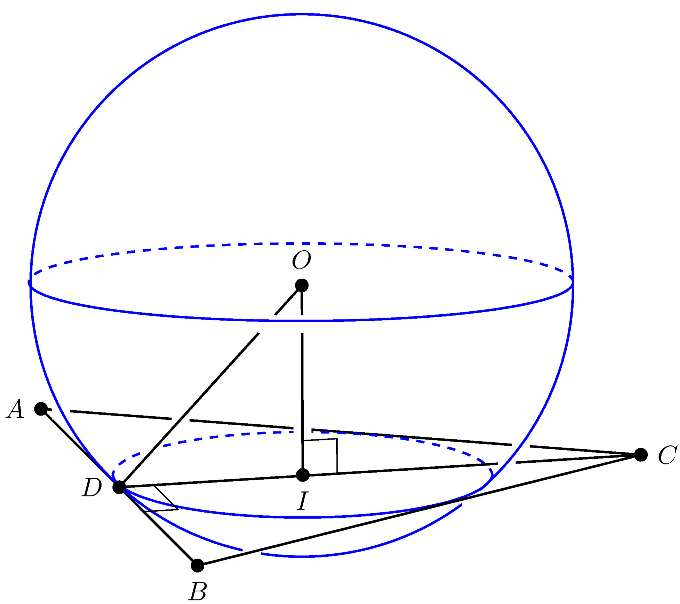
$$\begin{aligned}
(x+1)(x+2)(x+3)(x+4) &= \left(r-\frac{3}{2}\right)\left(r-\frac{1}{2}\right)\left(r+\frac{1}{2}\right)\left(r+\frac{3}{2}\right) \\
&= \left(r^2-\frac{1}{4}\right)\left(r^2-\frac{9}{4}\right) \\
&= \left(r^2-\frac{5}{4}\right)^2 - 1,
\end{aligned}$$

the minimum value of which is -1. Therefore the minimum value of the given expression is $2019 - 1 = 2018$.

20. **Answer (B):** The sum of three integers is odd exactly when either all of the integers are odd, or one is odd and two are even. Five of the numbers $1, 2, \ldots, 9$ are odd, so at least one row must contain two or more odd numbers. Thus one row must contain three odd

numbers and no even numbers, and the other two rows must contain one odd number and two even numbers. The same is true of the three columns. There are $3 \times 3 = 9$ ways to choose which row and which column contain all odd numbers, and then the remaining four squares must have even numbers. There are $\binom{9}{4} = 126$ ways in total to choose which squares have odd numbers and which have even numbers, so the desired probability is $\frac{9}{126} = \frac{1}{14}$.

21. **Answer (D):** Let $\triangle ABC$ be the given triangle, with $AB = 24$ and $AC = BC = 15$, and let D be the midpoint of $\overline{AB}$. The length of the altitude to $\overline{AB}$ is $CD = \sqrt{15^2 - 12^2} = 9$. The area of $\triangle ABC$ is $\frac{1}{2} \cdot 24 \cdot 9 = 108$. The plane of the triangle intersects the sphere in a circle, which is the inscribed circle for $\triangle ABC$. Let r and I be the radius and the center of the inscribed circle, respectively. The semiperimeter of the triangle is $\frac{1}{2}(AB + BC + AC) = 27$, so $r = \frac{108}{27} = 4$. In right triangle DIO the hypotenuse $\overline{OD}$ has length 6 (the radius of the sphere) and $DI = r = 4$, so $OI = \sqrt{36 - 16} = 2\sqrt{5}$, the requested distance between the center of the sphere and the plane determined by $\triangle ABC$.

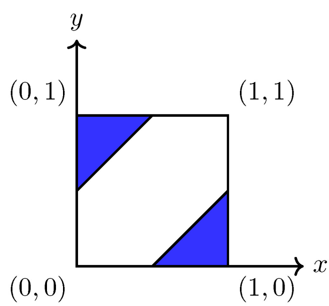


22. **Answer (B):** The probability that the first coin flip for both x and y is heads is $\frac{1}{4}$, and in half of these cases $|x - y|$ will be 0 and in the

other half of these cases $|x-y|$ will be 1. This contributes $\frac{1}{4} \cdot \frac{1}{2} = \frac{1}{8}$ to the probability that $|x-y| > \frac{1}{2}$.

The probability that the first coin flip for x is heads and the first coin flip for y is tails or vice versa is $\frac{1}{2}$. In such cases, one of the variables is 0 or 1, and the probability that $|x-y| > \frac{1}{2}$ is $\frac{1}{2}$. This contributes $\frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$ to the probability that $|x-y| > \frac{1}{2}$.

Finally, $\frac{1}{4}$ of the time both x and y will be chosen uniformly from $[0, 1]$. In this case, the situation can be modeled by the following diagram, in which the area of the shaded region gives the probability that $|x-y| > \frac{1}{2}$. This contributes $\frac{1}{4} \cdot \frac{1}{4} = \frac{1}{16}$ to the probability that $|x-y| > \frac{1}{2}$. The requested probability is $\frac{1}{8} + \frac{1}{4} + \frac{1}{16} = \frac{7}{16}$.



23. **Answer (C):** Define a round to be the recitations done during the successive turns of Tadd, Todd, and Tucker, in that order. Note that Tadd says 1 number in round 1, 4 numbers in round 2, 7 numbers in round 3, 10 numbers in round 4, and, in general, $3N-2$ numbers in round N. In turn, Todd says $3N-1$ numbers and Tucker says $3N$ numbers in round N. Therefore $9N-3$ numbers are recited by all three children in round N. During the course of the first N rounds Tadd recites a total of $1+4+7+10+\cdots+(3N-2)$ numbers. The sum of this arithmetic series is

$$N \cdot \frac{3N-1}{2} = \frac{3}{2}N^2 - \frac{1}{2}N.$$

During the first N rounds, all three children recite a total of $6+15+24+33+\cdots+(9N-3)$ numbers. The sum of this arithmetic series is

$$N \cdot \frac{9N+3}{2} = \frac{9}{2}N^2 + \frac{3}{2}N.$$

The number of rounds prior to the round during which Tadd says his 2019th number is the greatest value of N such that $\frac{3}{2}N^2 - \frac{1}{2}N < 2019$.

Multiplying by 2 yields $3N^2 - N < 4038$. If $N = 36$, then the left side is $3 \cdot 1296 - 36 = 3852$; and if $N = 37$, then the left side is $3 \cdot 1369 - 37 = 4070$. Therefore the N being sought is 36, and Tadd has recited $\frac{1}{2} \cdot 3852 = 1926$ numbers after 36 rounds. During round 37, Tadd will recite $3 \cdot 37 - 2 = 109$ numbers, including his 2019th number.

After 36 rounds have been completed, the children combined will have recited

$$\frac{9}{2} \cdot 36^2 + \frac{3}{2} \cdot 36 = \frac{9}{2} \cdot 1296 + 54 = 5832 + 54 = 5886$$

numbers, the integers from 1 through 5886.

Tadd will say his 2019th number when he has completed reciting $2019 - 1926 = 93$ numbers in round 37. This number is $5886 + 93 = 5979$.

24. **Answer (B):** Because

$$x^3 - 22x^2 + 80x - 67 = (x-p)(x-q)(x-r),$$

multiplying the given equation by the common denominator yields

$$1 = A(s-q)(s-r) + B(s-p)(s-r) + C(s-p)(s-q).$$

This is now a polynomial identity that holds for infinitely many values of s, so it must hold for all s. This means the condition that $s \notin \{p, q, r\}$ can be removed.

Setting $s = p$ yields $1 = A(p-q)(p-r)$, so $\frac{1}{A} = (p-q)(p-r)$. Similarly, $\frac{1}{B} = (q-p)(q-r)$ and $\frac{1}{C} = (r-p)(r-q)$. Hence

$$\begin{aligned}\frac{1}{A} + \frac{1}{B} + \frac{1}{C} &= (p-q)(p-r) + (q-p)(q-r) + (r-p)(r-q) \\ &= (p+q+r)^2 - 3(pq+qr+rp).\end{aligned}$$

By Viète's Formulas $p+q+r$ is the negative of the coefficient of x^2 in the polynomial and $pq+qr+rp$ is the coefficient of the x term, so the requested value is $22^2 - 3 \cdot 80 = 244$. (The numerical values (p, q, r, A, B, C) are approximately $(1.23, 3.08, 17.7, 0.0329, -0.0371, 0.00416)$.)

25. **Answer (D):** Let

$$A_n = \frac{(n^2-1)!}{(n!)^n}.$$

First, note that A_n is an integer when $n = 1$. Next, observe that if n is prime, then A_n is not an integer because the numerator has $n - 1$ factors of n but the denominator has n such factors. Note also that A_4 is not an integer, because the numerator, $15!$, has $7+3+1 = 11$ factors of 2, whereas the denominator, $(4!)^4$, has 12 factors of 2. Therefore for $n \geq 2$, in order for A_n to be an integer, a necessary condition is that n be composite and greater than 4. The following argument shows that this condition is also sufficient.

First note that
$$\frac{n!}{n^2} = \frac{(n-1)!}{n}.$$
If $n = ab$, where a and b are distinct positive integers greater than 1, then $\frac{(n-1)!}{n}$ is an integer because both a and b appear as factors in $(n-1)!$. Otherwise $n = p^2$ for some odd prime p. In this case $p^2 - 1 \geq 2p$, so $(n-1)!$ has at least two factors of p and again $\frac{(n-1)!}{n}$ is an integer.

Now the number
$$\frac{(n^2)!}{(n!)^{n+1}}$$
is an integer because this expression counts the number of ways to separate n^2 objects into n groups of size n without regard to the ordering of the groups (which accounts for the extra factor of $n!$ in the denominator).

By combining the previous two paragraphs, it follows that
$$A_n = \frac{(n^2-1)!}{(n!)^n} = \frac{(n^2)!}{(n!)^{n+1}} \cdot \frac{n!}{n^2}$$
is an integer if and only if $n = 1$ or n is composite and greater than 4. Thus the answer is 50 minus 1 minus the number of primes less than or equal to 50, which is $49 - 15 = 34$.

Problems and solutions were contributed by David Altizio, Risto Atanasov, Thomas Butts, Barbara Currier, Steven Davis, Zachary Franco, Peter Gilchrist, Ellina Grigorieva, Jerrold Grossman, Jonathan Kane, Joseph Li, Hugh Montgomery, Mohamed Omar, Albert Otto, Joachim Rebholz, Mehtaab Sawhney, Michael Tang, Roger Waggoner, Carl Yerger, and Paul Zeitz.

MAA Partner Organizations

We acknowledge the generosity of the following organizations in supporting the MAA AMC and Invitational Competitions:

Patron's Circle

Akamai Foundation

Innovator's Circle

The D. E. Shaw Group

Two Sigma

Winner's Circle

MathWorks

Tudor Investment Corporation

Achiever's Circle

Art of Problem Solving

Jane Street Capital

Sustainer's Circle

American Mathematical Society

Ansatz Capital

Army Educational Outreach Program

Collaborator's Circle

American Statistical Association

Casualty Actuarial Society

Conference Board of the Mathematical Sciences

Mu Alpha Theta

Society for Industrial and Applied Mathematics

MAA AMC

American Mathematics Competitions

Solutions Pamphlet

MAA American Mathematics Competitions

20th Annual

AMC 10B

American Mathematics Competition 10B

Wednesday, February 13, 2019

This Pamphlet gives at least one solution for each problem on this year's competition and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic versus geometric, computational versus conceptual, elementary versus advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *Copies of the problem booklet and solution pamphlet may be shared with your students for educational purposes. However, the publication, reproduction, or communication of the problems or solutions for this competition with anyone outside of the classroom is a violation of the competition rules. This includes dissemination via copier, telephone, email, internet, or media of any type.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

MAA American Mathematics Competitions
Attn: Publications, PO Box 471, Annapolis Junction, MD 20701
Phone 800.527.3690 | Fax 240.396.5647 | amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by MAA's Subcommittee on the AMC 10/AMC 12 Exams, under the direction of the co-chairs Jerrold W. Grossman and Carl Yerger.

1. **Answer (D):** Let x be the volume of the first container and y the volume of the second container. Then

$$\frac{5}{6}x = \frac{3}{4}y, \qquad \text{so} \qquad \frac{x}{y} = \frac{3}{4} \cdot \frac{6}{5} = \frac{9}{10}.$$

2. **Answer (E):** An implication is false if and only if the hypothesis is true but the conclusion is false. Choice **(E)**, $n = 27$, is a counterexample to the statement because the hypothesis is true (27 is not prime) but the conclusion is false ($27 - 2 = 25$ is not prime). For answer choices **(A)** and **(C)**, n is prime, so the hypothesis is false and these values of n do not provide a counterexample. For choices **(B)**, **(C)**, and **(D)**, $n-2$ is prime, so the conclusion is true and these values of n do not provide a counterexample.

3. **Answer (B):** Let n be the number of non-seniors. Because there are 500 students in all, there are $500 - n$ seniors. Because 40% of the seniors play a musical instrument, 60% of the seniors do not play a musical instrument. This leads to the equation

$$0.60(500 - n) + 0.30n = 0.468 \cdot 500.$$

The equation simplifies to $0.30n = 66$. Solving this equation for n gives $n = 220$. Because $100\% - 30\% = 70\%$ of the non-seniors play a musical instrument, there are $0.70 \cdot 220 = 154$ non-seniors who play a musical instrument.

4. **Answer (A):** If a, b, c form an arithmetic progression, then $a = b-d$ and $c = b + d$ for some number d. Then the given linear equation becomes $(b - d)x + by = b + d$, which is equivalent to

$$b(x + y - 1) - d(x + 1) = 0.$$

This will hold for all values of b and d if and only if $x + y - 1 = 0$ and $x + 1 = 0$, which means $x = -1$ and $y = 2$. Thus $(-1, 2)$ is the unique point through which all such lines pass.

OR

If a, b, c form an arithmetic progression, then $b - a = c - b$. This equation is equivalent to $a(-1) + b(2) = c$, so $x = -1$ and $y = 2$ give a point through which the line passes. Conversely, if $a = 1$, $b = 2$, and $c = 3$, then the equation of the line is $x + 2y = 3$, and none of the other four choices satisfies this equation.

OR

Letting $(a, b, c) = (0, 1, 2)$ yields $y = 2$, and letting $(a, b, c) = (1, 0, -1)$ yields $x = -1$. In fact, $(-1, 2)$ is on the line $ax + by = c$ if and only if $-a + 2b = c$, which is equivalent to $b - a = c - b$, which is the defining condition for a, b, c to be an arithmetic progression.

5. **Answer (E):** The reflection of the point (a, b) across the line $y = x$ is the point (b, a). Because the coordinates of the points in the original triangle are all positive, it follows that the coordinates of the images will also be all positive. Thus **(A)** is always true. It is a property of reflections that the line segment connecting a point not on the line of reflection and its image is perpendicular to the line of reflection. This fact shows that **(C)** and **(D)** are always true. Reflection is a rigid transformation, and therefore areas are preserved, so **(B)** is always true. The statement **(E)** is not true in general. As an example, consider a triangle ABC such that the side $\overline{AB}$ is parallel to the line $y = x$. Then the side $\overline{A'B'}$ in the image will also be parallel to $y = x$, which shows that lines AB and $A'B'$ are not perpendicular. Thus the statement **(E)** is not always true. In fact, because reflection across the line $y = x$ interchanges the roles of x and y, the slope of a non-vertical/non-horizontal line and the slope of its image are reciprocals, not negative reciprocals. A line will be perpendicular to its reflection across the line $y = x$ if and only if the line is horizontal or vertical, in which case its image will be vertical or horizontal, respectively.

6. **Answer (C):** Dividing the given equation by $n!$, simplifying, and completing the square yields

$$\begin{aligned}(n+1) + (n+2)(n+1) &= 440\\ n^2 + 4n + 3 &= 440\\ n^2 + 4n + 4 &= 441\\ (n+2)^2 &= 21^2.\end{aligned}$$

Thus $n + 2 = 21$ and $n = 19$. The requested sum of the digits of n is $1 + 9 = 10$.

7. **Answer (B):** Because Casper has exactly enough money to buy 12 pieces of red candy, the amount of money he has must be a multiple of 12 cents. Similarly, it must be a multiple of both 14 cents and 15 cents. Furthermore, this amount of money will buy a whole number of purple candies that cost 20 cents each, so the amount of money must also be a multiple of 20. The least common multiple of $12 = 2^2 \cdot 3$, $14 = 2 \cdot 7$, $15 = 3 \cdot 5$, and $20 = 2^2 \cdot 5$ is $2^2 \cdot 3 \cdot 5 \cdot 7 = 420$. Therefore the number of purple candies that Casper can buy, n, must be a multiple of $420 \div 20 = 21$. Thus the least possible value of n is 21. In this case the red candies cost $420 \div 12 = 35$ cents each, the green candies cost $420 \div 14 = 30$ cents each, and the blue candies cost $420 \div 15 = 28$ cents each.

8. **Answer (B):** The height of each equilateral triangle is $\sqrt{3}$, so the side length of the square is $2\sqrt{3}$. The area of the square is then $\left(2\sqrt{3}\right)^2 = 12$, and the area of the shaded region is

$$12 - 4 \cdot \frac{\sqrt{3}}{4} \cdot 2^2 = 12 - 4\sqrt{3}.$$

9. **Answer (A):** If $x \geq 0$, then $|x| = x$, so $\lfloor |x| \rfloor = \lfloor x \rfloor$. Furthermore, if $x \geq 0$, then $\lfloor x \rfloor \geq 0$, so $\left| \lfloor x \rfloor \right| = \lfloor x \rfloor$. Therefore $f(x) = \lfloor x \rfloor - \lfloor x \rfloor = 0$ when $x \geq 0$.

   Otherwise, $x < 0$, so $|x| = -x$.

   If $x < 0$ and x is an integer, then $\lfloor |x| \rfloor = \lfloor -x \rfloor = -x$ and $\left| \lfloor x \rfloor \right| = \left| x \right| = -x$. Therefore $f(x) = (-x) - (-x) = 0$ in this case.

   If $x < 0$ and x is not an integer, then $\lfloor |x| \rfloor = \lfloor -x \rfloor = -\lfloor x \rfloor - 1$ and $\left| \lfloor x \rfloor \right| = -\lfloor x \rfloor$. Therefore $f(x) = \left(-\lfloor x \rfloor - 1\right) - \left(-\lfloor x \rfloor\right) = -1$ in this case.

   Thus the range of $f(x)$ is $\{-1, 0\}$.

10. **Answer (A):** In order for the area of $\triangle ABC$ to be 100, the altitude to the base $\overline{AB}$ must be 20. Thus C must lie on one of the two lines parallel to and 20 units from line AB. In order for the perimeter of $\triangle ABC$ to be 50, the sum of the lengths of the other two sides must be 40, which implies that point C lies on an ellipse whose foci are A and B and whose semi-minor axis has length $\frac{1}{2}\sqrt{40^2 - 10^2} = \sqrt{375}$, which is less than 20. Therefore the ellipse does not intersect either of the parallel lines, and there are no such points C.

OR

As above, the altitude to the base $\overline{AB}$ of length 10 is 20. Therefore $CA \geq 20$ and $CB \geq 20$, and at least one of those sides has length greater than 20. This contradicts the fact that the perimeter is 50, so no such points C exist.

11. **Answer (A):** Let x be the number of green marbles in Jar 1. Then there are $95-x$ green marbles in Jar 2. Jar 1 contains $9x$ blue marbles and $10x$ marbles in all, and Jar 2 contains $8(95-x)$ blue marbles and $9(95-x)$ marbles in all. Because the jars contain the same number of marbles, $10x = 9(95-x)$, and this equation has the solution $x = 45$. Therefore Jar 1 contains $9 \cdot 45 = 405$ blue marbles, and Jar 2 contains $8(95-45) = 400$ blue marbles. Jar 1 contains $405 - 400 = 5$ more blue marbles than does Jar 2.

OR

The number of marbles in each jar must be a multiple of both 10 and 9, so it is a multiple of 90. If there are $90n$ marbles in each jar, then there are $9n$ green marbles in Jar 1 and $10n$ green marbles in Jar 2. Thus $9n + 10n = 95$, so $n = 5$. Therefore there are 5 fewer green marbles in Jar 1 than in Jar 2, so there are also 5 more blue marbles in Jar 1 than Jar 2.

12. **Answer (C):** One can convert 2018 to base seven by repeatedly dividing by 7; the successive remainders are the digits in the base-seven representation, from right to left. Thus $2018 = 5612_{\text{seven}}$. It follows that the base-seven representations of positive integers less than 2019 have at most four digits, each digit is at most 6, and the leftmost digit is at most 5. If the leftmost digit is 4, then the remaining digits can all be 6 for a sum of $4+6+6+6 = 22$. If the leftmost digit is 5, then the remaining digits cannot all be 6. Therefore the required sum of digits cannot exceed $5+5+6+6 = 22$. Because $5566_{\text{seven}} < 5612_{\text{seven}} < 2019$ (and $4666_{\text{seven}} < 5612_{\text{seven}} < 2019$), the requested maximum sum is 22.

13. **Answer (A):** The mean of the given numbers is

$$\frac{4+6+8+17+x}{5} = \frac{x+35}{5} = \frac{x}{5} + 7.$$

The median depends on the value of x.

If $x < 6$, then the median is 6. If the mean and median are equal, then $\frac{x}{5} + 7 = 6$, which is equivalent to $x = -5$.

If $6 \leq x \leq 8$, then the median is x. If the mean and median are equal, then $\frac{x}{5} + 7 = x$, which is equivalent to $x = \frac{35}{4}$. But this is outside of the given range.

If $x > 8$, then the median is 8. If the mean and median are equal then $\frac{x}{5} + 7 = 8$, which is equivalent to $x = 5$. Again this is outside of the given range.

Therefore the only value of x for which the mean equals the median is -5, so the requested sum is also -5.

14. **Answer (C):** Because 5, 10, and 15 all have a single factor of 5 in their prime factorization, 19! ends with 3 0s. Hence $H = 0$. To determine T and M, divisibility tests for 9 and 11 can be used. Because 19! is divisible by 9, its digit sum, $T + M + 33$, must also be divisible by 9, which implies $T + M = 3$ or $T + M = 12$. Similarly, because 19! is divisible by 11, its alternating digit sum, $(T + 13) - (M + 20) = T - M - 7$, must also be divisible by 11. This implies that $T - M = -4$ or $T - M = 7$. Combining these constraints results in only one solution in which T and M are digits, namely $T = 4$ and $M = 8$. Hence $T + M + H = 4 + 8 + 0 = 12$.

15. **Answer (A):** Let a and b, with $a < b$, be the shared side lengths. Then T_1 has hypotenuse b and legs a and $\sqrt{b^2 - a^2}$, and T_2 has hypotenuse $\sqrt{a^2 + b^2}$ and legs a and b. Thus $\frac{1}{2}a\sqrt{b^2 - a^2} = 1$ and $\frac{1}{2}ab = 2$. Multiplying the first equation by 2 and then squaring gives $a^2b^2 - a^4 = 4$. From the second equation, $a^2b^2 = 16$, so $16 - a^4 = 4$, which means $a^4 = 12$. Then

$$b^4 = \left(\frac{4}{a}\right)^4 = \frac{4^4}{a^4} = \frac{256}{12} = \frac{64}{3}.$$

Therefore the square of the product of the other sides is

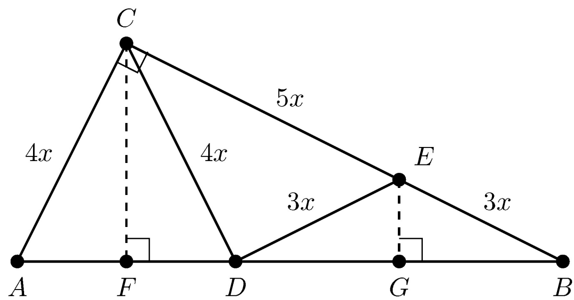
$$\left(\sqrt{b^2 - a^2} \cdot \sqrt{a^2 + b^2}\right)^2 = b^4 - a^4 = \frac{64}{3} - 12 = \frac{28}{3}.$$

16. **Answer (A):** Let $AC = DC = 4x$ and $DE = BE = 3x$. Because $\angle A \cong \angle ADC$, $\angle B \cong \angle EDB$, and $\angle A$ and $\angle B$ are complementary, it follows that $\angle CDE$ is a right angle. Thus $CE = 5x$. Let F and

G lie on $\overline{AB}$ so that $\overline{CF}$ and $\overline{EG}$ are perpendicular to $\overline{AB}$. Then it follows that

$$\frac{3}{8} = \frac{BE}{BC} = \frac{BG}{BF} = \frac{\frac{1}{2}BD}{BD + \frac{1}{2}AD} = \frac{BD}{2BD + AD},$$

so $8BD = 6BD + 3AD$. It follows that $AD : DB = 2 : 3$.



17. **Answer (C):** The probability that the two balls are tossed into the same bin is

$$\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{4} \cdot \frac{1}{4} + \frac{1}{8} \cdot \frac{1}{8} + \cdots = \sum_{n=1}^{\infty} \frac{1}{4^n} = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}.$$

Therefore the probability that the balls are tossed into different bins is $\frac{2}{3}$. By symmetry the probability that the red ball is tossed into a higher-numbered bin than the green ball is $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$.

18. **Answer (C):** By symmetry Henry's walks will converge toward walking between two points, one at a distance x from the gym and the other at the same distance x from his home. Because Henry would be $2 - x$ kilometers from home when he is closest to the gym and also because his trip toward home would take him to $\frac{1}{4}$ this distance from home, $x = \frac{1}{4}(2 - x)$. Solving this yields $x = \frac{2}{5}$. Therefore, Henry's walks will approach $2 - 2 \cdot \frac{2}{5} = 1\frac{1}{5}$ kilometers in length.

OR

If Henry is a_k kilometers from home after his kth walk toward the gym and b_k kilometers from home after his kth walk toward home, then $a_0 = b_0 = 0$, and for $k \geq 1$,

$$a_k = b_{k-1} + \frac{3}{4}(2 - b_{k-1}) = \frac{3}{2} + \frac{1}{4}b_{k-1}$$

and

$$b_k = \frac{1}{4}a_k = \frac{3}{8} + \frac{1}{16}b_{k-1}.$$

Iterating shows that the sequence (b_k) converges to

$$B = \frac{3}{8} + \frac{3}{8}\cdot\frac{1}{16} + \frac{3}{8}\cdot\left(\frac{1}{16}\right)^2 + \cdots = \frac{3}{8}\cdot\frac{1}{1-\frac{1}{16}} = \frac{2}{5},$$

from which it then follows that (a_k) converges to $A = \frac{3}{2} + \frac{1}{4}\cdot\frac{2}{5} = \frac{8}{5}$. The requested absolute difference is $\left|\frac{8}{5} - \frac{2}{5}\right| = 1\frac{1}{5}$.

OR

Let x_k denote Henry's distance from home after his kth walk. The following formulas give the value of x_k:

$$x_k = \frac{2}{5} - \frac{2}{5\cdot 4^k} \quad \text{when } k \text{ is even}$$

and

$$x_k = \frac{8}{5} - \frac{2}{5\cdot 4^k} \quad \text{when } k \text{ is odd.}$$

To prove this by mathematical induction, first note that indeed $x_0 = \frac{2}{5} - \frac{2}{5\cdot 4^0} = 0$ and $x_1 = \frac{8}{5} - \frac{2}{5\cdot 4^1} = \frac{3}{2} = \frac{3}{4}\cdot 2$. Then for even values of $k \geq 2$, Henry was heading home, so

$$x_k = \frac{1}{4}x_{k-1} = \frac{1}{4}\left(\frac{8}{5} - \frac{2}{5\cdot 4^{k-1}}\right) = \frac{2}{5} - \frac{2}{5\cdot 4^k};$$

and for odd values of $k \geq 3$, Henry was heading toward the gym, so

$$\begin{aligned} x_k &= x_{k-1} + \frac{3}{4}(2 - x_{k-1}) \\ &= \frac{3}{2} + \frac{1}{4}x_{k-1} \\ &= \frac{3}{2} + \frac{1}{4}\left(\frac{2}{5} - \frac{2}{5\cdot 4^{k-1}}\right) \\ &= \frac{8}{5} - \frac{2}{5\cdot 4^k}. \end{aligned}$$

As k approaches infinity, these values rapidly converge to $\frac{2}{5}$ and $\frac{8}{5}$, respectively, so Henry is essentially walking back and forth between two points that are $\frac{8}{5} - \frac{2}{5} = 1\frac{1}{5}$ kilometers apart.

19. **Answer (C):** Note that $100{,}000 = 2^5 \cdot 5^5$. This implies that for a number to be a product of two elements in S it must be of the form $2^a \cdot 5^b$ with $0 \le a \le 10$ and $0 \le b \le 10$. The corresponding product for the remainder of this solution will be denoted (a, b). Note that the pairs $(0, 0)$, $(0, 10)$, $(10, 0)$, and $(10, 10)$ cannot be obtained as the product of two distinct elements of S; these products can be obtained only as $1 \cdot 1 = 1$, $5^5 \cdot 5^5 = 5^{10}$, $2^5 \cdot 2^5 = 2^{10}$, and $10^5 \cdot 10^5 = 10^{10}$, respectively. This gives at most $11 \cdot 11 - 4 = 117$ possible products. To see that all these pairs can be achieved, consider four cases:

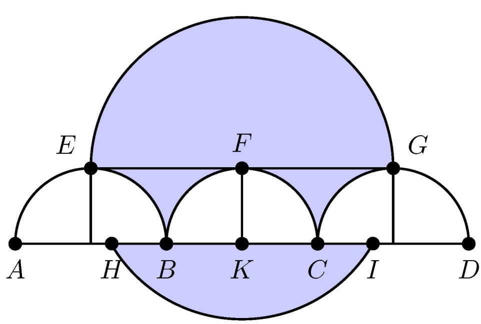
    If $0 \le a \le 5$ and $0 \le b \le 5$, other than $(0, 0)$, then (a, b) can be achieved with the divisors 1 and $2^a \cdot 5^b$.

    If $6 \le a \le 10$ and $0 \le b \le 5$, other than $(10, 0)$, then (a, b) can be achieved with the divisors 2^5 and $2^{a-5} \cdot 5^b$.

    If $0 \le a \le 5$ and $6 \le b \le 10$, other than $(0, 10)$, then (a, b) can be achieved with the divisors 5^5 and $2^a \cdot 5^{b-5}$.

    Finally, if $6 \le a \le 10$ and $6 \le b \le 10$, other than $(10, 10)$, then (a, b) can be achieved with the divisors $2^5 \cdot 5^5$ and $2^{a-5} \cdot 5^{b-5}$.

20. **Answer (E):** Let H and I be the intersections of $\overline{AD}$ with the circle centered at F, where H lies between A and B, and I lies between C and D; and let K be the foot of the perpendicular line segment from F to $\overline{AD}$. The specified region consists of three subregions: a semicircle of radius 2, a 4×1 rectangle with 4 quarter circles of radius 1 removed, and the segment of the circle cut off by chord $\overline{HI}$, as shown in the figure below.



    The semicircle of radius 2 has area 2π. The rectangle minus the 4 quarter circles has area $4 - \pi$. Because $FK = 1$ and $FI = 2$, it follows that $\angle KFI$ has measure 60°, and therefore the segment of the circle is a third of the circle with $\triangle HFI$ removed. The area of the segment is

$$\frac{4}{3}\pi - \frac{1}{2}\cdot 2\sqrt{3}\cdot 1 = \frac{4}{3}\pi - \sqrt{3}.$$

Adding the areas of the three subregions gives $\frac{7}{3}\pi - \sqrt{3} + 4$, and the requested sum is $7 + 3 + 3 + 4 = 17$.

21. **Answer (B):** With probability 1, either HH or TT will occur after a finite number of flips. The desired event will occur if and only if the sequence of flips is THTHH or THTHTHH or THTHTHTHH or .... The probabilities of these outcomes are $\left(\frac{1}{2}\right)^5$, $\left(\frac{1}{2}\right)^7$, $\left(\frac{1}{2}\right)^9$, $\ldots$, a geometric sequence with common ratio $\frac{1}{4}$. The requested probability is the sum of these probabilities,

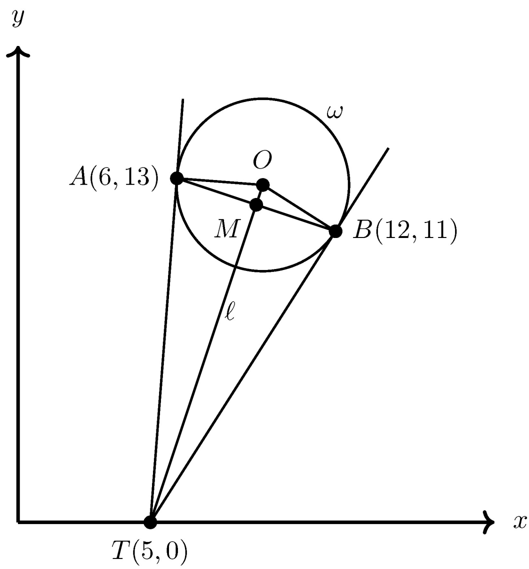
$$\frac{\left(\frac{1}{2}\right)^5}{1-\frac{1}{4}} = \frac{1}{32}\cdot\frac{4}{3} = \frac{1}{24}.$$

22. **Answer (B):** No player can ever end up with \$3 at the end of a round, because that player had to give away one of the dollars in play. Therefore the only two possible distributions of the money are 1-1-1 and 2-1-0. Suppose that a round of the game starts at 1-1-1. Without loss of generality, assume that Raashan gives his dollar to Sylvia. Then the only way for the round to end at 1-1-1 is for Ted to give his dollar to Raashan (otherwise Sylvia would end up with \$2) and for Sylvia to give her dollar to Ted; the probability of this is $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$. Next suppose that a round starts at 2-1-0; without loss of generality, assume that Raashan has \$2 and Sylvia has \$1. Then the only way for the round to end at 1-1-1 is for Sylvia to give her dollar to Ted (otherwise Raashan would end up with \$2) and for Raashan to give his dollar to Sylvia; the probability of this is $\frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}$. Thus no matter how the round starts, the probability that the round will end at 1-1-1 is $\frac{1}{4}$. In particular, the probability is $\frac{1}{4}$ that at the end of the 2019th round each player will have \$1.

Note: It may seem counterintuitive that an uneven distribution is more likely than an even distribution. But in a situation with larger initial bankrolls evenly distributed to a larger number of players, inequality reigns after many rounds. See this website:

`http://www.decisionsciencenews.com/?s=happens+next`

23. **Answer (C):** Let T be the point where the tangents at A and B intersect. By symmetry T lies on the perpendicular bisector ℓ of $\overline{AB}$, so in fact T is the (unique) intersection point of line ℓ with the x-axis. Computing the midpoint M of $\overline{AB}$ gives $(9, 12)$, and computing the slope of $\overline{AB}$ gives $\frac{13-11}{6-12} = -\frac{1}{3}$. This means that the slope of ℓ is 3, so the equation of ℓ is given by $y - 12 = 3(x - 9)$. Setting $y = 0$ yields that $T = (5, 0)$.



Now let O be the center of circle ω. Note that $\overline{OA} \perp \overline{AT}$ and $\overline{OB} \perp \overline{BT}$, so in fact M is the foot of the altitude from A to the hypotenuse of $\triangle OAT$. By the distance formula, $TM = 4\sqrt{10}$ and $MA = \sqrt{10}$. Then by the Altitude on Hypotenuse Theorem, $MO = \frac{1}{4}\sqrt{10}$, so by the Pythagorean Theorem radius AO of circle ω is $\frac{1}{4}\sqrt{170}$. As a result, the area of the circle is

$$\frac{1}{16} \cdot 170 \cdot \pi = \frac{85\pi}{8}.$$

OR

As above, the line $y - 12 = 3(x - 9)$ passes through $T(5, 0)$ and the center of the circle O. The slope of line AT is 13, so the slope of AO is $-\frac{1}{13}$. The equation of line OA is $13(y - 13) = 6 - x$. Thus the intersection of line AO and line OT is $O\left(\frac{37}{4}, \frac{51}{4}\right)$. Then the radius of the circle is

$$OA = \sqrt{\left(\frac{37}{4} - 6\right)^2 + \left(\frac{51}{4} - 13\right)^2} = \frac{1}{4}\sqrt{170}$$

as in the solution above.

24. **Answer (C):** First note that it suffices to study $y_n = x_n - 4$ and find the least positive integer m such that $y_m \le \frac{1}{2^{20}}$. Now $y_0 = 1$ and

$$y_{n+1} = \frac{y_n(y_n + 9)}{y_n + 10}.$$

Observe that (y_n) is a strictly decreasing sequence of positive numbers. Because

$$\frac{y_{n+1}}{y_n} = 1 - \frac{1}{y_n + 10},$$

it follows that

$$\frac{9}{10} \le \frac{y_{n+1}}{y_n} \le \frac{10}{11},$$

and because $y_0 = 1$,

$$\left(\frac{9}{10}\right)^k \le y_k \le \left(\frac{10}{11}\right)^k$$

for all integers $k \ge 2$.

Now note that

$$\left(\frac{1}{2}\right)^{\frac{1}{4}} < \frac{9}{10}$$

because this is equivalent to $0.5 < (0.9)^4 = (0.81)^2$. Therefore

$$\left(\frac{1}{2}\right)^{\frac{m}{4}} < y_m \le \frac{1}{2^{20}},$$

so $m > 80$. Now note that

$$\left(\frac{11}{10}\right)^{10} = \left(1 + \frac{1}{10}\right)^{10} > 1 + 10 \cdot \frac{1}{10} = 2,$$

so

$$\frac{10}{11} < \left(\frac{1}{2}\right)^{\frac{1}{10}}.$$

Thus

$$\frac{1}{2^{20}} < y_{m-1} < \left(\frac{10}{11}\right)^{m-1} < \left(\frac{1}{2}\right)^{\frac{m-1}{10}},$$

so $m < 201$. Thus m lies in the range **(C)**. (Numerical calculations will show that $m = 133$.)

25. **Answer (C):** For $n \geq 2$, let a_n be the number of sequences of length n that begin with a 0, end with a 0, contain no two consecutive 0s, and contain no three consecutive 1s. In order for the sequence to end with a 0 and satisfy the conditions, it must end either 010 or 0110. Thus $a_n = a_{n-2} + a_{n-3}$. The initial conditions for this recurrence relation are $a_2 = 0$, $a_3 = 1$ (the sequence 010), and $a_4 = 1$ (the sequence 0110). Then $a_5 = a_3 + a_2 = 1 + 0 = 1$, $a_6 = a_4 + a_3 = 1 + 1 = 2$, $a_7 = a_5 + a_4 = 1 + 1 = 2$, and so on. A bit of calculation produces the display below; $a_{19} = 65$.

n	2	3	4	5	6	7	8	9	10	11
a_n	0	1	1	1	2	2	3	4	5	7

n	12	13	14	15	16	17	18	19
a_n	9	12	16	21	28	37	49	65

OR

There are four cases, depending on the number of 0s in the string. If there are 10 0s, then there are 9 1s, and because no two 0s can be consecutive, the string must be 0101010101010101010. If there are 9 0s and 10 1s, then there are 8 gaps between the 0s into which at least one but no more than two 1s must be placed, and there are $\binom{8}{2}$ ways to choose the gaps into which to place two 1s. Similarly, if there are 8 0s and 11 1s, then there are $\binom{7}{4}$ ways to choose the gaps into which to place two 1s; and if there are 7 0s and 12 1s, then there are $\binom{6}{6}$ ways to choose the gaps into which to place two 1s. There cannot be more than 10 nor fewer than 7 0s. The number of possible strings is

$$1 + \binom{8}{2} + \binom{7}{4} + \binom{6}{6} = 1 + 28 + 35 + 1 = 65.$$

Problems and solutions were contributed by David Altizio, Chris Bolognese, Thomas Butts, Barbara Currier, Steven Davis, Marta Eso, Peter Gilchrist, Jerrold Grossman, Varun Kambhampati, Jonathan Kane, Hugh Montgomery, Mohamed Omar, Albert Otto, Joachim Rebholz, Mehtaab Sawhney, Michael Tang, Roger Waggoner, David Wells, Carl Yerger, and Paul Zeitz.

MAA Partner Organizations

We acknowledge the generosity of the following organizations in supporting the MAA AMC and Invitational Competitions:

Patron's Circle
Akamai Foundation

Innovator's Circle
The D. E. Shaw Group
Two Sigma

Winner's Circle
MathWorks
Tudor Investment Corporation

Achiever's Circle
Art of Problem Solving
Jane Street Capital

Sustainer's Circle
American Mathematical Society
Ansatz Capital
Army Educational Outreach Program

Collaborator's Circle
American Statistical Association
Casualty Actuarial Society
Conference Board of the Mathematical Sciences
Mu Alpha Theta
Society for Industrial and Applied Mathematics