AMERICAN MATHEMATICS COMPETITIONS

1st Annual Mathematics Contest $\rightarrow$10

AMC $\rightarrow$ 10

Solutions Pamphlet

TUESDAY, FEBRUARY 15, 2000

Sponsored by

Mathematical Association of America

Society of Actuaries   Mu Alpha Theta

National Council of Teachers of Mathematics

Casualty Actuarial Society   American Statistical Association

American Mathematical Association of Two-Year Colleges

American Mathematical Society

American Society of Pension Actuaries

Consortium for Mathematics and its Applications

Pi Mu Epsilon   National Association of Mathematicians

School Science and Mathematics Association

Clay Mathematics Institute   University of Nebraska-Lincoln

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC $\rightarrow$10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, eMail, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:

Professor Harold Reiter, AMC $\rightarrow$10 Chair
Department of Mathematics
University of North Carolina at Charlotte
Charlotte, NC 28223 USA

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:

Titu Andreescu, AMC Director
American Mathematics Competitions
University of Nebraska-Lincoln, P.O. Box 81606
Lincoln, NE 68501-1606

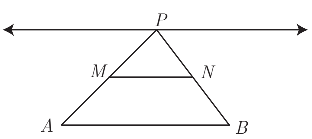
1. **Answer (E):** Factor 2001 into primes to get $2001 = 3 \cdot 23 \cdot 29$. The largest possible sum of three distinct factors whose product is the one which combines the two largest prime factors, namely $I = 23 \cdot 29 = 667$, $M = 3$, and $O = 1$, so the largest possible sum is $1 + 3 + 667 = 671$.

2. **Answer (A):** $2000(2000^{2000}) = (2000^1)(2000^{2000}) = 2000^{1+2000} = 2000^{2001}$. All the other options are greater than 2000^{2001}.

3. **Answer (B):** Since Jenny ate 20% of the jellybeans remaining each day, 80% of the jellybeans are left at the end of each day. If x is the number of jellybeans in the jar originally, then$(0.8)^2x = 32$. Thus $x = 50$.

4. **Answer (D):** Since Chandra paid extra \$5.06 in January, her December connect time must have cost her \$5.06. Therefore, her monthly fee is $\$12.48 - \$5.06 = \$7.42$.

5. **Answer (B):** Since $\triangle ABP$ is similar to $\triangle MNP$ and $PM = \frac{1}{2} \cdot AP$, it follows that $MN = \frac{1}{2} \cdot AB$. Since the base AB and the altitude to AB of $\triangle ABP$ do not change, the area does not change. The altitude of the trapezoid is half that of the triangle, and the bases do not change as P changes, so the area of the trapezoid does not change. Only the perimeter changes (reaching a minimum when $\triangle ABP$ is isosceles).

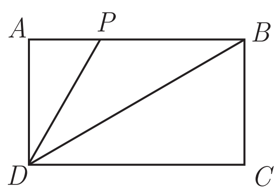


6. **Answer (C):** The sequence of units digits is

$$1, 1, 2, 3, 5, 8, 3, 1, 4, 5, 9, 4, 3, 7, 0, 7, 7, 4, 1, 5, 6, \ldots$$

The digit 6 is the last of the ten digits to appear.

7. **Answer (B):** Both triangles APD and CBD are $30-60-90^\circ$ triangles. Thus $DP = \frac{2\sqrt{3}}{3}$ and $DB = 2$. Since $\angle BDP = \angle PDB$, it follows that $PB = PD = \frac{2\sqrt{3}}{3}$. Hence the perimeter of $\triangle BDP$ is $\frac{2\sqrt{3}}{3} + \frac{2\sqrt{3}}{3} + 2 = 2 + \frac{4\sqrt{3}}{3}$.



8. **Answer (D):** Let f and s represent the numbers of freshmen and sophomores at the school, respectively. According to the given condition, $(2/5)f = (4/5)s$. Thus, $f = 2s$. That is, there are twice as many freshmen as sophomores.

9. **Answer (C):** Since $x < 2$, it follows that $|x - 2| = 2 - x$. If $2 - x = p$, then $x = 2 - p$. Thus $x - p = 2 - 2p$.

10. **Answer (D):** By the *Triangle Inequality*, each of x and y can be any number strictly between 2 and 10, so $0 \le |x - y| < 8$. Therefore, the smallest positive number that is not a possible value of $|x - y|$ is $10 - 2 = 8$.

11. **Answer (C):** There are five prime numbers between 4 and 18: 5,7,11,13, and 17. Hence the product of any two of these is odd and the sum is even. Because $xy - (x + y) = (x - 1)(y - 1) - 1$ increases as either x or y increases (since both x and y are bigger than 1), the answer must be an odd number that is no smaller than $23 = 5 \cdot 7 - (5 + 7)$ and no larger than $191 = 13 \cdot 17 - (13 + 17)$. The only possibility among the options is 119, and indeed $119 = 11 \cdot 13 - (11 + 13)$.

12. **Answer (C):** Calculating the number of squares in the first few figures uncovers a pattern. Figure 0 has $2(0) + 1 = 2(0^2) + 1$ squares, figure 1 has $2(1) + 3 = 2(1^2) + 3$ squares, figure 2 has $2(1 + 3) + 5 = 2(2^2) + 5$ squares, and figure 3 has $2(1 + 3 + 5) + 7 = 2(3^2) + 7$ squares. In general, the number of unit squares in figure n is

$$2(1 + 3 + 5 + \cdots + (2n - 1)) + 2n + 1 = 2(n^2) + 2n + 1.$$

Therefore, the figure 100 has $2(100^2) + 2 \cdot 100 + 1 = 20201$.

OR

Each figure can be considered as a large square with identical small pieces deleted from each of the four corners. Figure 1 has $3^2 - 4(1)$ unit squares, figure 2 has $5^2 - 4(1 + 2)$ unit squares, and figure 3 has $7^2 - 4(1 + 2 + 3)$ unit squares. In general, figure n has

$$(2n - 1)^2 - 4(1 + 2 + \cdots + n) = (2n + 1)^2 - 2n(n + 1) \text{ unit squares.}$$

Thus figure 100 has $201^2 - 200(101) = 20201$ unit squares.

OR

The number of unit squares in figure n is the sum of the first n positive odd integers plus the sum of the first $n + 1$ positive odd integers. Since the sum of the first k positive odd integers is k^2, figure n has $n^2 + (n + 1)^2$ unit squares. So figure 100 has $100^2 + 101^2 = 20201$ unit squares.

13. **Answer (B):** To avoid having two yellow pegs in the same row or column, there must be exactly one yellow peg in each row and in each column. Hence, starting at the top of the array, the peg in the first row must be yellow, the second peg of the second row must be yellow, the third peg of the third row must be yellow, etc. To avoid having two red pegs in some row, there must be a red peg in each of rows 2,3,4, and 5. The red pegs must be in the first position of the second row, the second position of the third row, etc. Continuation yields exactly one ordering that meets the requirements, as shown.

$$\begin{array}{ccccc} y & & & & \\ r & y & & & \\ g & r & y & & \\ b & g & r & y & \\ o & b & g & r & y \end{array}$$

14. **Answer (C):** Note that the integer average condition means that the sum of the scores of the first n students is a multiple of n. The scores of the first two students must be both even or both odd, and the sum of the scores of the first three students must be divisible by 3. The remainders when 71, 76, 80, 82, and 91 are divided by 3 are 2, 1, 2, 1, and 1, respectively. Thus the only sum of three scores divisible by 3 is $76 + 82 + 91 = 249$, so the first two scores entered are 76 and 82 (in some order), and the third score is 91. Since 249 is 1 larger than a multiple of 4, the fourth score must be 3 larger than a multiple of 4, and the only possible is 71, leaving 80 as the score of the fifth student.

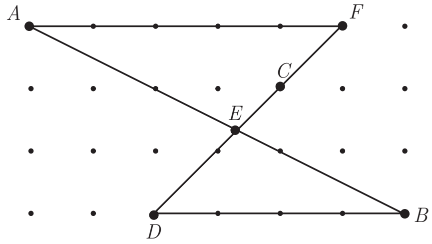
15. **Answer (E):** Find the common denominator and replace the ab in the numerator with $a - b$ to get

$$\begin{aligned} \frac{a}{b} + \frac{b}{a} - ab &= \frac{a^2 + b^2 - (ab)^2}{ab} \\ &= \frac{a^2 + b^2 - (a-b)^2}{ab} \\ &= \frac{a^2 + b^2 - (a^2 - 2ab + b^2)}{ab} \\ &= \frac{2ab}{ab} = 2. \end{aligned}$$

OR

Note that $a = a/b - 1$ and $b = 1 - b/a$. It follows that $\frac{a}{b} + \frac{b}{a} - ab = (a+1) + (1-b) - (a-b) = 2$.

16. **Answer (B):** Extend $\overline{DC}$ to F. Triangle FAE and DBE are similar with ratio $5:4$. Thus $AE = 5 \cdot AB/9$, $AB = \sqrt{3^2+6^2} = \sqrt{45} = 3\sqrt{5}$, and $AE = 5(3\sqrt{5})/9 = 5\sqrt{5}/3$.

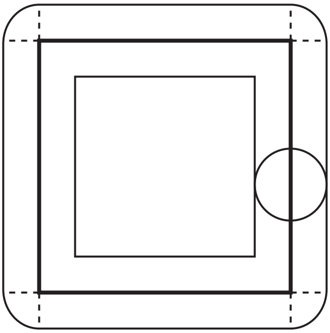


OR

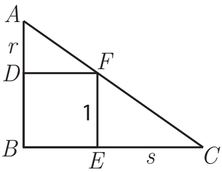
Coordinatize the points so that $A = (0,3)$, $B = (6,0)$, $C = (4,2)$, and $D = (2,0)$. Then the line through A and B is given by $x + 2y = 6$, and the line through C and D is given by $x - y = 2$. Solve these simultaneously to get $E = \left(\frac{10}{3}, \frac{4}{3}\right)$. Hence $AE = \sqrt{(\frac{10}{3}-0)^2 + (\frac{4}{3}-3)^2} = \sqrt{\frac{125}{9}} = \frac{5\sqrt{5}}{3}$.

17. **Answer (D):** Neither of the exchanges *quarter* $\rightarrow$ *five nickels* nor *nickel* $\rightarrow$ *five pennies* changes the total value of Boris's coins. The exchange *penny* $\rightarrow$ *five quarters* increase the total value of Boris's coins by \$1.24. Hence, Boris must have $\$.01 + \$1.24n$ after n uses of the last exchange. Only option D is of this form: $745 = 1 + 124 \cdot 6$. In cents, option A is 115 more than a multiple of 124, B is 17 more than a multiple of 124, C is 10 more than a multiple of 124, and E is 39 more than a multiple of 124.

18. **Answer (C):** At any point on Charlyn's walk, she can see all the points inside a circle of radius 1 km. The portion of the viewable region inside the square consists of the interior of the square except for a smaller square with side length 3 km. This portion of the viewable region has area $(25 - 9)$ km^2. The portion of the viewable region outside the square consists of four rectangles, each 5 km by 1 km, and four quarter-circles, each with a radius of 1 km. This portion of the viewable region has area $4(5 + \frac{\pi}{4}) = (20 + \pi)$ km^2. The area of the entire viewable region is $36 + \pi \approx 30$ km^2.

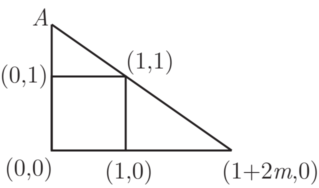


19. **Answer (D):** With out loss of generality, let the side of the square have length 1 unit and let the area of triangle ADF be m. Let $AD = r$ and $EC = s$. Because triangles ADF and FEC are similar, $s/1 = 1/r$. Since $\frac{1}{2}r = m$, the area of triangle FEC is $\frac{1}{2}s = \frac{1}{2r} = \frac{1}{4m}$.



OR

Let $B = (0,0)$, $E = (1,0)$, $F = (1,1)$ and $D = (0,1)$ be the vertices of the square. Let $C = (1 + 2m, 0)$, and notice that the area of $BEFD$ is 1 and the area of triangle FEC is m. The slope of the line through C and F is $-\frac{1}{2m}$; thus, it intersects the y-axis at $A = \left(0, 1 + \frac{1}{2m}\right)$. The area of triangle ADF is therefore $\frac{1}{4m}$.

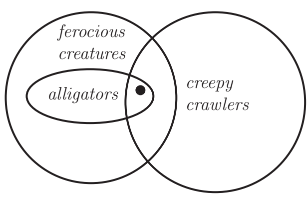


20. **Answer (C):** Note that

$$AMC+AM+MC+CA=(A+1)(M+1)(C+1)-(A+M+C)-1=pqr-11,$$

where p, q, and r are positive integers whose sum is 13. A case-by-case analysis shows that pqt is largest when two of the numbers p, q, r are 4 and the third is 5. Thus the answer is $4 \cdot 4 \cdot 5 - 11 = 69$.

21. **Answer (B):** From the conditions we can conclude that some creepy crawlers are ferocious (since some are alligators). Hence, there are some ferocious creatures that are creepy crawlers, and thus II must be true. The diagram below shows that the only conclusion that can be drawn is existence of an animal in the region with the dot. Thus, neither I nor III follows from the given conditions.



22. **Answer (C):** Suppose that the whole family drank x cups of milk and y cups of coffee. Let n denote the number of people in the family. The information given implies that $x/4 + y/6 = (x + y)/n$. This leads to

$$3x(n - 4) = 2y(6 - n).$$

Since x and y are positive, the only positive integer n for which both sides have the same sign is $n = 5$.

OR

If Angela drank c cups of coffee and m cups of mile, then $0 < c < 1$ and $m+c = 1$. The number of people in the family is $6c+4m = 4+2c$, which is an integer if and only if $c = \frac{1}{2}$. Thus, there are 5 people in the family.

23. **Answer (E):** If x were less than or equal to 2, then 2 would be both the median and the mode of the list. Thus $x > 2$. Consider the two cases $2 < x < 4$, and $x \geq 4$.
Case 1: If $2 < x < 4$, then 2 is the mode, x is the median, and $\frac{25+x}{7}$ is the mean, which must equal $2 - (x - 2)$, $\frac{x+2}{2}$, or $x + (x - 2)$, depending on the size of the mean relative to 2 and x. These give $x = \frac{3}{8}$, $x = \frac{36}{5}$, and $x = 3$, of which $x = 3$ is the only value between 2 and 4.
Case 2: If $x \geq 4$, then 4 is the median, 2 is the mode, and $\frac{25+x}{7}$ is the mean, which must be 0,3, or 6. Thus $x = -25$, -4, or 17, of which 17 is the only one of these values greater than or equal to 4.
Thus the x-value sum to $3 + 17 = 20$.

24. **Answer (B):** Let $x = 9z$. Then $f(3z) = f(9z/3) = f(3z) = (9z)^2 + 9z + 1 = 7$. Simplifying and solving the equation for z yields $81z^2 + 9z - 6 = 0$, so $3(3z + 1)(9z - 2) = 0$. Thus $z = -1/3$ or $z = 2/9$. The sum of these values is $-1/9$.
Note. The answer can also be obtained by using the sum-of-roots formula on $81z^2 + 9z - 6 = 0$. The sum of the roots is $-9/81 = -1/9$.

25. **Answer (A):** Note that, if a Tuesday is d days after a Tuesday, then d is a multiple of 7. Next, we need to consider whether any of the years $N - 1$, N, $N + 1$ is a leap year. If N is not a leap year, the 200^{th} day of year $N + 1$ is $365 - 300 + 200 = 265$ days after a Tuesday, and thus is a Monday, since 265 if 6 larger than a multiple of 7. Thus, year N is a leap year and the 200^{th} day of year $N + 1$ is another Tuesday (as given), being 266 days after a Tuesday. It follows that year $N - 1$ is not a leap year. Therefore, the 100^{th} day of year $N - 1$ precedes the given Tuesday in year N by $365 - 100 + 300 = 565$ days, and therefore is a Thursday, since $565 = 7 \cdot 80 + 5$ is 5 larger than a multiple of 7.

AMERICAN MATHEMATICS COMPETITIONS

2nd Annual Mathematics Contest 10

AMC 10
Solutions Pamphlet

TUESDAY, FEBRUARY 13, 2001

Sponsored by

Mathematical Association of America

University of Nebraska

American Statistical Association
Casualty Actuarial Society
Society of Actuaries
National Council of Teachers of Mathematics
American Society of Pension Actuaries
American Mathematical Society
American Mathematical Association of Two Year Colleges
Pi Mu Epsilon
Consortium for Mathematics and its Applications
Mu Alpha Theta
National Association of Mathematicians
Kappa Mu Epsilon
School Science and Mathematics Association
Clay Mathematics Institute

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC "10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:

Prof. Douglas Faires
Department of Mathematics
Youngstown State University
Youngstown, OH 44555-0001

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:

Prof. Titu Andreescu, AMC Director
American Mathematics Competitions
University of Nebraska-Lincoln, P.O. Box 81606
Lincoln, NE 68501-1606

A03t-200110

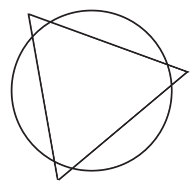
1. **(E)** The middle number in the 9-number list is $n+6$, which is given as 10. Thus $n = 4$. Add the terms together to get $9n + 63 = 9 \cdot 4 + 63 = 99$. Thus the mean is $99/9 = 11$.

2. **(C)** The reciprocal of x is $\frac{1}{x}$, and the additive inverse of x is $-x$. The product of these is $(\frac{1}{x}) \cdot (-x) = -1$. So $x = -1 + 2 = 1$, which is in the interval $0 < x \le 2$.

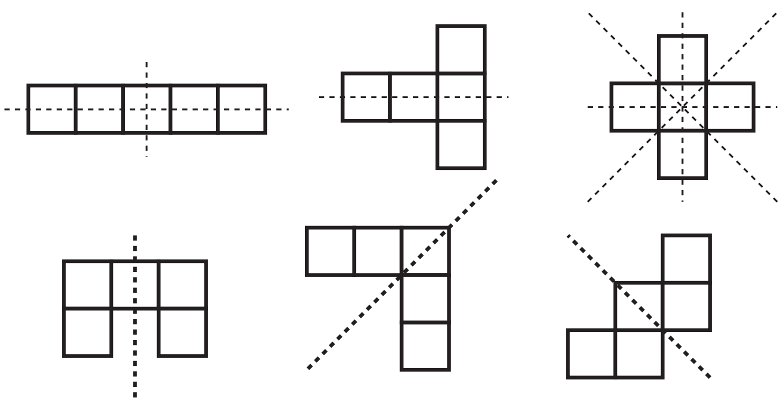
3. **(E)** Suppose the two numbers are a and b. Then the desired sum is

$$2(a_3) + 2(b + 3) = 2(a + b) + 12 = 2S + 12.$$

4. **(E)** The circle can intersect at most two points of each side of the triangle, so the number can be no greater than six. The figure shows that the number can indeed be six.



5. **(D)** Exactly six have at least one line of symmetry. They are:



6. **(E)** Suppose $N = 10a + b$. Then $10a + b = ab + (a + b)$. It follows that $9a = ab$, which implies that $b = 9$, since $a \ne 0$.

7. **(C)** If x is the number, then moving the decimal point four places to the right is the same as multiplying x by 10,000. That is, $10{,}000x = 4 \cdot (\frac{1}{x})$, which is equivalent to $x^2 = 4/10{,}000$. Since x is positive, it follows that $x = 2/100 = 0.02$.

8. **(B)** The number of school days until they will next be together is the least common multiple of 3, 4, 6, and 7, which is 84.

9. **(B)** If Kristin's annual income is $x \geq 28{,}000$ dollars, then

$$\frac{p}{100} \cdot 28{,}000 + \frac{p+2}{100} \cdot (x - 28{,}000) = \frac{p+0.25}{100} \cdot x.$$

Multiplying by 100 and expanding yields

$$28{,}000p + px + 2x - 28{,}000p - 56{,}000 = px + 0.25x.$$

So, $1.75x = \frac{7}{4}x = 56{,}000$ and $x = 32{,}000$.

10. **(D)** Since

$$x = \frac{24}{y} = 48z$$

we have $z = 2y$. So $72 = 2y^2$, which implies that $y = 6$, $x = 4$, and $z = 12$. Hence $x + y + z = 22$.

OR

Take the product of the equations to get $xy \cdot xz \cdot yz = 24 \cdot 48 \cdot 72$. Thus

$$(xyz)^2 = 2^3 \cdot 3 \cdot 2^4 \cdot 3 \cdot 2^3 \cdot 3^2 = 2^{10} \cdot 3^4.$$

So $(xyz)^2 = (2^5 \cdot 3^2)^2$, and we have $xyz = 2^5 \cdot 3^2$. Therefore,

$$x = \frac{xyz}{yz} = \frac{2^5 \cdot 3^2}{2^3 \cdot 3^2} = 4.$$

From this it follows that $y = 6$ and $z = 12$, so the sum is $4 + 6 + 12 = 22$.

11. **(C)** The n^{th} ring can be partitioned into four rectangles: two containing $2n+1$ unit squares and two containing $2n-1$ unit squares. So there are

$$2(2n+1) + 2(2n-1) = 8n$$

unit squares in the n^{th} ring. Thus, the 100^{th} ring has $8 \cdot 100 = 800$ unit squares.

OR

The n^{th} ring can be obtained by removing a square of side $2n-1$ from a square of side $2n+1$. So it contains

$$(2n+1)^2 - (2n-1)^2 = (4n^2 + 4n + 1) - (4n^2 - 4n + 1) = 8n$$

unit squares.

12. **(D)** In any triple of consecutive integers, at least one is even and one is a multiple of 3. Therefore, the product of the three integers is both even and a multiple of 3. Since 7 is adivisor of the product, the numbers 6, 14, 21, and 42 must also be divisors of the product. However, 28 contains two factors of 2, and n need not. For example, $5 \cdot 6 \cdot 7$ is divisible by 7, but not by 28.

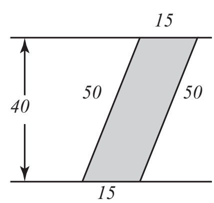
13. **(E)** The last four digits (GHIJ) are either 9753 or 7531, and the remaining odd digit (either 1 or 9) is A, B, or C. Since $A + B + C = 9$, the odd digit among A, B, and C must be 1. Thus the sum of the two even digits in ABC is 8. The three digits in DEF are 864, 642, or 420, leaving the pairs 2 and 0, 8 and 0, or 8 and 6, respectively, as the two even digits in ABC. Of those, only the pair 8 and 0 has sum 8, so ABC is 810, and the required first digit is 8. The only such telephone number is 810-642-9753.

14. **(A)** Let n be the number of full-price tickets and p be the price of each in dollars. Then

$$np + (140 - n) \cdot \frac{p}{2} = 2001, \text{ so } p(n + 140) = 4002.$$

Thus $n + 140$ must be a factor of $4002 = 2 \cdot 3 \cdot 23 \cdot 29$. Since $0 \le n \le 140$, we have $140 \le n + 140 \le 280$, and the only factor of 4002 that is in the required range for $n + 140$ is $174 = 2 \cdot 3 \cdot 29$. Therefore, $n + 140 = 174$, so $n = 34$ and $p = 23$. The money raised by the full-price tickets is $34 \cdot 23 = 782$ dollars.

15. **(C)** The crosswalk is in the shape of a parallelogram with base 15 feet and altitude 40 feet, so its area is $15 \times 40 = 600$ ft^2. But viewed another way, the parallelogram has base 50 feet and altitude equal to the distance between the stripes, so this distance must be $600/50 = 12$ feet.



16. **(D)** Since the median is 5, we can write the three numbers as x, 5, and y, where

$$\frac{1}{3}(x + 5 + y) = x + 10 \text{ and } \frac{1}{3}(x + 5 + y) + 15 = y.$$

If we add these equations, we get

$$\frac{2}{3}(x + 5 + y) + 15 = x + y + 10$$

and solving for $x+y$ gives $x+y = 25$. Hence the sum of the numbers $x+y+5 = 30$.

OR

Let m be the mean of the three numbers. Then the least of the numbers is $m - 10$ and the greatest is $m + 15$. The middle of the three numbers is the median, 5. So

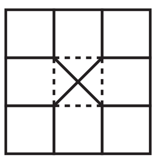
$$\frac{1}{3}((m - 10) + 5 + (m + 15)) = m$$

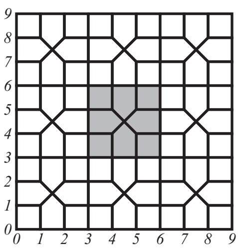
and $m = 10$. Hence, the sum of the three numbers is $3(10) = 30$.

17. **(C)** The slant height of the cone is 10, the radius of the sector. The circmference of the base of the cone is the same as the length of the secotr's arc. This is $252/360 = 7/10$ of the circumference, 20π, of the circle from which the sector is cut. The base circumference of the cone is 14π, so its radius is 7.

18. **(D)** The pattern shown at left is repeated in the plane. In fact, nine repetitions of it are shown in the statement of the problem. Note that four of the nine squres in the three-by-three square are not in the four pentagons that make up the three-by-three square. Therefore, the percentage of the plane that is enclosed by pentagons is

$$1 - \frac{4}{9} = \frac{5}{9} = 55\frac{5}{9}\%$$





19. **(D)** The number of possible selections is the number of solutions of the equation

$$g + c + p = 4$$

where g,c, and p represent, respectively, the number of glazed, chocolate, and powdered donuts. The 15 possible solutions to this equations are (4,0,0), (0,4,0), (0,0,4), (3,0,1),(3,1,0),(1,3,0),(0,3,1),(1,0,3),(0,1,3),(2,2,0),(2,0,2), (0,2,2), (2,1,1),(1,2 and (1,1,2).

OR

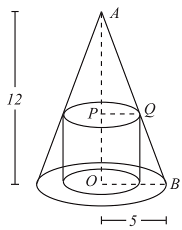
Code each selection as a sequence of four *'s and two —'s, where * represents a donut and each — denotes a "separator" between types of donuts. For example **—*—* represents two glazed donuts, one chocolate donut, and one powdered donut. From the six slots that can be occupied by a — or a *, we must choose two places for the —'s to determine a selection. Thus, there are $\binom{6}{2} \equiv C_2^6 \equiv 6C2 = 15$ selections.

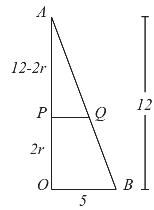
20. **(B)** Let x represent the length of each side of the octagon, which is also the length of the hypotenuse of each of the right triangles. Each leg of the right triangles has length $x\sqrt{2}/2$, so

$$2 \cdot \frac{x\sqrt{2}}{2} + x = 2000, \text{ and } x = \frac{2000}{\sqrt{2}+1} = 2000(\sqrt{2} - 1).$$

21. **(B)** Let the cylinder have radius r and height $2r$. Since $\triangle APQ$ is similar to $\triangle AOB$, we have

$$\frac{12-2r}{r} = \frac{12}{5}, \text{ so } r = \frac{30}{11}$$





22. **(D)** Since v appears in the first row, first column, and on diagonal, the sum of the remaining two numbers in each of these lines must be the same. Thus,

$$25 + 18 = 24 + w = 21 + x,$$

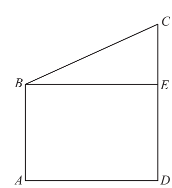
so $w = 19$ and $x = 22$. now 25,22, and 19 form a diagonal with a sum of 66, so we can find $v = 23$, $y = 26$, and $z = 20$. Hence $y + z = 46$.

23. **(D)** Think of continuing the drawing until all five chips are removed form the box. There are ten possible orderings of the colors: RRRWW, RRWRW, RWRRW, WRRRW, RRWWR, RWRWR, WRRWR, RWWRR, WRWRR, and WWRRR. The six orderings that end in R represent drawings that would have ended when the second white chip was drawn.

OR

Imagine drawing until only one chip remains. If the remaining chip is red, then that draw would have ended when the second white chip was removed. The last chip will be red with probability 3/5.

24. **(B)** Let E be the foot of the perpendicular from B to $\overline{CD}$. Then $AB = DE$ and $BE = AD = 7$. By the Pythagorean Theorem,



$$\begin{aligned} AD^2 = BE^2 &= BC^2 - CE^2 \\ &= (CD + AB)^2 - (CD - AB)^2 \\ &= (CD + AB + CD - AB)(CD + AB - CD + AB) \\ &= 4 \cdot CD \cdot AB. \end{aligned}$$

Hence, $AB \cdot CD = AD^2/4 = 7^2/4 = 49/4 = 12.25$.

25. **(B)** For integers not exceeding 2001, there are $\lfloor 2001/3 \rfloor = 667$ multiples of 3 and $\lfloor 2001/4 \rfloor = 500$ multiples of 4. The total, 1167, counts the $\lfloor 2001/12 \rfloor = 166$ multiples of 12 twice, so there are $1167 - 166 = 1001$ multiples of 3 or 4. From these we exclude the $\lfloor 2001/15 \rfloor = 133$ multiples of 15 and the $\lfloor 2001/20 \rfloor = 100$ multiples of 20, since these are multiples of 5. However, this excludes the $\lfloor 2001/60 \rfloor = 33$ multiples of 60 twice, so we must re-include these. The number of integers satisfying the conditions is $1001 - 133 - 100 + 33 = 801$.

The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

Presented by The Akamai Foundation

3rd Annual American Mathematics Contest 10

AMC 10 - Contest A

Solutions Pamphlet

Tuesday, FEBRUARY 12, 2002

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:

Prof. Douglas Faires

Department of Mathematics

Youngstown State University

Youngstown, OH 44555-0001

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:

Titu Andreescu, AMC Director

American Mathematics Competitions

University of Nebraska-Lincoln, P.O. Box 81606

Lincoln, NE 68501-1606

1. **(D)** We have

$$\frac{10^{2000}+10^{2002}}{10^{2001}+10^{2001}} = \frac{10^{2000}(1+100)}{10^{2000}(10+10)} = \frac{101}{20} \approx 5.$$

2. **(C)** We have

$$(2,12,9) = \frac{2}{12}+\frac{12}{9}+\frac{9}{2} = \frac{1}{6}+\frac{4}{3}+\frac{9}{2} = \frac{1+8+27}{6} = \frac{36}{6} = 6.$$

3. **(B)** No matter how the exponentiations are performed, 2^{2^2} always gives 16. Depending on which exponentiation is done last, we have

$$\left(2^{2^2}\right)^2 = 256, \quad 2^{\left(2^{2^2}\right)} = 65,536, \quad \text{or} \quad \left(2^2\right)^{\left(2^2\right)} = 256,$$

so there is one other possible value.

4. **(E)** When $n = 1$, the inequality becomes $m \le 1 + m$, which is satisfied by all integers m. Thus, there are infinitely many of the desired values of m.

5. **(C)** The large circle has radius 3, so its area is $\pi \cdot 3^2 = 9\pi$. The seven small circles have a total area of $7\left(\pi \cdot 1^2\right) = 7\pi$. So the shaded region has area $9\pi - 7\pi = 2\pi$.

6. **(A)** Let x be the number she was given. Her calculations produce

$$\frac{x-9}{3} = 43,$$

so

$$x - 9 = 129 \quad \text{and} \quad x = 138.$$

The correct answer is

$$\frac{138-3}{9} = \frac{135}{9} = 15.$$

7. **(A)** Let $C_A = 2\pi R_A$ be the circumference of circle A, let $C_B = 2\pi R_B$ be the circumference of circle B, and let L the common length of the two arcs. Then

$$\frac{45}{360}C_A = L = \frac{30}{360}C_B.$$

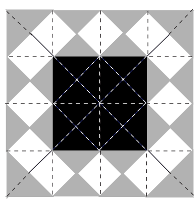
Therefore

$$\frac{C_A}{C_B} = \frac{2}{3} \quad \text{so} \quad \frac{2}{3} = \frac{2\pi R_A}{2\pi R_B} = \frac{R_A}{R_B}.$$

Thus, the ratio of the areas is

$$\frac{\text{Area of Circle }(A)}{\text{Area of Circle }(B)} = \frac{\pi R_A^2}{\pi R_B^2} = \left(\frac{R_A}{R_B}\right)^2 = \frac{4}{9}.$$

8. **(A)** Draw additional lines to cover the entire figure with congruent triangles. There are 24 triangles in the blue region, 24 in the white region, and 16 in the red region. Thus, $B = W$.



9. **(B)** Adding $1001C - 2002A = 4004$ and $1001B + 3003A = 5005$ yields $1001A + 1001B + 1001C = 9009$. So $A + B + C = 9$, and the average is

$$\frac{A + B + C}{3} = 3.$$

10. **(A)** Factor to get $(2x+3)(2x-10) = 0$, so the two roots are $-3/2$ and 5, which sum to $7/2$.

11. **(B)** First note that the amount of memory needed to store the 30 files is

$$3(0.8) + 12(0.7) + 15(0.4) = 16.8 \text{ mb},$$

so the number of disks is at least

$$\frac{16.8}{1.44} = 11 + \frac{2}{3}.$$

However, a disk that contains a 0.8-mb file can, in addition, hold only one 0.4-mb file, so on each of these disks at least 0.24 mb must remain unused. Hence, there is at least $3(0.24) = 0.72$ mb of unused memory, which is equivalent to half a disk. Since

$$\left(11 + \frac{2}{3}\right) + \frac{1}{2} > 12,$$

at least 13 disks are needed.
To see that 13 disks suffice, note that:
Six disks could be used to store the 12 files containing 0.7 mb;
Three disks could be used to store the three 0.8-mb files together with three of the 0.4-mb files;
Four disks could be used to store the remaining twelve 0.4-mb files.

12. **(B)** Let t be the number of hours Mr. Bird must travel to arrive on time. Since three minutes is the same as 0.05 hours, $40(t + 0.05) = 60(t - 0.05)$. Thus,

$$40t + 2 = 60t - 3, \quad \text{so } t = 0.25.$$

The distance from his home to work is $40(0.25 + 0.05) = 12$ miles. Therefore, his average speed should be $12/0.25 = 48$ miles per hour.

OR

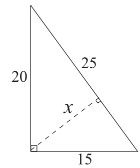
Let d be the distance from Mr. Bird's house to work, and let s be the desired average speed. Then the desired driving time is d/s. Since $d/60$ is three minutes too short and $d/40$ is three minutes too long, the desired time must be the average, so

$$\frac{d}{s} = \frac{1}{2}\left(\frac{d}{60} + \frac{d}{40}\right).$$

This implies that $s = 48$.

13. **(B)** First notice that this is a right triangle, so two of the altitudes are the legs, whose lengths are 15 and 20. The third altitude, whose length is x, is the one drawn to the hypotenuse. The area of the triangle is $\frac{1}{2}(15)(20) = 150$. Using 25 as the base and x as the altitude, we have

$$\frac{1}{2}(25)x = 150, \quad \text{so} \quad x = \frac{300}{25} = 12.$$



OR

Since the three right triangles in the figure are similar,

$$\frac{x}{15} = \frac{20}{25}, \quad \text{so} \quad x = \frac{300}{25} = 12.$$

14. **(B)** Let p and q be two primes that are roots of $x^2 - 63x + k = 0$. Then

$$x^2 - 63x + k = (x - p)(x - q) = x^2 - (p + q)x + p \cdot q,$$

so $p + q = 63$ and $p \cdot q = k$. Since 63 is odd, one of the primes must be 2 and the other 61. Thus, there is exactly one possible value for k, namely $k = p \cdot q = 2 \cdot 61 = 122$.

15. **(E)** The digits 2, 4, 5, and 6 cannot be the units digit of any two-digit prime, so these four digits must be the tens digits, and 1, 3, 7, and 9 are the units digits. The sum is thus

$$10(2 + 4 + 5 + 6) + (1 + 3 + 7 + 9) = 190.$$

(One set that satisfies the conditions is $\{23, 47, 59, 61\}$.)

16. **(B)** From the given information,

$$(a + 1) + (b + 2) + (c + 3) + (d + 4) = 4(a + b + c + d + 5),$$

so

$$(a + b + c + d) + 10 = 4(a + b + c + d) + 20$$

and $a + b + c + d = -\dfrac{10}{3}$.

OR

Note that $a = d + 3$, $b = d + 2$, and $c = d + 1$. So,

$$a + b + c + d = (d + 3) + (d + 2) + (d + 1) + d = 4d + 6.$$

Thus, $d + 4 = (4d + 6) + 5$, so $d = -7/3$, and

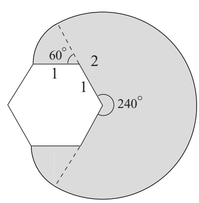
$$a + b + c + d = 4d + 6 = 4\left(-\frac{7}{3}\right) + 6 = -\frac{10}{3}.$$

17. **(D)** After the first transfer, the first cup contains two ounces of coffee, and the second cup contains two ounces of coffee and four ounces of cream. After the second transfer, the first cup contains $2 + (1/2)(2) = 3$ ounces of coffee and $(1/2)(4) = 2$ ounces of cream. Therefore, the fraction of the liquid in the first cup that is cream is $2/(2+3) = 2/5$.

18. **(D)** There are six dice that have a single face on the surface, and these dice can be oriented so that the face with the 1 is showing. They will contribute $6(1) = 6$ to the sum. There are twelve dice that have just two faces on the surface because they are along an edge but not at a vertex of the large cube. These dice can be oriented so that the 1 and 2 are showing, and they will contribute $12(1+2) = 36$ to the sum. There are eight dice that have three faces on the surface because they are at the vertices of the large cube, and these dice can be oriented so that the 1, 2, and 3 are showing. They will contribute $8(1+2+3) = 48$ to the sum. Consequently, the minimum sum of all the numbers showing on the large cube is $6 + 36 + 48 = 90$.

19. **(E)** Spot can go anywhere in a $240°$ sector of radius two yards and can cover a $60°$ sector of radius one yard around each of the adjoining corners. The total area is

$$\pi(2)^2 \cdot \frac{240}{360} + 2\left(\pi(1)^2 \cdot \frac{60}{360}\right) = 3\pi.$$



20. **(D)** Since $\triangle AGD$ is similar to $\triangle CHD$, we have $HC/1 = AG/3$. Also, $\triangle AGF$ is similar to $\triangle EJF$, so $JE/1 = AG/5$. Hence,

$$\frac{HC}{JE} = \frac{AG/3}{AG/5} = \frac{5}{3}.$$

21. **(D)** The values 6, 6, 6, 8, 8, 8, 8, 14 satisfy the requirements of the problem, so the answer is at least 14. If the largest number were 15, the collection would have the ordered form 7, __, __, 8, 8, __, __, 15. But $7 + 8 + 8 + 15 = 38$, and a mean of 8 implies that the sum of all values is 64. In this case, the four missing values would sum to $64 - 38 = 26$, and their average value would be 6.5. This implies that at least one would be less than 7, which is a contradiction. Therefore, the largest integer that can be in the set is 14.

22. **(C)** The first application removes ten tiles, leaving 90. The second and third applications each remove nine tiles leaving 81 and 72, respectively. Following this pattern, we consecutively remove $10, 9, 9, 8, 8, \ldots, 2, 2, 1$ tiles before we are left with only one. This requires $1 + 2(8) + 1 = 18$ applications.

OR

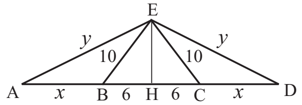
Starting with n^2 tiles, the first application leaves $n^2 - n$ tiles. The second application reduces the number to $n^2 - n - (n-1) = (n-1)^2$ tiles. Since two applications reduce the number from n^2 to $(n-1)^2$, it follows that $2(n-1)$ applications reduce the number from n^2 to $(n-(n-1))^2 = 1$, and $2(10-1) = 18$.

23. **(D)** Let H be the midpoint of $\overline{BC}$. Then $\overline{EH}$ is the perpendicular bisector of $\overline{AD}$, and $\triangle AED$ is isosceles. Segment $\overline{EH}$ is the common altitude of the two isosceles triangles $\triangle AED$ and $\triangle BEC$, and

$$EH = \sqrt{10^2 - 6^2} = 8.$$

Let $AB = CD = x$ and $AE = ED = y$. Then $2x+2y+12 = 2(32)$, so $y = 26-x$. Thus,

$$8^2 + (x+6)^2 = y^2 = (26-x)^2 \quad \text{and} \quad x = 9.$$

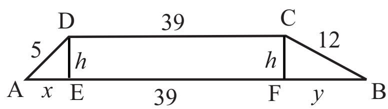


24. **(A)** There are ten ways for Tina to select a pair of numbers. The sums 9, 8, 4, and 3 can be obtained in just one way, and the sums 7, 6, and 5 can each be obtained in two ways. The probability for each of Sergio's choices is 1/10. Considering his selections in decreasing order, the total probability of Sergio's choice being greater is

$$\left(\frac{1}{10}\right)\left(1 + \frac{9}{10} + \frac{8}{10} + \frac{6}{10} + \frac{4}{10} + \frac{2}{10} + \frac{1}{10} + 0 + 0 + 0\right) = \frac{2}{5}.$$

25. **(C)** First drop perpendiculars from D and C to $\overline{AB}$. Let E and F be the feet of the perpendiculars to $\overline{AB}$ from D and C, respectively, and let

$$h = DE = CF, \quad x = AE, \quad \text{and} \quad y = FB.$$



Then

$$25 = h^2 + x^2, \quad 144 = h^2 + y^2, \quad \text{and} \quad 13 = x + y.$$

So

$$144 = h^2 + y^2 = h^2 + (13 - x)^2 = h^2 + x^2 + 169 - 26x = 25 + 169 - 26x,$$

which gives $x = 50/26 = 25/13$, and

$$h = \sqrt{5^2 - \left(\frac{25}{13}\right)^2} = 5\sqrt{1 - \frac{25}{169}} = 5\sqrt{\frac{144}{169}} = \frac{60}{13}.$$

Hence

$$\text{Area } (ABCD) = \frac{1}{2}(39 + 52) \cdot \frac{60}{13} = 210.$$

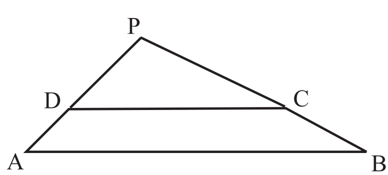
OR

Extend $\overline{AD}$ and $\overline{BC}$ to intersect at P. Since $\triangle PDC$ and $\triangle PAB$ are similar, we have

$$\frac{PD}{PD + 5} = \frac{39}{52} = \frac{PC}{PC + 12}.$$

So $PD = 15$ and $PC = 36$. Note that 15, 36, and 39 are three times 5, 12, and 13, respectively, so $\angle APB$ is a right angle. The area of the trapezoid is the difference of the areas of $\triangle PAB$ and $\triangle PDC$, so

$$\text{Area}(ABCD) = \frac{1}{2}(20)(48) - \frac{1}{2}(15)(36) = 210.$$



OR

Draw the line through D parallel to $\overline{BC}$, intersecting $\overline{AB}$ at E. Then $BCDE$ is a parallelogram, so $DE = 12$, $EB = 39$, and $AE = 52 - 39 = 13$. Thus $DE^2 + AD^2 = AE^2$, and $\triangle ADE$ is a right triangle. Let h be the altitude from D to $\overline{AE}$, and note that

$$\text{Area}(ADE) = \frac{1}{2}(5)(12) = \frac{1}{2}(13)(h),$$

so $h = 60/13$. Thus

$$\text{Area}(ABCD) = \frac{60}{13} \cdot \frac{1}{2}(39 + 52) = 210.$$

The

American Mathematics Contest 10 (AMC 10)

Sponsored by

Mathematical Association of America

The Akamai Foundation

University of Nebraska – Lincoln

Contributors

American Mathematical Association of Two Year Colleges

American Mathematical Society

American Society of Pension Actuaries

American Statistical Association

Casualty Actuarial Society

Clay Mathematics Institute

Consortium for Mathematics and its Applications

Institute for Operations Research and the Management Sciences

Kappa Mu Epsilon

Mu Alpha Theta

National Association of Mathematicians

National Council of Teachers of Mathematics

Pi Mu Epsilon

School Science and Mathematics Association

Society of Actuaries

The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions
Presented by The Akamai Foundation

3rd Annual American Mathematics Contest 10

AMC 10 - Contest B

Solutions Pamphlet

Wednesday, FEBRUARY 27, 2002

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:
Prof. Douglas Faires
Department of Mathematics
Youngstown State University
Youngstown, OH 44555-0001

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:
Titu Andreescu, AMC Director
American Mathematics Competitions
University of Nebraska-Lincoln, P.O. Box 81606
Lincoln, NE 68501-1606

1. **(E)** We have

$$\frac{2^{2001}\cdot 3^{2003}}{6^{2002}} = \frac{2^{2001}\cdot 3^{2003}}{(2\cdot 3)^{2002}} = \frac{2^{2001}\cdot 3^{2003}}{2^{2002}\cdot 3^{2002}} = \frac{3}{2}.$$

2. **(C)** We have

$$(2,4,6) = \frac{2\cdot 4\cdot 6}{2+4+6} = \frac{48}{12} = 4.$$

3. **(A)** The number M is equal to

$$\frac{1}{9}(9+99+999+\ldots+999{,}999{,}999) = 1+11+111+\ldots+111{,}111{,}111 = 123{,}456{,}789.$$

The number M does not contain the digit 0.

4. **(D)** Since

$$\begin{aligned}(3x-2)(4x+1)-(3x-2)4x+1 &= (3x-2)(4x+1-4x)+1\\ &= (3x-2)\cdot 1+1 = 3x-1,\end{aligned}$$

when $x=4$ we have the value $3\cdot 4-1=11$.

5. **(E)** The diameter of the large circle is $6+4=10$, so its radius is 5. Hence, the area of the shaded region is

$$\pi(5^2)-\pi(3^2)-\pi(2^2) = \pi(25-9-4) = 12\pi.$$

6. **(B)** If $n \geq 4$, then

$$n^2-3n+2 = (n-1)(n-2)$$

is the product of two integers greater than 1, and thus is not prime. For $n=1$, 2, and 3 we have, respectively,

$$(1-1)(1-2)=0, \quad (2-1)(2-2)=0, \quad \text{and} \quad (3-1)(3-2)=2.$$

Therefore, n^2-3n+2 is prime only when $n=3$.

7. **(E)** The number $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{n}$ is greater than 0 and less than $\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{1}<2$. Hence,
$$\frac{1}{2}+\frac{1}{3}+\frac{1}{7}+\frac{1}{n}=\frac{41}{42}+\frac{1}{n}$$
is an integer precisely when it is equal to 1. This implies that $n=42$, so the answer is (E).

8. **(D)** Since July has 31 days, Monday must be one of the last three days of July. Therefore, Thursday must be one of the first three days of August, which also has 31 days. So Thursday must occur five times in August.

9. **(D)** The last "word," which occupies position 120, is *USOMA*. Immediately preceding this we have *USOAM*, *USMOA*, *USMAO*, *USAOM*,and *USAMO*. The alphabetic position of the word *USAMO* is consequently 115.

10. **(C)** The given conditions imply that
$$x^2+ax+b=(x-a)(x-b)=x^2-(a+b)x+ab,$$
so
$$a+b=-a \quad \text{and} \quad ab=b.$$
Since $b\neq 0$, the second equation implies that $a=1$. The first equation gives $b=-2$, so $(a,b)=(1,-2)$.

11. **(B)** Let $n-1$, n, and $n+1$ denote the three integers. Then
$$(n-1)n(n+1)=8(3n).$$
Since $n\neq 0$, we have $n^2-1=24$. It follows that $n^2=25$ and $n=5$. Thus,
$$(n-1)^2+n^2+(n+1)^2=16+25+36=77.$$

12. **(E)** From the given equation we have $(x-1)(x-6) = (x-2)(x-k)$. This implies that
$$x^2 - 7x + 6 = x^2 - (2+k)x + 2k,$$
so
$$(k-5)x = 2k - 6 \quad \text{and} \quad x = \frac{2k-6}{k-5}.$$
Hence a value of x satisfying the equation occurs unless $k = 5$.
Note that when $k = 6$ there is also no solution for x, but this is not one of the answer choices.

13. **(D)** The given equation can be factored as
$$0 = 8xy - 12y + 2x - 3 = 4y(2x-3) + (2x-3) = (4y+1)(2x-3).$$
For this equation to be true for all values of y we must have $2x - 3 = 0$, that is, $x = 3/2$.

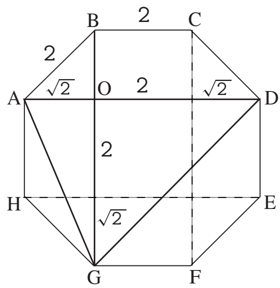
14. **(B)** We have
$$N = \sqrt{(5^2)^{64} \cdot (2^6)^{25}} = 5^{64} \cdot 2^{3\cdot 25} = (5\cdot 2)^{64} \cdot 2^{11} = 10^{64} \cdot 2048 = 2048\underbrace{000\cdots 0}_{64 \text{ digits}}.$$
The zeros do not contribute to the sum, so the sum of the digits of N is $2+4+8 = 14$.

15. **(E)** The numbers $A - B$ and $A + B$ are both odd or both even. However, they are also both prime, so they must both be odd. Therefore, one of A and B is odd and the other even. Because A is a prime between $A - B$ and $A + B$, A must be the odd prime. Therefore, $B = 2$, the only even prime. So $A - 2$, A, and $A + 2$ are consecutive odd primes and thus must be 3, 5, and 7. The sum of the four primes 2, 3, 5, and 7 is the prime number 17.

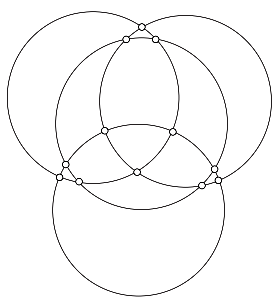
16. **(D)** If $\frac{n}{20-n} = k^2$, for some $k \ge 0$, then $n = \frac{20k^2}{k^2+1}$. Since k^2 and k^2+1 have no common factors and n is an integer, $k^2 + 1$ must be a factor of 20. This occurs only when $k = 0, 1, 2$, or 3. The corresponding values of n are $0, 10, 16$, and 18.

17. **(C)** Construct the right triangle $\triangle AOB$ as shown in the figure. Since $AB = 2$, we have $AO = \sqrt{2}$ and $AD = 2 + 2\sqrt{2}$. Similarly, we have $OG = 2 + \sqrt{2}$, so

$$\text{Area}(\triangle ADG) = \frac{1}{2}(2 + 2\sqrt{2})(2 + \sqrt{2}) = (1 + \sqrt{2})(2 + \sqrt{2}) = 4 + 3\sqrt{2}.$$



18. **(D)** Each pair of circles has at most two intersection points. There are $\binom{4}{2} = 6$ pairs of circles, so there are at most $6 \times 2 = 12$ points of intersection. The following configuration shows that 12 points of intersection are indeed possible:



19. **(C)** Let $d = a_2 - a_1$. Then $a_{k+100} = a_k + 100d$, and

$$\begin{aligned} a_{101} + a_{102} + \cdots + a_{200} &= (a_1 + 100d) + (a_2 + 100d) + \ldots + (a_{100} + 100d) \\ &= a_1 + a_2 + \ldots + a_{100} + 10{,}000d. \end{aligned}$$

Thus $200 = 100 + 10{,}000d$ and $d = \frac{100}{10{,}000} = 0.01$.

20. **(B)** We have $a + 8c = 4 + 7b$ and $8a - c = 7 - 4b$. Squaring both equations and adding the results yields

$$(a + 8c)^2 + (8a - c)^2 = (4 + 7b)^2 + (7 - 4b)^2.$$

Expanding gives $65(a^2+c^2) = 65(1+b^2)$. So $a^2+c^2 = 1+b^2$, and $a^2-b^2+c^2 = 1$.

21. **(B)** Let A be the number of square feet in Andy's lawn. Then $A/2$ and $A/3$ are the areas of Beth's lawn and Carlos' lawn, respectively, in square feet. Let R be the rate, in square feet per minute, that Carlos' lawn mower cuts. Then Beth's mower and Andy's mower cut at rates of $2R$ and $3R$ square feet per minute, respectively. Thus,

$$\text{Andy takes } \frac{A}{3R} \text{ minutes to mow his lawn,}$$

$$\text{Beth takes } \frac{A/2}{2R} = \frac{A}{4R} \text{ minutes to mow hers,}$$

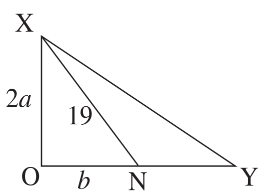
and

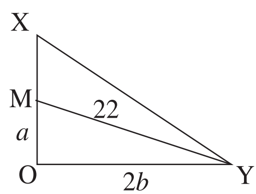
$$\text{Carlos takes } \frac{A/3}{R} = \frac{A}{3R} \text{ minutes to mow his.}$$

Since $\dfrac{A}{4R} < \dfrac{A}{3R}$, Beth will finish first.

22. **(B)** Let $OM = a$ and $ON = b$. Then

$$19^2 = (2a)^2 + b^2 \quad \text{and} \quad 22^2 = a^2 + (2b)^2.$$





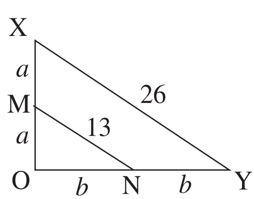
Hence

$$5(a^2 + b^2) = 19^2 + 22^2 = 845.$$

It follows that

$$MN = \sqrt{a^2 + b^2} = \sqrt{169} = 13.$$

Since $\triangle XOY$ is similar to $\triangle MON$ and $XO = 2 \cdot MO$, we have $XY = 2 \cdot MN = 26$.



23. **(D)** By setting $n = 1$ in the given recursive equation, we obtain $a_{m+1} = a_m + a_1 + m$, for all positive integers m. So $a_{m+1} - a_m = m+1$ for each $m = 1, 2, 3, \ldots$. Hence,

$$a_{12} - a_{11} = 12,\ a_{11} - a_{10} = 11,\ \ldots,\ a_2 - a_1 = 2.$$

Summing these equalities yields $a_{12} - a_1 = 12 + 11 + \cdots + 2$. So

$$a_{12} = 12 + 11 + \cdots + 2 + 1 = \frac{12(12+1)}{2} = 78.$$

OR

We have

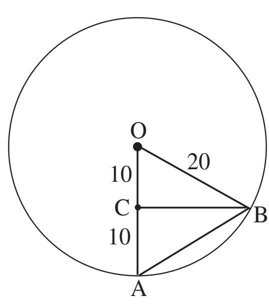
$$\begin{aligned}
a_2 &= a_{1+1} = a_1 + a_1 + 1 \cdot 1 = 1 + 1 + 1 = 3,\\
a_3 &= a_{2+1} = a_2 + a_1 + 2 \cdot 1 = 3 + 1 + 2 = 6,\\
a_6 &= a_{3+3} = a_3 + a_3 + 3 \cdot 3 = 6 + 6 + 9 = 21,
\end{aligned}$$

and

$$a_{12} = a_{6+6} = a_6 + a_6 + 6 \cdot 6 = 21 + 21 + 36 = 78.$$

24. **(D)** In the figure, the center of the wheel is at O, and the rider travels from A to B. Since $AC = 10$ and $OB = OA = 20$, the point C is the midpoint of $\overline{OA}$. In the right $\triangle OCB$, we have OC half of the length of the hypotenuse OB, so $m\angle COB = 60^\circ$. Since the wheel turns through an angle of 360° in 60 seconds, the time required to turn through an angle of 60° is

$$60\left(\frac{60}{360}\right) = 10 \text{ seconds}.$$



25. **(A)** Let n denote the number of integers in the original list, and m the original mean. Then the sum of the original numbers is mn. After 15 is appended to the list, we have the sum

$$(m+2)(n+1) = mn + 15, \quad \text{so} \quad m + 2n = 13.$$

After 1 is appended to the enlarged list, we have the sum

$$(m+1)(n+2) = mn + 16, \quad \text{so} \quad 2m + n = 14.$$

Solving $m + 2n = 13$ and $2m + n = 14$ gives $m = 5$ and $n = 4$.

This page left intentionally blank.

The

American Mathematics Contest 10 (AMC 10)

Sponsored by

Mathematical Association of America

The Akamai Foundation

University of Nebraska – Lincoln

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Casualty Actuarial Society
Clay Mathematics Institute
Consortium for Mathematics and its Applications
Institute for Operations Research and the Management Sciences
Kappa Mu Epsilon
Mu Alpha Theta
National Association of Mathematicians
National Council of Teachers of Mathematics
Pi Mu Epsilon
School Science and Mathematics Association
Society of Actuaries

The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

Presented by The Akamai Foundation

4th Annual American Mathematics Contest 10

AMC 10 - Contest A

Solutions Pamphlet

Tuesday, FEBRUARY 11, 2003

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:

Prof. Douglas Faires
Department of Mathematics
Youngstown State University
Youngstown, OH 44555-0001

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:

Titu Andreescu, AMC Director
American Mathematics Competitions
University of Nebraska-Lincoln, P.O. Box 81606
Lincoln, NE 68501-1606

1. (D) Each even counting number, beginning with 2, is one more than the preceding odd counting number. Therefore the difference is $(1)(2003) = 2003$.

2. (B) The cost for each member is the price of two pairs of socks, \$8, and two shirts, \$18, for a total of \$26. So there are $2366/26 = 91$ members.

3. (D) The total volume of the eight removed cubes is $8 \times 3^3 = 216$ cubic centimeters, and the volume of the original box is $15 \times 10 \times 8 = 1200$ cubic centimeters. Therefore the volume has been reduced by $\left(\frac{216}{1200}\right)(100\%) = 18\%$.

4. (A) Mary walks a total of 2 km in 40 minutes. Because 40 minutes is 2/3 hr, her average speed, in km/hr, is $2/(2/3) = 3$.

5. (B) Since

$$0 = 2x^2 + 3x - 5 = (2x+5)(x-1) \quad \text{we have} \quad d = -\frac{5}{2} \text{ and } e = 1.$$

So $(d-1)(e-1) = 0$.

OR

If $x = d$ and $x = e$ are the roots of the quadratic equation $ax^2 + bx + c = 0$, then

$$de = \frac{c}{a} \quad \text{and} \quad d + e = -\frac{b}{a}.$$

For our equation this implies that

$$(d-1)(e-1) = de - (d+e) + 1 = -\frac{5}{2} - \left(-\frac{3}{2}\right) + 1 = 0.$$

6. (C) For example, $-1\heartsuit 0 = |-1-0| = 1 \neq -1$. All the other statements are true:
(*A*) $x\heartsuit y = |x-y| = |-(y-x)| = |y-x| = y\heartsuit x$ for all x and y.
(*B*) $2(x\heartsuit y) = 2|x-y| = |2x-2y| = (2x)\heartsuit(2y)$ for all x and y.
(*D*) $x\heartsuit x = |x-x| = 0$ for all x.
(*E*) $x\heartsuit y = |x-y| > 0$ if $x \neq y$.

7. (B) The longest side cannot be greater than 3, since otherwise the remaining two sides would not be long enough to form a triangle. The only possible triangles have side lengths 1–3–3 or 2–2–3.

8. (E) The factors of 60 are

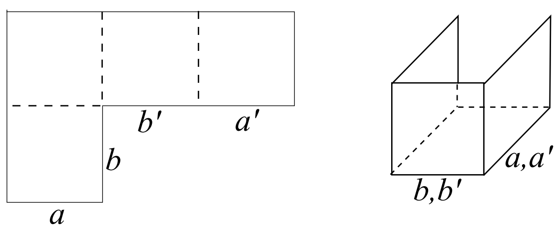
$$1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, \text{ and } 60.$$

Six of the twelve factors are less than 7, so the probability is 1/2.

9. (A) We have

$$\sqrt[3]{x\sqrt[3]{x\sqrt[3]{x\sqrt{x}}}} = (x(x(x \cdot x^{\frac{1}{2}})^{\frac{1}{3}})^{\frac{1}{3}})^{\frac{1}{3}}$$
$$= (x(x(x^{\frac{3}{2}})^{\frac{1}{3}})^{\frac{1}{3}})^{\frac{1}{3}}$$
$$= (x(x \cdot x^{\frac{1}{2}})^{\frac{1}{3}})^{\frac{1}{3}}$$
$$= (x(x^{\frac{3}{2}})^{\frac{1}{3}})^{\frac{1}{3}} = (x \cdot x^{\frac{1}{2}})^{\frac{1}{3}} = (x^{\frac{3}{2}})^{\frac{1}{3}} = x^{\frac{1}{2}} = \sqrt{x}.$$

10. (E) If the polygon is folded before the fifth square is attached, then edges a and a' must be joined, as must b and b'. The fifth face of the cube can be attached at any of the six remaining edges.

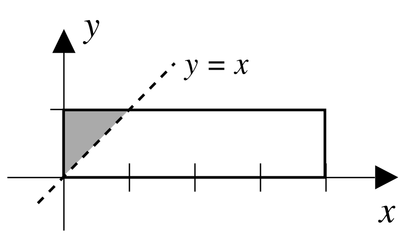


11. (E) Since the last two digits of $AMC10$ and $AMC12$ sum to 22, we have

$$AMC + AMC = 2(AMC) = 1234.$$

Hence $AMC = 617$, so $A = 6$, $M = 1$, $C = 7$, and $A + M + C = 6 + 1 + 7 = 14$.

12. (A) The point (x, y) satisfies $x < y$ if and only if it belongs to the shaded triangle bounded by the lines $x = y$, $y = 1$, and $x = 0$, the area of which is $1/2$. The ratio of the area of the triangle to the area of the rectangle is $\frac{1/2}{4} = \frac{1}{8}$.



13. (A) Let a, b, and c be the three numbers. Replace a by four times the sum of the other two to get

$$4(b + c) + b + c = 20, \quad \text{so} \quad b + c = 4.$$

Then replace b with $7c$ to get

$$7c + c = 4, \quad \text{so} \quad c = \frac{1}{2}.$$

The other two numbers are $b = 7/2$ and $a = 16$, and the product of the three is $16 \cdot 7/2 \cdot 1/2 = 28$.

OR

Let the first, second, and third numbers be x, $7x$, and $32x$, respectively. Then $40x = 20$ so $x = \frac{1}{2}$ and the product is

$$(32)(7)x^3 = (32)(7)\left(\frac{1}{8}\right) = 28.$$

14. (A) The largest single-digit primes are 5 and 7, but neither 75 nor 57 is prime. Using 3, 7, and 73 gives 1533, whose digits have a sum of 12.

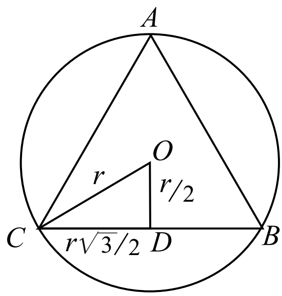
15. (C) Of the $\frac{100}{2} = 50$ integers that are divisible by 2, there are $\lfloor \frac{100}{6} \rfloor = 16$ that are divisible by both 2 and 3. So there are $50 - 16 = 34$ that are divisible by 2 and not by 3, and $34/100 = 17/50$.

16. (C) Powers of 13 have the same units digit as the corresponding powers of 3; and

$$3^1 = 3, \quad 3^2 = 9, \quad 3^3 = 27, \quad 3^4 = 81, \quad \text{and} \quad 3^5 = 243.$$

Since the units digit of 3^1 is the same as the units digit of 3^5, units digits of powers of 3 cycle through $3, 9, 7$, and 1. Hence the units digit of 3^{2000} is 1, so the units digit of 3^{2003} is 7. The same is true of the units digit of 13^{2003}.

17. (B) Let the triangle have vertices A, B, and C, let O be the center of the circle, and let D be the midpoint of $\overline{BC}$. Triangle COD is a 30–60–90 degree triangle. If r is the radius of the circle, then the sides of $\triangle COD$ are r, $r/2$, and $r\sqrt{3}/2$. The perimeter of $\triangle ABC$ is $6\left(\frac{r\sqrt{3}}{2}\right) = 3r\sqrt{3}$, and the area of the circle is πr^2. Thus $3r\sqrt{3} = \pi r^2$, and $r = (3\sqrt{3})/\pi$.



18. (B) Let $a = 2003/2004$. The given equation is equivalent to

$$ax^2 + x + 1 = 0.$$

If the roots of this equation are denoted r and s, then

$$rs = \frac{1}{a} \quad \text{and} \quad r + s = -\frac{1}{a},$$

so

$$\frac{1}{r} + \frac{1}{s} = \frac{r+s}{rs} = -1.$$

OR

If x is replaced by $1/y$, then the roots of the resulting equation are the reciprocals of the roots of the original equation. The new equation is

$$\frac{2003}{2004y} + 1 + y = 0 \quad \text{which is equivalent to} \quad y^2 + y + \frac{2003}{2004} = 0.$$

The sum of the roots of this equation is the opposite of the y-coefficient, which is -1.

19. (C) First note that the area of the region determined by the triangle topped by the semicircle of diameter 1 is

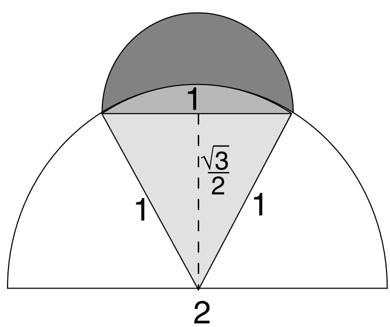
$$\frac{1}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2}\pi\left(\frac{1}{2}\right)^2 = \frac{\sqrt{3}}{4} + \frac{1}{8}\pi.$$

The area of the lune results from subtracting from this the area of the sector of the larger semicircle,

$$\frac{1}{6}\pi(1)^2 = \frac{1}{6}\pi.$$

So the area of the lune is

$$\frac{\sqrt{3}}{4} + \frac{1}{8}\pi - \frac{1}{6}\pi = \frac{\sqrt{3}}{4} - \frac{1}{24}\pi.$$



Note that the answer does not depend on the position of the lune on the semicircle.

20. (E) The largest base-9 three-digit number is $9^3 - 1 = 728$ and the smallest base-11 three-digit number is $11^2 = 121$. There are 608 integers that satisfy $121 \le n \le 728$, and 900 three-digit numbers altogether, so the probability is $608/900 \approx 0.7$.

21. (D) The numbers of the three types of cookies must have a sum of six. Possible sets of whole numbers whose sum is six are

$$0,0,6;\ \ 0,1,5;\ \ 0,2,4;\ \ 0,3,3;\ \ 1,1,4;\ \ 1,2,3;\ \ \text{and } 2,2,2.$$

Every ordering of each of these sets determines a different assortment of cookies. There are 3 orders for each of the sets

$$0,0,6;\ \ 0,3,3;\ \ \text{and } 1,1,4.$$

There are 6 orders for each of the sets

$$0,1,5;\ \ 0,2,4;\ \ \text{and } 1,2,3.$$

There is only one order for $2,2,2$. Therefore the total number of assortments of six cookies is $3 \cdot 3 + 3 \cdot 6 + 1 = 28$.

OR

Construct eight slots, six to place the cookies in and two to divide the cookies by type. Let the number of chocolate chip cookies be the number of slots to the left of the first divider, the number of oatmeal cookies be the number of slots between the two dividers, and the number of peanut butter cookies be the number of slots to the right of the second divider. For example, $111 \mid 11 \mid 1$ represents three chocolate chip cookies, two oatmeal cookies, and one peanut butter cookie. There are $\binom{8}{2} = 28$ ways to place the two dividers, so there are 28 ways to select the six cookies.

22. (B) We have $EA = 5$ and $CH = 3$. Triangles GCH and GEA are similar, so

$$\frac{GC}{GE} = \frac{3}{5} \quad \text{and} \quad \frac{CE}{GE} = \frac{GE - GC}{GE} = 1 - \frac{3}{5} = \frac{2}{5}.$$

Triangles GFE and CDE are similar, so

$$\frac{GF}{8} = \frac{CE}{GE} = \frac{5}{2}$$

and $FG = 20$.

OR

Place the figure in the coordinate plane with the origin at D, $\overline{DA}$ on the positive x-axis, and $\overline{DC}$ on the positive y-axis. Then $H = (3,8)$ and $A = (9,0)$, so line AG has the equation

$$y = -\frac{4}{3}x + 12.$$

Also, $C = (0,8)$ and $E = (4,0)$, so line EG has the equation

$$y = -2x + 8.$$

The lines intersect at $(-6, 20)$, so $FG = 20$.

23. (C) The base row of the large equilateral triangle has 1001 triangles pointing downward and 1002 pointing upward. This base row requires 3(1002) toothpicks since the downward pointing triangles require no additional toothpicks. Each succeeding row will require one less set of 3 toothpicks, so the total number of toothpicks required is

$$3(1002 + 1001 + 1000 + \cdots + 2 + 1) = 3 \cdot \frac{1002 \cdot 1003}{2} = 1,507,509.$$

OR

Create a table:

Number of Rows	Number of Triangles in Base Row	Number of Toothpicks in All Rows
1	1	3
2	3	$3+6$
3	5	$3+6+9$
$\vdots$	$\vdots$	$\vdots$
n	$2n-1$	$3(1+2+\cdots+n)$

Thus

$$2003 = 2n - 1 \quad \text{so} \quad n = 1002.$$

The number of toothpicks is

$$3(1 + 2 + \cdots + 1002) = 3\frac{(1002)(1003)}{2} = 1,507,509.$$

24. (E) Let R1, ..., R5 and B3, ..., B6 denote the numbers on the red and blue cards, respectively. Note that R4 and R5 divide evenly into only B4 and B5, respectively. Thus the stack must be R4, B4, ..., B5, R5, or the reverse. Since R2 divides evenly into only B4 and B6, we must have R4, B4, R2, B6, ..., B5, R5, or the reverse. Since R3 divides evenly into only B3 and B6, the stack must be R4, B4, R2, B6, R3, B3, R1, B5, R5, or the reverse. In either case, the sum of the middle three cards is 12.

25. (B) Note that $n = 100q + r = q + r + 99q$. Hence $q + r$ is divisible by 11 if and only if n is divisible by 11. Since $10,000 \le n \le 99,999$, there are

$$\left\lfloor \frac{99999}{11} \right\rfloor - \left\lfloor \frac{9999}{11} \right\rfloor = 9090 - 909 = 8181$$

such numbers.

The

American Mathematics Contest 12 (AMC 12)

Sponsored by

Mathematical Association of America

The Akamai Foundation

University of Nebraska – Lincoln

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Canada/USA Mathpath & Mathcamp
Casualty Actuarial Society
Clay Mathematics Institute
Consortium for Mathematics and its Applications
Institute for Operations Research and the Management Sciences
Kappa Mu Epsilon
Mu Alpha Theta
National Association of Mathematicians
National Council of Teachers of Mathematics
Pi Mu Epsilon
School Science and Mathematics Association
Society of Actuaries

The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

Presented by The Akamai Foundation

4th Annual American Mathematics Contest 10

AMC 10 - Contest B

Solutions Pamphlet

Wednesday, FEBRUARY 26, 2003

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:

Prof. Douglas Faires
Department of Mathematics
Youngstown State University
Youngstown, OH 44555-0001

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:

Titu Andreescu, AMC Director
American Mathematics Competitions
University of Nebraska-Lincoln, P.O. Box 81606
Lincoln, NE 68501-1606

1. (C) We have

$$\frac{2-4+6-8+10-12+14}{3-6+9-12+15-18+21} = \frac{2(1-2+3-4+5-6+7)}{3(1-2+3-4+5-6+7)} = \frac{2}{3}.$$

2. (D) The cost of each day's pills is $546/14 = 39$ dollars. If x denotes the cost of one green pill, then $x + (x-1) = 39$, so $x = 20$.

3. (B) Let n be the smallest of the even integers. Since

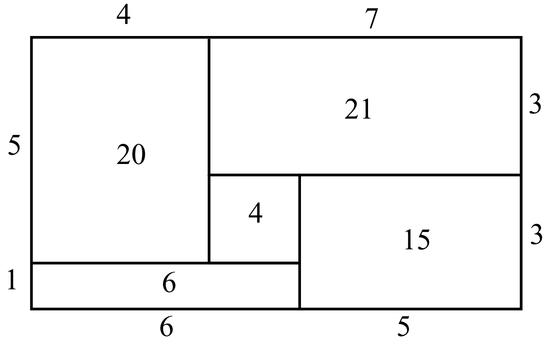
$$1+3+5+7+9+11+13+15 = 64,$$

we have

$$60 = n + (n+2) + (n+4) + (n+6) + (n+8) = 5n + 20, \quad \text{so} \quad n = 8.$$

4. (A) To minimize the cost, Rose should place the most expensive flowers in the smallest region, the next most expensive in the second smallest, etc. The areas of the regions are shown in the figure, so the minimal total cost, in dollars, is

$$(3)(4) + (2.5)(6) + (2)(15) + (1.5)(20) + (1)(21) = 108.$$



5. (C) The area of the lawn is

$$90 \cdot 150 = 13{,}500 \text{ ft}^2.$$

Moe cuts about two square feet for each foot he pushes the mower forward, so he cuts $2(5000) = 10{,}000$ ft^2 per hour. Therefore, it takes about $\frac{13{,}500}{10{,}000} = 1.35$ hours.

6. (D) The height, length, and diagonal are in the ratio $3 : 4 : 5$. The length of the diagonal is 27, so the horizontal length is

$$\frac{4}{5}(27) = 21.6 \text{ inches}.$$

7. (B) The first three values in the sum are 1, the next five are 2, the next seven are 3, and the final one is 4 for a total of

$$3 \cdot 1 + 5 \cdot 2 + 7 \cdot 3 + 1 \cdot 4 = 38.$$

8. (B) Let the sequence be denoted $a, ar, ar^2, ar^3, \ldots$, with $ar = 2$ and $ar^3 = 6$. Then $r^2 = 3$ and $r = \sqrt{3}$ or $r = -\sqrt{3}$. Therefore $a = \frac{2\sqrt{3}}{3}$ or $a = -\frac{2\sqrt{3}}{3}$.

9. (B) Write all the terms with the common base 5. Then

$$5^{-4} = 25^{-2} = \frac{5^{48/x}}{5^{26/x} \cdot 25^{17/x}} = \frac{5^{48/x}}{5^{26/x} \cdot 5^{34/x}} = 5^{(48-26-34)/x} = 5^{-12/x}.$$

It follows that $-\frac{12}{x} = -4$, so $x = 3$.

OR

First write 25 as 5^2. Raising both sides to the x power gives

$$5^{-4x} = \frac{5^{48}}{5^{26}5^{34}} = 5^{48-26-34} = 5^{-12}.$$

So $-4x = -12$ and $x = 3$.

10. (C) In the old scheme 26×10^4 different plates could be constructed. In the new scheme $26^3 \times 10^3$ different plates can be constructed. There are

$$\frac{26^3 \times 10^3}{26 \times 10^4} = \frac{26^2}{10}$$

times as many possible plates with the new scheme.

11. (A) The two lines have equations

$$y - 15 = 3(x - 10) \quad \text{and} \quad y - 15 = 5(x - 10).$$

The x-intercepts, obtained by setting $y = 0$ in the respective equations, are 5 and 7. The distance between the points $(5, 0)$ and $(7, 0)$ is 2.

12. (C) Denote the original portions for Al, Betty, and Clare as a, b, and c, respectively. Then

$$a + b + c = 1000 \quad \text{and} \quad a - 100 + 2(b + c) = 1500.$$

Substituting $b + c = 1000 - a$ in the second equation, we have

$$a - 100 + 2(1000 - a) = 1500.$$

This yields $a = 400$, which is Al's original portion.

Note that although we know that $b + c = 600$, we have no way of determining either b or c.

13. (E) Let $y = \clubsuit(x)$. Since $x \le 99$, we have $y \le 18$. Thus if $\clubsuit(y) = 3$, then $y = 3$ or $y = 12$. The 3 values of x for which $\clubsuit(x) = 3$ are 12, 21, and 30, and the 7 values of x for which $\clubsuit(x) = 12$ are 39, 48, 57, 66, 75, 84, and 93. There are 10 values in all.

14. (D) Since a must be divisible by 5, and $3^8 \cdot 5^2$ is divisible by 5^2, but not by 5^3, we have $b \le 2$. If $b = 1$, then

$$a^b = (3^8 5^2)^1 = (164,025)^1 \quad \text{and} \quad a + b = 164,026.$$

If $b = 2$, then

$$a^b = (3^4 5)^2 = 405^2 \quad \text{so} \quad a + b = 407,$$

which is the smallest value.

15. (E) In the first round $100 - 64 = 36$ players are eliminated, one per match. In the second round there are 32 matches, in the third 16, then 8, 4, 2, and 1. The total number of matches is:

$$36 + 32 + 16 + 8 + 4 + 2 + 1 = 99.$$

Note that 99 is divisible by 11, but 99 does not satisfy any of the other conditions given as answer choices.

OR

In each match, precisely one player is eliminated. Since there were 100 players in the tournament and all but one is eliminated, there must be 99 matches.

16. (E) Let m denote the number of main courses needed to meet the requirement. Then the number of dinners available is $3 \cdot m \cdot 2m = 6m^2$. Thus m^2 must be at least $365/6 \approx 61$. Since $7^2 = 49 < 61 < 64 = 8^2$, 8 main courses is enough, but 7 is not.

17. (B) Let r be the radius of the sphere and cone, and let h be the height of the cone. Then the conditions of the problem imply that

$$\frac{3}{4}\left(\frac{4}{3}\pi r^3\right) = \frac{1}{3}\pi r^2 h, \quad \text{so } h = 3r.$$

Therefore, the ratio of h to r is $3 : 1$.

18. (D) Among five consecutive odd numbers, at least one is divisible by 3 and exactly one is divisible by 5, so the product is always divisible by 15. The cases $n = 2$, $n = 10$, and $n = 12$ demonstrate that no larger common divisor is possible, since 15 is the greatest common divisor of $3 \cdot 5 \cdot 7 \cdot 9 \cdot 11$, $11 \cdot 13 \cdot 15 \cdot 17 \cdot 19$, and $13 \cdot 15 \cdot 17 \cdot 19 \cdot 21$.

19. (E) The area of the larger semicircle is

$$\frac{1}{2}\pi\,(2)^2 = 2\pi.$$

The region deleted from the larger semicircle consists of five congruent sectors and two equilateral triangles. The area of each of the sectors is

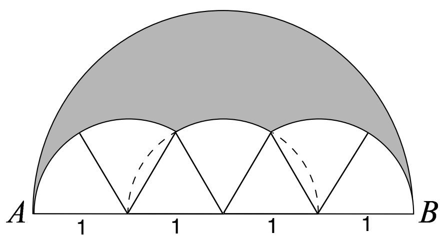
$$\frac{1}{6}\pi\,(1)^2 = \frac{\pi}{6}$$

and the area of each triangle is

$$\frac{1}{2}\cdot 1\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{4},$$

so the area of the shaded region is

$$2\pi - 5\cdot\frac{\pi}{6} - 2\cdot\frac{\sqrt{3}}{4} = \frac{7}{6}\pi - \frac{\sqrt{3}}{2}.$$

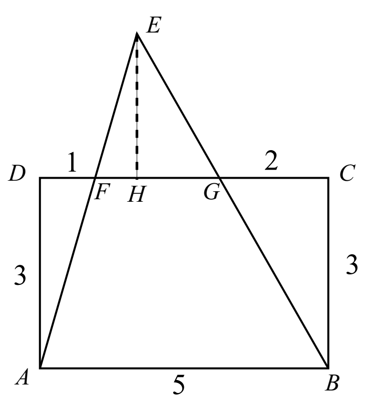


20. (D) Let H be the foot of the perpendicular from E to $\overline{DC}$. Since $CD = AB = 5$, $FG = 2$, and $\triangle FEG$ is similar to $\triangle AEB$, we have

$$\frac{EH}{EH+3} = \frac{2}{5}, \quad \text{so} \quad 5EH = 2EH + 6,$$

and $EH = 2$. Hence

$$\text{Area}(\triangle AEB) = \frac{1}{2}(2+3)\cdot 5 = \frac{25}{2}.$$



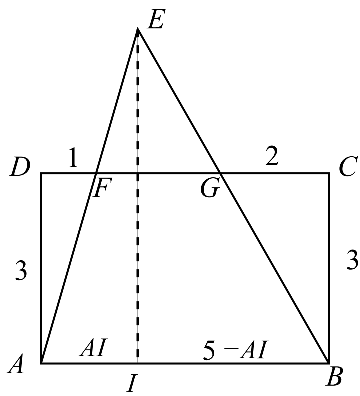
OR

Let I be the foot of the perpendicular from E to $\overline{AB}$. Since

$$\triangle EIA \text{ is similar to } \triangle ADF \quad \text{and} \quad \triangle EIB \text{ is similar to } \triangle BCG,$$

we have

$$\frac{AI}{EI} = \frac{1}{3} \quad \text{and} \quad \frac{5 - AI}{EI} = \frac{2}{3}.$$



Adding gives $5/EI = 1$, so $EI = 5$. The area of the triangle is $\frac{1}{2} \cdot 5 \cdot 5 = \frac{25}{2}$.

21. (C) The beads will all be red at the end of the third draw precisely when two green beads are chosen in the three draws. If the first bead drawn is green, then there will be one green and three red beads in the bag before the second draw. So the probability that green beads are drawn in the first two draws is

$$\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}.$$

The probability that a green bead is chosen, then a red bead, and then a green bead, is

$$\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{1}{4} = \frac{3}{32}.$$

Finally, the probability that a red bead is chosen then two green beads is

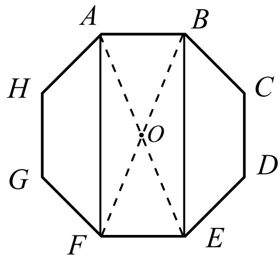
$$\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{16}.$$

The sum of these probabilities is

$$\frac{1}{8} + \frac{3}{32} + \frac{1}{16} = \frac{9}{32}.$$

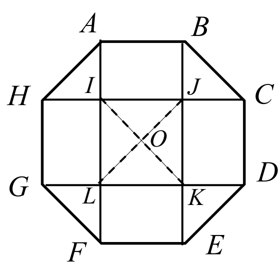
22. (B) In any twelve-hour period, there are 12 half-hour chimes and $1+2+3+\ldots+12 = 78$ on-the-hour chimes. Hence, a twelve-hour period results in 90 chimes. Dividing 2003 by 90 yields a quotient of $22.2\overline{5}$. Therefore the 2003^{rd} chime will occur a little more than 11 days later, on March 9.

23. (D) Let O be the intersection of the diagonals of $ABEF$. Since the octagon is regular, $\triangle AOB$ has area $1/8$. Since O is the midpoint of $\overline{AE}$, $\triangle OAB$ and $\triangle BOE$ have the same area. Thus $\triangle ABE$ has area $1/4$, so $ABEF$ has area $1/2$.



OR

Let O be the intersection of the diagonals of the square $IJKL$. Rectangles $ABJI$, $JCDK$, $KEFL$, and $LGHI$ are congruent. Also $IJ = AB = AH$, so the right isosceles triangles $\triangle AIH$ and $\triangle JOI$ are congruent. By symmetry, the area in the center square $IJKL$ is the sum of the areas of $\triangle AIH$, $\triangle CJB$, $\triangle EKD$, and $\triangle GLF$. Thus the area of rectangle $ABEF$ is half the area of the octagon.



24. (E) Since the difference of the first two terms is $-2y$, the third and fourth terms of the sequence must be $x - 3y$ and $x - 5y$. Thus

$$x - 3y = xy \quad \text{and} \quad x - 5y = \frac{x}{y},$$

so $xy - 5y^2 = x$. Combining these equations we obtain

$$(x - 3y) - 5y^2 = x \quad \text{and, therefore,} \quad -3y - 5y^2 = 0.$$

Since y cannot be 0, we have $y = -3/5$, and it follows that $x = -9/8$. The fifth term in the sequence is $x - 7y = 123/40$.

25. (B) A number is divisible by 3 if and only if the sum of its digits is divisible by 3. So a four-digit number $ab23$ is divisible by 3 if and only if the two-digit number ab leaves a remainder of 1 when divided by 3. There are 90 two-digit numbers, of which $90/3 = 30$ leave a remainder of 1 when divided by 3.

The

American Mathematics Contest 12 (AMC 12)

Sponsored by

Mathematical Association of America

The Akamai Foundation

University of Nebraska – Lincoln

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Canada/USA Mathpath & Mathcamp
Casualty Actuarial Society
Clay Mathematics Institute
Consortium for Mathematics and its Applications
Institute for Operations Research and the Management Sciences
Kappa Mu Epsilon
Mu Alpha Theta
National Association of Mathematicians
National Council of Teachers of Mathematics
Pi Mu Epsilon
School Science and Mathematics Association
Society of Actuaries

The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

5th Annual American Mathematics Contest 10

AMC 10 - Contest A

Solutions Pamphlet

Tuesday, FEBRUARY 10, 2004

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:

Prof. Douglas Faires,
Department of Mathematics,
Youngstown State University,
Youngstown, OH 44555-0001

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606

1. **(A)** Six people are fundraising, so each must raise $\$1500/6 = \$250$.

2. **(B)** Because

$$\P(1,2,3) = \frac{1}{2 \quad 3} = \quad 1, \quad \P(2,3,1) = \frac{2}{3 \quad 1} = 1, \text{ and}$$
$$\P(3,1,2) = \frac{3}{1 \quad 2} = \quad 3,$$

we have

$$\P\,(\P(1,2,3), \P(2,3,1), \P(3,1,2)) = \P(\quad 1, 1, \quad 3)$$
$$= \frac{1}{1 \quad (\quad 3)} = \quad \frac{1}{4}.$$

3. **(E)** Since $20 is 2000 cents, she pays $(0.0145)(2000) = 29$ cents per hour in local taxes.

4. **(D)** The equation implies that either

$$x \quad 1 = x \quad 2 \quad \text{or} \quad x \quad 1 = \quad (x \quad 2)$$

The rst equation has no solution, and the solution to the second equation is $x = 3/2$.

OR

Since $|x \quad a|$ is the distance of x from a, x must be equidistant from 1 and 2. Hence $x = 3/2$.

5. **(C)** The number of three-point sets that can be chosen from the nine grid points is

$$\begin{matrix} 9 \\ 3 \end{matrix} = \frac{9!}{3! \; 6!} = 84.$$

Eight of these sets consist of three collinear points:
3 sets of points lie on vertical lines, 3 on horizontal lines, and 2 on diagonals. Hence the probability is $8/84 = 2/21$.

6. **(E)** Bertha has $30 \quad 6 = 24$ granddaughters, none of whom have any daughters. The granddaughters are the children of $24/6 = 4$ of Bertha's daughters, so the number of women having no daughters is $30 \quad 4 = 26$.

7. **(C)** There are ve layers in the stack, and each of the top four layers has one less orange in its length and width than the layer on which it rests. Hence the total number of oranges in the stack is

$$5 \quad 8 + 4 \quad 7 + 3 \quad 6 + 2 \quad 5 + 1 \quad 4 = 100.$$

8. **(B)** After three rounds the players A, B, and C have 14, 13, and 12 tokens, respectively. Every subsequent three rounds of play reduces each player's supply of tokens by one. After 36 rounds they have 3, 2, and 1 token, respectively, and after the 37^{th} round Player A has no tokens.

9. **(B)** Let x, y, and z be the areas of $\triangle ADE$, $\triangle BDC$, and $\triangle ABD$, respectively. The area of $\triangle ABE$ is $(1/2)(4)(8) = 16 = x + z$, and the area of $\triangle BAC$ is $(1/2)(4)(6) = 12 = y + z$. The requested di erence is

$$x \quad y = (x + z) \quad (y + z) = 16 \quad 12 = 4.$$

10. **(D)** The result will occur when both A and B have either 0, 1, 2, or 3 heads, and these probabilities are shown in the table.

Heads	0	1	2	3
A	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$
B	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{6}{16}$	$\frac{4}{16}$

The probability of both coins having the same number of heads is

$$\frac{1}{8} \quad \frac{1}{16} + \frac{3}{8} \quad \frac{4}{16} + \frac{3}{8} \quad \frac{6}{16} + \frac{1}{8} \quad \frac{4}{16} = \frac{35}{128}.$$

11. **(C)** Let r, h, and V, respectively, be the radius, height, and volume of the jar that is currently being used. The new jar will have a radius of $1.25r$ and volume V. Let H be the height of the new jar. Then

$$r^2h = V = \quad (1.25r)^2H, \quad \text{so} \quad \frac{H}{h} = \frac{1}{(1.25)^2} = 0.64.$$

Thus H is 64% of h, so the height must be reduced by $(100 \quad 64)\% = 36\%$.

OR

Multiplying the diameter by $5/4$ multiplies the area of the base by $(5/4)^2 = 25/16$, so in order to keep the same volume, the height must be multiplied by $16/25$. Thus the height must be decreased by $9/25$, or 36%.

12. **(C)** A customer makes one of two choices for each condiment, to include it or not to include it. The choices are made independently, so there are $2^8 = 256$ possible combinations of condiments. For each of those combinations there are three choices regarding the number of meat patties, so there are altogether $(3)(256) = 768$ di erent kinds of hamburger.

13. **(D)** Because each man danced with exactly three women, there were $(12)(3) = 36$ pairs of men and women who danced together. Each woman had two partners, so the number of women who attended is $36/2 = 18$.

14. **(A)** If n is the number of coins in Paula's purse, then their total value is $20n$ cents. If she had one more quarter, she would have $n+1$ coins whose total value in cents could be expressed both as $20n + 25$ and as $21(n + 1)$. Therefore

$$20n + 25 = 21(n + 1), \quad \text{so} \quad n = 4.$$

Since Paula has four coins with a total value of 80 cents, she must have three quarters and one nickel, so the number of dimes is 0.

15. **(D)** Because

$$\frac{x+y}{x} = 1 + \frac{y}{x} \quad \text{and} \quad \frac{y}{x} < 0,$$

the value is maximized when $|y/x|$ is minimized, that is, when $|y|$ is minimized and $|x|$ is maximized. So $y = 2$ and $x = \;\; 4$ gives the largest value, which is $1 + (\;\; 1/2) = 1/2$.

16. **(D)** All of the squares of size 5 5, 4 4, and 3 3 contain the black square and there are

$$1^2 + 2^2 + 3^2 = 14$$

of these. In addition, 4 of the 2 2 squares and 1 of the 1 1 squares contain the black square, for a total of $14 + 4 + 1 = 19$.

17. **(C)** When they rst meet, they have run a combined distance equal to half the length of the track. Between their rst and second meetings, they run a combined distance equal to the full length of the track. Because Brenda runs at a constant speed and runs 100 meters before their rst meeting, she runs $2(100) = 200$ meters between their rst and second meetings. Therefore the length of the track is $200 + 150 = 350$ meters.

18. **(A)** The terms of the arithmetic progression are 9, $9 + d$, and $9 + 2d$ for some real number d. The terms of the geometric progression are 9, $11+d$, and $29+2d$. Therefore

$$(11 + d)^2 = 9(29 + 2d) \quad \text{so} \quad d^2 + 4d \quad 140 = 0.$$

Thus $d = 10$ or $d = \;\; 14$. The corresponding geometric progressions are 9, 21, 49 and 9, 3, 1, so the smallest possible value for the third term of the geometric progression is 1.

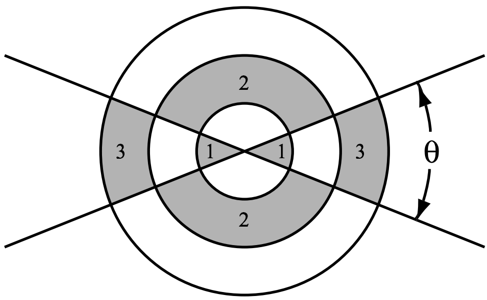
19. **(C)** If the stripe were cut from the silo and spread at, it would form a parallelogram 3 feet wide and 80 feet high. So the area of the stripe is $3(80) = 240$ square feet.

20. **(D)** First, assume that $AB = 1$, and let $ED = DF = x$. By the Pythagorean Theorem $x^2 + x^2 = EF^2 = EB^2 = 1^2 + (1 \quad x)^2$, so $x^2 = 2(1 \quad x)$. Hence the desired ratio of the areas is

$$\frac{\text{Area}(\triangle DEF)}{\text{Area}(\triangle ABE)} = \frac{x^2}{1 \quad x} = 2.$$

21. **(B)** Let θ be the acute angle between the two lines. The area of shaded Region 1 in the diagram is

$$2 \quad \frac{1}{2}\theta(1)^2 \quad = \theta.$$



The area of shaded Region 2 is

$$2 \quad \frac{1}{2}(\qquad \theta)(2^2 \quad 1^2) \quad = 3 \qquad 3\theta.$$

The area of shaded Region 3 is

$$2 \quad \frac{1}{2}\theta(3^2 \quad 2^2) \quad = 5\theta.$$

Hence the total area of the shaded regions is $3 \quad + 3\theta$. The area bounded by the largest circle is $9 \quad$, so

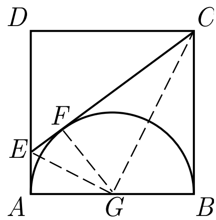
$$\frac{3 \quad + 3\theta}{9} = \frac{8}{8 + 13}.$$

Solving for θ gives $\theta = \quad /7$.

22. **(D)** Let F be the point at which $\overline{CE}$ is tangent to the semicircle, and let G be the midpoint of $\overline{AB}$. Because $\overline{CF}$ and $\overline{CB}$ are both tangents to the semicircle, $CF = CB = 2$. Similarly, $EA = EF$. Let $x = AE$. The Pythagorean Theorem applied to $\triangle CDE$ gives

$$(2 \quad x)^2 + 2^2 = (2 + x)^2.$$

It follows that $x = 1/2$ and $CE = 2 + x = 5/2$.



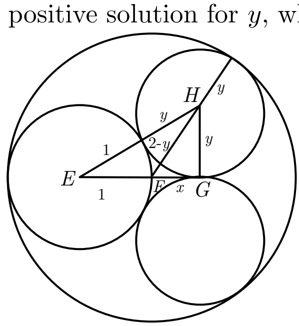
23. **(D)** Let E, H, and F be the centers of circles A, B, and D, respectively, and let G be the point of tangency of circles B and C. Let $x = FG$ and $y = GH$. Since the center of circle D lies on circle A and the circles have a common point of tangency, the radius of circle D is 2, which is the diameter of circle A. Applying the Pythagorean Theorem to right triangles EGH and FGH gives

$$(1 + y)^2 = (1 + x)^2 + y^2 \quad \text{and} \quad (2 \quad y)^2 = x^2 + y^2,$$

from which it follows that

$$y = x + \frac{x^2}{2} \quad \text{and} \quad y = 1 \quad \frac{x^2}{4}.$$

The solutions of this system are $(x, y) = (2/3, 8/9)$ and $(x, y) = (\quad 2, 0)$. The radius of circle B is the positive solution for y, which is $8/9$.



24. **(D)** Note that

$$\begin{aligned} a_{2^1} &= a_2 = a_{2\cdot 1} = 1 \quad a_1 = 2^0 \quad 2^0 = 2^0, \\ a_{2^2} &= a_4 = a_{2\cdot 2} = 2 \quad a_2 = 2^1 \quad 2^0 = 2^1, \\ a_{2^3} &= a_8 = a_{2\cdot 4} = 4 \quad a_4 = 2^2 \quad 2^1 = 2^{1+2}, \\ a_{2^4} &= a_{16} = a_{2\cdot 8} = 8 \quad a_8 = 2^3 \quad 2^{1+2} = 2^{1+2+3}, \end{aligned}$$

and, in general, $a_{2^n} = 2^{1+2+\cdots+(n \quad 1)}$. Because

$$1 + 2 + 3 + \qquad + (n \quad 1) = \frac{1}{2}n(n \quad 1),$$

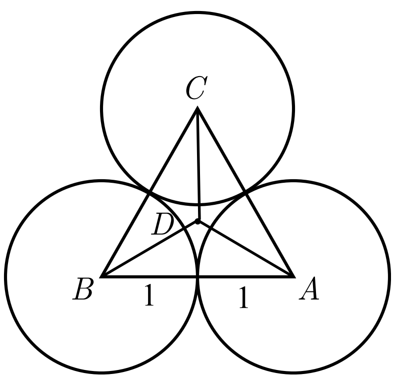
we have $a_{2^{100}} = 2^{(100)(99)/2} = 2^{4950}$.

25. **(B)** Let A, B, C and E be the centers of the three small spheres and the large sphere, respectively. Then $\triangle ABC$ is equilateral with side length 2. If D is the intersection of the medians of $\triangle ABC$, then E is directly above D. Because $AE = 3$ and $AD = 2\sqrt{3}/3$, it follows that

$$DE = \sqrt{3^2 \quad \left(\frac{2\sqrt{3}}{3}\right)^2} = \frac{\sqrt{69}}{3}.$$

Because D is 1 unit above the plane and the top of the larger sphere is 2 units above E, the distance from the plane to the top of the larger sphere is

$$3 + \frac{\sqrt{69}}{3}.$$



The

American Mathematics Contest 10 (AMC 10)

Sponsored by

Mathematical Association of America

University of Nebraska – Lincoln

Contributors

Akamai Foundation

American Mathematical Association of Two Year Colleges

American Mathematical Society

American Society of Pension Actuaries

American Statistical Association

Art of Problem Solving

Canada/USA Mathpath & Mathcamp

Casualty Actuarial Society

Clay Mathematics Institute

Consortium for Mathematics and its Applications

Institute for Operations Research and the Management Sciences

Kappa Mu Epsilon

Mu Alpha Theta

National Association of Mathematicians

National Council of Teachers of Mathematics

Pedagoguery Software Inc.

Pi Mu Epsilon

School Science and Mathematics Association

Society of Actuaries

The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

5th Annual American Mathematics Contest 10

AMC 10 - Contest B

Solutions Pamphlet

Wednesday, FEBRUARY 25, 2004

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction, or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results.* Duplication **at any time** via copier, phone, email, the Web or media of any type is a violation of the copyright law.

Correspondence about the problems and solutions should be addressed to:
Prof. Douglas Faires,
Department of Mathematics,
Youngstown State University,
Youngstown, OH 44555-0001

Orders for prior year Exam questions and Solutions Pamphlets should be addressed to:
American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606

1. **(C)** There are $22 \quad 12 + 1 = 11$ reserved rows. Because there are 33 seats in each row, there are $(33)(11) = 363$ reserved seats.

2. **(B)** There are 10 two-digit numbers with a 7 as their 10's digit, and 9 two-digit numbers with 7 as their units digit. Because 77 satis es both of these properties, the answer is $10 + 9 \quad 1 = 18$.

3. **(A)** At Jenny's fourth practice she made $\frac{1}{2}(48) = 24$ free throws. At her third practice she made 12, at her second practice she made 6, and at her rst practice she made 3.

4. **(B)** Since $6! = 720 = 2^4 \;\; 3^2 \;\; 5$, the prime factors of P can consist of at most 2's, 3's, and 5's. The least possible number of 2's is two, which occurs when 4 is not visible. The least possible number of 3's is one, which occurs when either 3 or 6 is not visible, and the least number of 5's is zero, when 5 is not visible. Thus P must be divisible by $2^2 \;\; 3 = 12$, but not necessarily by any larger number.

5. **(D)** If $d \neq 0$, the value of the expression can be increased by interchanging 0 with the value of d. Therefore the maximum value must occur when $d = 0$. If $a = 1$, the value is c, which is 2 or 3. If $b = 1$, the value is $c \;\; a = 6$. If $c = 1$, the value is a^b, which is $2^3 = 8$ or $3^2 = 9$. Thus the maximum value is 9.

6. **(C)** Note that for $m < n$ we have

$$m! \;\; n! = (m!)^2 \;\; (m+1) \;\; (m+2) \qquad n.$$

Therefore $m! \;\; n!$ is a perfect square if and only if

$$(m+1) \;\; (m+2) \qquad n$$

is a perfect square. For the ve answer choices, that quantity is

$$99, \; 99 \;\; 100, \; 100, \; 100 \;\; 101, \text{ and } \; 101,$$

and of those only 100 is a perfect square. Therefore the answer is $99! \;\; 100!$.

7. **(A)** Isabella received $10d/7$ Canadian dollars at the border and spent 60 of them. Thus $10d/7 \quad 60 = d$, from which it follows that $d = 140$, and the sum of the digits of d is 5.

8. **(A)** Let downtown St. Paul, downtown Minneapolis, and the airport be located at S, M, and A, respectively. Then $\triangle MAS$ has a right angle at A, so by the Pythagorean Theorem,

$$MS = \sqrt{10^2 + 8^2} = \sqrt{164} \qquad \sqrt{169} = 13.$$

9. **(B)** The areas of the regions enclosed by the square and the circle are $10^2 = 100$ and $\;(10)^2 = 100\;$, respectively. One quarter of the second region is also included in the rst, so the area of the union is

$$100 + 100 \qquad 25 \;\; = 100 + 75 \;\; .$$

10. **(D)** If there are n rows in the display, the bottom row contains $2n - 1$ cans. The total number of cans is therefore the sum of the arithmetic series

$$1 + 3 + 5 + \cdots + (2n - 1),$$

which is

$$\frac{n}{2}[(2n - 1) + 1] = n^2.$$

Thus $n^2 = 100$, so $n = 10$.

11. **(C)** There are $8 \times 8 = 64$ ordered pairs that can represent the top numbers on the two dice. Let m and n represent the top numbers on the dice. Then $mn > m+n$ implies that $mn - m - n > 0$, that is,

$$1 < mn - m - n + 1 = (m - 1)(n - 1).$$

This inequality is satisfied except when $m = 1$, $n = 1$, or when $m = n = 2$. There are 16 ordered pairs (m, n) excluded by these conditions, so the probability that the product is greater than the sum is

$$\frac{64 - 16}{64} = \frac{48}{64} = \frac{3}{4}.$$

12. **(A)** The area of the annulus is the difference between the areas of the two circles, which is $\pi b^2 - \pi c^2$. Because the tangent $\overline{XZ}$ is perpendicular to the radius $\overline{OZ}$, $b^2 - c^2 = a^2$, so the area is πa^2.

13. **(B)** The height in millimeters of any stack with an odd number of coins has a 5 in the hundredth place. The height of any two coins has an odd digit in the tenth place and a zero in the hundredth place. Therefore any stack with zeros in both its tenth and hundredth places must consist of a number of coins that is a multiple of 4. The highest stack of 4 coins has a height of $4(1.95) = 7.8$ mm, and the shortest stack of 12 coins has a height of $12(1.35) = 16.2$ mm, so no number other than 8 can work. Note that a stack of 8 quarters has a height of $8(1.75) = 14$ mm.

14. **(C)** If there are initially B blue marbles in the bag, after red marbles are added, then the total number of marbles in the bag must be $3B$. Then after the yellow marbles are added, the number of marbles in the bag must be $5B$. Finally, adding B blue marbles to the bag gives $2B$ blue marbles out of $6B$ total marbles. Thus $1/3$ of the marbles are blue.

OR

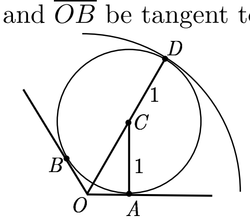
Just before the number of blue marbles is doubled, the ratio of blue marbles to non-blue marbles is 1 to 4. Doubling the number of blue marbles makes the ratio 2 to 4, so $1/3$ of the marbles are blue.

15. **(A)** Because the value of Patty's money would increase if the dimes and nickels were interchanged, she must have more nickels than dimes. Interchanging one nickel for a dime increases the amount by 5 cents, so she has $70/5 = 14$ more nickels than dimes. Therefore she has

$$\frac{1}{2}(20 - 14) = 3 \quad \text{dimes} \quad \text{and} \quad 20 - 3 = 17 \quad \text{nickels},$$

and her coins are worth $3 \cdot 10 + 17 \cdot 5 = 115$ cents $= \$1.15$.

16. **(D)** Let O be the center of the large circle, let C be the center of one of the small circles, and let $\overline{OA}$ and $\overline{OB}$ be tangent to the small circle at A and B.



By symmetry, $\angle AOB = 120°$ and $\angle AOC = 60°$. Thus $\triangle AOC$ is a 30-60-90 degree right triangle, and $AC = 1$, so

$$OC = \frac{2}{\sqrt{3}} AC = \frac{2\sqrt{3}}{3}.$$

If OD is a radius of the large circle through C, then

$$OD = CD + OC = 1 + \frac{2\sqrt{3}}{3} = \frac{3 + 2\sqrt{3}}{3}.$$

17. **(B)** Let Jack's age be $10x + y$ and Bill's age be $10y + x$. In five years Jack will be twice as old as Bill. Therefore

$$10x + y + 5 = 2(10y + x + 5),$$

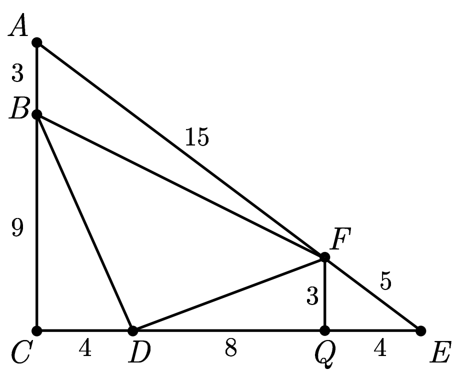
so $8x = 19y + 5$. The expression $19y + 5 = 16y + 8 + 3(y - 1)$ is a multiple of 8 if and only if $y - 1$ is a multiple of 8. Since both x and y are 9 or less, the only solution is $y = 1$ and $x = 3$. Thus Jack is 31 and Bill is 13, so the difference between their ages is 18.

18. **(E)** The area of $\triangle ACE$ is $(1/2)(12)(16) = 96$. Draw $\overline{FQ} \perp \overline{CE}$. By similar triangles, $FQ = 3$ and $QE = 4$. The area of trapezoid $BCQF$ is $(1/2)(3 + 9)(12) = 72$. Since $\triangle BCD$ and $\triangle FDQ$ have areas 18 and 12, respectively, the area of $\triangle BDF$ is $72 \quad 18 \quad 12 = 42$. The desired ratio is $42/96 = 7/16$.

OR

Note that each of $\triangle ABF$, $\triangle BCD$, and $\triangle DEF$ has a base-altitude pair where the base and altitude are, respectively, 3/4 and 1/4 that of a corresponding base and altitude for $\triangle ACE$. Hence

$$\frac{\text{Area of}\triangle BDF}{\text{Area of}\triangle ACE} = 1 \quad 3(1/4)(3/4) = 7/16.$$



19. **(C)** Let a_k be the k^{th} term of the sequence. For $k \geq 3$,

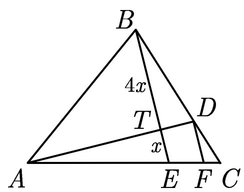
$$a_{k+1} = a_{k \quad 2} + a_{k \quad 1} \quad a_k, \quad \text{so} \quad a_{k+1} \quad a_{k \quad 1} = \quad (a_k \quad a_{k \quad 2}).$$

Because the sequence begins

$$2001, 2002, 2003, 2000, 2005, 1998, \ldots,$$

it follows that the odd-numbered terms and the even-numbered terms each form arithmetic progressions with common di erences of 2 and 2, respectively. The 2004^{th} term of the original sequence is the 1002^{nd} term of the sequence 2002, 2000, 1998,..., and that term is $2002 + 1001(\quad 2) = 0$.

20. **(D)** Let F be a point on $\overline{AC}$ such that $\overline{DF}$ is parallel to $\overline{BE}$. Let $BT = 4x$ and $ET = x$.



Because $\triangle ATE$ and $\triangle ADF$ are similar, we have

$$\frac{DF}{x} = \frac{AD}{AT} = \frac{4}{3}, \quad \text{and} \quad DF = \frac{4x}{3}.$$

Also, $\triangle BEC$ and $\triangle DFC$ are similar, so

$$\frac{CD}{BC} = \frac{DF}{BE} = \frac{4x/3}{5x} = \frac{4}{15}.$$

Thus

$$\frac{CD}{BD} = \frac{CD/BC}{1 - (CD/BC)} = \frac{4/15}{1 - 4/15} = \frac{4}{11}.$$

OR

Let $s = \text{Area}(\triangle ABC)$. Then

$$\text{Area}(\triangle TBC) = \frac{1}{4}s \quad \text{and} \quad \text{Area}(\triangle ATC) = \frac{1}{5}s,$$

so

$$\text{Area}(\triangle ATB) = \text{Area}(\triangle ABC) - \text{Area}(\triangle TBC) - \text{Area}(\triangle ATC) = \frac{11}{20}s.$$

Hence

$$\frac{CD}{BD} = \frac{\text{Area}(\triangle ADC)}{\text{Area}(\triangle ABD)} = \frac{\text{Area}(\triangle ATC)}{\text{Area}(\triangle ATB)} = \frac{s/5}{11s/20} = \frac{4}{11}.$$

21. **(A)** The smallest number that appears in both sequences is 16. The two sequences have common differences 3 and 7, whose least common multiple is 21, so a number appears in both sequences if and only if it is in the form

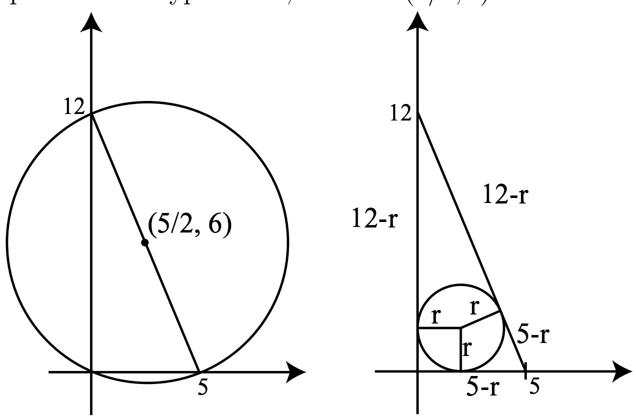
$$16 + 21k,$$

where k is a nonnegative integer. Such a number is in the first 2004 terms of both sequences if and only if

$$16 + 21k \le 1 + 2003(3) = 6010.$$

Thus $0 \le k \le 285$, so there are 286 duplicate numbers. Therefore the number of distinct numbers is $4008 - 286 = 3722$.

22. **(D)** The triangle is a right triangle that can be placed in a coordinate system with vertices at $(0,0)$, $(5,0)$, and $(0,12)$. The center of the circumscribed circle is the midpoint of the hypotenuse, which is $(5/2,6)$.



To determine the radius r of the inscribed circle notice that the hypotenuse of the triangle is

$$(12-r)+(5-r)=13 \quad \text{so} \quad r=2.$$

So the center of the inscribed circle is $(2,2)$, and the distance between the two centers is

$$\sqrt{\left(\frac{5}{2}-2\right)^2+(6-2)^2}=\frac{\sqrt{65}}{2}.$$

23. **(B)** If the orientation of the cube is fixed, there are $2^6=64$ possible arrangements of colors on the faces. There are

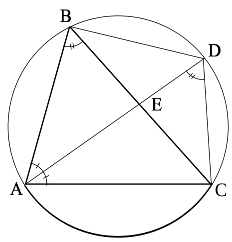
$$2\binom{6}{6}=2$$

arrangements in which all six faces are the same color and

$$2\binom{6}{5}=12$$

arrangements in which exactly five faces have the same color. In each of these cases the cube can be placed so that the four vertical faces have the same color. The only other suitable arrangements have four faces of one color, with the other color on a pair of opposing faces. Since there are three pairs of opposing faces, there are $2(3)=6$ such arrangements. The total number of suitable arrangements is therefore $2+12+6=20$, and the probability is $20/64=5/16$.

24. **(B)** Suppose that AD and BC intersect at E.



Since $\angle ADC$ and $\angle ABC$ cut the same arc of the circumscribed circle, the Inscribed Angle Theorem implies that

$$\angle ABC = \angle ADC.$$

Also, $\angle EAB = \angle CAD$, so $\triangle ABE$ is similar to $\triangle ADC$, and

$$\frac{AD}{CD} = \frac{AB}{BE}.$$

By the Angle Bisector Theorem,

$$\frac{BE}{EC} = \frac{AB}{AC},$$

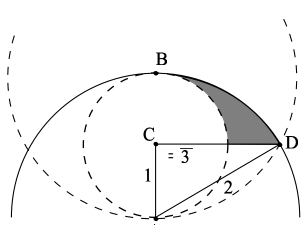
so

$$BE = \frac{AB}{AC} \quad EC = \frac{AB}{AC}(BC \quad BE) \quad \text{and} \quad BE = \frac{AB \quad BC}{AB + AC}.$$

Hence

$$\frac{AD}{CD} = \frac{AB}{BE} = \frac{AB + AC}{BC} = \frac{7+8}{9} = \frac{5}{3}.$$

25. **(B)** The centers of the two larger circles are at A and B. Let C be the center of the smaller circle, and let D be one of the points of intersection of the two larger circles.



Then $\triangle ACD$ is a right triangle with $AC = 1$ and $AD = 2$, so $CD = \sqrt{3}$, $\angle CAD = 60°$, and the area of $\triangle ACD$ is $\sqrt{3}/2$. The area of 1/4 of the shaded region, as shown in the Figure, is the area of sector BAD of the circle centered at A, minus the area of $\triangle ACD$, minus the area of 1/4 of the smaller circle. That area is

$$\frac{2}{3} \quad\quad \frac{\sqrt{3}}{2} \quad \frac{1}{4} = \frac{5}{12} \quad\quad \frac{\sqrt{3}}{2},$$

so the area of the entire shaded region is

$$4\left(\frac{5}{12} \quad\quad \frac{\sqrt{3}}{2}\right) = \frac{5}{3} \quad\quad 2\sqrt{3}.$$

The

American Mathematics Contest 10 (AMC 10)

Sponsored by

Mathematical Association of America

University of Nebraska – Lincoln

Contributors

Akamai Foundation
American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Canada/USA Mathpath & Mathcamp
Casualty Actuarial Society
Clay Mathematics Institute
Consortium for Mathematics and its Applications
Institute for Operations Research and the Management Sciences
Kappa Mu Epsilon
Mu Alpha Theta
National Association of Mathematicians
National Council of Teachers of Mathematics
Pedagoguery Software Inc.
Pi Mu Epsilon
School Science and Mathematics Association
Society of Actuaries

The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

6th Annual American Mathematics Contest 10

AMC 10 – Contest A

Solutions Pamphlet

TUESDAY, FEBRUARY 1, 2005

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, the World Wide Web or media of any type is a violation of the competition rules.*

Correspondence about the problems and solutions for this AMC 10 and orders for any of the publications listed below should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Prof. Douglas Faires, Department of Mathematics
Youngstown State University, Youngstown, OH 44555-0001

1. **(D)** Since Mike tipped \$2, which was $10\% = 1/10$ of his bill, his bill must have been $2 \cdot 10 = 20$ dollars. Similarly, Joe tipped \$2, which was $20\% = 1/5$ of his bill, so his bill must have been $2 \cdot 5 = 10$ dollars. The difference between their bills is therefore \$10.

2. **(C)** First we have
$$(1 \star 2) = \frac{1+2}{1-2} = -3.$$
Then
$$((1 \star 2) \star 3) = (-3 \star 3) = \frac{-3+3}{-3-3} = \frac{0}{-6} = 0.$$

3. **(B)** Since $2x + 7 = 3$ we have $x = -2$. Hence
$$-2 = bx - 10 = -2b - 10, \quad \text{so} \quad 2b = -8, \text{ and } b = -4.$$

4. **(B)** Let w be the width of the rectangle. Then the length is $2w$, and
$$x^2 = w^2 + (2w)^2 = 5w^2.$$
The area is consequently $w(2w) = 2w^2 = \frac{2}{5}x^2$.

5. **(A)** If Dave buys seven windows separately he will purchase six and receive one free, for a cost of \$600. If Doug buys eight windows separately, he will purchase seven and receive one free, for a total cost of \$700. The total cost to Dave and Doug purchasing separately will be \$1300. If they purchase fifteen windows together, they will need to purchase only 12 windows, for a cost of \$1200, and will receive 3 free. This will result in a savings of \$100.

6. **(B)** The sum of the 50 numbers is $20 \cdot 30 + 30 \cdot 20 = 1200$. Their average is $1200/50 = 24$.

7. **(B)** Because (rate)(time) = (distance), the distance Josh rode was $(4/5)(2) = 8/5$ of the distance that Mike rode. Let m be the number of miles that Mike had ridden when they met. Then the number of miles between their houses is
$$13 = m + \frac{8}{5}m = \frac{13}{5}m.$$
Thus $m = 5$.

8. **(C)** The symmetry of the figure implies that $\triangle ABH$, $\triangle BCE$, $\triangle CDF$, and $\triangle DAG$ are congruent right triangles. So
$$BH = CE = \sqrt{BC^2 - BE^2} = \sqrt{50 - 1} = 7,$$
and $EH = BH - BE = 7 - 1 = 6$. Hence the square $EFGH$ has area $6^2 = 36$.

OR

As in the first solution, $BH = 7$. Now note that $\triangle ABH$, $\triangle BCE$, $\triangle CDF$, and $\triangle DAG$ are congruent right triangles, so
$$\text{Area}(EFGH) = \text{Area}(ABCD) - 4\text{Area}(\triangle ABH) = 50 - 4\left(\tfrac{1}{2} \cdot 1 \cdot 7\right) = 36.$$

9. **(B)** There are three X's and two O's, and the tiles are selected without replacement, so the probability is

$$\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}\cdot\frac{1}{2}\cdot\frac{1}{1}=\frac{1}{10}.$$

OR

The three tiles marked X are equally likely to lie in any of $\binom{5}{3}=10$ positions, so the probability of this arrangement is $1/10$.

10. **(A)** The quadratic formula yields

$$x=\frac{-(a+8)\pm\sqrt{(a+8)^2-4\cdot 4\cdot 9}}{2\cdot 4}.$$

The equation has only one solution precisely when the value of the discriminant, $(a+8)^2-144$, is 0. This implies that $a=-20$ or $a=4$, and the sum is -16.

OR

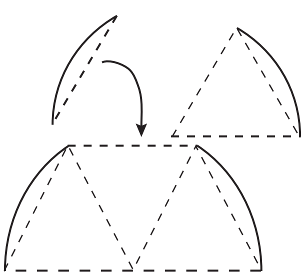
The equation has one solution if and only if the polynomial is the square of a binomial with linear term $\pm\sqrt{4x^2}=\pm 2x$ and constant term $\pm\sqrt{9}=\pm 3$. Because $(2x\pm 3)^2$ has a linear term $\pm 12x$, it follows that $a+8=\pm 12$. Thus a is either -20 or 4, and the sum of those values is -16.

11. **(B)** The unit cubes have a total of $6n^3$ faces, of which $6n^2$ are red. Therefore

$$\frac{1}{4}=\frac{6n^2}{6n^3}=\frac{1}{n},\quad\text{so}\quad n=4.$$

12. **(B)** The trefoil is constructed of four equilateral triangles and four circular segments, as shown. These can be combined to form four 60° circular sectors. Since the radius of the circle is 1, the area of the trefoil is

$$\frac{4}{6}\left(\pi\cdot 1^2\right)=\frac{2}{3}\pi.$$



13. **(E)** The condition is equivalent to

$$130n>n^2>2^4=16,\quad\text{so}\quad 130n>n^2\ \text{ and }\ n^2>16.$$

This implies that $130>n>4$. So n can be any of the 125 integers strictly between 130 and 4.

14. **(E)** The first and last digits must be both odd or both even for their average to be an integer. There are $5 \cdot 5 = 25$ odd-odd combinations for the first and last digits. There are $4 \cdot 5 = 20$ even-even combinations that do not use zero as the first digit. Hence the total is 45.

15. **(E)** Written as a product of primes, we have

$$3! \cdot 5! \cdot 7! = 2^8 \cdot 3^4 \cdot 5^2 \cdot 7.$$

A cube that is a factor has a prime factorization of the form $2^p \cdot 3^q \cdot 5^r \cdot 7^s$, where p, q, r, and s are all multiples of 3. There are 3 possible values for p, which are 0, 3, and 6. There are 2 possible values for q, which are 0 and 3. The only value for r and for s is 0. Hence there are $6 = 3 \cdot 2 \cdot 1 \cdot 1$ distinct cubes that divide $3! \cdot 5! \cdot 7!$. They are

$$1 = 2^0 3^0 5^0 7^0, \quad 8 = 2^3 3^0 5^0 7^0, \quad 27 = 2^0 3^3 5^0 7^0,$$
$$64 = 2^6 3^0 5^0 7^0, \quad 216 = 2^3 3^3 5^0 7^0, \quad \text{and} \quad 1728 = 2^6 3^3 5^0 7^0.$$

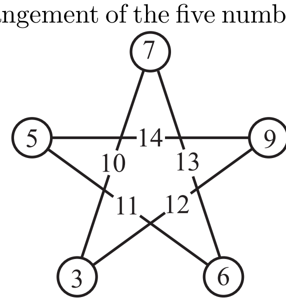
16. **(D)** Let $10a + b$ be the two-digit number. When $a + b$ is subtracted the result is $9a$. The only two-digit multiple of 9 that ends in 6 is $9 \cdot 4 = 36$, so $a = 4$. The ten numbers between 40 and 49, inclusive, have this property.

17. **(D)** Each number appears in two sums, so the sum of the sequence is

$$2(3 + 5 + 6 + 7 + 9) = 60.$$

The middle term of a five-term arithmetic sequence is the mean of its terms, so $60/5 = 12$ is the middle term.

The figure shows an arrangement of the five numbers that meets the requirement.



18. **(A)**

There are four possible outcomes,

$$\text{ABAA}, \quad \text{ABABA}, \quad \text{ABBAA}, \quad \text{and} \quad \text{BBAAA},$$

but they are not equally likely. This is because, in general, the probability of any specific four-game series is $(1/2)^4 = 1/16$, whereas the probability of any specific five-game series is $(1/2)^4 = 1/32$. Thus the first listed outcome is twice as likely as each of the other three. Let p be the probability of the occurrence $ABBAA$. Then the probability of $ABABA$ is also p, as is the probability of $BBAAA$, whereas the probability of $ABAA$ is $2p$. So

$$2p + p + p + p = 1, \quad \text{and} \quad p = \frac{1}{5}.$$

The only outcome in which team B wins the first game is $BBAAA$, so the probability of this occurring is $1/5$.

OR

To consider equally-likely cases, suppose that all five games are played, even if team A has won the series before the fifth game. Then the possible ways that team A can win the series, given that team B wins the second game, are

BBAAA, ABBAA, ABABA, ABAAB, and ABAAA.

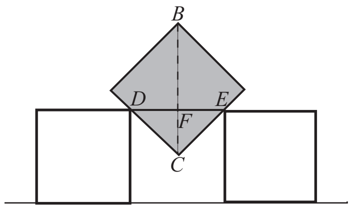
In only the first case does team B win the first game, so the probability of this occurring is $1/5$.

19. **(D)** Consider the rotated middle square shown in the figure. It will drop until length DE is 1 inch. Thus

$$FC = DF = FE = \frac{1}{2} \quad \text{and} \quad BC = \sqrt{2}.$$

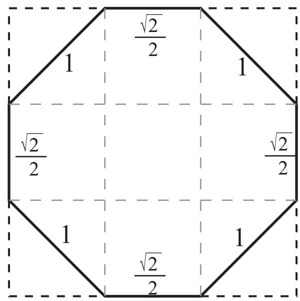
Hence $BF = \sqrt{2} - 1/2$. This is added to the 1 inch height of the supporting squares, so the overall height of point B above the line is

$$1 + BF = \sqrt{2} + \frac{1}{2} \text{ inches.}$$



20. **(A)** The octagon can be partitioned into five squares and four half squares, each with side length $\sqrt{2}/2$, so its area is

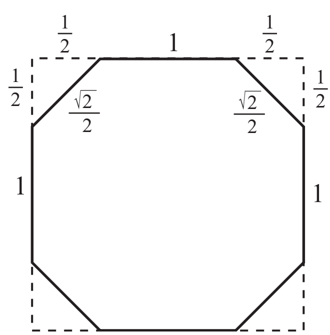
$$\left(5+4\cdot\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)^2=\frac{7}{2}.$$



OR

The octagon can be obtained by removing four isosceles right triangles with legs of length $1/2$ from a square with sides of length 2. Thus its area is

$$2^2-4\cdot\frac{1}{2}\left(\frac{1}{2}\right)^2=\frac{7}{2}.$$



21. **(B)** Because

$$1+2+\cdots+n=\frac{n(n+1)}{2},$$

$1+2+\cdots+n$ divides the positive integer $6n$ if and only if

$$\frac{6n}{n(n+1)/2}=\frac{12}{n+1}\text{ is an integer.}$$

There are five such positive values of n, namely, 1, 2, 3, 5, and 11.

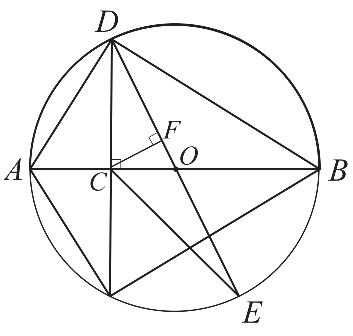
22. **(D)** The sets S and T consist, respectively, of the positive multiples of 4 that do not exceed $2005 \cdot 4 = 8020$ and the positive multiples of 6 that do not exceed $2005 \cdot 6 = 12{,}030$. Thus $S \bigcap T$, the set of numbers that are common to S and to T, consists of the positive multiples of 12 that do not exceed 8020. Let $\lfloor x \rfloor$ represent the largest integer that is less than or equal to x. Then the number of elements in the set $S \bigcap T$ is

$$\left\lfloor \frac{8020}{12} \right\rfloor = \left\lfloor 668 + \frac{1}{3} \right\rfloor = 668.$$

23. **(C)** Let O be the center of the circle. Each of $\triangle DCE$ and $\triangle ABD$ has a diameter of the circle as a side. Thus the ratio of their areas is the ratio of the two altitudes to the diameters. These altitudes are $\overline{DC}$ and the altitude from C to $\overline{DO}$ in $\triangle DCE$. Let F be the foot of this second altitude. Since $\triangle CFO$ is similar to $\triangle DCO$,

$$\frac{CF}{DC} = \frac{CO}{DO} = \frac{AO - AC}{DO} = \frac{\frac{1}{2}AB - \frac{1}{3}AB}{\frac{1}{2}AB} = \frac{1}{3},$$

which is the desired ratio.



OR

Because $AC = AB/3$ and $AO = AB/2$, we have $CO = AB/6$. Triangles DCO and DAB have a common altitude to $\overline{AB}$ so the area of $\triangle DCO$ is $\frac{1}{6}$ the area of $\triangle ADB$. Triangles DCO and ECO have equal areas since they have a common base $\overline{CO}$ and their altitudes are equal. Thus the ratio of the area of $\triangle DCE$ to the area of $\triangle ABD$ is $1/3$.

24. **(B)** The conditions imply that both n and $n+48$ are squares of primes. So for each successful value of n we have primes p and q with $p^2 = n+48$ and $q^2 = n$, and

$$48 = p^2 - q^2 = (p+q)(p-q).$$

The pairs of factors of 48 are

$$48 \text{ and } 1, \quad 24 \text{ and } 2, \quad 16 \text{ and } 3, \quad 12 \text{ and } 4, \quad \text{and} \quad 8 \text{ and } 6.$$

These give pairs (p, q), respectively, of

$$\left(\frac{49}{2}, \frac{47}{2}\right), \quad (13, 11), \quad \left(\frac{19}{2}, \frac{13}{2}\right), \ (8, 4), \quad \text{and} \quad (7, 1).$$

Only $(p, q) = (13, 11)$ gives prime values for p and for q, with $n = 11^2 = 121$ and $n + 48 = 13^2 = 169$.

25. **(D)** We have

$$\frac{\text{Area}(ADE)}{\text{Area}(ABE)} = \frac{AD}{AB} = \frac{19}{25} \quad \text{and} \quad \frac{\text{Area}(ABE)}{\text{Area}(ABC)} = \frac{AE}{AC} = \frac{14}{42} = \frac{1}{3},$$

so

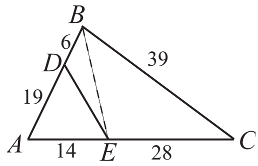
$$\frac{\text{Area}(ABC)}{\text{Area}(ADE)} = \frac{25}{19} \cdot \frac{3}{1} = \frac{75}{19},$$

and

$$\frac{\text{Area}(BCED)}{\text{Area}(ADE)} = \frac{\text{Area}(ABC) - \text{Area}(ADE)}{\text{Area}(ADE)} = \frac{75}{19} - 1 = \frac{56}{19}.$$

Thus

$$\frac{\text{Area}(ADE)}{\text{Area}(BCED)} = \frac{19}{56}.$$



The

American Mathematics Contest 10 (AMC 10)

Sponsored by

The Mathematical Association of America

University of Nebraska – Lincoln

Contributors

Akamai Foundation
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
Institute for Operations Research and the Management Sciences
L. G. Balfour & Company
Mu Alpha Theta
National Council of Teachers of Mathematics
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
USA Math Talent Search
W. H. Freeman & Company

The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

6th Annual American Mathematics Contest 10

AMC 10 – Contest B

Solutions Pamphlet

Wednesday, FEBRUARY 16, 2005

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, the World Wide Web or media of any type is a violation of the competition rules.*

Correspondence about the problems and solutions for this AMC 10 and orders for any of the publications listed below should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Prof. Douglas Faires, Department of Mathematics
Youngstown State University, Youngstown, OH 44555-0001

1. **(A)** The scouts bought $1000/5 = 200$ groups of 5 candy bars at a total cost of $200 \cdot 2 = 400$ dollars. They sold $1000/2 = 500$ groups of 2 candy bars for a total of $500 \cdot 1 = 500$ dollars. Their profit was $\$500 - \$400 = \$100$.

2. **(D)** We have
$$\frac{x}{100} \cdot x = 4, \quad \text{so} \quad x^2 = 400.$$
Because $x > 0$, it follows that $x = 20$.

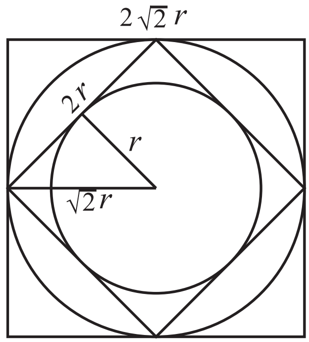
3. **(D)** After the first day,
$$1 - \frac{1}{3} = \frac{2}{3}$$
of the paint remains. On the second day,
$$\frac{1}{3} \cdot \frac{2}{3} = \frac{2}{9}$$
of the paint is used. So for the third day
$$1 - \frac{1}{3} - \frac{2}{9} = \frac{4}{9}$$
of the original gallon of paint is available.

4. **(D)** It follows from the definition that
$$\begin{aligned}(5 \diamond 12) \diamond ((-12) \diamond (-5)) &= \sqrt{5^2 + 12^2} \diamond \sqrt{(-12)^2 + (-5)^2} \\ &= 13 \diamond 13 = \sqrt{13^2 + 13^2} = 13\sqrt{2}.\end{aligned}$$

5. **(C)** The number of CDs that Brianna will finally buy is three times the number she has already bought. The fraction of her money that will be required for all the purchases is $(3)(1/5) = 3/5$. The fraction she will have left is $1 - 3/5 = 2/5$.

6. **(B)** To earn an A on at least 80% of her quizzes, Lisa needs to receive an A on at least $(0.8)(50) = 40$ quizzes. Thus she must earn an A on at least $40 - 22 = 18$ of the remaining 20. So she can earn a grade lower than an A on at most 2 of the remaining quizzes.

7. **(B)** Let the radius of the smaller circle be r. Then the side length of the smaller square is $2r$. The radius of the larger circle is half the length of the diagonal of the smaller square, so it is $\sqrt{2}r$. Hence the larger square has sides of length $2\sqrt{2}r$. The ratio of the area of the smaller circle to the area of the larger square is therefore
$$\frac{\pi r^2}{\left(2\sqrt{2}r\right)^2} = \frac{\pi}{8}.$$



8. **(A)** The four white quarter circles in each tile have the same area as a whole circle of radius $1/2$, that is, $\pi(1/2)^2 = \pi/4$ square feet. So the area of the shaded portion of each tile is $1 - \pi/4$ square feet. Since there are $8 \cdot 10 = 80$ tiles in the entire floor, the area of the total shaded region in square feet is

$$80\left(1 - \frac{\pi}{4}\right) = 80 - 20\pi.$$

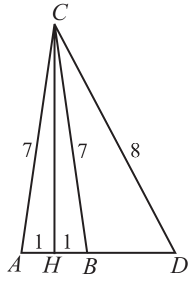
9. **(D)** An odd sum requires either that the first die is even and the second is odd or that the first die is odd and the second is even. The probability is

$$\frac{1}{3} \cdot \frac{1}{3} + \frac{2}{3} \cdot \frac{2}{3} = \frac{1}{9} + \frac{4}{9} = \frac{5}{9}.$$

10. **(A)** Let $\overline{CH}$ be an altitude of $\triangle ABC$. Applying the Pythagorean Theorem to $\triangle CHB$ and to $\triangle CHD$ produces

$$8^2 - (BD+1)^2 = CH^2 = 7^2 - 1^2 = 48, \quad \text{so} \quad (BD+1)^2 = 16.$$

Thus $BD = 3$.



11. **(E)** The sequence begins 2005, 133, 55, 250, 133, $\ldots$. Thus after the initial term 2005, the sequence repeats the cycle 133, 55, 250. Because $2005 = 1 + 3 \cdot 668$, the 2005^{th} term is the same as the last term of the repeating cycle, 250.

12. **(E)** Exactly one die must have a prime face on top, and the other eleven must have 1's. The prime die can be any one of the twelve, and the prime can be 2, 3, or 5. Thus the probability of a prime face on any one die is 1/2, and the probability of a prime product is

$$12\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)^{11} = \left(\frac{1}{6}\right)^{10}.$$

13. **(C)** Between 1 and 2005, there are 668 multiples of 3, 501 multiples of 4, and 167 multiples of 12. So there are

$$(668 - 167) + (501 - 167) = 835$$

numbers between 1 and 2005 that are integer multiples of 3 or of 4 but not of 12.

14. **(C)** Drop $\overline{MQ}$ perpendicular to $\overline{BC}$. Then $\triangle MQC$ is a 30–60–90° triangle, so $MQ = \sqrt{3}/2$, and the area of $\triangle CDM$ is

$$\frac{1}{2}\left(2 \cdot \frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}.$$

OR

Triangles ABC and CDM have equal bases. Because M is the midpoint of $\overline{AC}$, the ratio of the altitudes from M and from A is 1/2. So the area of $\triangle CDM$ is half of the area of $\triangle ABC$. Since

$$\text{Area}(\triangle ABC) = \frac{\sqrt{3}}{4} \cdot 2^2 = \sqrt{3}, \quad \text{we have} \quad \text{Area}(\triangle CDM) = \frac{\sqrt{3}}{2}.$$

15. **(D)** There are

$$\binom{8}{2} = \frac{8!}{6! \cdot 2!} = 28$$

ways to choose the bills. A sum of at least \$20 is obtained by choosing both \$20 bills, one of the \$20 bills and one of the six smaller bills, or both \$10 bills. Hence the probability is

$$\frac{1 + 2 \cdot 6 + 1}{28} = \frac{14}{28} = \frac{1}{2}.$$

16. **(D)** Let r_1 and r_2 be the roots of $x^2+px+m = 0$. Since the roots of $x^2+mx+n = 0$ are $2r_1$ and $2r_2$, we have the following relationships:

$$m = r_1r_2, \quad n = 4r_1r_2, \quad p = -(r_1 + r_2), \quad \text{and} \quad m = -2(r_1 + r_2).$$

So

$$n = 4m, \quad p = \frac{1}{2}m, \quad \text{and} \quad \frac{n}{p} = \frac{4m}{\frac{1}{2}m} = 8.$$

OR

The roots of

$$\left(\frac{x}{2}\right)^2 + p\left(\frac{x}{2}\right) + m = 0$$

are twice those of $x^2 + px + m = 0$. Since the first equation is equivalent to $x^2 + 2px + 4m = 0$, we have

$$m = 2p \quad \text{and} \quad n = 4m, \quad \text{so} \quad \frac{n}{p} = 8.$$

17. **(B)** Because

$$4^{a \cdot b \cdot c \cdot d} = \left(\left(\left(4^a\right)^b\right)^c\right)^d = \left(\left(5^b\right)^c\right)^d = \left(6^c\right)^d = 7^d = 8 = 4^{3/2},$$

we have $a \cdot b \cdot c \cdot d = 3/2$.

18. **(D)** The last seven digits of the phone number use seven of the eight digits $\{2, 3, 4, 5, 6, 7, 8, 9\}$, so all but one of these digits is used. The unused digit can be chosen in eight ways. The remaining seven digits are then placed in increasing order to obtain a possible phone number. Thus there are 8 possible phone numbers.

19. **(B)** The percentage of students getting 95 points is

$$100 - 10 - 25 - 20 - 15 = 30,$$

so the mean score on the exam is

$$0.10(70) + 0.25(80) + 0.20(85) + 0.15(90) + 0.30(95) = 86.$$

Since fewer than half of the scores were less than 85, and fewer than half of the scores were greater than 85, the median score is 85. The difference between the mean and the median score on this exam is $86 - 85 = 1$.

20. **(C)** Each digit appears the same number of times in the 1's place, the 10's place, . . ., and the 10,000's place. The average of the digits in each place is

$$\frac{1}{5}(1 + 3 + 5 + 7 + 8) = \frac{24}{5} = 4.8.$$

Hence the average of all the numbers is

$$4.8(1 + 10 + 100 + 1000 + 10000) = 4.8(11111) = 53332.8.$$

21. **(A)** The total number of ways that the numbers can be chosen is $\binom{40}{4}$. Exactly 10 of these possibilities result in the four slips having the same number.

Now we need to determine the number of ways that two slips can have a number a and the other two slips have a number b, with $b \neq a$. There are $\binom{10}{2}$ ways to choose the distinct numbers a and b. For each value of a there are $\binom{4}{2}$ to choose the two slips with a and for each value of b there are $\binom{4}{2}$ to choose the two slips with b. Hence the number of ways that two slips have some number a and the other two slips have some distinct number b is

$$\binom{10}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} = 45 \cdot 6 \cdot 6 = 1620.$$

So the probabilities q and p are $\dfrac{10}{\binom{40}{4}}$ and $\dfrac{1620}{\binom{40}{4}}$, respectively, which implies that

$$\frac{p}{q} = \frac{1620}{10} = 162.$$

22. **(C)** Since

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2},$$

the condition is equivalent to having an integer value for

$$\frac{n!}{n(n+1)/2}.$$

This reduces, when $n \geq 1$, to having an integer value for

$$\frac{2(n-1)!}{n+1}.$$

This fraction is an integer unless $n+1$ is an odd prime. There are 8 odd primes less than or equal to 25, so there are $24 - 8 = 16$ numbers less than or equal to 24 that satisfy the condition.

23. **(C)** First note that $FE = (AB + DC)/2$. Because trapezoids $ABEF$ and $FECD$ have the same height, the ratio of their areas is equal to the ratio of the averages of their parallel sides. Since

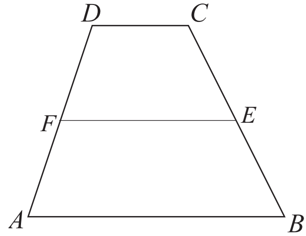
$$AB + \frac{AB + DC}{2} = \frac{3AB + DC}{2}$$

and

$$\frac{AB + DC}{2} + DC = \frac{AB + 3DC}{2},$$

we have

$$3AB + DC = 2(AB + 3DC) = 2AB + 6DC, \quad \text{and} \quad \frac{AB}{DC} = 5.$$



24. **(E)** By the given conditions, it follows that $x > y$. Let $x = 10a + b$ and $y = 10b + a$, where $a > b$. Then

$$m^2 = x^2 - y^2 = (10a + b)^2 - (10b + a)^2 = 99a^2 - 99b^2 = 99(a^2 - b^2).$$

Since $99(a^2 - b^2)$ must be a perfect square,

$$a^2 - b^2 = (a + b)(a - b) = 11k^2,$$

for some positive integer k. Because a and b are distinct digits, we have $a - b \leq 9 - 1 = 8$ and $a + b \leq 9 + 8 = 17$. It follows that $a + b = 11$, $a - b = k^2$, and k is either 1 or 2.

If $k = 2$, then $(a, b) = (15/2, 7/2)$, which is impossible. Thus $k = 1$ and $(a, b) = (6, 5)$. This gives $x = 65$, $y = 56$, $m = 33$, and $x + y + m = 154$.

25. **(C)** Several pairs of numbers from 1 to 100 sum to 125. These pairs are $(25, 100)$, $(26, 99), \ldots, (62, 63)$. Set B can have at most one number from each of these $62 - 25 + 1 = 38$ pairs. In addition, B can contain all of the numbers $1, 2, \ldots, 24$ since these cannot be paired with any of the available numbers to sum to 125. So B has at most $38 + 24 = 62$ numbers. The set containing the first 62 positive integers, for example, is one of these maximum sets.

The

American Mathematics Contest 10 (AMC 10)

Sponsored by

The Mathematical Association of America
University of Nebraska – Lincoln

Contributors

Akamai Foundation
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
Institute for Operations Research and the Management Sciences
L. G. Balfour & Company
Mu Alpha Theta
National Council of Teachers of Mathematics
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
USA Math Talent Search
W. H. Freeman & Company

The MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

57th Annual American Mathematics Contest 10

AMC 10 – Contest A

Solutions Pamphlet

Tuesday, JANUARY 31, 2006

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, the World Wide Web or media of any type is a violation of the competition rules*

Correspondence about the problems and solutions for this AMC 10 should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Prof. Douglas Faires, Department of Mathematics
Youngstown State University, Youngstown, OH 44555-0001

1. **(A)** Five sandwiches cost $5\cdot 3 = 15$ dollars and eight sodas cost $8\cdot 2 = 16$ dollars. Together they cost $15 + 16 = 31$ dollars.

2. **(C)** By the definition we have

$$h \otimes (h \otimes h) = h \otimes (h^3 - h) = h^3 - (h^3 - h) = h.$$

3. **(B)** Mary is $(3/5)(30) = 18$ years old.

4. **(E)** The largest possible sum of the two digits representing the minutes is $5+9 = 14$, occurring at 59 minutes past each hour. The largest possible single digit that can represent the hour is 9. This exceeds the largest possible sum of two digits that can represent the hour, which is $1 + 2 = 3$. Therefore, the largest possible sum of all the digits is $14 + 9 = 23$, occurring at 9:59.

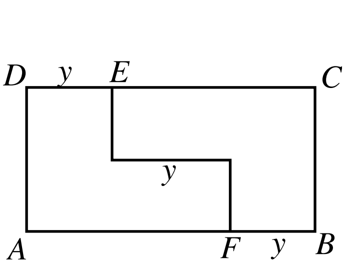
5. **(D)** Each slice of plain pizza cost \$1. Dave paid \$2 for the anchovies in addition to \$5 for the five slices of pizza that he ate, for a total of \$7. Doug paid only \$3 for the three slices of pizza that he ate. Hence Dave paid $7 - 3 = 4$ dollars more than Doug.

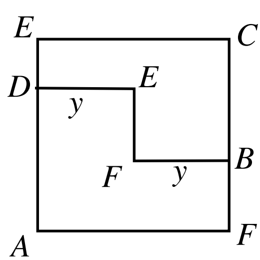
6. **(B)** Take the seventh root of both sides to get $(7x)^2 = 14x$. Then $49x^2 = 14x$, and because $x \neq 0$ we have $49x = 14$. Thus $x = 2/7$.

7. **(A)** Let E represent the end of the cut on $\overline{DC}$, and let F represent the end of the cut on $\overline{AB}$. For a square to be formed, we must have

$$DE = y = FB \quad \text{and} \quad DE + y + FB = 18, \quad \text{so} \quad y = 6.$$

The rectangle that is formed by this cut is indeed a square, since the original rectangle has area $8 \cdot 18 = 144$, and the rectangle that is formed by this cut has a side of length $2 \cdot 6 = 12 = \sqrt{144}$.





8. **(E)** Substitute $(2,3)$ and $(4,3)$ into the equation to give

$$3 = 4 + 2b + c \quad \text{and} \quad 3 = 16 + 4b + c.$$

Subtracting corresponding terms in these equations gives $0 = 12 + 2b$. So

$$b = -6 \quad \text{and} \quad c = 3 - 4 - 2(-6) = 11.$$

OR

The parabola is symmetric about the vertical line through its vertex, and the points $(2,3)$ and $(4,3)$ have the same y-coordinate. The vertex has x-coordinate $(2+4)/2 = 3$, so the equation has the form

$$y = (x-3)^2 + k$$

for some constant k. Since $y = 3$ when $x = 4$, we have $3 = 1^2 + k$ and $k = 2$. Consequently the constant term c is

$$(-3)^2 + k = 9 + 2 = 11.$$

9. **(C)** First note that, in general, the sum of n consecutive integers is n times their median. If the sum is 15, we have the following cases:

   if $n = 2$, then the median is 7.5 and the two integers are 7 and 8;

   if $n = 3$, then the median is 5 and the three integers are 4, 5, and 6;

   if $n = 5$, then the median is 3 and the five integers are 1, 2, 3, 4, and 5.

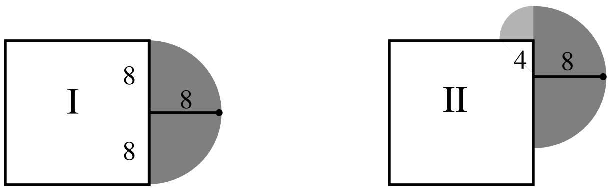
   Because the sum of four consecutive integers is even, 15 cannot be written in such a manner. Also, the sum of more than five consecutive integers must be more than $1+2+3+4+5 = 15$. Hence there are 3 sets satisfying the condition.

   Note: It can be shown that the number of sets of two or more consecutive positive integers having a sum of k is equal to the number of odd positive divisors of k, excluding 1.

10. **(E)** Suppose that $k = \sqrt{120 - \sqrt{x}}$ is an integer. Then $0 \le k \le \sqrt{120}$, and because k is an integer, we have $0 \le k \le 10$. Thus there are 11 possible integer values of k. For each such k, the corresponding value of x is $\left(120 - k^2\right)^2$. Because $\left(120 - k^2\right)^2$ is positive and decreasing for $0 \le k \le 10$, the 11 values of x are distinct.

11. **(C)** The equation $(x+y)^2 = x^2 + y^2$ is equivalent to $x^2 + 2xy + y^2 = x^2 + y^2$, which reduces to $xy = 0$. Thus the graph of the equation consists of the two lines that are the coordinate axes.

12. **(C)** The regions in which the dog can roam for each arrangement are shaded in the figure. For arrangement I, the area of this region is $\frac{1}{2}\pi \cdot 8^2 = 32\pi$ square feet. The area of the shaded region in arrangement II exceeds this by the area of a quarter-circle of radius 4 feet, that is, by $\frac{1}{4}\pi \cdot 4^2 = 4\pi$ square feet.



13. **(D)** Let x represent the amount the player wins if the game is fair. The chance of an even number is $1/2$, and the chance of matching this number on the second roll is $1/6$. So the probability of winning is $(1/2)(1/6) = 1/12$. Therefore $(1/12)x = \$5$ and $x = \$60$.

14. **(B)** The top of the largest ring is 20 cm above its bottom. That point is 2 cm below the top of the next ring, so it is 17 cm above the bottom of the next ring. The additional distances to the bottoms of the remaining rings are $16\,\text{cm}, 15\,\text{cm}, \ldots, 1\,\text{cm}$. Thus the total distance is

$$20 + (17 + 16 + \cdots + 2 + 1) = 20 + \frac{17 \cdot 18}{2} = 20 + 17 \cdot 9 = 173\,\text{cm}.$$

OR

The required distance is the sum of the outside diameters of the 18 rings minus a 2-cm overlap for each of the 17 pairs of consecutive rings. This equals

$$(3+4+5+\cdots+20)-2\cdot 17 = (1+2+3+4+5+\cdots+20)-3-34 = \frac{20 \cdot 21}{2}-37 = 173\,\text{cm}.$$

15. **(D)** Since Odell's rate is $5/6$ that of Kershaw, but Kershaw's lap distance is $6/5$ that of Odell, they each run a lap in the same time. Hence they pass twice each time they circle the track. Odell runs

$$(30\text{ min})\left(250\frac{\text{m}}{\text{min}}\right)\left(\frac{1}{100\pi}\,\frac{\text{laps}}{\text{m}}\right) = \frac{75}{\pi}\text{ laps} \approx 23.87\text{ laps},$$

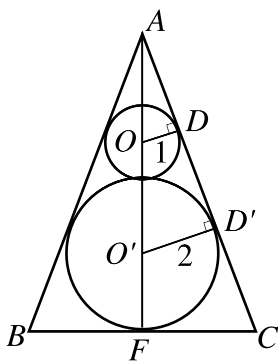
as does Kershaw. Because $23.5 < 23.87 < 24$, they pass each other $2(23.5) = 47$ times.

16. **(D)** Let O and O' denote the centers of the smaller and larger circles, respectively. Let D and D' be the points on $\overline{AC}$ that are also on the smaller and larger circles, respectively. Since $\triangle ADO$ and $\triangle AD'O'$ are similar right triangles, we have

$$\frac{AO}{1} = \frac{AO'}{2} = \frac{AO+3}{2}, \quad \text{so} \quad AO = 3.$$

As a consequence,

$$AD = \sqrt{AO^2 - OD^2} = \sqrt{9-1} = 2\sqrt{2}.$$



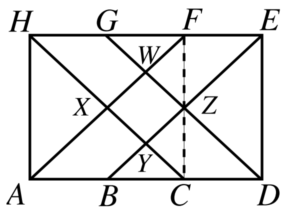
Let F be the midpoint of $\overline{BC}$. Since $\triangle ADO$ and $\triangle AFC$ are similar right triangles, we have

$$\frac{FC}{1} = \frac{AF}{AD} = \frac{AO + OO' + O'F}{AD} = \frac{3+3+2}{2\sqrt{2}} = 2\sqrt{2}.$$

So the area of $\triangle ABC$ is

$$\frac{1}{2} \cdot BC \cdot AF = \frac{1}{2} \cdot 4\sqrt{2} \cdot 8 = 16\sqrt{2}.$$

17. **(A)** First note that since points B and C trisect $\overline{AD}$, and points G and F trisect $\overline{HE}$, we have $HG = GF = FE = AB = BC = CD = 1$. Also, $\overline{HG}$ is parallel to $\overline{CD}$ and $HG = CD$, so $CDGH$ is a parallelogram. Similarly, $\overline{AB}$ is parallel to $\overline{FE}$ and $AB = FE$, so $ABEF$ is a parallelogram. As a consequence, $WXYZ$ is a parallelogram, and since $HG = CD = AB = FE$, it is a rhombus.



Since $AH = AC = 2$, the rectangle $ACFH$ is a square of side length 2. Its diagonals $\overline{AF}$ and $\overline{CH}$ have length $2\sqrt{2}$ and form a right angle at X. As a consequence, $WXYZ$ is a square. In isosceles $\triangle HXF$ we have $HX = XF = \sqrt{2}$. In addition, $HG = \frac{1}{2}HF$. So $XW = \frac{1}{2}XF = \frac{1}{2}\sqrt{2}$, and the square $WXYZ$ has area $XW^2 = 1/2$.

18. **(C)** Since the two letters have to be next to each other, think of them as forming a two-letter word w. So each license plate consists of 4 digits and w. For each digit there are 10 choices. There are $26 \cdot 26$ choices for the letters of w, and there are 5 choices for the position of w. So the total number of distinct license plates is $5 \cdot 10^4 \cdot 26^2$.

19. **(C)** Let $n - d$, n, and $n + d$ be the angles in the triangle. Then

$$180 = n - d + n + n + d = 3n, \quad \text{so} \quad n = 60.$$

Because the sum of the degree measures of two angles of a triangle is less than 180, we have

$$180 > n + (n + d) = 120 + d, \quad \text{which implies that } 0 < d < 60.$$

There are 59 triangles with this property.

20. **(E)** Place each of the integers in a pile based on the remainder when the integer is divided by 5. Since there are only 5 piles but there are 6 integers, at least one of the piles must contain two or more integers. The difference of two integers in the same pile is divisible by 5. Hence the probability is 1.

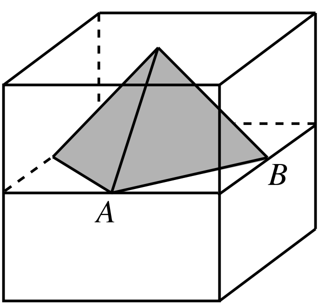
We have applied what is called the Pigeonhole Principle. This states that if you have more pigeons than boxes and you put each pigeon in a box, then at least one of the boxes must have more than one pigeon. In this problem the pigeons are integers and the boxes are piles.

21. **(E)** There are 9000 four-digit positive integers. For those without a 2 or 3, the first digit could be one of the seven numbers 1, 4, 5, 6, 7, 8, or 9, and each of the other digits could be one of the eight numbers 0, 1, 4, 5, 6, 7, 8, or 9. So there are
$$9000 - 7 \cdot 8 \cdot 8 \cdot 8 = 5416$$
four-digit numbers with at least one digit that is a 2 or a 3.

22. **(C)** If a debt of D dollars can be resolved in this way, then integers p and g must exist with
$$D = 300p + 210g = 30(10p + 7g).$$
As a consequence, D must be a multiple of 30, and no positive debt less than \$30 can be resolved. A debt of \$30 can be resolved since
$$30 = 300(-2) + 210(3).$$
This is done by giving 3 goats and receiving 2 pigs.

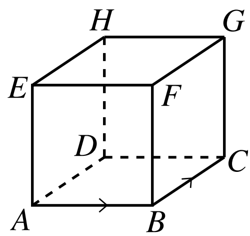
23. **(B)** Radii $\overline{AC}$ and $\overline{BD}$ are each perpendicular to $\overline{CD}$. By the Pythagorean Theorem,
$$CE = \sqrt{5^2 - 3^2} = 4.$$
Because $\triangle ACE$ and $\triangle BDE$ are similar,
$$\frac{DE}{CE} = \frac{BD}{AC}, \quad \text{so} \quad DE = CE \cdot \frac{BD}{AC} = 4 \cdot \frac{8}{3} = \frac{32}{3}.$$
Therefore
$$CD = CE + DE = 4 + \frac{32}{3} = \frac{44}{3}.$$

24. **(B)** Two pyramids with square bases form the octahedron. The upper pyramid is shown.



Since the length of $\overline{AB}$ is $\sqrt{2}/2$, the base area of the pyramid is $(\sqrt{2}/2)^2 = 1/2$. The altitude of the pyramid is 1/2, so its volume is
$$\frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{12}.$$
The volume of the octahedron is $2(1/12) = 1/6$.

25. **(C)** At each vertex there are three possible locations that the bug can travel to in the next move, so the probability that the bug will visit three different vertices after two moves is $2/3$. Label the first three vertices that the bug visits as A to B to C, as shown in the diagram. In order to visit every vertex, the bug must travel from C to either G or D.



The bug travels to G with probability $1/3$, and from there the bug must visit the vertices F, E, H, D in that order. Each of these choices has probability $1/3$ of occurring. So the probability that the path continues in the form

$$C \to G \to F \to E \to H \to D$$

is $\left(\frac{1}{3}\right)^5$.

Alternatively, the bug could travel from C to D with probability $1/3$, and then travel to H, which also occurs with probability $1/3$. From H the bug could go either to G or to E, with probability $2/3$, and from there to the two remaining vertices, each with probability $1/3$. So the probability that the path continues in one of the forms

$$C \to D \to H \begin{array}{l} \nearrow E \to F \to G \\ \searrow G \to F \to E \end{array}$$

is $\left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4$.

Hence the bug will visit every vertex in seven moves with probability

$$\left(\frac{2}{3}\right)\left[\left(\frac{1}{3}\right)^5 + \left(\frac{2}{3}\right)\left(\frac{1}{3}\right)^4\right] = \left(\frac{2}{3}\right)\left(\frac{1}{3}+\frac{2}{3}\right)\left(\frac{1}{3}\right)^4 = \frac{2}{243}.$$

OR

From a given starting point there are 3^7 possible walks of seven moves for the bug, all of them equally likely. If such a walk visits every vertex exactly once, there are three choices for the first move and, excluding a return to the start, two choices for the second. Label the first three vertices visited as $A, B,$ and C, in that order, and label the other vertices as shown. The bug must go to either G or D on its third move. In the first case it must then visit vertices $F, E, H,$ and D in order. In the second case it must visit either $H, E, F,$ and G or $H, G, F,$ and E in order. Thus there are $3 \cdot 2 \cdot 3 = 18$ walks that visit every vertex exactly once, so the required probability is $18/3^7 = 2/243$.

The

AMERICAN MATHEMATICS COMPETITIONS

are Sponsored by

The Mathematical Association of America
University of Nebraska-Lincoln
The Akamai Foundation

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Mu Alpha Theta
National Council of Teachers of Mathematics
National Assessment & Testing
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.

The MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

7th Annual American Mathematics Contest 10

AMC 10 – Contest B

Solutions Pamphlet

Wednesday, FEBRUARY 15, 2006

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, the World Wide Web or media of any type is a violation of the competition rules.*

Correspondence about the problems and solutions for this AMC 10 should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Prof. Douglas Faires, Department of Mathematics
Youngstown State University, Youngstown, OH 44555-0001

1. **(C)** Because

$$(-1)^k = \begin{cases} 1, & \text{if } k \text{ is even,} \\ -1, & \text{if } k \text{ is odd,} \end{cases}$$

the sum can be written as

$$(-1+1)+(-1+1)+\cdots+(-1+1) = 0+0+\cdots+0 = 0.$$

2. **(A)** Because $4 \spadesuit 5 = (4+5)(4-5) = -9$, it follows that

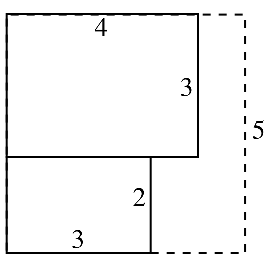
$$3 \spadesuit (4 \spadesuit 5) = 3 \spadesuit (-9) = (3+(-9))(3-(-9)) = (-6)(12) = -72.$$

3. **(A)** Let c and p represent the number of points scored by the Cougars and the Panthers, respectively. The two teams scored a total of 34 points, so $c+p=34$. The Cougars won by 14 points, so $c-p=14$. The solution is $c=24$ and $p=10$, so the Panthers scored 10 points.

4. **(D)** The circle with diameter 3 has area $\pi\left(\frac{3}{2}\right)^2$. The circle with diameter 1 has area $\pi\left(\frac{1}{2}\right)^2$. Therefore the ratio of the blue-painted area to the red-painted area is

$$\frac{\pi\left(\frac{3}{2}\right)^2 - \pi\left(\frac{1}{2}\right)^2}{\pi\left(\frac{1}{2}\right)^2} = 8.$$

5. **(B)** The side length of the square is at least equal to the sum of the smaller dimensions of the rectangles, which is $2+3=5$.



If the rectangles are placed as shown, it is in fact possible to contain them within a square of side length 5. Thus the smallest possible area is $5^2 = 25$.

6. **(D)** Since the square has side length $2/\pi$, the diameter of each circular section is $2/\pi$. The boundary of the region consists of 4 semicircles, whose total perimeter is twice the circumference of a circle having diameter $2/\pi$. Hence the perimeter of the region is

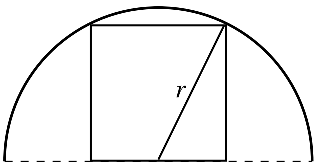
$$2 \cdot \left(\pi \cdot \frac{2}{\pi}\right) = 4.$$

7. **(A)** We have

$$\sqrt{\frac{x}{1-\frac{x-1}{x}}} = \sqrt{\frac{x}{\frac{x-x+1}{x}}} = \sqrt{\frac{x}{\frac{1}{x}}} = \sqrt{x^2} = |x|.$$

When $x < 0$, the given expression is equivalent to $-x$.

8. **(B)**The square has side length $\sqrt{40}$.



Let r be the radius of the semicircle. Then

$$r^2 = \left(\sqrt{40}\right)^2 + \left(\frac{\sqrt{40}}{2}\right)^2 = 40 + 10 = 50,$$

so the area of the semicircle is $\frac{1}{2}\pi r^2 = 25\pi$.

9. **(B)** Francesca's 600 grams of lemonade contains $25 + 386 = 411$ calories, so 200 grams of her lemonade contains $411/3 = 137$ calories.

10. **(A)** Let the sides of the triangle have lengths x, $3x$, and 15. The Triangle Inequality implies that $3x < x + 15$, so $x < 7.5$. Because x is an integer, the greatest possible perimeter is $7 + 21 + 15 = 43$.

11. **(C)** Since $n!$ contains the product $2 \cdot 5 \cdot 10 = 100$ whenever $n \geq 10$, it suffices to determine the tens digit of

$$7! + 8! + 9! = 7!(1 + 8 + 8 \cdot 9) = 5040(1 + 8 + 72) = 5040 \cdot 81.$$

This is the same as the units digit of $4 \cdot 1$, which is 4.

12. **(E)** Substituting $x = 1$ and $y = 2$ into the equations gives

$$1 = \frac{2}{4} + a \quad \text{and} \quad 2 = \frac{1}{4} + b.$$

It follows that

$$a + b = \left(1 - \frac{2}{4}\right) + \left(2 - \frac{1}{4}\right) = 3 - \frac{3}{4} = \frac{9}{4}.$$

OR

Because

$$a = x - \frac{y}{4} \quad \text{and} \quad b = y - \frac{x}{4} \quad \text{we have} \quad a + b = \frac{3}{4}(x + y).$$

Since $x = 1$ when $y = 2$, this implies that $a + b = \frac{3}{4}(1 + 2) = \frac{9}{4}$.

13. **(E)** Joe has 2 ounces of cream in his cup. JoAnn has drunk 2 ounces of the 14 ounces of coffee-cream mixture in her cup, so she has only $12/14 = 6/7$ of her 2 ounces of cream in her cup. Therefore the ratio of the amount of cream in Joe's coffee to that in JoAnn's coffee is

$$\frac{2}{\frac{6}{7}\cdot 2} = \frac{7}{6}.$$

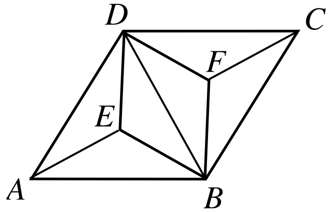
14. **(D)** Since a and b are roots of $x^2 - mx + 2 = 0$, we have

$$x^2 - mx + 2 = (x-a)(x-b) \quad \text{and} \quad ab = 2.$$

In a similar manner, the constant term of $x^2 - px + q$ is the product of $a + (1/b)$ and $b + (1/a)$, so

$$q = \left(a + \frac{1}{b}\right)\left(b + \frac{1}{a}\right) = ab + 1 + 1 + \frac{1}{ab} = \frac{9}{2}.$$

15. **(C)** Since $\angle BAD = 60^\circ$, isosceles $\triangle BAD$ is also equilateral. As a consequence, $\triangle AEB$, $\triangle AED$, $\triangle BED$, $\triangle BFD$, $\triangle BFC$, and $\triangle CFD$ are congruent. These six triangles have equal areas and their union forms rhombus $ABCD$, so each has area $24/6 = 4$. Rhombus $BFDE$ is the union of $\triangle BED$ and $\triangle BFD$, so its area is 8.

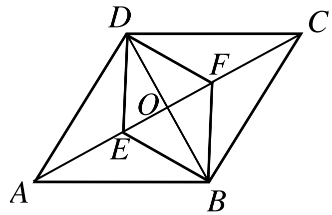


OR

Let the diagonals of rhombus $ABCD$ intersect at O. Since the diagonals of a rhombus intersect at right angles, $\triangle ABO$ is a $30-60-90^\circ$ triangle. Therefore $AO = \sqrt{3}\cdot BO$. Because AO and BO are half the length of the longer diagonals of rhombi $ABCD$ and $BFDE$, respectively, it follows that

$$\frac{\text{Area}(BFDE)}{\text{Area}(ABCD)} = \left(\frac{BO}{AO}\right)^2 = \frac{1}{3}.$$

Thus the area of rhombus $BFDE$ is $(1/3)(24) = 8$.



16. **(E)** In the years from 2004 through 2020, Each Leap Day occurs $3 \cdot 365 + 366 = 1461$ days after the preceding Leap Day. When 1461 is divided by 7 the remainder is 5. So the day of the week advances 5 days for each 4-year cycle. In the four cycles from 2004 to 2020, the Leap Day will advance 20 days. So Leap Day in 2020 will occur one day of the week earlier than in 2004, that is, on a Saturday.

17. **(D)** After Alice puts the ball into Bob's bag, his bag will contain six balls: two of one color and one of each of the other colors. After Bob selects a ball and places it into Alice's bag, the two bags will have the same contents if and only if Bob picked one of the two balls in his bag that are the same color. Because there are six balls in the bag when Bob makes his selection, the probability of selecting one of the same colored pair is $2/6 = 1/3$.

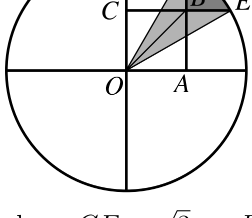
18. **(E)** Note that the first several terms of the sequence are:

$$2,\ 3,\ \frac{3}{2},\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ 2,\ 3,\ \ldots,$$

so the sequence consists of a repeating cycle of 6 terms. Since $2006 = 334 \cdot 6 + 2$, we have $a_{2006} = a_2 = 3$.

19. **(A)** Since $OC = 1$ and $OE = 2$, it follows that $\angle EOC = 60°$ and $\angle EOA = 30°$. The area of the shaded region is the area of the $30°$ sector DOE minus the area of congruent triangles OBD and OBE. First note that

$$\text{Area(Sector } DOE) = \frac{1}{12}(4\pi) = \frac{\pi}{3}.$$



In right triangle OCE, we have $CE = \sqrt{3}$, so $BE = \sqrt{3} - 1$. Therefore

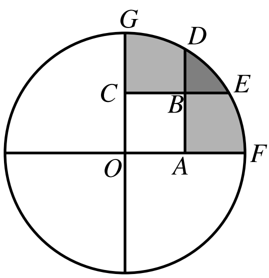
$$\text{Area}(\triangle OBE) = \frac{1}{2}(\sqrt{3} - 1)(1).$$

The required area is consequently

$$\frac{\pi}{3} - 2\left(\frac{\sqrt{3}-1}{2}\right) = \frac{\pi}{3} + 1 - \sqrt{3}.$$

OR

Let F be the point where ray OA intersects the circle, and let G be the point where ray OC intersects the circle.



Let a be the area of the shaded region described in the problem, and b be the area of the region bounded by $\overline{AD}$, $\overline{AF}$, and the minor arc from D to F. Then b is also the area of the region bounded by $\overline{CE}$, $\overline{CG}$, and the minor arc from G to E. By the Inclusion-Exclusion Principle,

$$2b - a = \text{Area (Quartercircle } OFG) - \text{Area (Square } OABC) = \pi - 1.$$

Since b is the area of a 60° sector from which the area of $\triangle OAD$ has been deleted, we have

$$b = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}.$$

Hence the area of the shaded region described in the problem is

$$a = 2b - \pi + 1 = 2\left(\frac{2\pi}{3} - \frac{\sqrt{3}}{2}\right) - \pi + 1 = \frac{\pi}{3} + 1 - \sqrt{3}.$$

20. **(E)** The slope of line AB is $(178-(-22))/(2006-6) = 1/10$. Since the line AD is perpendicular to the line AB, its slope is -10. This implies that

$$-10 = \frac{y-(-22)}{8-6}, \quad \text{so} \quad y = -10(2) - 22 = -42, \quad \text{and} \quad D = (8, -42).$$

As a consequence,

$$AB = \sqrt{2000^2 + 200^2} = 200\sqrt{101} \quad \text{and} \quad AD = \sqrt{2^2 + 20^2} = 2\sqrt{101}.$$

Thus

$$\text{Area}(ABCD) = AB \cdot AD = 400 \cdot 101 = 40{,}400.$$

21. **(C)** On each die the probability of rolling k, for $1 \le k \le 6$, is

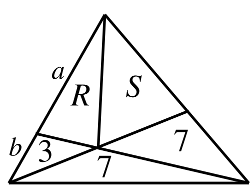
$$\frac{k}{1+2+3+4+5+6} = \frac{k}{21}.$$

There are six ways of rolling a total of 7 on the two dice, represented by the ordered pairs $(1,6)$, $(2,5)$, $(3,4)$, $(4,3)$, $(5,2)$, and $(6,1)$. Thus the probability of rolling a total of 7 is

$$\frac{1\cdot 6+2\cdot 5+3\cdot 4+4\cdot 3+5\cdot 2+6\cdot 1}{21^2} = \frac{56}{21^2} = \frac{8}{63}.$$

22. **(D)** The total cost of the peanut butter and jam is $N(4B+5J) = 253$ cents, so N and $4B+5J$ are factors of $253 = 11\cdot 23$. Because $N > 1$, the possible values of N are 11, 23, and 253. If $N = 253$, then $4B+5J = 1$, which is impossible since B and J are positive integers. If $N = 23$, then $4B+5J = 11$, which also has no solutions in positive integers. Hence $N = 11$ and $4B+5J = 23$, which has the unique positive integer solution $B = 2$ and $J = 3$. So the cost of the jam is $11(3)(5¢) = \$1.65$.

23. **(D)** Partition the quadrilateral into two triangles and let the areas of the triangles be R and S as shown. Then the required area is $T = R+S$.



Let a and b, respectively, be the bases of the triangles with areas R and 3, as indicated. If two triangles have the same altitude, then the ratio of their areas is the same as the ratio of their bases. Thus

$$\frac{a}{b} = \frac{R}{3} = \frac{R+S+7}{3+7}, \quad \text{so} \quad \frac{R}{3} = \frac{T+7}{10}.$$

Similarly,

$$\frac{S}{7} = \frac{S+R+3}{7+7}, \quad \text{so} \quad \frac{S}{7} = \frac{T+3}{14}.$$

Thus

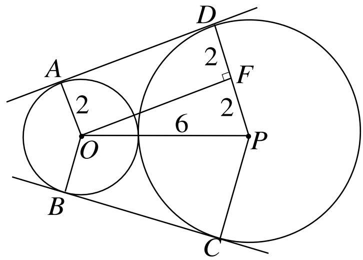
$$T = R+S = 3\left(\frac{T+7}{10}\right) + 7\left(\frac{T+3}{14}\right).$$

From this we obtain

$$10T = 3(T+7) + 5(T+3) = 8T + 36,$$

and it follows that $T = 18$.

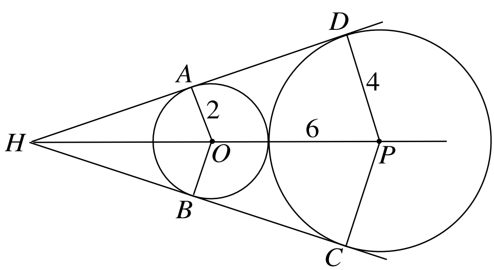
24. **(B)** Through O draw a line parallel to $\overline{AD}$ intersecting $\overline{PD}$ at F.



Then $AOFD$ is a rectangle and OPF is a right triangle. Thus $DF = 2$, $FP = 2$, and $OF = 4\sqrt{2}$. The area of trapezoid $AOPD$ is $12\sqrt{2}$, and the area of hexagon $AOBCPD$ is $2 \cdot 12\sqrt{2} = 24\sqrt{2}$.

OR

Lines $AD, BC,$ and OP intersect at a common point H.



Because $\angle PDH = \angle OAH = 90^\circ$, triangles PDH and OAH are similar with ratio of similarity 2. Thus $2HO = HP = HO + OP = HO + 6$, so $HO = 6$ and $AH = \sqrt{HO^2 - OA^2} = 4\sqrt{2}$. Hence the area of $\triangle OAH$ is $(1/2)(2)(4\sqrt{2}) = 4\sqrt{2}$, and the area of $\triangle PDH$ is $(2^2)(4\sqrt{2}) = 16\sqrt{2}$. The area of the hexagon is twice the area of $\triangle PDH$ minus twice the area of $\triangle OAH$, so it is $24\sqrt{2}$.

25. **(B)** The 4-digit number on the license plate has the form *aabb* or *abab* or *baab*, where a and b are distinct integers from 0 to 9. Because Mr. Jones has a child of age 9, the number on the license plate is divisible by 9. Hence the sum of the digits, $2(a + b)$, is also divisible by 9. Because of the restriction on the digits a and b, this implies that $a + b = 9$. Moreover, since Mr. Jones must have either a 4-year-old or an 8-year-old, the license plate number is divisible by 4. These conditions narrow the possibilities for the number to 1188, 2772, 3636, 5544, 6336, 7272, and 9900. The last two digits of 9900 could not yield Mr. Jones's age, and none of the others is divisible by 5, so he does not have a 5-year-old.

Note that 5544 is divisible by each of the other eight non-zero digits.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
University of Nebraska-Lincoln
The Akamai Foundation

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Mu Alpha Theta
National Council of Teachers of Mathematics
National Assessment & Testing
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.

THE MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

8th Annual American Mathematics Contest 10

AMC 10
CONTEST A

Solutions Pamphlet

Tuesday, FEBRUARY 6, 2007

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Prof. Douglas Faires, Department of Mathematics
Youngstown State University, Youngstown, OH 44555-0001

1. **Answer (C):** Susan pays $(4)(0.75)(20) = 60$ dollars. Pam pays $(5)(0.70)(20) = 70$ dollars, so she pays $70 - 60 = 10$ more dollars than Susan.

2. **Answer (A):** The value of $6@2$ is $6 \cdot 2 - 2^2 = 12 - 4 = 8$, and the value of $6\#2$ is $6 + 2 - 6 \cdot 2^2 = 8 - 24 = -16$. Thus
$$\frac{6@2}{6\#2} = \frac{8}{-16} = -\frac{1}{2}.$$

3. **Answer (D):** The brick has a volume of $40 \cdot 20 \cdot 10 = 8000$ cubic centimeters. Suppose that after the brick is placed in the tank, the water level rises by h centimeters. Then the additional volume occupied in the aquarium is $100 \cdot 40 \cdot h = 4000h$ cubic centimeters. Since this must be the same as the volume of the brick, we have
$$8000 = 4000h \quad \text{and} \quad h = 2 \text{ centimeters}$$

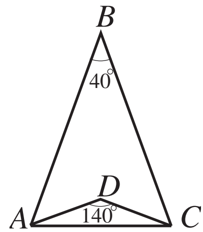
4. **Answer (A):** Let the smaller of the integers be x. Then the larger is $x + 2$. So $x + 2 = 3x$, from which $x = 1$. Thus the two integers are 1 and 3, and their sum is 4.

5. **Answer (B):** Let p be the cost in cents of a pencil and n be the cost in cents of a notebook. Then
$$7p + 8n = 415 \quad \text{and} \quad 5p + 3n = 177.$$
The solution of this pair of equations is $p = 9$ and $n = 44$. So the cost of 16 pencils and 10 notebooks is $16(9) + 10(44) = 584$ cents, or \$5.84.

6. **Answer (A):** Between 2002 and 2003, the increase was
$$\frac{6}{60} = \frac{1}{10} = 10\%.$$
Between the other four pairs of consecutive years, the increases were
$$\frac{4}{66} < \frac{4}{40} = \frac{1}{10}, \quad \frac{6}{70} < \frac{6}{60} = \frac{1}{10}, \quad \frac{2}{76} < \frac{2}{20} = \frac{1}{10}, \quad \text{and} \quad \frac{7}{78} < \frac{7}{70} = \frac{1}{10}.$$
Therefore the largest percentage increase occurred between 2002 and 2003.

7. **Answer (D):** After paying the federal taxes, Mr. Public had 80% of his inheritance money left. He paid 10% of that, or 8% of his inheritance, in state taxes. Hence his total tax bill was 28% of his inheritance, and his inheritance was \$10,500/0.28 = \$37,500.

8. **Answer (D):** Because $\triangle ABC$ is isosceles, $\angle BAC = \frac{1}{2}(180^\circ - \angle ABC) = 70^\circ$.



Similarly,

$$\angle DAC = \frac{1}{2}(180^\circ - \angle ADC) = 20^\circ.$$

Thus $\angle BAD = \angle BAC - \angle DAC = 50^\circ$.

OR

Because $\triangle ABC$ and $\triangle ADC$ are isosceles triangles and $\overline{BD}$ bisects $\angle ABC$ and $\angle ADC$, applying the Exterior Angle Theorem to $\triangle ABD$ gives $\angle BAD = 70^\circ - 20^\circ = 50^\circ$.

9. **Answer (E):** The given equations are equivalent, respectively, to

$$3^a = 3^{4(b+2)} \quad \text{and} \quad 5^{3b} = 5^{a-3}.$$

Therefore $a = 4(b+2)$ and $3b = a - 3$. The solution of this system is $a = -12$ and $b = -5$, so $ab = 60$.

10. **Answer (E):** Let N represent the number of children in the family and T represent the sum of the ages of all the family members. The average age of the members of the family is 20, and the average age of the members when the 48-year-old father is not included is 16, so

$$20 = \frac{T}{N+2} \quad \text{and} \quad 16 = \frac{T-48}{N+1}.$$

This implies that

$$20N + 40 = T \quad \text{and} \quad 16N + 16 = T - 48,$$

so

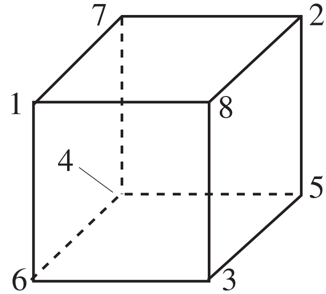
$$20N + 40 = 16N + 64.$$

Hence $4N = 24$ and $N = 6$.

11. **Answer (C):** Each vertex appears on exactly three faces, so the sum of the numbers on all the faces is

$$3(1 + 2 + \cdots + 8) = 3 \cdot \frac{8 \cdot 9}{2} = 108.$$

There are six faces for the cube, so the common sum must be $108/6 = 18$. A possible numbering is shown in the figure.



12. **Answer (D):** The first guide can take any combination of tourists except all the tourists or none of the tourists. Therefore the number of possibilities is

$$\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4} + \binom{6}{5} = 6 + 15 + 20 + 15 + 6 = 62.$$

OR

If each guide did not need to take at least one tourist, then each tourist could choose one of the two guides independently. In this case there would be $2^6 = 64$ possible arrangements. The two arrangements for which all tourists choose the same guide must be excluded, leaving a total of $64 - 2 = 62$ possible arrangements.

13. **Answer (B):** Let w be Yan's walking speed, and let x and y be the distances from Yan to his home and to the stadium, respectively. The time required for Yan to walk to the stadium is y/w, and the time required for him to walk home is x/w. Because he rides his bicycle at a speed of $7w$, the time required for him to ride his bicycle from his home to the stadium is $(x+y)/(7w)$. Thus

$$\frac{y}{w} = \frac{x}{w} + \frac{x+y}{7w} = \frac{8x+y}{7w}.$$

As a consequence, $7y = 8x + y$, so $8x = 6y$. The required ratio is $x/y = 6/8 = 3/4$.

OR

Because we are interested only in the ratio of the distances, we may assume that the distance from Yan's home to the stadium is 1 mile. Let x be his present distance from his home. Imagine that Yan has a twin, Nay. While Yan walks to the stadium, Nay walks to their home and continues 1/7 of a mile past their home. Because walking 1/7 of a mile requires the same amount of time as riding 1 mile, Yan and Nay will complete their trips at the same time. Yan has walked $1-x$ miles while Nay has walked $x + \frac{1}{7}$ miles, so $1 - x = x + \frac{1}{7}$. Thus $x = 3/7$, $1 - x = 4/7$, and the required ratio is $x/(1-x) = 3/4$.

14. **Answer (A):** Let the sides of the triangle have lengths $3x$, $4x$, and $5x$. The triangle is a right triangle, so its hypotenuse is a diameter of the circle. Thus $5x = 2 \cdot 3 = 6$, so $x = 6/5$. The area of the triangle is

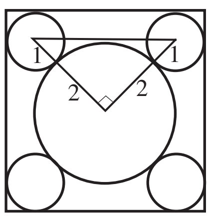
$$\frac{1}{2} \cdot 3x \cdot 4x = \frac{1}{2} \cdot \frac{18}{5} \cdot \frac{24}{5} = \frac{216}{25} = 8.64.$$

OR

A right triangle with side lengths 3, 4, and 5 has area $(1/2)(3)(4) = 6$. Because the given right triangle is inscribed in a circle with diameter 6, the hypotenuse of this triangle has length 6. Thus the sides of the given triangle are $6/5$ as long as those of a $3-4-5$ triangle, and its area is $(6/5)^2$ times that of a $3-4-5$ triangle. The area of the given triangle is

$$\left(\frac{6}{5}\right)^2 (6) = \frac{216}{25} = 8.64.$$

15. **Answer (B):** Let s be the length of a side of the square. Consider an isosceles right triangle with vertices at the centers of the circle of radius 2 and two of the circles of radius 1. This triangle has legs of length 3, so its hypotenuse has length $3\sqrt{2}$.



The length of a side of the square is 2 more than the length of this hypotenuse, so $s = 2 + 3\sqrt{2}$. Hence the area of the square is

$$s^2 = (2 + 3\sqrt{2})^2 = 22 + 12\sqrt{2}.$$

OR

The distance from a vertex of the square to the center of the nearest small circle is $\sqrt{1^2 + 1^2} = \sqrt{2}$, and the distance between the centers of two small circles in opposite corners of the square is $1 + 4 + 1 = 6$. Therefore each diagonal of the square has length $6 + 2\sqrt{2}$, and each side has length

$$s = \frac{6 + 2\sqrt{2}}{\sqrt{2}} = 2 + 3\sqrt{2}.$$

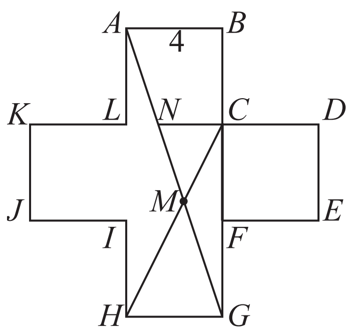
The area of the square is consequently $s^2 = \left(2 + 3\sqrt{2}\right)^2 = 22 + 12\sqrt{2}$.

16. **Answer (E):** The number $ad - bc$ is even if and only if ad and bc are both odd or are both even. Each of ad and bc is odd if both of its factors are odd, and even otherwise. Exactly half of the integers from 0 to 2007 are odd, so each of ad and bc is odd with probability $(1/2) \cdot (1/2) = 1/4$ and are even with probability $3/4$. Hence the probability that $ad - bc$ is even is

$$\frac{1}{4} \cdot \frac{1}{4} + \frac{3}{4} \cdot \frac{3}{4} = \frac{5}{8}.$$

17. **Answer (D):** An integer is a cube if and only if, in the prime factorization of the number, each prime factor occurs a multiple of three times. Because $n^3 = 75m = 3 \cdot 5^2 \cdot m$, the minimum value for m is $3^2 \cdot 5 = 45$. In that case $n = 15$, and $m + n = 60$.

18. **Answer (C):** Extend $\overline{CD}$ past C to meet $\overline{AG}$ at N.



Since $\triangle ABG$ is similar to $\triangle NCG$,

$$NC = AB \cdot \frac{CG}{BG} = 4 \cdot \frac{8}{12} = \frac{8}{3}.$$

This implies that trapezoid $ABCN$ has area

$$\frac{1}{2} \cdot \left(\frac{8}{3} + 4\right) \cdot 4 = \frac{40}{3}.$$

Let v denote the length of the perpendicular from M to $\overline{NC}$. Since $\triangle CMN$ is similar to $\triangle HMG$, and

$$\frac{GH}{NC} = \frac{4}{8/3} = \frac{3}{2},$$

the length of the perpendicular from M to $\overline{HG}$ is $\frac{3}{2}v$. Because

$$v + \frac{3}{2}v = 8, \quad \text{we have} \quad v = \frac{16}{5}.$$

Hence the area of $\triangle CMN$ is

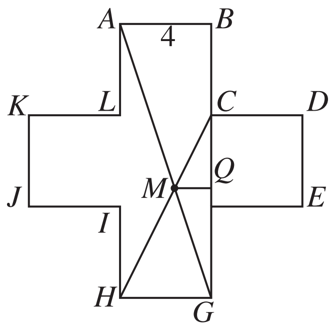
$$\frac{1}{2} \cdot \frac{8}{3} \cdot \frac{16}{5} = \frac{64}{15}.$$

So

$$\text{Area}(ABCM) = \text{Area}(ABCN) + \text{Area}(\triangle CMN) = \frac{40}{3} + \frac{64}{15} = \frac{88}{5}.$$

OR

Let Q be the foot of the perpendicular from M to $\overline{BG}$.



Since $\triangle MQG$ is similar to $\triangle ABG$, we have

$$\frac{MQ}{QG} = \frac{AB}{BG} = \frac{4}{12} = \frac{1}{3}.$$

Also, $\triangle MCQ$ is similar to $\triangle HCG$, so

$$\frac{MQ}{CQ} = \frac{HG}{CG} = \frac{4}{8} = \frac{1}{2}.$$

Thus

$$QG = 3MQ = 3\left(\frac{1}{2}CQ\right) = \frac{3}{2}(8 - QG),$$

which implies that

$$QG = \frac{24}{5} \quad \text{and} \quad MQ = \frac{1}{3}QG = \frac{8}{5}.$$

Hence

$$\text{Area}(ABCM) = \text{Area}(\triangle ABG) - \text{Area}(\triangle CMG) = \frac{1}{2}\cdot 4 \cdot 12 - \frac{1}{2}\cdot 8 \cdot \frac{8}{5} = \frac{88}{5}.$$

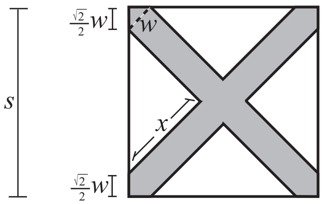
19. **Answer (C):** Let s be the side length of the square, let w be the width of the brush, and let x be the leg length of one of the congruent unpainted isosceles right triangles. Since the unpainted area is half the area of the square, the area of each unpainted triangle is 1/8 of the area of the square. So

$$\frac{1}{2}x^2 = \frac{1}{8}s^2 \quad \text{and} \quad x = \frac{1}{2}s.$$

The leg length x plus the brush width w is equal to half the diagonal of the square, so $x + w = (\sqrt{2}/2)s$. Thus

$$w = \frac{\sqrt{2}}{2}s - \frac{1}{2}s \quad \text{and} \quad \frac{s}{w} = \frac{2}{\sqrt{2}-1} = 2\sqrt{2}+2.$$

OR



The painted stripes have isosceles right triangles with hypotenuse w at each vertex of the square, and the legs of these triangles have length $(\sqrt{2}/2)w$. Since the total area of the four congruent unpainted triangles is half the area of the original square, we have

$$s - \sqrt{2}w = \frac{s}{\sqrt{2}}, \quad \text{so} \quad \sqrt{2}s - 2w = s.$$

and

$$\frac{s}{w} = \frac{2}{\sqrt{2}-1} = 2\sqrt{2}+2.$$

20. **Answer (D):** Squaring each side of the equation $4 = a + a^{-1}$ gives

$$16 = a^2 + 2a \cdot a^{-1} + \left(a^{-1}\right)^2 = a^2 + 2 + a^{-2}, \quad \text{so} \quad 14 = a^2 + a^{-2}.$$

Squaring again gives

$$196 = a^4 + 2a^2 \cdot a^{-2} + \left(a^{-2}\right)^2 = a^4 + 2 + a^{-4}, \quad \text{so} \quad 194 = a^4 + a^{-4}.$$

21. **Answer (C):** Since the surface area of the original cube is 24 square meters, each face of the cube has a surface area of $24/6 = 4$ square meters, and the side length of this cube is 2 meters. The sphere inscribed within the cube has diameter 2 meters, which is also the length of the diagonal of the cube inscribed in the sphere. Let l represent the side length of the inscribed cube. Applying the Pythagorean Theorem twice gives

$$l^2 + l^2 + l^2 = 2^2 = 4.$$

Hence each face has surface area

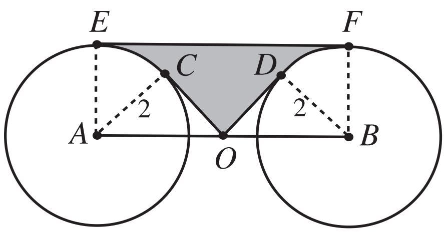
$$l^2 = \frac{4}{3} \text{ square meters.}$$

So the surface area of the inscribed cube is $6 \cdot (4/3) = 8$ square meters.

22. **Answer (D):** A given digit appears as the hundreds digit, the tens digit, and the units digit of a term the same number of times. Let k be the sum of the units digits in all the terms. Then $S = 111k = 3 \cdot 37k$, so S must be divisible by 37. To see that S need not be divisible by any larger prime, note that the sequence 123, 231, 312 gives $S = 666 = 2 \cdot 3^2 \cdot 37$.

23. **Answer (B):** Let x and y be, respectively, the larger and smaller of the integers. Then $96 = x^2 - y^2 = (x+y)(x-y)$. Because 96 is even, x and y are both even or are both odd. In either case $x+y$ and $x-y$ are both even. Hence there are four possibilities for $(x+y, x-y)$, which are $(48,2)$, $(24,4)$, $(16,6)$, and $(12,8)$. The four corresponding values of (x,y) are $(25,23)$, $(14,10)$, $(11,5)$, and $(10,2)$.

24. **Answer (B):** Rectangle $ABFE$ has area $AE \cdot AB = 2 \cdot 4\sqrt{2} = 8\sqrt{2}$. Right triangles ACO and BDO each have hypotenuse $2\sqrt{2}$ and one leg of length 2.



Hence they are each isosceles, and each has area $(1/2)\left(2^2\right) = 2$. Angles CAE and DBF are each $45°$, so sectors CAE and DBF each have area

$$\frac{1}{8} \cdot \pi \cdot 2^2 = \frac{\pi}{2}.$$

Thus the area of the shaded region is

$$8\sqrt{2} - 2 \cdot 2 - 2 \cdot \frac{\pi}{2} = 8\sqrt{2} - 4 - \pi.$$

25. **Answer (D):** If $n \le 2007$, then $S(n) \le S(1999) = 28$. If $n \le 28$, then $S(n) \le S(28) = 10$. Therefore if n satisfies the required condition it must also satisfy

$$n \ge 2007 - 28 - 10 = 1969.$$

In addition, $n, S(n),$ and $S(S(n))$ all leave the same remainder when divided by 9. Because 2007 is a multiple of 9, it follows that $n, S(n),$ and $S(S(n))$ must all be multiples of 3. The required condition is satisfied by 4 multiples of 3 between 1969 and 2007, namely 1977, 1980, 1983, and 2001.

Note: There appear to be many cases to check, that is, all the multiples of 3 between 1969 and 2007. However, for $1987 \le n \le 1999$, we have $n + S(n) \ge 1990 + 19 = 2009$, so these numbers are eliminated. Thus we need only check 1971, 1974, 1977, 1980, 1983, 1986, 2001, and 2004.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Mu Alpha Theta
National Assessment & Testing
National Council of Teachers of Mathematics
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.

THE MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

8th Annual American Mathematics Contest 10

AMC 10
CONTEST B

Solutions Pamphlet

Wednesday, FEBRUARY 21, 2007

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Prof. Douglas Faires, Department of Mathematics
Youngstown State University, Youngstown, OH 44555-0001

1. **Answer (E):** The perimeter of each bedroom is $2(12 + 10) = 44$ feet, so the surface to be painted in each bedroom has an area of $44 \cdot 8 - 60 = 292$ square feet. Since there are 3 bedrooms, Isabella must paint $3 \cdot 292 = 876$ square feet.

2. **Answer (E):** Since $3\star5 = (3+5)5 = 8\cdot5 = 40$ and $5\star3 = (5+3)3 = 8\cdot3 = 24$, we have
$$3 \star 5 - 5 \star 3 = 40 - 24 = 16.$$

3. **Answer (B):** The student used $120/30 = 4$ gallons on the trip home and $120/20 = 6$ gallons on the trip back to school. So the average gas mileage for the round trip was
$$\frac{240 \text{ miles}}{10 \text{ gallons}} = 24 \text{ miles per gallon.}$$

4. **Answer (D):** Since $OA = OB = OC$, triangles AOB, BOC, and COA are all isosceles. Hence
$$\angle ABC = \angle ABO + \angle OBC = \frac{180^\circ - 140^\circ}{2} + \frac{180^\circ - 120^\circ}{2} = 50^\circ.$$

OR

Since
$$\angle AOC = 360^\circ - 140^\circ - 120^\circ = 100^\circ,$$
the Central Angle Theorem implies that
$$\angle ABC = \frac{1}{2}\angle AOC = 50^\circ.$$

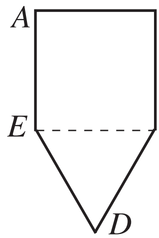
5. **Answer (D):** Let A, B, C, and D represent the following statements about a person in the land.
$$A : \text{Is an Arog.} \qquad B : \text{Is a Braf.} \qquad C : \text{Is a Crup.} \qquad D : \text{Is a Dramp.}$$
Then the statement in the first sentence of the problem can be expressed as:
$$A \implies B, \qquad C \implies B, \qquad D \implies A \qquad \text{and} \qquad C \implies D.$$
The most we can conclude is that
$$C \implies D \implies A \implies B.$$
So the only statement listed that we are certain is true is that Crups are both Arogs and Brafs.

6. **Answer (D):** Sarah will receive 4.5 points for the three questions she leaves unanswered, so she must earn at least $100 - 4.5 = 95.5$ points on the first 22 problems. Because

$$15 < \frac{95.5}{6} < 16,$$

she must solve at least 16 of the first 22 problems correctly. This would give her a score of 100.5.

7. **Answer (E):** Because $AB = BC = EA$ and $\angle A = \angle B = 90°$, quadrilateral $ABCE$ is a square, so $\angle AEC = 90°$.



Also $CD = DE = EC$, so $\triangle CDE$ is equilateral and $\angle CED = 60°$. Therefore

$$\angle E = \angle AEC + \angle CED = 90° + 60° = 150°.$$

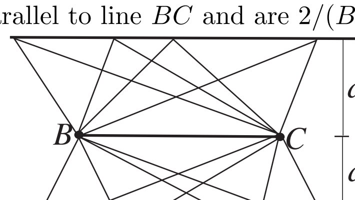
8. **Answer (D):** Once a and c are chosen, the integer b is determined. For $a = 0$, we could have $c = 2$, 4, 6, or 8. For $a = 2$, we could have $c = 4$, 6, or 8. For $a = 4$, we could have $c = 6$ or 8, and for $a = 6$ the only possibility is $c = 8$. Thus there are $1 + 2 + 3 + 4 = 10$ possibilities when a is even. Similarly, there are 10 possibilities when a is odd, so the number of possibilities is 20.

9. **Answer (D):** The last s is the 12th appearance of this letter in the message, so it will be replaced by the letter that is

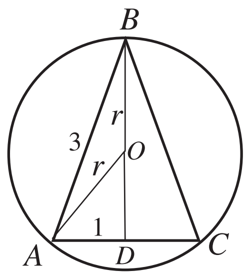
$$1 + 2 + 3 + \cdots + 12 = \frac{1}{2}(12 \cdot 13) = 3 \cdot 26$$

letters to the right of s. Since the alphabet has 26 letters, this letter s is coded as s.

10. **Answer (A):** If the altitude from A has length d, then $\triangle ABC$ has area $(1/2)(BC)d$. The area is 1 if and only if $d = 2/(BC)$. Thus S consists of the two lines that are parallel to line BC and are $2/(BC)$ units from it, as shown.



11. **Answer (C):** Let $\overline{BD}$ be an altitude of the isosceles $\triangle ABC$, and let O denote the center of the circle with radius r that passes through A, B, and C, as shown.



Then

$$BD = \sqrt{3^2 - 1^2} = 2\sqrt{2} \quad \text{and} \quad OD = 2\sqrt{2} - r.$$

Since $\triangle ADO$ is a right triangle, we have

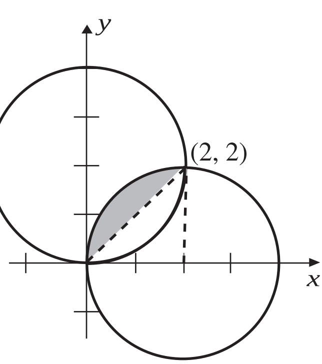
$$r^2 = 1^2 + \left(2\sqrt{2} - r\right)^2 = 1 + 8 - 4\sqrt{2}r + r^2, \quad \text{and} \quad r = \frac{9}{4\sqrt{2}} = \frac{9}{8}\sqrt{2}.$$

As a consequence, the circle has area

$$\left(\frac{9}{8}\sqrt{2}\right)^2 \pi = \frac{81}{32}\pi.$$

12. **Answer (D):** Tom's age N years ago was $T - N$. The sum of his three children's ages at that time was $T - 3N$. Therefore $T - N = 2(T - 3N)$, so $5N = T$ and $T/N = 5$. The conditions of the problem can be met, for example, if Tom's age is 30 and the ages of his children are 9, 10, and 11. In that case $T = 30$ and $N = 6$.

13. **Answer (D):** The two circles intersect at $(0,0)$ and $(2,2)$, as shown.



Half of the region described is formed by removing an isosceles right triangle of leg length 2 from a quarter of one of the circles. Because the quarter-circle has area $(1/4)\pi(2)^2 = \pi$ and the triangle has area $(1/2)(2)^2 = 2$, the area of the region is $2(\pi - 2)$.

14. **Answer (C):** Let g be the number of girls and b the number of boys initially in the group. Then $g = 0.4(g + b)$. After two girls leave and two boys arrive, the size of the entire group is unchanged, so $g - 2 = 0.3(g + b)$. The solution of the system of equations

$$g = 0.4(g + b) \quad \text{and} \quad g - 2 = 0.3(g + b)$$

is $g = 8$ and $b = 12$, so there were initially 8 girls.

OR

After two girls leave and two boys arrive, the size of the group is unchanged. So the two girls who left represent $40\% - 30\% = 10\%$ of the group. Thus the size of the group is 20, and the original number of girls was 40% of 20, or 8.

15. **Answer (D):** Let x be the degree measure of $\angle A$. Then the degree measures of angles B, C, and D are $x/2$, $x/3$, and $x/4$, respectively. The degree measures of the four angles have a sum of 360, so

$$360 = x + \frac{x}{2} + \frac{x}{3} + \frac{x}{4} = \frac{25x}{12}.$$

Thus $x = (12 \cdot 360)/25 = 172.8 \approx 173$.

16. **Answer (C):** Let N be the number of students in the class. Then there are $0.1N$ juniors and $0.9N$ seniors. Let s be the score of each junior. The scores totaled $84N = 83(0.9N) + s(0.1N)$, so

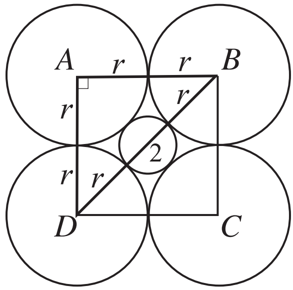
$$s = \frac{84N - 83(0.9N)}{0.1N} = 93.$$

Note: In this problem, we could assume that the class has one junior and nine seniors. Then

$$9 \cdot 83 + s = 10 \cdot 84 = 9 \cdot 84 + 84 \quad \text{and} \quad s = 9(84 - 83) + 84 = 93.$$

17. **Answer (D):** Let the side length of $\triangle ABC$ be s. Then the areas of $\triangle APB$, $\triangle BPC$, and $\triangle CPA$ are, respectively, $s/2$, s, and $3s/2$. The area of $\triangle ABC$ is the sum of these, which is $3s$. The area of $\triangle ABC$ may also be expressed as $(\sqrt{3}/4)s^2$, so $3s = (\sqrt{3}/4)s^2$. The unique positive solution for s is $4\sqrt{3}$.

18. **Answer (B):** Construct the square $ABCD$ by connecting the centers of the large circles, as shown, and consider the isosceles right $\triangle BAD$.



Since $AB = AD = 2r$ and $BD = 2 + 2r$, we have $2(2r)^2 = (2 + 2r)^2$. So

$$1 + 2r + r^2 = 2r^2, \quad \text{and} \quad r^2 - 2r - 1 = 0.$$

Applying the quadratic formula gives $r = 1 + \sqrt{2}$.

19. **Answer (C):** The first remainder is even with probability $2/6 = 1/3$ and odd with probability $2/3$. The second remainder is even with probability $3/6 = 1/2$ and odd with probability $1/2$. The shaded squares are those that indicate that both remainders are odd or both are even. Hence the square is shaded with probability

$$\frac{1}{3} \cdot \frac{1}{2} + \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{2}.$$

20. **Answer (C):** After one of the 25 blocks is chosen, 16 of the remaining blocks do not share its row or column. After the second block is chosen, 9 of the remaining blocks do not share a row or column with either of the first two. Because the three blocks can be chosen in any order, the number of different combinations is

$$\frac{25 \cdot 16 \cdot 9}{3!} = 25 \cdot 8 \cdot 3 = 600.$$

21. **Answer (B):** Let s be the side length of the square, and let h be the length of the altitude of $\triangle ABC$ from B. Because $\triangle ABC$ and $\triangle WBZ$ are similar, it follows that

$$\frac{h - s}{s} = \frac{h}{AC} = \frac{h}{5}, \quad \text{so} \quad s = \frac{5h}{5 + h}.$$

Because $h = 3 \cdot 4/5 = 12/5$, the side length of the square is

$$s = \frac{5(12/5)}{5 + 12/5} = \frac{60}{37}.$$

OR

Because $\triangle WBZ$ is similar to $\triangle ABC$, we have

$$BZ = \frac{4}{5}s \quad \text{and} \quad CZ = 4 - \frac{4}{5}s.$$

Because $\triangle ZYC$ is similar to $\triangle ABC$, we have

$$\frac{s}{4-(4/5)s} = \frac{3}{5}.$$

Thus

$$5s = 12 - \frac{12}{5}s \quad \text{and} \quad s = \frac{60}{37}.$$

22. **Answer (B):** The probability of the number appearing 0, 1, and 2 times is

$$P(0) = \frac{3}{4}\cdot\frac{3}{4} = \frac{9}{16}, \quad P(1) = 2\cdot\frac{1}{4}\cdot\frac{3}{4} = \frac{6}{16}, \quad \text{and} \quad P(2) = \frac{1}{4}\cdot\frac{1}{4} = \frac{1}{16},$$

respectively. So the expected return, in dollars, to the player is

$$P(0)\cdot(-1) + P(1)\cdot(1) + P(2)\cdot(2) = \frac{-9+6+2}{16} = -\frac{1}{16}.$$

23. **Answer (E):** Let h be the altitude of the original pyramid. Then the altitude of the smaller pyramid is $h-2$. Because the two pyramids are similar, the ratio of their altitudes is the square root of the ratio of their surface areas. Thus $h/(h-2) = \sqrt{2}$, so

$$h = \frac{2\sqrt{2}}{\sqrt{2}-1} = 4 + 2\sqrt{2}.$$

24. **Answer (C):** Since n is divisible by 9, the sum of the digits of n must be a multiple of 9. At least one digit of n is 4, so at least nine digits must be 4, and at least one digit must be 9. For n to be divisible by 4, the last two digits of n must each be 4. These conditions are satisfied by several ten-digit numbers, of which the smallest is 4,444,444,944.

25. **Answer (A):** Let $u = a/b$. Then the problem is equivalent to finding all positive rational numbers u such that

$$u + \frac{14}{9u} = k$$

for some integer k. This equation is equivalent to $9u^2 - 9uk + 14 = 0$, whose solutions are

$$u = \frac{9k \pm \sqrt{81k^2 - 504}}{18} = \frac{k}{2} \pm \frac{1}{6}\sqrt{9k^2 - 56}.$$

Hence u is rational if and only if $\sqrt{9k^2 - 56}$ is rational, which is true if and only if $9k^2 - 56$ is a perfect square. Suppose that $9k^2 - 56 = s^2$ for some positive integer s. Then $(3k - s)(3k + s) = 56$. The only factors of 56 are 1, 2, 4, 7, 8, 14, 28, and 56, so $(3k - s, 3k + s)$ is one of the ordered pairs $(1, 56)$, $(2, 28)$, $(4, 14)$, or $(7, 8)$. The cases $(1, 56)$ and $(7, 8)$ yield no integer solutions. The cases $(2, 28)$ and $(4, 14)$ yield $k = 5$ and $k = 3$, respectively. If $k = 5$, then $u = 1/3$ or $u = 14/3$. If $k = 3$, then $u = 2/3$ or $u = 7/3$. Therefore there are four pairs (a, b) that satisfy the given conditions, namely $(1, 3)$, $(2, 3)$, $(7, 3)$, and $(14, 3)$.

OR

Rewrite the equation

$$\frac{a}{b} + \frac{14b}{9a} = k$$

in two different forms. First, multiply both sides by b and subtract a to obtain

$$\frac{14b^2}{9a} = bk - a.$$

Because a, b, and k are integers, $14b^2$ must be a multiple of a, and because a and b have no common factors greater than 1, it follows that 14 is divisible by a. Next, multiply both sides of the original equation by $9a$ and subtract $14b$ to obtain

$$\frac{9a^2}{b} = 9ak - 14b.$$

This shows that $9a^2$ is a multiple of b, so 9 must be divisible by b. Thus if (a, b) is a solution, then $b = 1$, 3, or 9, and $a = 1$, 2, 7, or 14. This gives a total of twelve possible solutions (a, b), each of which can be checked quickly. The only such pairs for which

$$\frac{a}{b} + \frac{14b}{9a}$$

is an integer are when (a, b) is $(1, 3)$, $(2, 3)$, $(7, 3)$, or $(14, 3)$.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Mu Alpha Theta
National Assessment & Testing
National Council of Teachers of Mathematics
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.

THE MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

9th Annual American Mathematics Contest 10

AMC 10 CONTEST A

Solutions Pamphlet

Tuesday, FEBRUARY 12, 2008

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Chair: LeRoy Wenstrom, Columbus, MS

1. **Answer (D):** The machine worked for 2 hours and 40 minutes, or 160 minutes, to complete one third of the job, so the entire job will take $3 \cdot 160 = 480$ minutes, or 8 hours. Hence the doughnut machine will complete the job at 4:30 PM.

2. **Answer (A):** Let x be the side length of the square. Then the area of the square is x^2. The rectangle has sides of length $2x$ and $4x$, and hence area $8x^2$. The fraction of the rectangle's area inside the square is $\frac{x^2}{8x^2} = \frac{1}{8}$ or 12.5%.

3. **Answer (A):** The positive divisors of 6, other than 6, are 1, 2, and 3, so $<6> = 1 + 2 + 3 = 6$. As a consequence, we also have $<<<6>>> = 6$.

   Note: A positive integer whose divisors other than itself add up to that positive integer is called a perfect number. The two smallest perfect numbers are 6 and 28.

4. **Answer (C):** Note that $\frac{2}{3}$ of 10 bananas is $\frac{20}{3}$ bananas, which are worth as much as 8 oranges. So one banana is worth as much as $8 \cdot \frac{3}{20} = \frac{6}{5}$ oranges. Therefore $\frac{1}{2}$ of 5 bananas are worth as much as $\frac{5}{2} \cdot \frac{6}{5} = 3$ oranges.

5. **Answer (B):** Because each denominator except the first can be canceled with the previous numerator, the product is $\frac{2008}{4} = 502$.

6. **Answer (D):** Let x be the length of one segment, in kilometers. To complete the race, the triathlete takes

$$\frac{x}{3} + \frac{x}{20} + \frac{x}{10} = \frac{29}{60}x$$

   hours to cover the distance of $3x$ kilometers. The average speed is therefore

$$\frac{3x}{\frac{29}{60}x} \approx 6 \text{ kilometers per hour.}$$

7. **Answer (E):** First note that

$$\frac{\left(3^{2008}\right)^2 - \left(3^{2006}\right)^2}{\left(3^{2007}\right)^2 - \left(3^{2005}\right)^2} = \frac{9^{2008} - 9^{2006}}{9^{2007} - 9^{2005}}.$$

   Factoring 9^{2005} from each of the terms on the right side produces

$$\frac{9^{2008} - 9^{2006}}{9^{2007} - 9^{2005}} = \frac{9^{2005} \cdot 9^3 - 9^{2005} \cdot 9^1}{9^{2005} \cdot 9^2 - 9^{2005} \cdot 1} = \frac{9^{2005}}{9^{2005}} \cdot \frac{9^3 - 9}{9^2 - 1} = 9 \cdot \frac{9^2 - 1}{9^2 - 1} = 9.$$

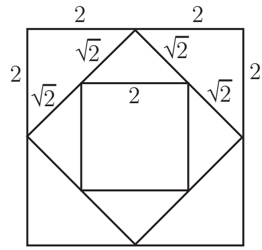
8. **Answer (A):** Let x denote the sticker price, in dollars. Heather pays $0.85x - 90$ dollars at store A and would have paid $0.75x$ dollars at store B. Thus the sticker price x satisfies $0.85x - 90 = 0.75x - 15$, so $x = 750$.

9. **Answer (B):** Because

$$\frac{2x}{3} - \frac{x}{6} = \frac{x}{2}$$

is an integer, x must be even. The case $x = 4$ shows that x is not necessarily a multiple of 3 and that none of the other statements must be true.

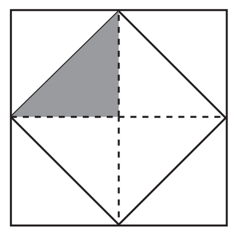
10. **Answer (E):** The sides of S_1 have length 4, so by the Pythagorean Theorem the sides of S_2 have length $\sqrt{2^2 + 2^2} = 2\sqrt{2}$. By similar reasoning the sides of S_3 have length $\sqrt{(\sqrt{2})^2 + (\sqrt{2})^2} = 2$. Thus the area of S_3 is $2^2 = 4$.



OR

Connect the midpoints of the opposite sides of S_1. This cuts S_1 into 4 congruent squares as shown. Each side of S_2 cuts one of these squares into two congruent triangles, one inside S_2 and one outside.

Thus the area of S_2 is half that of S_1. By similar reasoning, the area of S_3 is half that of S_2, and one fourth that of S_1.



11. **Answer (D):** At the rate of 4 miles per hour, Steve can row a mile in 15 minutes. During that time $15 \cdot 10 = 150$ gallons of water will enter the boat. LeRoy must bail $150 - 30 = 120$ gallons of water during that time. So he must bail at the rate of at least $\frac{120}{15} = 8$ gallons per minute.

OR

Steve must row for 15 minutes to reach the shore, so the amount of water in the boat can increase by at most $\frac{30}{15} = 2$ gallons per minute. Therefore LeRoy must bail out at least $10 - 2 = 8$ gallons per minute.

12. **Answer (C):** Let b and g represent the number of blue and green marbles, respectively. Then $r = 1.25b$ and $g = 1.6r$. Thus the total number of red, blue, and green marbles is

$$r + b + g = r + \frac{r}{1.25} + 1.6r = r + 0.8r + 1.6r = 3.4r.$$

13. **Answer (D):** In one hour Doug can paint $\frac{1}{5}$ of the room, and Dave can paint $\frac{1}{7}$ of the room. Working together, they can paint $\frac{1}{5}+\frac{1}{7}$ of the room in one hour. It takes them t hours to do the job, but because they take an hour for lunch, they work for only $t-1$ hours. The fraction of the room that they paint in this time is

$$\left(\frac{1}{5}+\frac{1}{7}\right)(t-1),$$

which must be equal to 1. It may be checked that the solution, $t=\frac{47}{12}$, does not satisfy the equation in any of the other answer choices.

14. **Answer (D):** Let h and w be the height and width of the screen, respectively, in inches. By the Pythagorean Theorem, $h{:}w{:}27 = 3{:}4{:}5$, so

$$h=\frac{3}{5}\cdot 27 = 16.2 \quad \text{and} \quad w=\frac{4}{5}\cdot 27 = 21.6.$$

The height of the non-darkened portion of the screen is half the width, or 10.8 inches. Therefore the height of each darkened strip is

$$\frac{1}{2}(16.2-10.8) = 2.7 \quad \text{inches.}$$

OR

The screen has dimensions $4a\times 3a$ for some a. The portion of the screen not covered by the darkened strips has aspect ratio 2:1, so it has dimensions $4a\times 2a$. Thus the darkened strips each have height $\frac{a}{2}$. By the Pythagorean Theorem, the diagonal of the screen is $5a=27$ inches. Hence the height of each darkened strip is $\frac{27}{10}=2.7$ inches.

15. **Answer (D):** Suppose that Ian drove for t hours at an average speed of r miles per hour. Then he covered a distance of rt miles. The number of miles Han covered by driving 5 miles per hour faster for 1 additional hour is

$$(r+5)(t+1) = rt+5t+r+5.$$

Since Han drove 70 miles more than Ian,

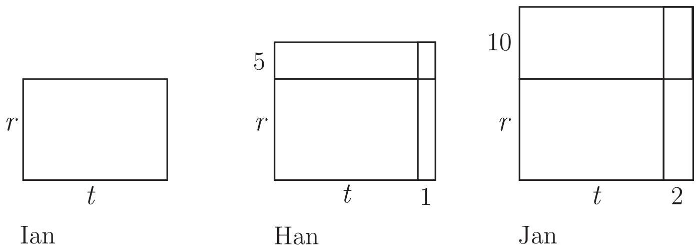
$$70=(r+5)(t+1)-rt = 5t+r+5, \quad \text{so} \quad 5t+r=65.$$

The number of miles Jan drove more than Ian is consequently

$$(r+10)(t+2)-rt = 10t+2r+20 = 2(5t+r)+20 = 2\cdot 65+20 = 150.$$

OR

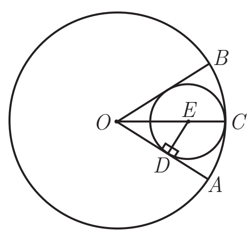
Represent the time traveled, average speed, and distance for Ian as length, width, and area, respectively, of a rectangle as shown. A similarly formed rectangle for Han would include an additional 1 unit of length and 5 units of width as compared to Ian's rectangle. Jan's rectangle would have an additional 2 units of length and 10 units of width as compared to Ian's rectangle.



Given that Han's distance exceeds that of Ian by 70 miles, and Jan's $10 \times t$ and $2 \times r$ rectangles are twice the size of Ian's $5 \times t$ and $1 \times r$ rectangles, respectively, it follows that Jan's distance exceeds that of Ian by

$$2(70 - 5) + 20 = 150 \text{ miles.}$$

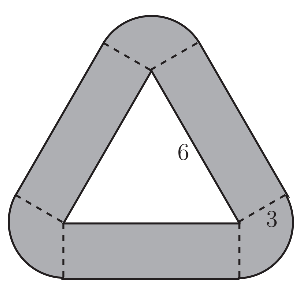
16. **Answer (B):** Let r and R be the radii of the smaller and larger circles, respectively. Let E be the center of the smaller circle, let $\overline{OC}$ be the radius of the larger circle that contains E, and let D be the point of tangency of the smaller circle to $\overline{OA}$. Then $OE = R - r$, and because $\triangle EDO$ is a $30-60-90°$ triangle, $OE = 2DE = 2r$. Thus $2r = R - r$, so $\frac{r}{R} = \frac{1}{3}$. The ratio of the areas is $(\frac{1}{3})^2 = \frac{1}{9}$.



17. **Answer (B):** The region consists of three rectangles with length 6 and width 3 together with three 120° sectors of circles with radius 3.

The combined area of the three 120° sectors is the same as the area of a circle with radius 3, so the area of the region is

$$3 \cdot 6 \cdot 3 + \pi \cdot 3^2 = 54 + 9\pi.$$



18. **Answer (B):** Let x be the length of the hypotenuse, and let y and z be the lengths of the legs. The given conditions imply that

$$y^2 + z^2 = x^2, \quad y + z = 32 - x, \quad \text{and} \quad yz = 40.$$

Thus

$$(32-x)^2 = (y+z)^2 = y^2+z^2+2yz = x^2+80,$$

from which $1024-64x = 80$, and $x = \frac{59}{4}$.

Note: Solving the system of equations yields leg lengths of

$$\frac{1}{8}(69+\sqrt{2201}) \quad \text{and} \quad \frac{1}{8}(69-\sqrt{2201}),$$

so a triangle satisfying the given conditions does in fact exist.

19. **Answer (C):** Let P' and S' denote the positions of P and S, respectively, after the rotation about R, and let P'' denote the final position of P. In the rotation that moves P to position P', the point P rotates 90° on a circle with center R and radius $PR = \sqrt{2^2+6^2} = 2\sqrt{10}$. The length of the arc traced by P is $(1/4)\left(2\pi \cdot 2\sqrt{10}\right) = \pi\sqrt{10}$. Next, P' rotates to P'' through a 90° arc on a circle with center S' and radius $S'P' = 6$. The length of this arc is $\frac{1}{4}(2\pi \cdot 6) = 3\pi$. The total distance traveled by P is

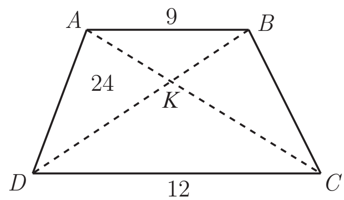
$$\pi\sqrt{10}+3\pi = \left(3+\sqrt{10}\right)\pi.$$

20. **Answer (D):** Note that $\triangle ABK$ is similar to $\triangle CDK$. Because $\triangle AKD$ and $\triangle KCD$ have collinear bases and share a vertex D,

$$\frac{\text{Area}(\triangle KCD)}{\text{Area}(\triangle AKD)} = \frac{KC}{AK} = \frac{CD}{AB} = \frac{4}{3},$$

so $\triangle KCD$ has area 32.

By a similar argument, $\triangle KAB$ has area 18. Finally, $\triangle BKC$ has the same area as $\triangle AKD$ since they are in the same proportion to each other of the other two triangles. The total area is $24+32+18+24 = 98$.



OR

Let h denote the height of the trapezoid. Then

$$24+\text{Area}(\triangle AKB) = \frac{9h}{2}.$$

Because $\triangle CKD$ is similar to $\triangle AKB$ with similarity ratio $\frac{12}{9} = \frac{4}{3}$,

$$\text{Area}(\triangle CKD) = \frac{16}{9}\text{Area}(\triangle AKB), \quad \text{so} \quad 24 + \frac{16}{9}\text{Area}(\triangle AKB) = \frac{12h}{2}.$$

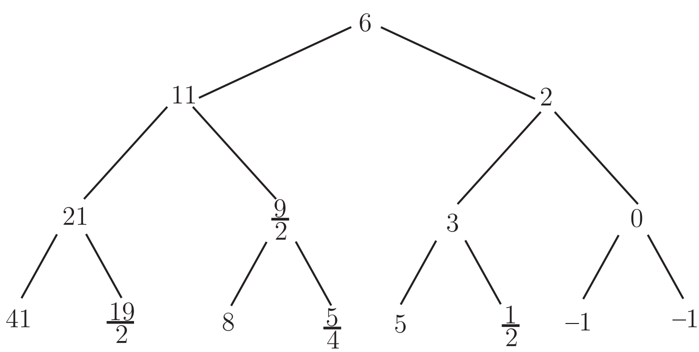
Solving the two equations simultaneously yields $h = \frac{28}{3}$. This implies that the area of the trapezoid is

$$\frac{1}{2} \cdot \frac{28}{3}(9 + 12) = 98.$$

21. **Answer (A):** All sides of $ABCD$ are of equal length, so $ABCD$ is a rhombus. Its diagonals have lengths $AC = \sqrt{3}$ and $BD = \sqrt{2}$, so its area is

$$\frac{1}{2}\sqrt{3} \cdot \sqrt{2} = \frac{\sqrt{6}}{2}.$$

22. **Answer (D):** The tree diagram below gives all possible sequences of four terms. In the diagram, each left branch from a number corresponds to a head, and each right branch to a tail.

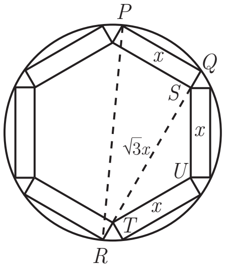


Because the coin is fair, each of the eight possible outcomes in the bottom row of the diagram is equally likely. Five of those numbers are integers, so the required probability is $\frac{5}{8}$.

23. **Answer (B):** Let the two subsets be A and B. There are $\binom{5}{2} = 10$ ways to choose the two elements common to A and B. There are then $2^3 = 8$ ways to assign the remaining three elements to A or B, so there are 80 ordered pairs (A, B) that meet the required conditions. However, the ordered pairs (A, B) and (B, A) represent the same pair $\{A, B\}$ of subsets, so the conditions can be met in $\frac{80}{2} = 40$ ways.

24. **Answer (D):** The units digit of 2^n is 2, 4, 8, and 6 for $n = 1$, 2, 3, and 4, respectively. For $n > 4$, the units digit of 2^n is equal to that of 2^{n-4}. Thus for every positive integer j the units digit of 2^{4j} is 6, and hence 2^{2008} has a units digit of 6. The units digit of 2008^2 is 4. Therefore the units digit of k is 0, so the units digit of k^2 is also 0. Because 2008 is even, both 2008^2 and 2^{2008} are multiples of 4. Therefore k is a multiple of 4, so the units digit of 2^k is 6, and the units digit of $k^2 + 2^k$ is also 6.

25. **Answer (C):** Select one of the mats. Let P and Q be the two corners of the mat that are on the edge of the table, and let R be the point on the edge of the table that is diametrically opposite P as shown. Then R is also a corner of a mat and $\triangle PQR$ is a right triangle with hypotenuse $PR = 8$. Let S be the inner corner of the chosen mat that lies on $\overline{QR}$, T the analogous point on the mat with corner R, and U the corner common to the other mat with corner S and the other mat with corner T. Then $\triangle STU$ is an isosceles triangle with two sides of length x and vertex angle $120°$. It follows that $ST = \sqrt{3}x$, so $QR = QS + ST + TR = \sqrt{3}x + 2$. Apply the Pythagorean Theorem to $\triangle PQR$ to obtain $\left(\sqrt{3}x + 2\right)^2 + x^2 = 8^2$, from which $x^2 + \sqrt{3}x - 15 = 0$. Solve for x and ignore the negative root to obtain



$$x = \frac{3\sqrt{7} - \sqrt{3}}{2}.$$

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
IDEA Math
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Mu Alpha Theta
National Assessment & Testing
National Council of Teachers of Mathematics
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.

THE MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

9th Annual American Mathematics Contest 10

AMC 10
CONTEST B

Solutions Pamphlet

Wednesday, FEBRUARY 27, 2008

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Chair: LeRoy Wenstrom, Columbus, MS

1. **Answer (E):** The number of points could be any integer between $5 \cdot 2 = 10$ and $5 \cdot 3 = 15$, inclusive. The number of possibilities is $15 - 10 + 1 = 6$.

2. **Answer (B):** The two sums are $1 + 10 + 17 + 22 = 50$ and $4+9+16+25 = 54$, so the positive difference between the sums is $54 - 50 = 4$.

1	2	3	4
11	10	9	8
15	16	17	18
25	24	23	22

   Query: If a different 4×4 block of dates had been chosen, the answer would be unchanged. Why?

3. **Answer (D):** The properties of exponents imply that

$$\sqrt[3]{x\sqrt{x}} = \left(x \cdot x^{\frac{1}{2}}\right)^{\frac{1}{3}} = \left(x^{\frac{3}{2}}\right)^{\frac{1}{3}} = x^{\frac{1}{2}}.$$

4. **Answer (C):** A single player can receive the largest possible salary only when the other 20 players on the team are each receiving the minimum salary of \$15,000. Thus the maximum salary for any player is $\$700{,}000 - 20 \cdot \$15{,}000 = \$400{,}000$.

5. **Answer (A):** Note that $(y - x)^2 = (x - y)^2$, so

$$(x-y)^2\$(y-x)^2 = (x-y)^2\$(x-y)^2 = \left((x-y)^2 - (x-y)^2\right)^2 = 0^2 = 0.$$

6. **Answer (C):** Because $AB + BD = AD$ and $AB = 4BD$, it follows that $BD = \frac{1}{5} \cdot AD$. By similar reasoning, $CD = \frac{1}{10} \cdot AD$. Thus

$$BC = BD - CD = \frac{1}{5} \cdot AD - \frac{1}{10} \cdot AD = \frac{1}{10} \cdot AD.$$

7. **Answer (C):** The side length of the large triangle is 10 times the side length of each small triangle, so the area of the large triangle is $10^2 = 100$ times the area of each small triangle.

8. **Answer (C):** The total cost of the carnations must be an even number of dollars. The total number of dollars spent is the even number 50, so the number of roses purchased must also be even. In addition, the number of roses purchased cannot exceed $\frac{50}{3}$. Therefore the number of roses purchased must be one of the even integers between 0 and 16, inclusive. This gives 9 possibilities for the number of roses purchased, and consequently 9 possibilities for the number of bouquets.

9. **Answer (A):** The quadratic formula implies that the two solutions are

$$x_1 = \frac{2a + \sqrt{4a^2 - 4ab}}{2a} \quad \text{and} \quad x_2 = \frac{2a - \sqrt{4a^2 - 4ab}}{2a},$$

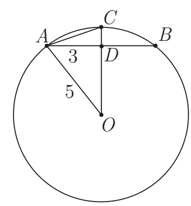
so the average is

$$\frac{1}{2}(x_1 + x_2) = \frac{1}{2}\left(\frac{2a}{2a} + \frac{2a}{2a}\right) = 1.$$

OR

The sum of the solutions of a quadratic equation is the negative of the coefficient of the linear term divided by the coefficient of the quadratic term. In this case the sum of the solution is $\dfrac{-(-2a)}{a} = 2$. Hence the average of the solutions is 1 .

10. **Answer (A):** Let O be the center of the circle, and let D be the intersection of $\overline{OC}$ and $\overline{AB}$. Because $\overline{OC}$ bisects minor arc AB, $\overline{OD}$ is a perpendicular bisector of chord $\overline{AB}$. Hence $AD = 3$, and applying the Pythagorean Theorem to $\triangle ADO$ yields $OD = \sqrt{5^2 - 3^3} = 4$. Therefore $DC = 1$, and applying the Pythagorean Theorem to $\triangle ADC$ yields $AC = \sqrt{3^2 + 1^2} = \sqrt{10}$.



11. **Answer (B):** Note that $u_5 = 2u_4 + 9$ and $128 = u_6 = 2u_5 + u_4 = 5u_4 + 18$. Thus $u_4 = 22$, and it follows that $u_5 = 2 \cdot 22 + 9 = 53$.

12. **Answer (A):** During the year Pete takes

$$44 \times 10^5 + 5 \times 10^4 = 44.5 \times 10^5$$

steps. At 1800 steps per mile, the number of miles Pete walks is

$$\frac{44.5 \times 10^5}{18 \times 10^2} = \frac{44.5}{18} \times 10^3 \approx 2.5 \times 10^3 = 2500.$$

13. **Answer (B):** Because the mean of the first n terms is n, their sum is n^2. Therefore the nth term is $n^2 - (n-1)^2 = 2n - 1$, and the 2008th term is $2 \cdot 2008 - 1 = 4015$.

14. **Answer (B):** Because $\triangle OAB$ is a $30-60-90°$ triangle, we have $BA = \frac{5\sqrt{3}}{3}$. Let A' and B' be the images of A and B, respectively, under the rotation. Then

$B' = (0,5)$, $\overline{B'A'}$ is horizontal, and $B'A' = BA = \frac{5\sqrt{3}}{3}$. Hence A' is in the second quadrant and

$$A' = \left(-\frac{5}{3}\sqrt{3}, 5\right).$$

15. **Answer (A):** By the Pythagorean Theorem we have $a^2 + b^2 = (b+1)^2$, so

$$a^2 = (b+1)^2 - b^2 = 2b + 1.$$

Because b is an integer with $b < 100$, a^2 is an odd perfect square between 1 and 201, and there are six of these, namely, 9, 25, 49, 81, 121, and 169. Hence a must be 3, 5, 7, 9, 11, or 13, and there are 6 triangles that satisfy the given conditions.

16. **Answer (A):** If one die is rolled, 3 of the 6 possible numbers are odd. If two dice are rolled, 18 of the 36 possible outcomes have odd sums. In each of these cases, the probability of an odd sum is $\frac{1}{2}$. If no die is rolled, the sum is 0, which is not odd. The probability that no die is rolled is equal to the probability that both coin tosses are tails, which is $(\frac{1}{2})^2 = \frac{1}{4}$. Thus the requested probability is

$$\left(1 - \frac{1}{4}\right) \cdot \frac{1}{2} = \frac{3}{8}.$$

17. **Answer (B):** The responses on these three occasions, in order, must be YNN, NYN, or NNY, where Y indicates approval and N indicates disapproval. The probability of each of these is $(0.7)(0.3)(0.3) = 0.063$, so the requested probability is $3(0.063) = 0.189$.

18. **Answer (B):** Let n be the number of bricks in the chimney. Then the number of bricks per hour Brenda and Brandon can lay working alone is $\frac{n}{9}$ and $\frac{n}{10}$, respectively. Working together they can lay $(\frac{n}{9} + \frac{n}{10} - 10)$ bricks in an hour, or

$$5\left(\frac{n}{9} + \frac{n}{10} - 10\right)$$

bricks in 5 hours to complete the chimney. Thus

$$5\left(\frac{n}{9} + \frac{n}{10} - 10\right) = n,$$

and the number of bricks in the chimney is $n = 900$.

OR

Suppose that Brenda can lay x bricks in an hour and Brandon can lay y bricks in an hour. Then the number of bricks in the chimney can be expressed as $9x$,

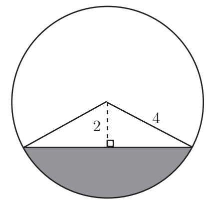
$10y$, or $5(x+y-10)$. The equality of these expressions leads to the system of equations

$$4x - 5y = -50$$

$$-5x + 5y = -50.$$

It follows that $x = 100$, so the number of bricks in the chimney is $9x = 900$.

19. **Answer (E):** The portion of each end of the tank that is under water is a circular sector with two right triangles removed as shown. The hypotenuse of each triangle is 4, and the vertical leg is 2, so each is a $30-60-90°$ triangle. Therefore the sector has a central angle of $120°$, and the area of the sector is



$$\frac{120}{360} \cdot \pi(4)^2 = \frac{16}{3}\pi.$$

The area of each triangle is $\frac{1}{2}(2)\left(2\sqrt{3}\right)$, so the portion of each end that is underwater has area $\frac{16}{3}\pi - 4\sqrt{3}$. The length of the cylinder is 9, so the volume of the water is

$$9\left(\frac{16}{3}\pi - 4\sqrt{3}\right) = 48\pi - 36\sqrt{3}.$$

20. **Answer (B):** Of the 36 possible outcomes, the four pairs $(1,4)$, $(2,3)$, $(2,3)$, and $(4,1)$ yield a sum of 5. The six pairs $(1,6)$, $(2,5)$, $(2,5)$, $(3,4)$, $(3,4)$, and $(4,3)$ yield a sum of 7. The four pairs $(1,8)$, $(3,6)$, $(3,6)$, and $(4,5)$ yield a sum of 9. Thus the probability of getting a sum of 5, 7, or 9 is $(4+6+4)/36 = 7/18$.

Note: The dice described here are known as Sicherman dice. The probability of obtaining each sum between 2 and 12 is the same as that on a pair of standard dice.

21. **Answer (C):** Let the women be seated first. The first woman may sit in any of the 10 chairs. Because men and women must alternate, the number of choices for the remaining women is 4, 3, 2, and 1. Thus the number of possible seating arrangements for the women is $10 \cdot 4! = 240$. Without loss of generality, suppose that a woman sits in chair 1. Then this woman's spouse must sit in chair 4 or chair 8. If he sits in chair 4 then the women sitting in chairs 7, 3, 9, and 5 must have their spouses sitting in chairs 10, 6, 2, and 8, respectively. If he sits in chair 8 then the women sitting in chairs 5, 9, 3, and 7 must have their spouses sitting in chairs 2, 6, 10, and 4, respectively. So for each possible seating arrangement for the women there are two arrangements for the men. Hence, there are $2 \cdot 240 = 480$ possible seating arrangements.

22. **Answer (C):** There are $6!/(3!2!1!) = 60$ distinguishable orders of the beads on the line. To meet the required condition, the red beads must be placed in

one of four configurations: positions 1, 3, and 5, positions 2, 4, and 6, positions 1, 3, and 6, or positions 1, 4, and 6. In the first two cases, the blue bead can be placed in any of the three remaining positions. In the last two cases, the blue bead can be placed in either of the two adjacent remaining positions. In each case, the placement of the white beads is then determined. Hence there are $2 \cdot 3 + 2 \cdot 2 = 10$ orders that meet the required condition, and the requested probability is $\frac{10}{60} = \frac{1}{6}$.

23. **Answer (B):** Because the area of the border is half the area of the floor, the same is true of the painted rectangle. The painted rectangle measures $a-2$ by $b-2$ feet. Hence $ab = 2(a-2)(b-2)$, from which $0 = ab - 4a - 4b + 8$. Add 8 to each side of the equation to produce

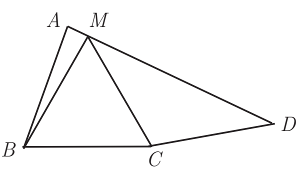
$$8 = ab - 4a - 4b + 16 = (a-4)(b-4).$$

Because the only integer factorizations of 8 are

$$8 = 1 \cdot 8 = 2 \cdot 4 = (-4) \cdot (-2) = (-8) \cdot (-1),$$

and because $b > a > 0$, the only possible ordered pairs satisfying this equation for $(a-4, b-4)$ are $(1,8)$ and $(2,4)$. Hence (a,b) must be one of the two ordered pairs $(5,12)$, or $(6,8)$.

24. **Answer (C):** Let M be on the same side of line BC as A, such that $\triangle BMC$ is equilateral. Then $\triangle ABM$ and $\triangle MCD$ are isosceles with $\angle ABM = 10°$ and $\angle MCD = 110°$. Hence $\angle AMB = 85°$ and $\angle CMD = 35°$. Therefore



$$\begin{aligned}\angle AMD &= 360° - \angle AMB - \angle BMC - \angle CMD \\ &= 360° - 85° - 60° - 35° = 180°.\end{aligned}$$

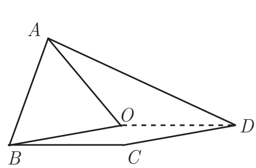
It follows that M lies on $\overline{AD}$ and $\angle BAD = \angle BAM = 85°$.

OR

Let $\triangle ABO$ be equilateral as shown.

Then

$$\angle OBC = \angle ABC - \angle ABO = 70° - 60° = 10°.$$



Because $\angle BCD = 170°$ and $OB = BC = CD$, the quadrilateral $BCDO$ is a parallelogram. Thus

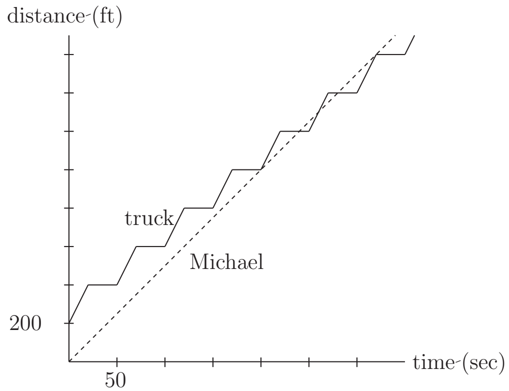
$OD = BC = AO$ and $\triangle AOD$ is isosceles. Let $\alpha = \angle ODA = \angle OAD$. The sum of the interior angles of $ABCD$ is $360°$, so we have

$$360 = (\alpha + 60) + 70 + 170 + (\alpha + 10) \quad \text{and} \quad \alpha = 25.$$

Thus $\angle DAB = 60 + \alpha = 85°$.

25. **Answer (B):** Number the pails consecutively so that Michael is presently at pail 0 and the garbage truck is at pail 1. Michael takes $200/5 = 40$ seconds to walk between pails, so for $n \geq 0$ he passes pail n after $40n$ seconds. The truck takes 20 seconds to travel between pails and stops for 30 seconds at each pail. Thus for $n \geq 1$ it leaves pail n after $50(n-1)$ seconds, and for $n \geq 2$ it arrives at pail n after $50(n-1) - 30$ seconds. Michael will meet the truck at pail n if and only if

$$50(n-1) - 30 \leq 40n \leq 50(n-1) \quad \text{or, equivalently, } 5 \leq n \leq 8.$$



Hence Michael first meets the truck at pail 5 after 200 seconds, just as the truck leaves the pail. He passes the truck at pail 6 after 240 seconds and at pail 7 after 280 seconds. Finally, Michael meets the truck just as it arrives at pail 8 after 320 seconds. These conditions imply that the truck is ahead of Michael between pails 5 and 6 and that Michael is ahead of the truck between pails 7 and 8. However, the truck must pass Michael at some point between pails 6 and 7, so they meet a total of five times.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

American Mathematical Association of Two Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Canada/USA Mathpath
Casualty Actuarial Society
Clay Mathematics Institute
IDEA Math
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Mu Alpha Theta
National Assessment & Testing
National Council of Teachers of Mathematics
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.

THE MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

10th Annual American Mathematics Contest 10

AMC 10 CONTEST A

Solutions Pamphlet

Tuesday, FEBRUARY 10, 2009

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom
Columbia, MD 21044

1. **Answer (E):** Because $\frac{128}{12} = 10\frac{2}{3}$, there must be 11 cans.

2. **Answer (A):** The value of any combination of four coins that includes pennies cannot be a multiple of 5 cents, and the value of any combination of four coins that does not include pennies must exceed 15 cents. Therefore the total value cannot be 15 cents. The other four amounts can be made with, respectively, one dime and three nickels; three dimes and one nickel; one quarter, one dime and two nickels; and one quarter and three dimes.

3. **Answer (C):** Simplifying the expression,

$$1 + \frac{1}{1 + \frac{1}{1+1}} = 1 + \frac{1}{1 + \frac{1}{2}} = 1 + \frac{1}{\frac{3}{2}} = 1 + \frac{2}{3} = \frac{5}{3}.$$

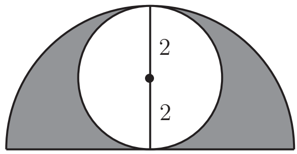
4. **Answer (A):** Eric can complete the swim in $\frac{1/4}{2} = \frac{1}{8}$ of an hour. He can complete the run in $\frac{3}{6} = \frac{1}{2}$ of an hour. This leaves $2 - \frac{1}{8} - \frac{1}{2} = \frac{11}{8}$ hours to complete the bicycle ride. His average speed for the ride must be $\frac{15}{11/8} = \frac{120}{11}$ miles per hour.

5. **Answer (E):** The square of 111,111,111 is

$$\begin{array}{r}
1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\
\times 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\
\hline
1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\
1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \\
1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \\
1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \\
1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \\
1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \\
1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \\
1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \\
1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ \\
\hline
1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9\ 8\ 7\ 6\ 5\ 4\ 3\ 2\ 1
\end{array}$$

Hence the sum of the digits of the square of 111,111,111 is 81.

6. **Answer (A):** The semicircle has radius 4 and total area $\frac{1}{2} \cdot \pi \cdot 4^2 = 8\pi$. The area of the circle is $\pi \cdot 2^2 = 4\pi$. The fraction of the area that is not shaded is $\frac{4\pi}{8\pi} = \frac{1}{2}$, and hence the fraction of the area that is shaded is also $\frac{1}{2}$.



7. **Answer (C):** Suppose whole milk is $x\%$ fat. Then 60% of x is equal to 2. Thus

$$x = \frac{2}{0.6} = \frac{20}{6} = \frac{10}{3}.$$

8. **Answer (B):** Grandfather Wen's ticket costs \$6, which is $\frac{3}{4}$ of the full price, so each ticket at full price costs $\frac{4}{3} \cdot 6 = 8$ dollars, and each child's ticket costs $\frac{1}{2} \cdot 8 = 4$ dollars. The cost of all the tickets is $2(\$6 + \$8 + \$4) = \$36$.

9. **Answer (B):** Let the ratio be r. Then $ar^2 = 2009 = 41 \cdot 7^2$. Because r must be an integer greater than 1, the only possible value of r is 7, and $a = 41$.

10. **Answer (B):** By the Pythagorean Theorem, $AB^2 = BD^2+9$, $BC^2 = BD^2+16$, and $AB^2 + BC^2 = 49$. Adding the first two equations and substituting gives $2 \cdot BD^2 + 25 = 49$. Then $BD = 2\sqrt{3}$, and the area of $\triangle ABC$ is $\frac{1}{2} \cdot 7 \cdot 2\sqrt{3} = 7\sqrt{3}$.

OR

Because $\triangle ADB$ and $\triangle BDC$ are similar, $\frac{BD}{3} = \frac{4}{BD}$, from which $BD = 2\sqrt{3}$. Therefore the area of $\triangle ABC$ is $\frac{1}{2} \cdot 7 \cdot 2\sqrt{3} = 7\sqrt{3}$.

11. **Answer (D):** Let x be the side length of the cube. Then the volume of the cube was x^3, and the volume of the new solid is $x(x + 1)(x - 1) = x^3 - x$. Therefore $x^3 - x = x^3 - 5$, from which $x = 5$, and the volume of the cube was $5^3 = 125$.

12. **Answer (C):** Let x be the length of $\overline{BD}$. By the triangle inequality on $\triangle BCD$, $5 + x > 17$, so $x > 12$. By the triangle inequality on $\triangle ABD$, $5 + 9 > x$, so $x < 14$. Since x must be an integer, $x = 13$.

13. **Answer (E):** Note that

$$12^{mn} = (2^2 \cdot 3)^{mn} = 2^{2mn} \cdot 3^{mn} = (2^m)^{2n} \cdot (3^n)^m = P^{2n}Q^m.$$

Remark: The pair of integers $(2, 1)$ shows that the other choices are not possible.

14. **Answer (A):** Let the lengths of the shorter and longer side of each rectangle be x and y, respectively. The outer and inner squares have side lengths $y + x$ and $y - x$, respectively, and the ratio of their side lengths is $\sqrt{4} = 2$. Therefore $y + x = 2(y - x)$, from which $y = 3x$.

15. **Answer (E):** The outside square for F_n has 4 more diamonds on its boundary than the outside square for F_{n-1}. Because the outside square of F_2 has 4 diamonds, the outside square of F_n has $4(n-2)+4 = 4(n-1)$ diamonds. Hence the number of diamonds in figure F_n is the number of diamonds in F_{n-1} plus $4(n - 1)$, or

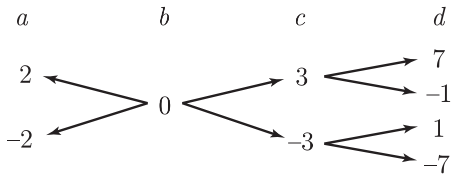
$$\begin{aligned} &1 + 4 + 8 + 12 + \cdots + 4(n-2) + 4(n-1) \\ =&1 + 4(1 + 2 + 3 + \cdots + (n-2) + (n-1)) \\ =&1 + 4\frac{(n-1)n}{2} \\ =&1 + 2(n-1)n. \end{aligned}$$

Therefore figure F_{20} has $1 + 2 \cdot 19 \cdot 20 = 761$ diamonds.

16. **Answer (D):** The given conditions imply that $b = a \pm 2$, $c = b \pm 3 = a \pm 2 \pm 3$, and $d = c \pm 4 = a \pm 2 \pm 3 \pm 4$, where the signs can be combined in all possible ways. Therefore the possible values of $|a - d|$ are $2 + 3 + 4 = 9$, $2 + 3 - 4 = 1$, $2 - 3 + 4 = 3$, and $-2 + 3 + 4 = 5$. The sum of all possible values of $|a - d|$ is $9 + 1 + 3 + 5 = 18$.

OR

The equations in the problem statement are true for numbers a, b, c, d if and only if they are true for $a+r$, $b+r$, $c+r$, $d+r$, where r is any real number. The value of $|a - d|$ is also unchanged with this substitution. Therefore there is no loss of generality in letting $b = 0$, and we can then write down the possibilities for the other variables:



The different possible values for $|a - d|$ are

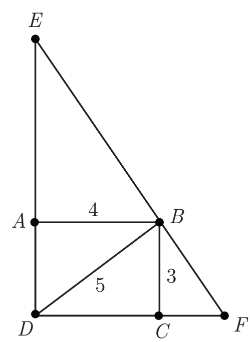
$$|2 - 7| = 5, \quad |2 - (-1)| = 3, \quad |2 - 1| = 1, \quad |2 - (-7)| = 9.$$

The sum of these possible values is 18.

17. **Answer (C):** Note that $DB = 5$ and $\triangle EBA$, $\triangle DBC$, and $\triangle BFC$ are all similar.

    Therefore $\frac{4}{EB} = \frac{3}{5}$, so $EB = \frac{20}{3}$. Similarly, $\frac{3}{BF} = \frac{4}{5}$, so $BF = \frac{15}{4}$.

    Thus
    $$EF = EB + BF = \frac{20}{3} + \frac{15}{4} = \frac{125}{12}.$$



18. **Answer (D):** For every 100 children, 60 are soccer players and 40 are non-soccer players. Of the 60 soccer players, 40% or $60 \times \frac{40}{100} = 24$ are also swimmers, so 36 are non-swimmers. Of the 100 children, 30 are swimmers and 70 are non-swimmers. The fraction of non-swimmers who play soccer is $\frac{36}{70} \approx .51$, or 51%.

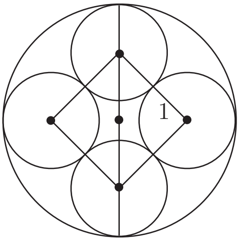
19. **Answer (B):** Circles A and B have circumferences 200π and $2\pi r$, respectively. After circle B begins to roll, its initial point of tangency with circle A touches circle A again a total of
    $$\frac{200\pi}{2\pi r} = \frac{100}{r}$$
    times. In order for this to be an integer greater than 1, r must be one of the integers 1, 2, 4, 5, 10, 20, 25, or 50. Hence there are a total of 8 possible values of r.

20. **Answer (D):** Let r be the rate that Lauren bikes, in kilometers per minute. Then $r + 3r = 1$, so $r = \frac{1}{4}$. In the first 5 minutes, the distance between Andrea and Lauren decreases by $5 \cdot 1 = 5$ kilometers, leaving Lauren to travel the remaining 15 kilometers between them. This requires

$$\frac{15}{\frac{1}{4}} = 60$$

minutes, so the total time since they started biking is $5 + 60 = 65$ minutes.

21. **Answer (C):** It may be assumed that the smaller circles each have radius 1. Their centers form a square with side length 2 and diagonal length $2\sqrt{2}$. Thus the diameter of the large circle is $2 + 2\sqrt{2}$, so its area is $(1 + \sqrt{2})^2\pi = (3 + 2\sqrt{2})\pi$. The desired ratio is



$$\frac{4\pi}{(3 + 2\sqrt{2})\pi} = 4(3 - 2\sqrt{2}).$$

22. **Answer (D):** Suppose that the two dice originally had the numbers 1, 2, 3, 4, 5, 6 and $1'$, $2'$, $3'$, $4'$, $5'$, $6'$, respectively. The process of randomly picking the numbers, randomly affixing them to the dice, rolling the dice, and adding the top numbers is equivalent to picking two of the twelve numbers at random and adding them. There are $\binom{12}{2} = 66$ sets of two elements taken from $S = \{1, 1', 2, 2', 3, 3', 4, 4', 5, 5', 6, 6'\}$. There are 4 ways to use a 1 and 6 to obtain 7, namely, $\{1, 6\}$, $\{1, 6'\}$, $\{1', 6\}$, and $\{1', 6'\}$. Similarly there are 4 ways to obtain the sum of 7 using a 2 and 5, and 4 ways using a 3 and 4. Hence there are 12 pairs taken from S whose sum is 7. Therefore the requested probability is $\frac{12}{66} = \frac{2}{11}$.

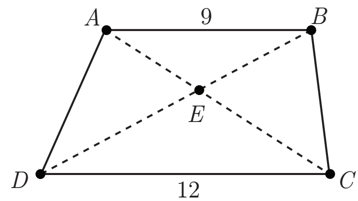
OR

Because the process is equivalent to picking two of the twelve numbers at random and then adding them, suppose we first pick number N. Then the second choice must be number $7 - N$. For any value of N, there are two "removable numbers" equal to $7 - N$ out of the remaining 11, so the probability of rolling a 7 is $\frac{2}{11}$.

23. **Answer (E):** Because $\triangle AED$ and $\triangle BEC$ have equal areas, so do $\triangle ACD$ and $\triangle BCD$. Side $\overline{CD}$ is common to $\triangle ACD$ and $\triangle BCD$, so the altitudes from A and B to $\overline{CD}$ have the same length. Thus $\overline{AB} \parallel \overline{CD}$, so $\triangle ABE$ is similar to $\triangle CDE$ with similarity ratio

$$\frac{AE}{EC} = \frac{AB}{CD} = \frac{9}{12} = \frac{3}{4}.$$

Let $AE = 3x$ and $EC = 4x$. Then $7x = AE + EC = AC = 14$, so $x = 2$, and $AE = 3x = 6$.



24. **Answer (C):** A plane that intersects at least three vertices of a cube either cuts into the cube or is coplanar with a cube face. Therefore the three randomly chosen vertices result in a plane that does not contain points inside the cube if and only if the three vertices come from the same face of the cube. There are 6 cube faces, so the number of ways to choose three vertices on the same cube face is $6 \cdot \binom{4}{3} = 24$. The total number of ways to choose the distinct vertices without restriction is $\binom{8}{3} = 56$. Hence the probability is $1 - \frac{24}{56} = \frac{4}{7}$.

25. **Answer (B):** Note that $I_k = 2^{k+2} \cdot 5^{k+2} + 2^6$. For $k < 4$, the first term is not divisible by 2^6, so $N(k) < 6$. For $k > 4$, the first term is divisible by 2^7, but the second term is not, so $N(k) < 7$. For $k = 4$, $I_4 = 2^6(5^6 + 1)$, and because the second factor is even, $N(4) \geq 7$. In fact the second factor is a sum of cubes so

$$(5^6 + 1) = ((5^2)^3 + 1^3) = (5^2 + 1)((5^2)^2 - 5^2 + 1).$$

The factor $5^2 + 1 = 26$ is divisible by 2 but not 4, and the second factor is odd, so $5^6 + 1$ contributes one more factor of 2. Hence the maximum value for $N(k)$ is 7.

The problems and solutions in this contest were proposed by Steve Blasberg, Thomas Butts, Steven Davis, Steve Dunbar, Douglas Faires, Jerrold Grossman, John Haverhals, Elgin Johnston, Joe Kennedy, Bonnie Leitch, David Wells, LeRoy Wenstrom, Woody Wenstrom, and Ron Yannone.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

Academy of Applied Sciences
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Casualty Actuarial Society
Clay Mathematics Institute
IDEA Math
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Math Zoom Academy
Mu Alpha Theta
National Assessment & Testing
National Council of Teachers of Mathematics
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.

THE MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

10th Annual American Mathematics Contest 10

AMC 10 CONTEST B

Solutions Pamphlet

Wednesday, FEBRUARY 25, 2009

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom
Columbia, MD 21044

1. **Answer (B):** Make a table for the cost of the muffins and bagels:

Cost of Muffins	Cost of Bagels	Total Cost
$0 \cdot 0.50 = 0.00$	$5 \cdot 0.75 = 3.75$	3.75
$1 \cdot 0.50 = 0.50$	$4 \cdot 0.75 = 3.00$	3.50
$2 \cdot 0.50 = 1.00$	$3 \cdot 0.75 = 2.25$	3.25
$3 \cdot 0.50 = 1.50$	$2 \cdot 0.75 = 1.50$	3.00
$4 \cdot 0.50 = 2.00$	$1 \cdot 0.75 = 0.75$	2.75
$5 \cdot 0.50 = 2.50$	$0 \cdot 0.75 = 0.00$	2.50

The only combination which is a whole number of dollars is the cost of 3 muffins and 2 bagels.

2. **Answer (C):** The least common multiple of 2, 3, and 4 is 12, and

$$\frac{\frac{1}{3}-\frac{1}{4}}{\frac{1}{2}-\frac{1}{3}} \cdot \frac{12}{12} = \frac{4-3}{6-4} = \frac{1}{2}.$$

3. **Answer (C):** The loss of 3 cans of paint resulted in 5 fewer rooms being painted, so the ratio of cans of paint to rooms painted is $3:5$. Hence for 25 rooms she would require $\frac{3}{5} \cdot 25 = 15$ cans of paint.

OR

If she used x cans of paint for 25 rooms, then $\frac{x+3}{30} = \frac{x}{25}$. Hence $25x + 75 = 30x$, and $x = 15$.

4. **Answer (C):** Each triangle has leg length $\frac{1}{2} \cdot (25 - 15) = 5$ meters and area $\frac{1}{2} \cdot 5^2 = \frac{25}{2}$ square meters. Thus the flower beds have a total area of 25 square meters. The entire yard has length 25 and width 5, so its area is 125. The fraction of the yard occupied by the flower beds is $\frac{25}{125} = \frac{1}{5}$.

5. **Answer (D):** Twenty percent less than 60 is $\frac{4}{5} \cdot 60 = 48$. One-third more than a number n is $\frac{4}{3}n$. Therefore $\frac{4}{3}n = 48$, and the number is 36.

6. **Answer (D):** The age of each person is a factor of $128 = 2^7$. So the twins could be $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$ years of age and, consequently, Kiana could be $\frac{128}{1^2} = 128$, $\frac{128}{2^2} = 32$, $\frac{128}{4^2} = 8$, or $\frac{128}{8^2} = 2$ years old, respectively. Because Kiana is younger than her brothers, she must be 2 years old. The sum of their ages is $2 + 8 + 8 = 18$.

7. **Answer (C):** The three operations can be performed in any of $3! = 6$ orders. However, if the addition is performed either first or last, then multiplying in either order produces the same result. Thus at most four distinct values can be obtained. It is easily checked that the values of the four expressions

$$(2 \times 3) + (4 \times 5) = 26,$$
$$((2 \times 3 + 4) \times 5) = 50,$$
$$2 \times (3 + (4 \times 5)) = 46,$$
$$2 \times (3 + 4) \times 5 = 70$$

are in fact all distinct.

8. **Answer (B):** Let p denote the price at the beginning of January. The price at the end of March was $(1.2)(0.8)(1.25)p = 1.2p$. Because the price at the end of April was p, the price decreased by $0.2p$ during April, and the percent decrease was

$$x = 100 \cdot \frac{0.2p}{1.2p} = \frac{100}{6} \approx 16.7\,.$$

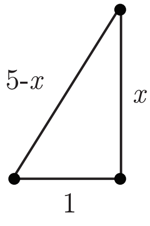
To the nearest integer, x is 17.

9. **Answer (A):** Because $\triangle ABC$ is isosceles, $\angle A = \angle C$. Because $\angle A = \frac{5}{2}\angle B$, we have $\frac{5}{2}\angle B + \frac{5}{2}\angle B + \angle B = 180°$, so $\angle B = 30°$. Therefore $\angle ACB = \angle DCE = 75°$. Because $\triangle CDE$ is isosceles, $2\angle D + 75° = 180°$, so $\angle D = 52.5°$.

10. **Answer (E):** Let x be the height of the stump. Then $5 - x$ is the height of the snapped part, now forming the hypotenuse of a right triangle. By the Pythagorean Theorem,

$$x^2 + 1^2 = (5 - x)^2 = x^2 - 10x + 25$$

from which $x = 2.4$.



11. **Answer (A):** Because the digit 5 appears three times, 5 must be the middle digit of any such palindrome. In the first three digits each of 2, 3, and 5 must appear once and the order in which they appear determines the last three digits. Since there are $3! = 6$ ways to order three distinct digits the number of palindromes is 6.

12. **Answer (A):** The base of the triangle can be 1, 2, or 3, and its altitude is the distance between the two parallel lines, so there are three possible values for the area.

13. **Answer (C):** Define a rotation of the pentagon to be a sequence that starts with $\overline{AB}$ on the x-axis and ends when $\overline{AB}$ is on the x-axis the first time thereafter. Because the pentagon has perimeter 23 and $2009 = 23 \cdot 87 + 8$, it follows that after 87 rotations, point A will be at $x = 23 \cdot 87 = 2001$ and point B will be at $x = 2001 + 3 = 2004$. Points C and D will next touch the x-axis at $x = 2004 + 4 = 2008$ and $x = 2008 + 6 = 2014$, respectively. Therefore a point on $\overline{CD}$ will touch $x = 2009$.

14. **Answer (D):** On Monday, day 1, the birds find $\frac{1}{4}$ quart of millet in the feeder. On Tuesday they find

$$\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4}$$

quarts of millet. On Wednesday, day 3, they find

$$\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4}$$

quarts of millet. The number of quarts of millet they find on day n is

$$\frac{1}{4} + \frac{3}{4} \cdot \frac{1}{4} + \left(\frac{3}{4}\right)^2 \cdot \frac{1}{4} + \cdots + \left(\frac{3}{4}\right)^{n-1} \cdot \frac{1}{4} = \frac{(\frac{1}{4})(1 - (\frac{3}{4})^n)}{1 - \frac{3}{4}} = 1 - \left(\frac{3}{4}\right)^n.$$

The birds always find $\frac{3}{4}$ quart of other seeds, so more than half the seeds are millet if $1 - (\frac{3}{4})^n > \frac{3}{4}$, that is, when $(\frac{3}{4})^n < \frac{1}{4}$. Because $(\frac{3}{4})^4 = \frac{81}{256} > \frac{1}{4}$ and $(\frac{3}{4})^5 = \frac{243}{1024} < \frac{1}{4}$, this will first occur on day 5 which is Friday.

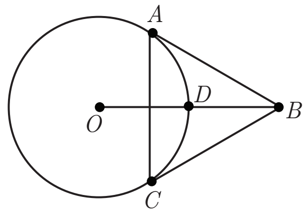
15. **Answer (E):** Let x be the weight of the bucket and let y be the weight of the water in a full bucket. Then we are given that $x + \frac{2}{3}y = a$ and $x + \frac{1}{2}y = b$. Hence $\frac{1}{6}y = a - b$, so $y = 6a - 6b$. Thus $x = b - \frac{1}{2}(6a - 6b) = -3a + 4b$. Finally, $x + y = 3a - 2b$.

OR

The difference between a kg and b kg is the weight of water that would fill $\frac{1}{6}$ of a bucket. So the weight of water that would fill $\frac{1}{2}$ of a bucket is $3(a - b)$. Therefore the weight of a bucket filled with water is $b + 3(a - b) = 3a - 2b$.

16. **Answer (B):** Let the radius of the circle be r. Because $\triangle BCO$ is a right triangle with a 30° angle at B, the hypotenuse $\overline{BO}$ is twice as long as $\overline{OC}$, so $BO = 2r$. It follows that $BD = 2r - r = r$, and

$$\frac{BD}{BO} = \frac{r}{2r} = \frac{1}{2}.$$

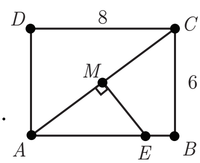


17. **Answer (C):** The area of the entire region is 5. The shaded region consists of a triangle with base $3-a$ and altitude 3, with one unit square removed. Therefore

$$\frac{3(3-a)}{2} - 1 = \frac{5}{2}.$$

Solving this equation yields $a = \frac{2}{3}$.

18. **Answer (D):** By the Pythagorean Theorem, $AC = 10$, so $AM = 5$. Triangles AME and ABC are similar, so $\frac{ME}{AM} = \frac{6}{8}$ and $ME = \frac{15}{4}$. The area of $\triangle AME$ is $\frac{1}{2} \cdot 5 \cdot \frac{15}{4} = \frac{75}{8}$.



OR

As above, $AM = 5$ and $\triangle AME$ and $\triangle ABC$ are similar with similarity ratio 5:8. Therefore

$$Area(\triangle AME) = \left(\frac{5}{8}\right)^2 \cdot Area(\triangle ABC) = \frac{5^2}{8^2} \cdot \frac{8 \cdot 6}{2} = \frac{75}{8}.$$

19. **Answer (A):** The clock will display the incorrect time for the entire hours of 1, 10, 11, and 12. So the correct hour is displayed correctly $\frac{2}{3}$ of the time. The minutes will not display correctly whenever either the tens digit or the ones digit is a 1, so the minutes that will not display correctly are 10, 11, 12, ..., 19, and 01, 21, 31, 41, and 51. This is 15 of the 60 possible minutes for a given hour. Hence the fraction of the day that the clock shows the correct time is $\frac{2}{3} \cdot (1 - \frac{15}{60}) = \frac{2}{3} \cdot \frac{3}{4} = \frac{1}{2}$.

20. **Answer (B):** By the Pythagorean Theorem, $AC = \sqrt{5}$. By the Angle Bisector Theorem, $\frac{BD}{AB} = \frac{CD}{AC}$. Therefore $CD = \sqrt{5} \cdot BD$ and $BD + CD = 2$, from which

$$BD = \frac{2}{1+\sqrt{5}} = \frac{\sqrt{5}-1}{2}.$$

OR

Let $\overline{DE}$ be an altitude of $\triangle ADC$. Then note that $\triangle ABD$ is congruent to $\triangle AED$, and so $AE = 1$. As in the first solution $AC = \sqrt{5}$. Let $x = BD$. Then $DE = x$, $EC = \sqrt{5} - 1$, and $DC = 2 - x$. Applying the Pythagorean Theorem to $\triangle DEC$ yields $x^2 + \left(\sqrt{5} - 1\right)^2 = (2 - x)^2$, from which $x = \frac{\sqrt{5}-1}{2}$.

21. **Answer (D):** The sum of any four consecutive powers of 3 is divisible by $3^0 + 3^1 + 3^2 + 3^3 = 40$ and hence is divisible by 8. Therefore

$$(3^2 + 3^3 + 3^4 + 3^5) + \cdots + (3^{2006} + 3^{2007} + 3^{2008} + 3^{2009})$$

is divisible by 8. So the required remainder is $3^0 + 3^1 = 4$.

22. **Answer (B):** The area of triangle A is 1, and its hypotenuse has length $\sqrt{5}$. Triangle B is similar to triangle A and has a hypotenuse of 2, so its area is $(\frac{2}{\sqrt{5}})^2 = \frac{4}{5}$. The volume of the required piece is $c = \frac{4}{5} \cdot 2 = \frac{8}{5}$ cubic inches. The icing on this piece has an area of $s = \frac{4}{5} + 2^2 = \frac{24}{5}$ square inches. Therefore $c + s = \frac{8}{5} + \frac{24}{5} = \frac{32}{5}$.

23. **Answer (C):** After 10 min. = 600 sec., Rachel will have completed 6 laps and be 30 seconds from the finish line. Because Rachel runs one-fourth of a lap in 22.5 seconds, she will be in the picture taking region between

$$30 - \frac{22.5}{2} = 18.75 \quad \text{and} \quad 30 + \frac{22.5}{2} = 41.25$$

seconds of the 10th minute. After 10 minutes Robert will have completed 7 laps and will be 40 seconds from the starting line. Because Robert runs one-fourth of a lap in 20 seconds, he will be in the picture taking region between 30 and 50 seconds of the 10th minute. Hence both Rachel and Robert will be in the picture if it is taken between 30 and 41.25 seconds of the 10th minute. The probability that the picture is snapped during this time is

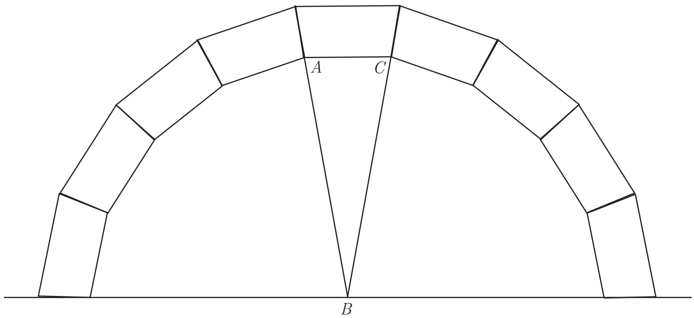
$$\frac{41.25 - 30}{60} = \frac{3}{16}.$$

24. **Answer (A):** Add a symmetric arch to the given arch to create a closed loop of trapezoids. Consider the regular 18-sided polygon created by the interior of the completed loop. Each interior angle of a regular 18-gon measures

$$(18 - 2) \cdot 180^\circ/18 = 160^\circ.$$

Then $x + x + 160^\circ = 360^\circ$, so $x = 100^\circ$.

OR



Extend two sides of a trapezoid until they meet at the center of the arch, as shown. Then $\triangle ABC$ is isosceles and by symmetry $\angle ABC = \frac{180}{9} = 20°$, and $\angle BAC = 80°$. The requested angle is supplementary to $\angle BAC$, so $x = 180 - 80 = 100°$.

25. **Answer (B):** The stripe on each face of the cube will be oriented in one of two possible directions, so there are $2^6 = 64$ possible stripe combinations on the cube. There are 3 pairs of parallel faces so, if there is an encircling stripe, then the pair of faces that do not contribute uniquely determine the stripe orientation for the remaining faces. In addition, the stripe on each face that does not contribute may be oriented in 2 different ways. Thus a total of $3 \cdot 2 \cdot 2 = 12$ stripe combinations on the cube result in a continuous stripe around the cube, and the requested probability is $\frac{12}{64} = \frac{3}{16}$.

OR

Without loss of generality, orient the cube so that the stripe on the top face goes from front to back. There are two mutually exclusive ways for there to be an encircling stripe: either the front, bottom, and back faces are painted to complete an encircling stripe with the top face's stripe, or the front, right, back, and left faces are painted to form an encircling stripe. The probability of the first cases is $(\frac{1}{2})^3 = \frac{1}{8}$, and the probability of the second case is $(\frac{1}{2})^4 = \frac{1}{16}$, so the answer is $\frac{1}{8} + \frac{1}{16} = \frac{3}{16}$.

OR

There are three possible orientations of an encircling stripe. For any one of these to appear, the four faces through which the stripe is to pass must be properly aligned. The probability of one such stripe alignment is $\left(\frac{1}{2}\right)^4 = \frac{1}{16}$. Because there are 3 such possibilities, and these events are disjoint, the total probability is $3\left(\frac{1}{16}\right) = \frac{3}{16}$.

The problems and solutions in this contest were proposed by Steve Blasberg, Thomas Butts, Gerald Bergum, Steve Dunbar, Douglas Faires, Sister Josanne Furey, Gregory Galperin, Jerrold Grossman, Elgin Johnston, Joe Kennedy, David Wells, LeRoy Wenstrom, Woody Wenstrom.

MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

11th Annual

AMC 10 A

American Mathematics Contest 10A

Solutions Pamphlet

Tuesday, February 9, 2010

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom
lwenstrom@gmail.com

1. **Answer (D):** The average of the five values is

$$\frac{6+0.5+1+2.5+10}{5}=\frac{20}{5}=4.$$

2. **Answer (B):** Let s be the side length of the smaller square. Then the length of the rectangle is $4s$, and the width is $4s-s=3s$. Hence the rectangle length is $\frac{4s}{3s}=\frac{4}{3}$ times as large as its width.

3. **Answer (D):** Let x be the number of marbles that Tyrone gave to Eric. Then $97-x=2(11+x)$. Solving this equation yields $x=25$.

4. **Answer (B):** Because $412\div 56$ is between 7 and 8, the reading will need 8 discs. Therefore each disc will contain $412\div 8=51.5$ minutes of reading.

5. **Answer (E):** Because the circumference is $2\pi r=24\pi$, the radius r is 12. Therefore the area is $\pi r^2=144\pi$, and $k=144$.

6. **Answer (C):** Note that $\spadesuit(2,2)=2-\frac{1}{2}=\frac{3}{2}$. Therefore

$$\spadesuit(2,\spadesuit(2,2))=\spadesuit\left(2,\frac{3}{2}\right)=2-\frac{2}{3}=\frac{4}{3}.$$

7. **Answer (C):** When Crystal travels one mile northeast she travels $\frac{\sqrt{2}}{2}$ miles north and $\frac{\sqrt{2}}{2}$ miles east. Similarly, when she travels southeast for one mile she travels $\frac{\sqrt{2}}{2}$ miles south and $\frac{\sqrt{2}}{2}$ miles east. Just before the last portion of her run she has traveled a net of $1+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}=1$ miles north, and $\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}=\sqrt{2}$ miles east. By the Pythagorean Theorem, the last portion of her run is

$$\sqrt{1^2+(\sqrt{2})^2}=\sqrt{1+2}=\sqrt{3}\quad\text{miles.}$$

8. **Answer (D):** Tony worked for $2\cdot 50=100$ hours. His average earnings per hour during this period is $\frac{\$630}{100}=\$6.30$. Hence his average age during this period was $\frac{\$6.30}{\$0.50}=12.6$, and so at the end of the six month period he was 13.

9. **Answer (E):** Let $x+32$ be written in the form $CDDC$. Because x has three digits, $1000 < x+32 < 1032$, and so $C = 1$ and $D = 0$. Hence $x = 1001 - 32 = 969$, and the sum of the digits of x is $9+6+9 = 24$.

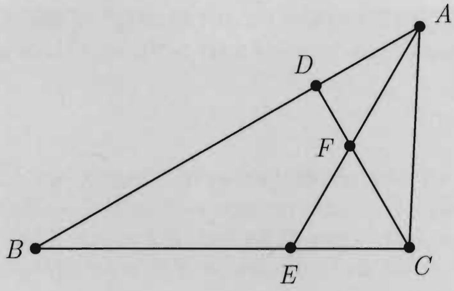
10. **Answer (E):** A non-leap year has 365 days, and $365 = 52 \cdot 7 + 1$, so there are 52 weeks and 1 day in a non-leap year. Because May 27 was after leap day in 2008, Marvin's birthday fell on Wednesday in 2009, and will fall on Thursday in 2010 and Friday in 2011. His birthday will be on Sunday in the leap year 2012, Monday in 2013, Tuesday in 2014, Wednesday in 2015, Friday in 2016, and Saturday in 2017.

11. **Answer (D):** The solution of the inequality is
$$\frac{a-3}{2} \le x \le \frac{b-3}{2}.$$
If $\frac{b-3}{2} - \frac{a-3}{2} = 10$, then $b - a = 20$.

12. **Answer (C):** The volume scale for Logan's model is $0.1 : 100{,}000 = 1 : 1{,}000{,}000$. Therefore the linear scale is $1 : \sqrt[3]{1{,}000{,}000}$, which is $1 : 100$. Logan's water tower should stand $\frac{40}{100} = 0.4$ meters tall.

13. **Answer (A):** Angelina drove $80t$ km before she stopped. After her stop, she drove $(3 - \frac{1}{3} - t)$ hours at an average rate of 100 kph, so she covered $100(\frac{8}{3} - t)$ km in that time. Therefore $80t + 100(\frac{8}{3} - t) = 250$. Note that $t = \frac{5}{6}$.

14. **Answer (C):** Let $\alpha = \angle BAE = \angle ACD = \angle ACF$. Because $\triangle CFE$ is equilateral, it follows that $\angle CFA = 120°$ and then
$$\angle FAC = 180° - 120° - \angle ACF = 60° - \alpha.$$
Therefore
$$\angle BAC = \angle BAE + \angle FAC = \alpha + (60° - \alpha) = 60°.$$
Because $AB = 2 \cdot AC$, it follows that $\triangle BAC$ is a $30-60-90°$ triangle, and thus $\angle ACB = 90°$.



15. **Answer (D):** LeRoy and Chris cannot both be frogs, because their statements would be true and frogs lie. Also LeRoy and Chris cannot both be toads, because then their statements would be false, and toads tell the truth. Hence between LeRoy and Chris, exactly one must be a toad.

    If Brian is a toad, then Mike must be a frog, but this is a contradiction as Mike's statement would then be true. Hence Brian is a frog, so Brian's statement must be false, and Mike must be a frog. Altogether there are 3 frogs: Brian, Mike, and either LeRoy or Chris.

16. **Answer (B):** By the Angle Bisector Theorem, $8 \cdot BA = 3 \cdot BC$. Thus BA must be a multiple of 3. If $BA = 3$, the triangle is degenerate. If $BA = 6$, then $BC = 16$, and the perimeter is $6 + 16 + 11 = 33$.

17. **Answer (A):** The volume of the solid cube is 27 in^3. The first hole to be cut removes $2 \times 2 \times 3 = 12$ in^3 from the volume. The other holes remove $2 \times 2 \times 0.5 = 2$ in^3 from each of the four remaining faces. The volume of the remaining solid is $27 - 12 - 4(2) = 7$ in^3.

18. **Answer (B):** The probability that Bernardo picks a 9 is $\frac{3}{9} = \frac{1}{3}$. In this case, his three-digit number will begin with a 9 and will be larger than Silvia's three-digit number.

    If Bernardo does not pick a 9, then Bernardo and Silvia will form the same number with probability
    $$\frac{1}{\binom{8}{3}} = \frac{1}{56}.$$

    If they do not form the same number then Bernardo's number will be larger $\frac{1}{2}$ of the time.

Hence the probability is

$$\frac{1}{3}+\frac{2}{3}\cdot\frac{1}{2}(1-\frac{1}{56})=\frac{111}{168}=\frac{37}{56}.$$

19. **Answer (E):** Triangles ABC, CDE and EFA are congruent, so $\triangle ACE$ is equilateral. Let X be the intersection of the lines AB and EF and define Y and Z similarly as shown in the figure. Because $ABCDEF$ is equiangular, it follows that $\angle XAF = \angle AFX = 60°$. Thus $\triangle XAF$ is equilateral. Let H be the midpoint of $\overline{XF}$. By the Pythagorean Theorem,

$$AE^2 = AH^2 + HE^2 = (\frac{\sqrt{3}}{2}r)^2 + (\frac{r}{2}+1)^2 = r^2 + r + 1$$

Thus, the area of $\triangle ACE$ is

$$\frac{\sqrt{3}}{4}AE^2 = \frac{\sqrt{3}}{4}(r^2+r+1).$$

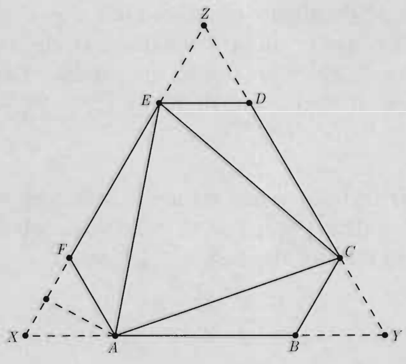
The area of hexagon $ABCDEF$ is equal to

$$[XYZ]-[XAF]-[YCB]-[ZED] = \frac{\sqrt{3}}{4}\left((2r+1)^2-3r^2\right) = \frac{\sqrt{3}}{4}(r^2+4r+1)$$

Because $[ACE] = \frac{7}{10}[ABCDEF]$, it follows that

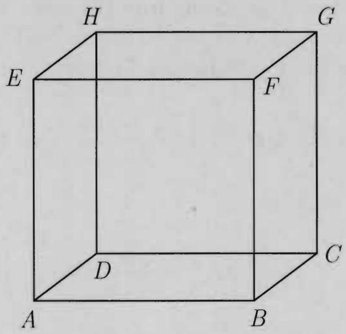
$$r^2+r+1 = \frac{7}{10}(r^2+4r+1)$$

from which $r^2 - 6r + 1 = 0$ and $r = 3 \pm 2\sqrt{2}$. The sum of all possible values of r is 6.



20. **Answer (D):** Each of the 8 lines segments on the fly's path is an edge, a face diagonal, or an interior diagonal of the cube. These three type of line segments have lengths 1, $\sqrt{2}$, and $\sqrt{3}$, respectively. Because each vertex of the cube is visited only once, the two line segments that meet at a vertex have a combined length of at most $\sqrt{2}+\sqrt{3}$. Therefore the sum of the lengths of the 8 segments is at most $4\sqrt{2}+4\sqrt{3}$. This maximum is achieved by the path

$$A \to G \to B \to H \to C \to E \to D \to F \to A.$$



21. **Answer (A):** Let the polynomial be $(x-r)(x-s)(x-t)$ with $0 < r \le s \le t$. Then $rst = 2010 = 2 \cdot 3 \cdot 5 \cdot 67$, and $r+s+t=a$. If $t = 67$, then $rs = 30$, and $r+s$ is minimized when $r=5$ and $s=6$. In that case $a = 67+5+6 = 78$. If $t \ne 67$, then $a > t \ge 2 \cdot 67 = 134$, so the minimum value of a is 78.

22. **Answer (A):** Three chords create a triangle if and only if they intersect pairwise inside the circle. Two chords intersect inside the circle if and only if their endpoints alternate in order around the circle. Therefore, if points A, B, C, D, E, and F are in order around the circle, then only the chords $\overline{AD}$, $\overline{BE}$, $\overline{CF}$ all intersect pairwise inside the circle. Thus every set of 6 points determines a unique triangle, and there are $\binom{8}{6} = 28$ such triangles.

23. **Answer (A):** If Isabella reaches the k^{th} box, she will draw a white marble from it with probability $\frac{k}{k+1}$. For $n \ge 2$, the probability that she will draw white marbles from each of the first $n-1$ boxes is

$$\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n-1}{n} = \frac{1}{n},$$

so the probability that she will draw her first red marble from the n^{th} box is $P(n) = \frac{1}{n(n+1)}$. The condition $P(n) < 1/2010$ is equivalent to $n^2+n-2010 > 0$, from which $n > \frac{1}{2}(-1+\sqrt{8041})$ and $(2n+1)^2 > 8041$. The smallest positive odd integer whose square exceeds 8041 is 91, and the corresponding value of n is 45.

24. **Answer (A):** There are 18 factors of 90! that are multiples of 5, 3 factors that are multiples of 25, and no factors that are multiples of higher powers of 5. Also, there are more than 45 factors of 2 in 90!. Thus $90! = 10^{21}N$ where N is an integer not divisible by 10, and if $N \equiv n \pmod{100}$ with $0 < n \le 99$, then n is a multiple of 4.

 Let $90! = AB$ where A consists of the factors that are relatively prime to 5 and B consists of the factors that are divisible by 5. Note that $\prod_{j=1}^{4}(5k+j) \equiv 5k(1+2+3+4)+1\cdot 2\cdot 3\cdot 4 \equiv 24 \pmod{25}$, thus

$$\begin{aligned} A &= (1\cdot 2\cdot 3\cdot 4)\cdot(6\cdot 7\cdot 8\cdot 9)\cdot\cdots\cdot(86\cdot 87\cdot 88\cdot 89) \\ &\equiv 24^{18} \equiv (-1)^{18} \equiv 1 \pmod{25}. \end{aligned}$$

 Similarly,

$$B = (5\cdot 10\cdot 15\cdot 20)\cdot(30\cdot 35\cdot 40\cdot 45)\cdot(55\cdot 60\cdot 65\cdot 70)\cdot(80\cdot 85\cdot 90)\cdot(25\cdot 50\cdot 75),$$

 thus

$$\begin{aligned} \frac{B}{5^{21}} &= (1\cdot 2\cdot 3\cdot 4)\cdot(6\cdot 7\cdot 8\cdot 9)\cdot(11\cdot 12\cdot 13\cdot 14)\cdot(16\cdot 17\cdot 18)\cdot(1\cdot 2\cdot 3) \\ &\equiv 24^3\cdot(-9)\cdot(-8)\cdot(-7)\cdot 6 \equiv (-1)^3\cdot 1 \equiv -1 \pmod{25}. \end{aligned}$$

 Finally, $2^{21} = 2\cdot(2^{10})^2 = 2\cdot(1024)^2 \equiv 2\cdot(-1)^2 \equiv 2 \pmod{25}$, so $13\cdot 2^{21} \equiv 13\cdot 2 \equiv 1 \pmod{25}$. Therefore

$$\begin{aligned} N &\equiv (13\cdot 2^{21})N = 13\cdot\frac{90!}{5^{21}} = 13\cdot A\cdot\frac{B}{5^{21}} \equiv 13\cdot 1\cdot(-1) \pmod{25} \\ &\equiv -13 \equiv 12 \pmod{25}. \end{aligned}$$

 Thus n is equal to 12, 37, 62, or 87, and because n is a multiple of 4, it follows that $n = 12$.

25. **Answer (B):** Let the sequence be $(a_1, a_2, \ldots, a_8)$. For $j > 1$, $a_{j-1} = a_j + m^2$ for some m such that $a_j < (m+1)^2 - m^2 = 2m+1$. To minimize the value of a_1, construct the sequence in reverse order and choose the smallest possible value of m for each j, $2 \le j \le 8$. The terms in reverse order are $a_8 = 0$, $a_7 = 1$, $a_6 = 1+1^2 = 2$, $a_5 = 2+1^2 = 3$, $a_4 = 3+2^2 = 7$, $a_3 = 7+4^2 = 23$, $a_2 = 23+12^2 = 167$, and $N = a_1 = 167+84^2 = 7223$, which has the unit digit 3.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steven Blasberg, Steven Davis, Sundeep Desai, Steven Dunbar, Sister Josannae Furey, David Grabiner, Michelle Ghrist, Jerrold Grossman, Brian Hartwig, Dan Kennedy, Joe Kennedy, Mike Korn, Leon La Spina, Glen Marr, Raymond Scacalossi, David Wells, LeRoy Wenstrom, Woody Wenstrom, and Ron Yannone.

MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

11th Annual

AMC 10 B

American Mathematics Contest 10B

Solutions Pamphlet

Wednesday, February 24, 2010

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom
lwenstrom@gmail.com

1. **Answer (C):** Simplifying gives

$$\begin{aligned}100(100-3)-(100\cdot 100-3) &= 100\cdot 100-100\cdot 3-100\cdot 100+3\\ &= -300+3\\ &= -297.\end{aligned}$$

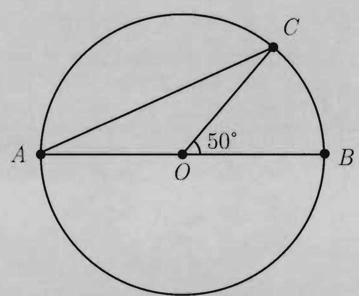
2. **Answer (C):** Makayla spent $45+2\cdot 45=135$ minutes, or $\frac{135}{60}=\frac{9}{4}$ hours in meetings. Hence she spent $100\cdot\frac{9/4}{9}=25$ percent of her time in meetings.

3. **Answer (C):** If a set of 4 socks does not contain a pair, there must be one of each color. The fifth sock must match one of the others and guarantee a matching pair.

4. **Answer (C):** Note that $\heartsuit(1)=\dfrac{1+1^2}{2}=1$, $\heartsuit(2)=\dfrac{2+2^2}{2}=3$, and $\heartsuit(3)=\dfrac{3+3^2}{2}=6$. Thus $\heartsuit(1)+\heartsuit(2)+\heartsuit(3)=1+3+6=10$.

5. **Answer (B):** A month with 31 days has 3 successive days of the week appearing five times and 4 successive days of the week appearing four times. If Monday and Wednesday appear five times then Monday must be the first day of the month. If Monday and Wednesday appear only four times then either Thursday or Friday must be the first day of the month. Hence there are 3 days of the week that could be the first day of the month.

6. **Answer (B):** Note that $\angle AOC=180°-50°=130°$. Because $\triangle AOC$ is isosceles, $\angle CAB=\frac{1}{2}(180°-130°)=25°$.



OR

By the Inscribed Angle Theorem, $\angle CAB = \frac{1}{2}(\angle COB) = \frac{1}{2}(50^\circ) = 25^\circ$.

7. **Answer (D):** Let the triangle be ABC with $AB = 12$, and let D be the foot of the altitude from C. Then $\triangle ACD$ is a right triangle with hypotenuse $AC = 10$ and one leg $AD = \frac{1}{2}AB = 6$. By the Pythagorean Theorem $CD = \sqrt{10^2 - 6^2} = 8$, and the area of $\triangle ABC$ is $\frac{1}{2}(AB)(CD) = \frac{1}{2}(12)(8) = 48$. The rectangle has length $\frac{48}{4} = 12$ and perimeter $2(12+4) = 32$.

8. **Answer (E):** The cost of an individual ticket must divide 48 and 64. The common factors of 48 and 64 are $1, 2, 4, 8$, and 16. Each of these may be the cost of one ticket, so there are 5 possible values for x.

9. **Answer (D):** The correct answer was $1-(2-(3-(4+e))) = 1-2+3-4-e = -2-e$. Larry's answer was $1-2-3-4+e = -8+e$. Therefore $-2-e = -8+e$, so $e = 3$.

10. **Answer (C):** Let t be the number of minutes Shelby spent driving in the rain. Then she traveled $20\frac{t}{60}$ miles in the rain, and $30\frac{40-t}{60}$ miles in the sun. Solving $20\frac{t}{60} + 30\frac{40-t}{60} = 16$ results in $t = 24$ minutes.

11. **Answer (A):** Let p dollars be the purchase price of the stem. The savings provided by Coupon A, B, and C respectively are $0.15p$, 30, and $0.25(p-100)$. Coupon A saves at least as much as Coupon B if $0.15p \geq 30$, so $p \geq 200$. Coupon A saves at least as much as Coupon C if $0.15p \geq 0.25(p-100)$, so $p \leq 250$. Therefore $x = 200$, $y = 250$, and $y - x = 50$.

12. **Answer (D):** Assume there are 100 students in Mr. Wells' class. Then at least $70 - 50 = 20$ students answered "No" at the beginning of the school year and "Yes" at the end, so $x \geq 20$. Because only 30 students answered "No" at the end of the school year, at least $50-30 = 20$ students who answered "Yes" at the beginning of the year gave the same answer at the end, so $x \leq 80$. The difference between the maximum and minimum possible values of x is $80 - 20 = 60$. The minimum $x = 20$ is achieved if exactly 20 students answered "No" at the beginning and "Yes" at the end of the school year. The maximum $x = 80$ is achieved if exactly 20 students answered "Yes" at the beginning and the end.

13. **Answer (C):** If $60 - 2x > 0$, then $|2x - |60 - 2x|| = |4x - 60|$. Solving $x = 4x - 60$, and $x = -(4x - 60)$ results in $x = 20$, and $x = 12$, respectively, both of which satisfy the original equation.
If $60 - 2x < 0$, then $|2x - |60 - 2x|| = |2x + 60 - 2x| = 60$. Note that $x = 60$ satisfies the original equation. The sum of the solutions is $12 + 20 + 60 = 92$.

14. **Answer (B):** The average of the numbers is

$$\frac{1+2+\cdots+99+x}{100} = \frac{\frac{99\cdot100}{2}+x}{100} = \frac{99\cdot50+x}{100} = 100x.$$

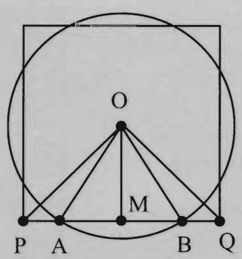
This equation is equivalent to $9999x = (99 \cdot 101)x = 99 \cdot 50$, so $x = \frac{50}{101}$.

15. **Answer (C):** If Jesse answered R questions correctly and W questions incorrectly, then $R + W \leq 50$, and Jesse's score is $99 = 4R - W \geq 4R - (50 - R) = 5R - 50$. Thus $5R \leq 149$, and because R is an integer, $R \leq 29$, Jesse could achieve a score of 99 by answering 29 questions correctly and 17 incorrectly, leaving 4 answers blank.

16. **Answer (B):** Let O be the common center of the circle and the square. Let M be the midpoint of a side of the square and P and Q be the vertices of the square on the side containing M. Since

$$OM^2 = \left(\frac{1}{2}\right)^2 < \left(\frac{\sqrt{3}}{3}\right)^2 < \left(\frac{\sqrt{2}}{2}\right)^2 = OP^2 = OQ^2,$$

the midpoint of each side is inside the circle and the vertices of the square are outside the circle. Therefore the circle intersects the square in two points along each side.



Let A and B be the intersection points of the circle with $\overline{PQ}$. Then M is also the midpoint of $\overline{AB}$ and $\triangle OMA$ is a right triangle. By the Pythagorean Theorem $AM = \frac{1}{2\sqrt{3}}$, so $\triangle OMA$ is a $30-60-90^\circ$ right triangle. Then $\angle AOB = 60^\circ$, and the area of the sector corresponding to $\angle AOB$ is $\frac{1}{6} \cdot \pi \cdot \left(\frac{\sqrt{3}}{3}\right)^2 = \frac{\pi}{18}$. The area of $\triangle AOB$ is $2 \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2\sqrt{3}} = \frac{\sqrt{3}}{12}$. The area outside the square but inside the circle is $4 \cdot (\frac{\pi}{18} - \frac{\sqrt{3}}{12}) = \frac{2\pi}{9} - \frac{\sqrt{3}}{3}$.

17. **Answer (B):** If there are n schools in the city, then there are $3n$ contestants, so $3n \geq 64$, and $n \geq 22$. Because Andrea received the median score and each student received a different score, n is odd, so $n \geq 23$. Andrea's position is $\frac{3n+1}{2}$, and Andrea finished ahead of Beth, so $\frac{3n+1}{2} < 37$, and $3n < 73$. Because n is an odd integer, $n \leq 23$. Therefore $n = 23$.

18. **Answer (E):** Let $N = abc + ab + a = a(bc + b + 1)$. If a is divisible by 3, then N is divisible by 3. Note that 2010 is divisible by 3, so the probability that a is divisible by 3 is $\frac{1}{3}$.

    If a is not divisible by 3 then N is divisible by 3 if $bc + b + 1$ is divisible by 3. Define b_0 and b_1 so that $b = 3b_0 + b_1$ is an integer and b_1 is equal to 0, 1, or 2. Note that each possible value of b_1 is equally likely. Similarly define c_0 and c_1. Then

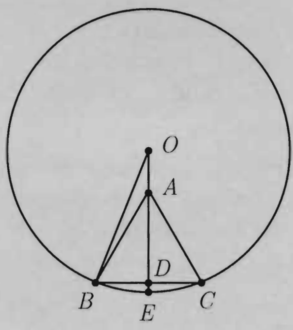
$$\begin{aligned} bc + b + 1 &= (3b_0 + b_1)(3c_0 + c_1) + 3b_0 + b_1 + 1 \\ &= 3(3b_0c_0 + c_0b_1 + c_1b_0 + b_0) + b_1c_1 + b_1 + 1. \end{aligned}$$

    Hence $bc + b + 1$ is divisible by 3 if and only if $b_1 = 1$ and $c_1 = 1$, or $b_1 = 2$ and $c_1 = 0$. The probability of this occurrence is $\frac{1}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{1}{3} = \frac{2}{9}$.
    Therefore the requested probability is $\frac{1}{3} + \frac{2}{3} \cdot \frac{2}{9} = \frac{13}{27}$.

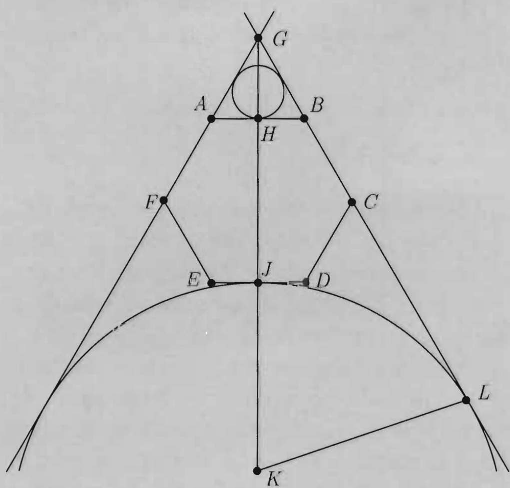
19. **Answer (B):** The radius of circle O is $\sqrt{156} > 4\sqrt{3} = OA$, so A is inside the circle. Let s be the side length of $\triangle ABC$, let D be the foot of the altitude from A, and let $\overline{OE}$ be the radius through A. This radius is perpendicular to $\overline{BC}$ and contains D, so $OD = \sqrt{OB^2 - BD^2} = \sqrt{156 - \frac{1}{4}s^2}$. If A is on $\overline{DE}$, then $\angle BAC > \angle BEC > 90°$, an impossibility. Therefore A lies on $\overline{OD}$, and $OA = OD - AD$, that is,

$$4\sqrt{3} = \sqrt{156 - \frac{1}{4}s^2} - \frac{\sqrt{3}}{2}s.$$

    Rearranging terms and squaring both sides leads to the quadratic equation $s^2 + 12s - 108 = 0$, and the positive solution is $s = 6$.



20. **Answer (D):** It may be assumed that hexagon $ABCDEF$ has side length 1. Let lines BC and FA intersect at G, let H and J be the midpoints of $\overline{AB}$ and $\overline{DE}$, respectively, let K be the center of the second circle, and let that circle be tangent to line BC at L. Equilateral $\triangle ABG$ has side length 1, so the first circle, which is the inscribed circle of $\triangle ABG$, has radius $\frac{\sqrt{3}}{6}$. Let r be the radius of the second circle. Then $\triangle GLK$ is a $30-60-90°$ right triangle with $LK = r$ and $2r = GK = GH + HJ + JK = \frac{\sqrt{3}}{2} + \sqrt{3} + r$. Therefore $r = \frac{3\sqrt{3}}{2} = 9(\frac{\sqrt{3}}{6})$. The ratio of the radii of the two circles is 9, and the ratio of their areas is $9^2 = 81$.



21. **Answer (E):** Each four-digit palindrome has digit representation $abba$ with $1 \le a \le 9$ and $0 \le b \le 9$. The value of the palindrome is $1001a + 110b$. Because 1001 is divisible by 7 and 110 is not, the palindrome is divisible by 7 if and only if $b = 0$ or $b = 7$. Thus the requested probability is $\frac{2}{10} = \frac{1}{5}$.

22. **Answer (C):** If there were no restrictions on the number of candies per bag, then each piece of candy could be distributed in 3 ways. In this case there would be 3^7 ways to distribute the candy. However, this counts the cases where the red bag or blue bag is empty.

    If the red bag remained empty then the candy could be distributed in 2^7 ways. The same is true for the blue bag. Both totals include the case in which all the candy is put into the white bag. Hence there are $2^7 + 2^7 - 1$ ways to distribute the candy such that either the red or blue bag is empty.

    The number of ways to distribute the candy, subject to the given conditions, is $3^7 - (2^7 + 2^7 - 1) = 1932$.

23. **Answer (D):** Let a_{ij} denote the entry in row i and column j. he given conditions imply that $a_{11} = 1$, $a_{33} = 9$, and $a_{22} = 4, 5$, or 6. If $a_{22} = 4$, then $\{a_{12}, a_{21}\} = \{2, 3\}$, and the sets $\{a_{31}, a_{32}\}$ and $\{a_{13}, a_{23}\}$ are complementary subsets of $\{5, 6, 7, 8\}$. There are $\binom{4}{2} = 6$ ways to choose $\{a_{31}, a_{32}\}$ and $\{a_{13}, a_{23}\}$, and only one way to order the entries. There are 2 ways to order $\{a_{12}, a_{21}\}$, so 12 arrays with $a_{22} = 4$ meet the given conditions. Similarly, the conditions are met by 12 arrays with $a_{22} = 6$. If $a_{22} = 5$, then $\{a_{12}, a_{13}, a_{23}\}$ and $\{a_{21}, a_{31}, a_{32}\}$ are complementary subsets of $\{2, 3, 4, 6, 7, 8\}$ subject to the conditions $a_{12} < 5$, $a_{21} < 5$, $a_{32} > 5$, and $a_{23} > 5$. Thus $\{a_{12}, a_{13}, a_{23}\} \neq \{2, 3, 4\}$ or $\{6, 7, 8\}$, so its elements can be chosen in $\binom{6}{3} - 2 = 18$ ways. Both the remaining entries and the ordering of all entries are then determined, so 18 arrays with $a_{22} = 5$ meet the given conditions.

    Altogether, the conditions are met by $12 + 12 + 18 = 42$ arrays.

24. **Answer (E):** The Raiders' score was $a(1 + r + r^2 + r^3)$, where a is a positive integer and $r > 1$. Because ar is also an integer, $r = m/n$ for relatively prime positive integers m and n with $m > n$. Moreover $ar^3 = a \cdot \frac{m^3}{n^3}$ is an integer, so n^3 divides a. Let $a = n^3 A$. Then the Raiders' score was $R = A(n^3 + mn^2 + m^2 n + m^3)$, and the Wildcats' score was $R-1 = a+(a+d)+(a+2d)+(a+3d) = 4a+6d$ for some positive integer d. Because $A \geq 1$, the condition $R \leq 100$ implies that $n \leq 2$ and $m \leq 4$. The only possibilities are $(m, n) = (4, 1), (3, 2), (3, 1)$, or $(2, 1)$. The corresponding values of R are, respectively, $85A$, $65A$, $40A$, and $15A$. In the first two cases $A = 1$, and the corresponding values of $R - 1$ are, respectively, $64 = 32 + 6d$ and $84 = 4 + 6d$. In neither case is d an integer. In the third case $40A = 40a = 4a + 6d + 1$ which is impossible in integers. In the last case $15a = 4a + 6d + 1$, from which $11a = 6d + 1$. The only solution in positive integers for which $4a+6d \leq 100$ is $(a, d) = (5, 9)$. Thus $R = 5+10+20+40 = 75$, $R - 1 = 5 + 14 + 23 + 32 = 74$, and the number of points scored in the first half was $5 + 10 + 5 + 14 = 34$.

25. **Answer (B):** Because $1, 3, 5$, and 7 are roots of the polynomial $P(x) - a$, it follows that

$$P(x) - a = (x-1)(x-3)(x-5)(x-7)Q(x),$$

where $Q(x)$ is a polynomial with integer coefficients. The previous identity must hold for $x = 2, 4, 6$, and 8, thus

$$-2a = -15Q(2) = 9Q(4) = -15Q(6) = 105Q(8).$$

Therefore $315 = \text{lcm}(15, 9, 105)$ divides a, that is a is an integer multiple of 315. Let $a = 315A$. Because $Q(2) = Q(6) = 42A$, it follows that $Q(x) - 42A = (x-2)(x-6)R(x)$ where $R(x)$ is a polynomial with integer coefficients. Because $Q(4) = -70A$ and $Q(8) = -6A$ it follows that $-112A = -4R(4)$ and $-48A = 12R(8)$, that is $R(4) = 28A$ and $R(8) = -4A$. Thus $R(x) = 28A + (x-4)(-6A + (x-8)T(x))$ where $T(x)$ is a polynomial with integer coefficients. Moreover, for any polynomial $T(x)$ and any integer A, the polynomial $P(x)$ constructed this way satisfies the required conditions. The required minimum is obtained when $A = 1$ and so $a = 315$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steven Blasberg, Steven Davis, Steven Dunbar, Douglas Faires, Michelle Ghrist, Jerrold Grossman, Joe Kennedy, Leon La Spina, Raymond Scacalossi, William Wardlaw, David Wells, LeRoy Wenstrom, Woody Wenstrom and Ron Yannone.

The

AMERICAN MATHEMATICS COMPETITIONS

are Sponsored by

The Mathematical Association of America

The Akamai Foundation

Contributors

Academy of Applied Sciences

American Mathematical Association of Two-Year Colleges

American Mathematical Society

American Statistical Association

Art of Problem Solving

Awesome Math

Canada/USA Mathcamp

Casualty Actuarial Society

IDEA Math

Institute for Operations Research and the Management Sciences

MathPath

Math Zoom Academy

Mu Alpha Theta

National Council of Teachers of Mathematics

Pi Mu Epsilon

Society of Actuaries

U.S.A. Math Talent Search

W. H. Freeman and Company

Wolfram Research Inc.

MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

12th Annual

AMC 10 A

American Mathematics Contest 10 A

Tuesday, February 8, 2011

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom
lwenstrom@gmail.com

1. **Answer (D):** The text messages cost $\$0.05 \cdot 100 = \$5.00$, and the 30 minutes of excess chatting cost $\$0.10 \cdot 30 = \$3.00$. Therefore the total bill came to $\$5 + \$3 + \$20 = \$28$.

2. **Answer (E):** Because $14 \cdot 35 = 490 < 500$ and $15 \cdot 35 = 525 \geq 500$, the minimum number of bottles that she needs to buy is 15.

3. **Answer (D):** First note that $\{1\ 1\ 0\} = \frac{2}{3}$ and $[0\ 1] = \frac{1}{2}$. Therefore
$$\{\{1\ 1\ 0\}\ [0\ 1]\ 0\} = \{\frac{2}{3}\ \frac{1}{2}\ 0\} = \frac{\frac{2}{3} + \frac{1}{2} + 0}{3} = \frac{7}{18}.$$

4. **Answer (A):** Every term in X except 10 appears in Y. Every term in Y except 102 appears in X. Therefore $Y - X = 102 - 10 = 92$.

   OR

   The sum X has 46 terms because it includes all 50 even positive integers less than or equal to 100 except for 2, 4, 6, and 8. The sum Y has the same number of terms, and every term in Y exceeds the corresponding term in X by 2. Therefore $Y - X = 46 \cdot 2 = 92$.

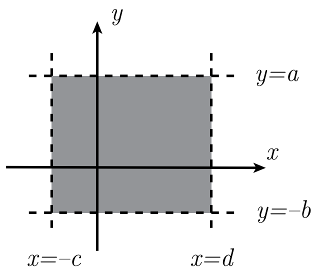
5. **Answer (C):** Let N equal the number of fifth graders. Then there are $2N$ fourth graders and $4N$ third graders. The total number of minutes run per day by the students is $4N \cdot 12 + 2N \cdot 15 + N \cdot 10 = 88N$. There are a total of $4N + 2N + N = 7N$ students, so the average number of minutes run by the students per day is $\frac{88N}{7N} = \frac{88}{7}$.

6. **Answer (C):** The union must contain all of the elements of A, so it has at least 20 elements. It is possible that B is a subset of A, in which case there are no additional elements.

7. **Answer (B):** Because $|-3x|+5$ is strictly positive, the equation $|-3x|+5 = 0$ has no solution. The solutions of equations **(A)**, **(C)**, **(D)**, and **(E)** are -7, -4, 64, and $\pm\frac{4}{3}$, respectively.

8. **Answer (C):** Because 75% of the birds were not swans and 30% of the birds were geese, it follows that $\frac{30}{75} \cdot 100\% = 40\%$ of the birds that were not swans were geese.

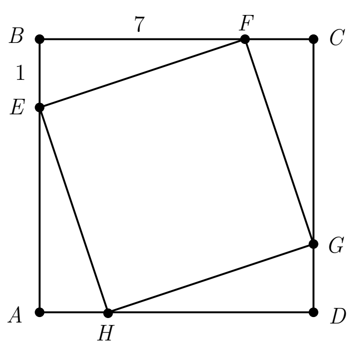
9. **Answer (A):** Because a, b, c, and d are positive numbers, $a > -b$ and $d > -c$. Therefore the height of the rectangle is $a + b$ and the width is $c + d$. The area of the region is $(a + b)(c + d) = ac + ad + bc + bd$.



10. **Answer (B):** Let C be the cost of a pencil in cents, N be the number of pencils each student bought, and S be the number of students who bought pencils. Then $C \cdot N \cdot S = 1771 = 7 \cdot 11 \cdot 23$, and $C > N > 1$. Because a majority of the students bought pencils, $30 \geq S > \frac{30}{2} = 15$. Therefore $S = 23, N = 7$, and $C = 11$.

11. **Answer (B):** Without loss of generality, assume that F lies on $\overline{BC}$ and that $EB = 1$. Then $AE = 7$ and $AB = 8$. Because $EFGH$ is a square, $BF = AE = 7$, so the hypotenuse $\overline{EF}$ of $\triangle EBF$ has length $\sqrt{1^2 + 7^2} = \sqrt{50}$. The ratio of the area of $EFGH$ to that of $ABCD$ is therefore

$$\frac{EF^2}{AB^2} = \frac{50}{64} = \frac{25}{32}.$$



12. **Answer (A):** Let x, y, and z be the number of successful three-point shots, two-point shots, and free throws, respectively. Then the given conditions imply

$$\begin{aligned} &3x + 2y + z = 61, \\ &2y = 3x, \text{ and} \\ &y + 1 = z. \end{aligned}$$

Solving results in $x = 8$, $y = 12$, and $z = 13$. Hence the team made 13 free throws.

13. **Answer (A):** Because the numbers are even, they must end in either 2 or 8. If the last digit is 2, the first digit must be 5 and thus there are four choices remaining for the middle digit. If the last digit is 8, then there are two choices for the first digit, either 2 or 5, and for each choice there are four possibilities for the middle digit. The total number of choices is then $4 + 2 \cdot 4 = 12$.

14. **Answer (B):** Let d be the sum of the numbers rolled. The conditions are satisfied if and only if $\pi\left(\frac{d}{2}\right)^2 < \pi d$, that is, $d < 4$. Of the 36 equally likely outcomes for the roll of the two dice, one has a sum of 2 and two have sums of 3. Thus the desired probability is $\frac{1+2}{36} = \frac{1}{12}$.

15. **Answer (C):** Let x be the number of miles driven exclusively on gasoline. Then the total number of miles traveled is $x + 40$, and the amount of gas used is $0.02x$ gallons. Therefore the average number of miles per gallon is

$$\frac{x+40}{0.02x} = 55\,.$$

Solving results in $x = 400$, so the total number of miles traveled is 440.

16. **Answer (B):** Let $k = \sqrt{9 - 6\sqrt{2}} + \sqrt{9 + 6\sqrt{2}}$. Squaring both sides and simplifying results in

$$\begin{aligned} k^2 &= 9 - 6\sqrt{2} + 2\sqrt{(9 - 6\sqrt{2})(9 + 6\sqrt{2})} + 9 + 6\sqrt{2} \\ &= 18 + 2\sqrt{81 - 72} \\ &= 18 + 2\sqrt{9} \\ &= 24 \end{aligned}$$

Because $k > 0$, $k = 2\sqrt{6}$.

17. **Answer (C):** Note that for any four consecutive terms, the first and last terms must be equal. For example, consider B, C, D, and E; because

$$B + C + D = 30 = C + D + E\,,$$

we must have $B = E$. Hence $A = D = G$, and $C = F = 5$. The required sum $A + H = G + (30 - G - F) = 30 - 5 = 25$.

OR

Note that

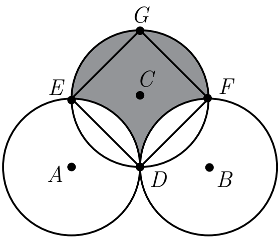
$$\begin{aligned} A + C + H &= (A + B + C) - (B + C + D) + (C + D + E) \\ &\quad - (E + F + G) + (F + G + H) \\ &= 3 \cdot 30 - 2 \cdot 30 = 30. \end{aligned}$$

Hence $A + H = 30 - C = 25$.

18. **Answer (C):** Let D be the midpoint of $\overline{AB}$, and let circle C intersect circles A and B at E and F, respectively, distinct from D. The shaded portion of unit square $ADCE$ has area $1 - \frac{\pi}{4}$, as does the shaded portion of unit square $BDCF$. The portion of the shaded region which is outside these squares is a semicircle of radius 1 and has area $\frac{\pi}{2}$. The total shaded area is $2\left(1 - \frac{\pi}{4}\right) + \frac{\pi}{2} = 2$.

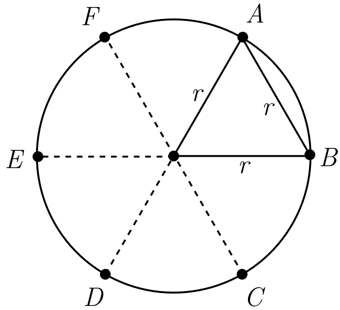
OR

Let D, E, and F be defined as in the first solution, and let G be diametrically opposite D on circle C. The shaded area is equal to the area of square $DFGE$, which has diagonal length 2. Its side length is $\sqrt{2}$, and its area is $\left(\sqrt{2}\right)^2 = 2$.



19. **Answer (E):** Let p^2, $q^2 + 9$, and $r^2 = p^2 + 300$ be the populations of the town in 1991, 2001, and 2011, respectively. Then $q^2 + 9 = p^2 + 150$, so $q^2 - p^2 = 141$. Therefore $(q - p)(q + p) = 141$, and so either $q - p = 3$ and $q + p = 47$, or $q - p = 1$ and $q + p = 141$. These give $p = 22$ or $p = 70$. Note that if $p = 70$, then $70^2 + 300 = 5200 = 52 \cdot 10^2$, which is not a perfect square. Thus $p = 22$, $p^2 = 484$, $p^2 + 150 = 634 = 25^2 + 9$, and $p^2 + 300 = 784 = 28^2$. The percent growth from 1991 to 2011 was $\frac{784-484}{484} \approx 62\%$.

20. **Answer (D):** Let point A be the first point chosen, and let point B be the opposite endpoint of the corresponding chord. Drawing a radius to each endpoint of this chord of length r results in an equilateral triangle. Hence a chord of length r subtends an arc $\frac{1}{6}$ the circumference of the circle. Let diameter $\overline{FC}$ be parallel to $\overline{AB}$, and divide the circle into six equal portions as shown. The second point chosen will result in a chord that intersects $\overline{AB}$ if and only if the point is chosen from minor $\widehat{FB}$. Hence the probability is $\frac{1}{3}$.



21. **Answer (D):** The weights of the two pairs of coins are equal if each pair contains the same number of counterfeit coins. Therefore either the first pair and the second pair both contain only genuine coins, or the first pair and the second pair both contain one counterfeit coin. The number of ways to choose the coins in the first case is $\binom{8}{2} \cdot \binom{6}{2} = 420$. The number of ways to choose the coins in the second case is $8 \cdot 2 \cdot 7 \cdot 1 = 112$. Therefore the requested probability is $\frac{420}{112+420} = \frac{15}{19}$.

22. **Answer (C):** If five distinct colors are used, then there are $\binom{6}{5} = 6$ different color choices possible. They may be arranged in $5! = 120$ ways on the pentagon, resulting in $120 \cdot 6 = 720$ colorings.

If four distinct colors are used, then there is one duplicated color, so there are $\binom{6}{4}\binom{4}{1} = 60$ different color choices possible. The duplicated color must appear on neighboring vertices. There are 5 neighbor choices and $3! = 6$ ways to color the remaining three vertices, resulting in a total of $60 \cdot 5 \cdot 6 = 1800$ colorings.

If three distinct colors are used, then there must be two duplicated colors, so there are $\binom{6}{3}\binom{3}{2} = 60$ different color choices possible. The non-duplicated color may appear in 5 locations. As before, a duplicated color must appear on neighboring vertices, so there are 2 ways left to color the remaining vertices. In this case there are $60 \cdot 5 \cdot 2 = 600$ colorings possible.

There are no colorings with two or fewer colors. The total number of colorings is $720 + 1800 + 600 = 3120$.

23. **Answer (C):** After each person counts, the numbers left for the next person form an arithmetic progression. For example, Alice leaves all of the numbers $2, 5, 8, 11, 14, \ldots, 2 + 3 \cdot 332$ for Barbara. If a student leaves the progression $a, a+d, a+2d, a+3d, a+4d, \ldots$, then the next student leaves the progression $a+d, (a+d)+3d, (a+d)+6d, \ldots$.

This implies that in the following table, each number in the third column is three times the previous entry in the third column, and each entry in the second column is the sum of the two entries in the row above:

Left for	First Term	Common Difference
Alice	1	1
Barbara	2	3
Candice	5	9
Debbie	14	27
Eliza	41	81
Fatima	122	243
George	365	729

George is left with the single term 365.

OR

The numbers skipped by Alice are the middle numbers in each consecutive group of 3, that is, $2, 5, 8$, and so on. The numbers skipped by Alice and Barbara are the middle numbers in each group of 9, that is, $5, 14, 23$, and so on. In general, the numbers skipped by all of the first n students are the middle numbers in each group of 3^n. Because $3^6 = 729$, the only number not exceeding 1000 that is skipped by the first six students is $\frac{729+1}{2} = 365$. That is the number that George says.

24. **Answer (D):** Let the tetrahedra be T_1 and T_2, and let R be their intersection. Let squares $ABCD$ and $EFGH$, respectively, be the top and bottom faces of the unit cube, with E directly under A and F directly under B. Without loss of generality, T_1 has vertices A, C, F, and H, and T_2 has vertices B, D, E, and G. One face of T_1 is $\triangle ACH$, which intersects edges of T_2 at the midpoints J, K, and L of $\overline{AC}$, $\overline{CH}$, and $\overline{HA}$, respectively. Let S be the tetrahedron with vertices J, K, L, and D. Then S is similar to T_2 and is contained in T_2, but not in R. The other three faces of T_1 each cut off from T_2 a tetrahedron congruent to S. Therefore the volume of R is equal to the volume of T_2 minus four times the volume of S.

A regular tetrahedron of edge length s has base area $\frac{\sqrt{3}}{4}s^2$ and altitude $\frac{\sqrt{6}}{3}s$, so its volume is $\frac{1}{3}\left(\frac{\sqrt{3}}{4}s^2\right)\left(\frac{\sqrt{6}}{3}s\right) = \frac{\sqrt{2}}{12}s^3$. Because the edges of tetrahedron T_2 are face diagonals of the cube, T_2 has edge length $\sqrt{2}$. Because J and K are centers of adjacent faces of the cube, tetrahedron S has edge length $\frac{\sqrt{2}}{2}$. Thus the volume of R is

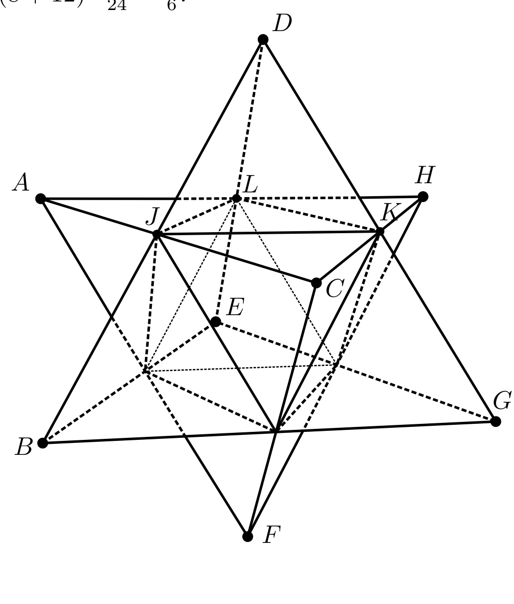
$$\frac{\sqrt{2}}{12}\left((\sqrt{2})^3 - 4\left(\frac{\sqrt{2}}{2}\right)^3\right) = \frac{1}{6}.$$

OR

Let T_1 and T_2 be labeled as in the previous solution. The cube is partitioned by T_1 and T_2 into 8 tetrahedra congruent to $DJKL$ (one for every vertex of the cube), 12 tetrahedra congruent to $AJLD$ (one for every edge of the cube), and the solid $T_1 \cap T_2$. Because the bases AJL and JLK are equilateral triangles with the same area, and the altitudes to vertex D of the tetrahedra $AJLD$ and $DJKL$ are the same, it follows that the volumes of $AJLD$ and $DJKL$ are equal. Moreover,

$$\text{Volume}(AJLD) = \frac{1}{3}\text{Area}(ALD) \cdot h_J,$$

where $h_J = \frac{1}{2}$ is the distance from J to the face ALD, and $\text{Area}(ALD) = \frac{1}{4}$. Therefore $\text{Volume}(AJLD) = \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{2} = \frac{1}{24}$, and thus the volume of $T_1 \cap T_2$ is equal to $1 - (8+12) \cdot \frac{1}{24} = \frac{1}{6}$.



25. **Answer (C):** Assume without loss of generality that R is bounded by the square with vertices $A = (0,0)$, $B = (1,0)$, $C = (1,1)$, and $D = (0,1)$, and let $X = (x,y)$ be n-ray partitional. Because the n rays partition R into triangles, they must include the rays from X to A, B, C, and D. Let the number of rays intersecting the interiors of $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ be n_1, n_2, n_3, and n_4, respectively. Because $\triangle ABX \cup \triangle CDX$ has the same area as $\triangle BCX \cup \triangle DAX$,

it follows that $n_1+n_3 = n_2+n_4 = \frac{n}{2}-2$, so n is even. Furthermore, the n_1+1 triangles with one side on $\overline{AB}$ have equal area, so each has area $\frac{1}{2}\cdot\frac{1}{n_1+1}\cdot y$. Similarly, the triangles with sides on $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ have areas $\frac{1}{2}\cdot\frac{1}{n_2+1}\cdot(1-x)$, $\frac{1}{2}\cdot\frac{1}{n_3+1}\cdot(1-y)$, and $\frac{1}{2}\cdot\frac{1}{n_4+1}\cdot x$, respectively. Setting these expressions equal to each other gives

$$x = \frac{n_4+1}{n_2+n_4+2} = \frac{2(n_4+1)}{n} \text{ and } y = \frac{n_1+1}{n_1+n_3+2} = \frac{2(n_1+1)}{n}.$$

Thus an n-ray partitional point must have the form $X = (\frac{2a}{n}, \frac{2b}{n})$ with $1 \le a < \frac{n}{2}$ and $1 \le b < \frac{n}{2}$. Conversely, if X has this form, R is partitioned into n triangles of equal area by the rays from X that partition $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ into b, $\frac{n}{2}-a$, $\frac{n}{2}-b$, and a congruent segments, respectively.

Assume X is 100-ray partitional. If X is also 60-ray partitional, then $X = (\frac{a}{50}, \frac{b}{50}) = (\frac{c}{30}, \frac{d}{30})$ for some integers $1 \le a, b \le 49$ and $1 \le c, d \le 29$. Thus $3a = 5c$ and $3b = 5d$; that is, both a and b are multiples of 5. Conversely, if a and b are multiples of 5, then

$$X = \left(\frac{a}{50}, \frac{b}{50}\right) = \left(\frac{\frac{3a}{5}}{30}, \frac{\frac{3b}{5}}{30}\right)$$

is 60-ray partitional. Because there are exactly 9 multiples of 5 between 1 and 49, the required number of points X is equal to $49^2 - 9^2 = 40 \cdot 58 = 2320$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steve Dunbar, Michelle Ghrist, Peter Gilchrist, Jerrold Grossman, Dan Kennedy, Joe Kennedy, David Torney, David Wells, LeRoy Wenstrom, and Ron Yannone.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

Academy of Applied Sciences
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Casualty Actuarial Society
D.E. Shaw & Co.
IDEA Math
Institute for Operations Research and the Management Sciences
Jane Street
MathPath
Math Zoom Academy
Mu Alpha Theta
National Council of Teachers of Mathematics
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company

MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

12th Annual

AMC 10 B

American Mathematics Contest 10 B

Wednesday, February 23, 2011

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 10 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 10 and orders for any publications should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606, Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AMC 10 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 10 Subcommittee Chair:

Dr. Leroy Wenstrom
lwenstrom@gmail.com

1. **Answer (C):** The given expression is equal to

$$\frac{12}{9} - \frac{9}{12} = \frac{4}{3} - \frac{3}{4} = \frac{16-9}{12} = \frac{7}{12}.$$

2. **Answer (E):** The sum of her first 5 test scores is 385, yielding an average of 77. To raise her average to 80, her 6^{th} test score must be the difference between $6 \cdot 80 = 480$ and 385, which is 95.

3. **Answer (A):** The smallest possible width for the rectangle is $2 - 0.5 = 1.5$ inches. Similarly the smallest possible length is 2.5 inches. Hence the minimum area is $(1.5)(2.5) = 3.75$ square inches.

4. **Answer (C):** Bernardo has paid $B - A$ dollars more than LeRoy. If LeRoy gives Bernardo half of that difference, $\frac{B-A}{2}$, then each will have paid the same amount.

5. **Answer (E):** Because $161 = 23 \cdot 7$, the only two digit factor of 161 is 23. The correct product must have been $32 \cdot 7 = 224$.

6. **Answer (A):** Let x be Casper's original number of candies. After the first day he was left with $x - (\frac{1}{3}x + 2) = \frac{2}{3}x - 2$ candies. On the second day he ate $\frac{1}{3}(\frac{2}{3}x - 2)$ candies, gave away 4 candies, and was left with 8 candies. Therefore

$$\frac{2}{3}x - 2 - \left(\frac{1}{3}\left(\frac{2}{3}x - 2\right) + 4\right) = 8.$$

Solving for x results in $x = 30$.

OR

Before giving 4 candies to his sister, Casper had 12. This was $\frac{2}{3}$ of the number he had after the first day, so he had 18 after the first day. Before giving 2 candies to his brother, he had 20, and this was $\frac{2}{3}$ of the number he had originally. Therefore he had 30 candies at the beginning.

7. **Answer (B):** The degree measures of two of the angles have a sum of $\frac{6}{5} \cdot 90 = 108$ and a positive difference of 30, so their measures are 69 and 39. The remaining angle has a degree measure of $180 - 108 = 72$, which is the largest angle.

8. **Answer (B):** Because the beach was not crowded on June 10, at least one of the conditions was not met. That is, the weather might have been cooler than $80\,^\circ F$ and sunny, at least $80\,^\circ F$ and cloudy, or cooler than $80\,^\circ F$ and cloudy. The first possibility shows that **(A)** and **(E)** are invalid, the second shows that **(C)** is invalid, and the third shows that **(D)** is invalid. Only conclusion **(B)** is consistent with all three possibilities.

9. **Answer (D):** The area of $\triangle ABC$ is $\frac{1}{2} \cdot 3 \cdot 4 = 6$, so the area of $\triangle EBD$ is $\frac{1}{3} \cdot 6 = 2$. Note that $\triangle ABC$ and $\triangle EBD$ are right triangles with an angle in common, so they are similar. Therefore BD and DE are in the ratio 4 to 3. Let $BD = x$ and $DE = \frac{3}{4}x$. Then the area of $\triangle EBD$ can be expressed as $\frac{1}{2} \cdot x \cdot \frac{3}{4}x = \frac{3}{8}x^2$. Because $\triangle EBD$ has area 2, solving yields $BD = \frac{4\sqrt{3}}{3}$.

OR

Because $\triangle EBD$ and $\triangle ABC$ are similar triangles, their areas are in the ratio of the squares of their corresponding linear parts. Therefore $\left(\frac{BD}{4}\right)^2 = \frac{1}{3}$ and $BD = \frac{4\sqrt{3}}{3}$.

10. **Answer (B):** The sum of the smallest ten elements is

$$1 + 10 + 100 + \cdots + 1{,}000{,}000{,}000 = 1{,}111{,}111{,}111.$$

Hence the desired ratio is

$$\frac{10{,}000{,}000{,}000}{1{,}111{,}111{,}111} = \frac{9{,}999{,}999{,}999 + 1}{1{,}111{,}111{,}111} = 9 + \frac{1}{1{,}111{,}111{,}111} \approx 9.$$

OR

The sum of a finite geometric series of the form $a(1 + r + r^2 + \cdots + r^n)$ is $\frac{a}{1-r}(1 - r^{n+1})$. The desired denominator $1 + 10 + 10^2 + \cdots + 10^9$ is a finite geometric series with $a = 1, r = 10$, and $n = 9$. Therefore the ratio is

$$\frac{10^{10}}{1 + 10 + 10^2 + \cdots + 10^9} = \frac{10^{10}}{\frac{1}{1-10}(1 - 10^{10})} = \frac{10^{10}}{10^{10} - 1} \cdot 9 \approx \frac{10^{10}}{10^{10}} \cdot 9 = 9.$$

11. **Answer (D):** If no more than 4 people have birthdays in any month, then at most 48 people would be accounted for. Therefore the statement is true for $n = 5$. The statement is false for $n \geq 6$ if, for example, 5 people have birthdays in each of the first 4 months of the year, and 4 people have birthdays in each of the last 8 months, for a total of $5 \cdot 4 + 4 \cdot 8 = 52$ people.

OR

The average number of birthdays per month is $\frac{52}{12}$, which is strictly between 4 and 5. Therefore at least one month must contain at least 5 birthdays, and, as above, it is possible to distribute the birthdays so that all months contain 4 or 5 birthdays.

12. **Answer (A):** The only parts of the track that are longer walking on the outside edge rather than the inside edge are the two semicircular portions. If the radius of the inner semicircle is r, then the difference in the lengths of the two paths is $2\pi(r+6) - 2\pi r = 12\pi$. Let x be her speed in meters per second. Then $36x = 12\pi$, and $x = \frac{\pi}{3}$.

13. **Answer (D):** Consider all ordered pairs (a, b) with each of the numbers a and b in the closed interval $[-20, 10]$. These pairs fill a 30×30 square in the coordinate plane, with an area of 900 square units. Ordered pairs in the first and third quadrants have the desired property, namely $a \cdot b > 0$. The areas of the portions of the 30×30 square in the first and third quadrants are $10^2 = 100$ and $20^2 = 400$, respectively. Therefore the probability of a positive product is $\frac{100+400}{900} = \frac{5}{9}$.

OR

Each of the numbers is positive with probability $\frac{1}{3}$ and negative with probability $\frac{2}{3}$. Their product is positive if and only if both numbers are positive or both are negative, so the requested probability is $\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{5}{9}$.

14. **Answer (C):** Let x and y be the length and width of the parking lot, respectively. Then $xy = 168$ and $x^2 + y^2 = 25^2$. Note that

$$(x+y)^2 = x^2 + y^2 + 2xy = 25^2 + 2 \cdot 168 = 961.$$

Hence the perimeter is $2(x+y) = 2 \cdot \sqrt{961} = 62$.

Note that the dimensions of the parking lot are 7 and 24 meters.

15. **Answer (E):** If $x \neq 0$, then I is false:

$$x \mathbin{@} (y+z) = \frac{x+(y+z)}{2} \neq \frac{x+y+x+z}{2} = \frac{x+y}{2} + \frac{x+z}{2} = (x \mathbin{@} y) + (x \mathbin{@} z).$$

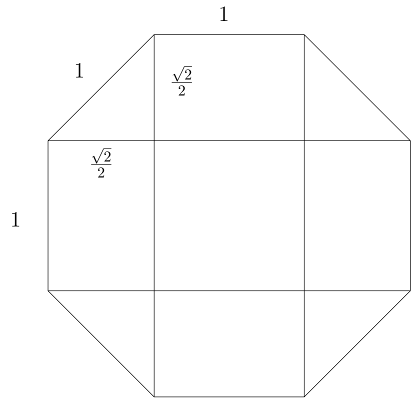
On the other hand, II and III are true for all values of x, y and z:

$$x + (y \mathbin{@} z) = x + \frac{y+z}{2} = \frac{2x+y+z}{2} = \frac{(x+y)+(x+z)}{2} = (x+y) \mathbin{@} (x+z),$$

and

$$x @ (y @ z) = \frac{x + \frac{y+z}{2}}{2} = \frac{\left(\frac{2x+y+z}{2}\right)}{2} = \frac{\frac{x+y}{2} + \frac{x+z}{2}}{2} = (x @ y) @ (x @ z)$$

16. **Answer (A):** Assume the octagon's edge is 1. Then the corner triangles have hypotenuse 1 and thus legs $\frac{\sqrt{2}}{2}$ and area $\frac{1}{4}$ each; the four rectangles are 1 by $\frac{\sqrt{2}}{2}$ and have area $\frac{\sqrt{2}}{2}$ each, and the center square has area 1. The total area is $4 \cdot \frac{1}{4} + 4 \cdot \frac{\sqrt{2}}{2} + 1 = 2 + 2\sqrt{2}$. The probability that the dart hits the center square is $\frac{1}{2+2\sqrt{2}} = \frac{\sqrt{2}-1}{2}$.



17. **Answer (C):** Angle EAB is 90° because it subtends a diameter. Therefore angles BEA and ABE are 40° and 50°, respectively. Angle DEB is 50° because $\overline{AB}$ is parallel to $\overline{ED}$. Also, $\angle DEB$ is supplementary to $\angle CDE$, so $\angle CDE = 130^\circ$. Because $\overline{EB}$ and $\overline{DC}$ are parallel chords, $ED = BC$ and $EBCD$ is an isosceles trapezoid. Thus $\angle BCD = \angle CDE = 130^\circ$.

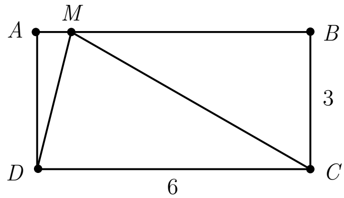
OR

Let O be the center of the circle. Establish, as in the first solution, that $\angle EAB = 90^\circ$, $\angle BEA = 40^\circ$, $\angle ABE = 50^\circ$, and $\angle DEB = 50^\circ$. Thus $\overline{AD}$ is a diameter and $\angle AOE = 100^\circ$. By the Inscribed Angle Theorem

$$\angle BCD = \frac{1}{2}(\angle BOA + \angle AOE + \angle EOD) = \frac{1}{2}(80^\circ + 100^\circ + 80^\circ) = 130^\circ .$$

18. **Answer (E):** Sides $\overline{AB}$ and $\overline{CD}$ are parallel, so $\angle CDM = \angle AMD$. Because $\angle AMD = \angle CMD$, it follows that $\triangle CMD$ is isosceles and $CD = CM = 6$.

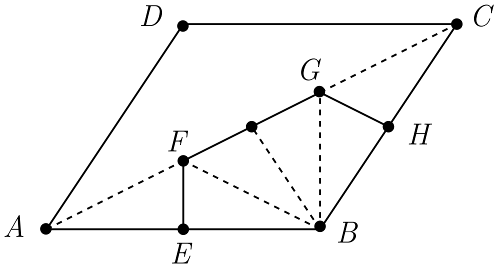
Therefore $\triangle MCB$ is a $30-60-90°$ right triangle with $\angle BMC = 30°$. Finally, $2 \cdot \angle AMD + 30° = \angle AMD + \angle CMD + 30° = 180°$, so $\angle AMD = 75°$.



19. **Answer (A):** The right side of the equation is defined only when $|x| \geq 4$. If $x \geq 4$, the equation is equivalent to $5x + 8 = x^2 - 16$, and the only solution with $x \geq 4$ is $x = 8$. If $x \leq -4$, the equation is equivalent to $8 - 5x = x^2 - 16$, and the only solution with $x \leq -4$ is $x = -8$. The product of the solutions is $-8 \cdot 8 = -64$.

20. **Answer (C):** Let E and H be the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. The line drawn perpendicular to $\overline{AB}$ through E divides the rhombus into two regions: points that are closer to vertex A than B, and points that are closer to vertex B than A. Let F be the intersection of this line with diagonal $\overline{AC}$. Similarly, let point G be the intersection of the diagonal $\overline{AC}$ with the perpendicular to $\overline{BC}$ drawn from the midpoint of $\overline{BC}$. Then the desired region R is the pentagon $BEFGH$.

Note that $\triangle AFE$ is a $30-60-90°$ triangle with $AE = 1$. Hence the area of $\triangle AFE$ is $\frac{1}{2} \cdot 1 \cdot \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{6}$. Both $\triangle BFE$ and $\triangle BGH$ are congruent to $\triangle AFE$, so they have the same areas. Also $\angle FBG = 120° - \angle FBE - \angle GBH = 60°$, so $\triangle FBG$ is an equilateral triangle. In fact, the altitude from B to $\overline{FG}$ divides $\triangle FBG$ into two triangles, each congruent to $\triangle AFE$. Hence the area of $BEFGH$ is $4 \cdot \frac{\sqrt{3}}{6} = \frac{2\sqrt{3}}{3}$.



21. **Answer (B):** The largest pairwise difference is 9, so $w - z = 9$. Let n be either x or y. Because n is between w and z,

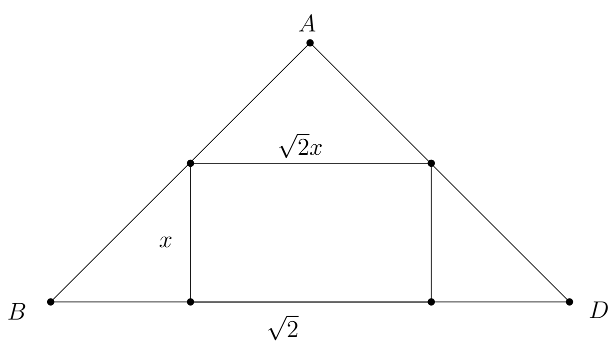
$$9 = w - z = (w - n) + (n - z).$$

Therefore the positive differences $w - n$ and $n - z$ must sum to 9. The given pairwise differences that sum to 9 are $3 + 6$ and $4 + 5$. The remaining pairwise difference must be $x - y = 1$.

The second largest pairwise difference is 6, so either $w - y = 6$ or $x - z = 6$. In the first case the set of four numbers may be expressed as $\{w, w-5, w-6, w-9\}$. Hence $4w - 20 = 44$, so $w = 16$. In the second case $w - x = 3$, and the four numbers may be expressed as $\{w, w-3, w-4, w-9\}$. Therefore $4w - 16 = 44$, so $w = 15$. The sum of the possible values for w is $16 + 15 = 31$.

Note: The possible sets of four numbers are $\{16, 11, 10, 7\}$ and $\{15, 12, 11, 6\}$.

22. **Answer (A):** Let A be the apex of the pyramid, and let the base be the square $BCDE$. Then $AB = AD = 1$ and $BD = \sqrt{2}$, so $\triangle BAD$ is an isosceles right triangle. Let the cube have edge length x. The intersection of the cube with the plane of $\triangle BAD$ is a rectangle with height x and width $\sqrt{2}x$. It follows that $\sqrt{2} = BD = 2x + \sqrt{2}x$, from which $x = \sqrt{2} - 1$.



Hence the cube has volume

$$(\sqrt{2} - 1)^3 = (\sqrt{2})^3 - 3(\sqrt{2})^2 + 3\sqrt{2} - 1 = 5\sqrt{2} - 7.$$

OR

Let A be the apex of the pyramid, let O be the center of the base, let P be the midpoint of one base edge, and let the cube intersect $\overline{AP}$ at Q. Let a coordinate plane intersect the pyramid so that O is the origin, A on the positive y-axis, and $P = \left(\frac{1}{2}, 0\right)$. Segment AP is an altitude of a lateral side of the pyramid, so $AP = \frac{\sqrt{3}}{2}$, and it follows that $A = \left(0, \frac{\sqrt{2}}{2}\right)$. Thus the equation of line AP

is $y = \frac{\sqrt{2}}{2} - \sqrt{2}x$. If the side length of the cube is s, then $Q = (\frac{s}{2}, s)$, so $s = \frac{\sqrt{2}}{2} - \sqrt{2} \cdot \frac{s}{2}$. Solving gives $s = \sqrt{2} - 1$, and the result follows that in the first solution.

23. **Answer (D):** In the expansion of $(2000 + 11)^{2011}$, all terms except 11^{2011} are divisible by 1000, so the hundreds digit of 2011^{2011} is equal to that of 11^{2011}. Furthermore, in the expansion of $(10 + 1)^{2011}$, all terms except 1^{2011}, $\binom{2011}{1}(10)\left(1^{2010}\right)$, and $\binom{2011}{2}(10)^2\left(1^{2009}\right)$ are divisible by 1000. Thus the hundreds digit of 2011^{2011} is equal to that of

$$\begin{aligned} &1 + \binom{2011}{1}(10)\left(1^{2010}\right) + \binom{2011}{2}(10)^2\left(1^{2009}\right) \\ &= 1 + 2011 \cdot 10 + 2011 \cdot 1005 \cdot 100 \\ &= 1 + 2011 \cdot 100510. \end{aligned}$$

Finally, the hundreds digit of this number is equal to that of $1 + 11 \cdot 510 = 5611$, so the requested hundreds digit is 6.

24. **Answer (B):** For $0 < x \le 100$, the nearest lattice point directly above the line $y = \frac{1}{2}x + 2$ is $\left(x, \frac{1}{2}x + 3\right)$ if x is even and $\left(x, \frac{1}{2}x + \frac{5}{2}\right)$ if x is odd. The slope of the line that contains this point and $(0, 2)$ is $\frac{1}{2} + \frac{1}{x}$ if x is even and $\frac{1}{2} + \frac{1}{2x}$ if x is odd. The minimum value of the slope is $\frac{51}{100}$ if x is even and $\frac{50}{99}$ if x is odd. Therefore the line $y = mx + 2$ contains no lattice point with $0 < x \le 100$ for $\frac{1}{2} < m < \frac{50}{99}$.

25. **Answer (D):** Let $T_n = \triangle ABC$. Suppose $a = BC$, $b = AC$, and $c = AB$. Because $\overline{BD}$ and $\overline{BE}$ are both tangent to the incircle of $\triangle ABC$, it follows that $BD = BE$. Similarly, $AD = AF$ and $CE = CF$. Then

$$\begin{aligned} 2BE &= BE + BD = BE + CE + BD + AD - (AF + CF) \\ &= a + c - b, \end{aligned}$$

that is, $BE = \frac{1}{2}(a + c - b)$. Similarly $AD = \frac{1}{2}(b + c - a)$ and $CF = \frac{1}{2}(a + b - c)$. In the given $\triangle ABC$, suppose that $AB = x + 1$, $BC = x - 1$, and $AC = x$. Using the formulas for BE, AD, and CF derived before, it must be true that

$$\begin{aligned} BE &= \frac{1}{2}((x - 1) + (x + 1) - x) = \frac{1}{2}x, \\ AD &= \frac{1}{2}(x + (x + 1) - (x - 1)) = \frac{1}{2}x + 1, \text{ and} \\ CF &= \frac{1}{2}((x - 1) + x - (x + 1)) = \frac{1}{2}x - 1. \end{aligned}$$

Hence both (BC, CA, AB) and (CF, BE, AD) are of the form $(y-1, y, y+1)$. This is independent of the values of a, b, and c, so it holds for all T_n. Furthermore, adding the formulas for BE, AD, and CF shows that the perimeter of T_{n+1} equals $\frac{1}{2}(a+b+c)$, and consequently the perimeter of the last triangle T_N in the sequence is

$$\frac{1}{2^{N-1}}(2011 + 2012 + 2013) = \frac{1509}{2^{N-3}}.$$

The last member T_N of the sequence will fail to define a successor if for the first time the new lengths fail the Triangle Inequality, that is, if

$$-1 + \frac{2012}{2^N} + \frac{2012}{2^N} \leq 1 + \frac{2012}{2^N}.$$

Equivalently, $2012 \leq 2^{N+1}$ which happens for the first time when $N = 10$. Thus the required perimeter of T_N is $\frac{1509}{2^7} = \frac{1509}{128}$.

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Steven Davis, Steve Dunbar, Doug Faires, Sister Josannae Furey, Michelle Ghrist, Peter Gilchrist, Jerrold Grossman, Joe Kennedy, Eugene Veklerov, David Wells, LeRoy Wenstrom, and Ron Yannone.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

Academy of Applied Sciences
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Casualty Actuarial Society
D.E. Shaw & Co.
IDEA Math
Institute for Operations Research and the Management Sciences
Jane Street
MathPath
Math Zoom Academy
Mu Alpha Theta
National Council of Teachers of Mathematics
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company