AIME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

1st ANNUAL AMERICAN INVITATIONAL MATHEMATICS EXAMINATION 1983

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COMMITTEE ON HIGH SCHOOL CONTESTS

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1. (Answer: 60)

   Converting each of the given logarithms into exponential form gives $x^{24} = w$, $y^{40} = w$, $(xyz)^{12} = w$. It follows that

$$z^{12} = \frac{w}{x^{12}y^{12}} = \frac{w}{w^{1/2}\,w^{3/10}} = w^{1/5}.$$

   Thus $w = z^{60}$ and $\log_z w = 60$.

2. (15)

   Since $0 < p \le x \le 15$, then $|x-p| = x-p$, $|x-15| = 15-x$, and $|x-(p+15)| = p+15-x$. Thus

$$f(x) = (x-p)+(15-x)+(p+15-x) = 30-x.$$

   It follows that $f(x)$ is least when x is greatest, and that the answer is 15.

3. (20)

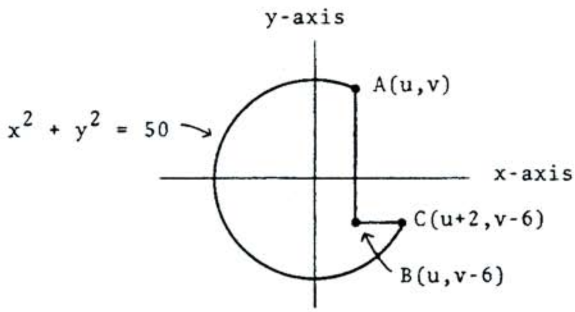
   Substitute to simplify. Several choices work well; the following substitution, which eliminates the radical immediately, is perhaps best. Define u to be the nonnegative number such that $u^2 = x^2+18x+45$. (There is such a u, for if $x^2+18x+45$ were negative, the right-hand side of the original equation would be undefined.) So

$$\begin{aligned} u^2-15 &= 2\sqrt{u^2} = 2u \quad (\text{since } u \ge 0), \\ u^2-2u-15 &= 0, \\ (u-5)(u+3) &= 0. \end{aligned}$$

   Since $u \ge 0$, we have $u = 5$. That is, x is a solution of the original equation iff (if and only if) $x^2+18x+45 = 5^2$, i.e., iff $x^2+18x+20 = 0$. Both solutions to this last equation are real (why?) and their product is the constant term, 20. (Note: by being careful about the sign of u and by using iff-arguments we have avoided introducing any extraneous roots for x.)

4\. (26)

We introduce a coordinate system as shown in the figure: WLOG (Without Loss Of Generality) we have put the top corner of the notch in the First Quadrant and have positioned the sides of the notch parallel to the axes. Letting $A = (u,v)$ it follows, using the given dimensions, that $B = (u,v-6)$ and $C = (u+2,v-6)$. Our problem is to determine $u^2 + (v-6)^2$. Since A and C are



both on the circle $x^2 + y^2 = 50$, we may substitute their coordinates into this equation to get the system

$$u^2 + v^2 = 50,$$
$$(u + 2)^2 + (v - 6)^2 = 50.$$

Subtracting the first equation from the second leads to $u - 3v + 10 = 0$. Solving for u and substituting into the first equation gives

$$(3v - 10)^2 + v^2 = 50,$$
$$v^2 - 6v + 5 = 0,$$
$$v = 1,\ u = -7 \quad \text{or} \quad v = 5,\ u = 5.$$

Since $A = (u,v)$ is in the First Quadrant, we have $A = (5,5)$ and $B = (5,-1)$. Finally, the square of the distance from B to the origin is $25 + 1 = 26$.

5. (4)

We are given that $x^2 + y^2 = 7$ and $x^3 + y^3 = 10$. Because we are asked to find the sum $x + y$, rather than x or y individually, we are moved to rewrite these equations so as to exhibit the sum:

$$(1) \qquad \begin{aligned} x^2 + y^2 &= (x + y)^2 - 2xy = 7, \\ x^3 + y^3 &= (x + y)^3 - 3xy(x + y) = 10. \end{aligned}$$

This further suggests defining

$$(2) \qquad s = x + y, \quad p = xy.$$

Substituting (2) into (1) gives

$$(3) \qquad \begin{aligned} s^2 - 2p &= 7, \\ s^3 - 3ps &= 10. \end{aligned}$$

Solving for p in the first line of (3) and substituting the result into the second line gives

$$(4) \quad s^3 - 3s\left(\frac{s^2 - 7}{2}\right) = 10 \iff s^3 - 21s + 20 = 0.$$

An obvious root of this is $s = 1$. Dividing the cubic by $s - 1$ gives the factorization $(s - 1)(s^2 + s - 20) = 0$ and thence

$$(s - 1)(s - 4)(s + 5) = 0.$$

Thus <u>all</u> values of $s = x + y$ are real and the largest is 4.

There is a gap in this argument. Clearly any solution to (1) has led by substitution to some solution to (4), but we must also show that the largest solution for s in (4) is one which arises this way. We show more, that every solution for s in (4) arises this way. This converse is hardly automatic, as the equations are not linear.

Given any s satisfying (4), set $p = \frac{s^2 - 7}{2}$. Then (4)$\Rightarrow$(3). Next, for this pair (s,p), there necessarily exists a pair (x,y) satisfying (2), namely, the (possibly complex) roots of $z^2 - sz + p = 0$. Finally, substituting (2) into (3) gets us back to (1). That is, there is a solution (x,y) to (1) with $x + y = s$.

Alternate Solution. We use the method of Newton sums for determining

$$S_n = x^n + y^n, \quad n \geq 0,$$

where x and y are any two complex numbers. Setting $s = x+y$, $p = xy$, we have again that x and y are the roots of $z^2 - sz + p = 0$. We claim that

$$(1) \qquad S_{n+2} - s\,S_{n+1} + p\,S_n = 0, \quad n \geq 0.$$

This is shown by adding $x^n(x^2 - sx + p) = x^n \cdot 0 = 0$ and $y^n(y^2 - sy + p) = 0$. Now let x and y be the desired solutions to

$$(2) \qquad x^2 + y^2 = 7, \quad x^3 + y^3 = 10.$$

That is, we are given $S_2 = 7$ and $S_3 = 10$. We seek $S_1 = s$. We also know that $S_0 = x^0 + y^0 = 2$. Thus setting $n = 0$ and then $n = 1$ in (1), we obtain

$$(3) \qquad \begin{aligned} 7 - s^2 + 2p &= 0, \\ 10 - 7s + ps &= 0. \end{aligned}$$

Thus $p = \dfrac{s^2 - 7}{2}$ and

$$(4) \quad 10 - 7s + \frac{s^3 - 7s}{2} = 0 \iff s^3 - 21s + 20 = 0.$$

Thus, as before, $s = -5, 1, 4$ and the largest value is 4.

Also as before, the argument is not complete: every solution to (2) yields a solution to (4), but we also want the converse. The converse can be proved by methods similar to those in the first solution but a little more involved. Can the reader do it?

6. (35)

For n odd we may write

$$\begin{aligned} a_n &= (7-1)^n + (7+1)^n \\ &= \left[7^n - \tbinom{n}{1}7^{n-1} + \cdots -1\right] + \left[7^n + \tbinom{n}{1}7^{n-1} + \cdots + 1\right] \\ &= 2\left(7^n + \tbinom{n}{2}7^{n-2} + \cdots + \tbinom{n}{n-3}7^3 + \tbinom{n}{n-1}7\right) \end{aligned}$$

$$= 2\cdot 49\left[7^{n-2} + \binom{n}{2}7^{n-4} + \cdots + \binom{n}{n-3}7\right] + 14n.$$

It follows that $a_{83} = 49k + 14\cdot 83 = 49k + 1162$, where k is an integer. Thus, the remainder on dividing a_{83} by 49 is the same as the remainder on dividing 1162 by 49. That remainder is 35.

7. (57)

It is perhaps easier to think in terms of n knights, where $n > 4$. We observe first that there are $\binom{n}{3}$ ways of selecting 3 knights if there are no restrictions. But how many of these threesomes include at least two table neighbors?

First, there are n ways to pick three neighboring knights. (Consider each knight along with the two knights to his immediate right.) Second, there are n ways to pick two neighboring knights (as with three) followed by $(n-4)$ ways of picking a third non-neighboring knight. (We must avoid the pair and the two knights on either side.) Thus, there are $n(n-4)$ threesomes that include exactly two neighbors.

Letting P_n be the probability that at least two of the three chosen knights had been neighbors, it follows that

$$P_n = \frac{n + n(n-4)}{\binom{n}{3}} = \frac{6(n-3)}{(n-1)(n-2)} .$$

Then $P = P_{25} = \frac{11}{46}$ and the required sum is 57.

8. (61)

Let p be a prime less than 100. Then

$$n = \frac{1\cdot 2\cdot 3\cdots p\cdots 2p\cdots 3p\cdots\cdots kp\cdots 200}{(1\cdot 2\cdot 3\cdots p\cdots 2p\cdots\cdots jp\cdots 100)^2} .$$

Thus, if in addition $p^2 > 200$, we have

$$n = (\text{a } p\text{-free integer})\cdot\frac{p^k}{p^{2j}} .$$

Now, such a prime divides n iff $k > 2j$. Once we show that there is at least one prime meeting these conditions, our answer will be the largest such prime, because any p with $p^2 \le 200$ will be smaller. Since $kp \le 200$, the requirement that p be as large as possible leads to our choosing k as small as possible. Thus we take $k = 3$ and $j = 1$, for which the largest p is 61. Since $61^2 > 200$, the second display above is correct for $p = 61$. Thus 61 divides n and is the largest 2-digit prime to do so.

9. (12)

Dividing out gives

$$f(x) = 9x\sin x + \frac{4}{x\sin x} \quad .$$

Call the first term on the right u, the second v, and note that uv is constant. This suggests using the geometric-arithmetic mean inequality:

$$\frac{u+v}{2} \ge \sqrt{uv} \ ,$$

where u, v are any nonnegative numbers and equality holds iff $u = v$. Applying this inequality to our u and v gives

$$\frac{f(x)}{2} \ge \sqrt{9\cdot 4} \ , \quad \text{or} \quad f(x) \ge 12.$$

The value 12 is actually attained iff there is an x for which

$$9x\sin x = \frac{4}{x\sin x}, \quad \text{i.e.,} \quad x^2\sin^2 x = \frac{4}{9}.$$

Since $x^2\sin^2 x$ is 0 when $x = 0$ and it exceeds 1 when $x = \pi/2$, it follows that it equals the intermediate value 4/9 somewhere between 0 and $\pi/2$. Thus the minimum value of $f(x)$ is indeed 12.

10. (432)

We split the set of numbers of the type described into two subsets and make two separate subcounts:

1) Those numbers having two 1's. The second may be placed in any one of three positions and the two other numbers (distinct) may be placed in $9\cdot 8$ ways. Thus, there are $3\cdot 9\cdot 8 = 216$ such numbers.
2) Those numbers having two of some digit other than 1. The pair of identical digits may be selected in 9 ways and then

placed in 3 ways (positions 2 and 3, 2 and 4, or 3 and 4). The remaining number may be selected in 8 ways. Thus there are $9 \cdot 3 \cdot 8 = 216$ such numbers.

Thus, the answer is $216 + 216 = 432$. (It is coincidental, perhaps remarkable, that the two subtotals are the same. This is not so for, say, the 5-digit version of this problem. Try it!)

Alternate Solution. Consider 4 cells (for the 4-digit number) as below:

Select two of the cells as the ones that are to receive the equal digits. Call the left of these cells L and the right one R. This can be done in $\binom{4}{2} = 6$ ways. For instance, one might have

	L	R	

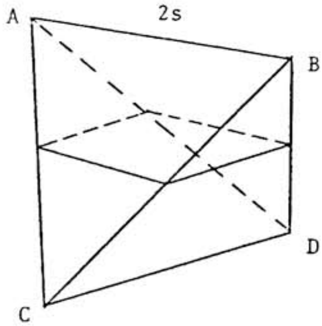
Starting with the first cell, fill in all cells except R with distinct numbers. This can be done in $1 \cdot 9 \cdot 8$ ways (since there is only one choice, the number 1, for the first cell, which is never R). Finally, fill R with the same number as was inserted in L. This can be done in just 1 way. Thus, the "product of ways" is $6 \cdot 1 \cdot 9 \cdot 8 \cdot 1 = 432$.

11. (288)

Consider the regular tetrahedron ABCD, shown on the next page, that has edge length $2s$. Connect the midpoints of AC, BC, AD and BD. The quadrilateral thus determined is a square (why?) and the plane of this square divides the tetrahedron into two solids that are identical with the solid given in the problem. Thus, we have only to find the volume of the tetrahedron and then divide by 2.

The formula for the volume of a regular tetrahedron of side length e — its derivation requires only right-triangle trigonometry — is

$$V = \frac{\sqrt{2}}{12} e^3 .$$



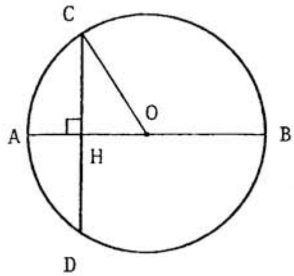
Thus, substituting $e = 2s = 12\sqrt{2}$ and taking $\frac{1}{2}V$ gives 288.

12. (65)

Let $AB = 10t + u$, where t and u are digits. Then $CD = 10u + t$. We must now find t and u for which OH will be a positive rational. Evidently, $t \geq u$; to insure that OH is positive we take $t > u$.

Since ΔOCH is a right triangle, we may use the Pythagorean Theorem to express OH in terms of AB and CD:

$$\begin{aligned} OH &= \sqrt{(OC)^2 - (CH)^2} \\ &= \frac{1}{2}\sqrt{(10t+u)^2 - (10u+t)^2} \\ &= \frac{3}{2}\sqrt{11(t^2 - u^2)} \ . \end{aligned}$$



It follows that OH is rational iff $\sqrt{11(t^2 - u^2)}$ is rational. But the square root of an integer is rational only if it is integral. Thus, we must find t and u for which $11(t^2 - u^2)$ is a perfect square, and this will be the case only if there is a positive integer m such that

$$t^2 - u^2 = 11m^2 ,$$
$$(t-u)(t+u) = 11m^2 .$$

Now 11 cannot divide $t-u$. (Why?) Therefore, it must divide $t+u$. But this is possible only if $t+u$ is 11. (Why?) It follows that $t-u = m^2$. Thus, we seek two numbers whose sum is 11 and whose difference is a perfect square.

Finally, we examine all t and u $(t>u)$ for which $t+u = 11$: $t=9$, $u=2$; $t=8$, $u=3$; $t=7$, $u=4$; and $t=6$, $u=5$. Only in the last case is $t-u$ a perfect square. Thus $AB = 65$.

Note. Without the condition that OH is positive rational, the problem has many answers. For example, take $AB = 52$ and $CD = 25$ (OH is irrational), or $AB = 77 = CD$ (OH is zero). It happens that (in base ten) there is just one case in which OH is a positive rational.

13. (448)

It is easier, perhaps, to generalize the problem (ever so slightly) by considering the alternating sums for <u>all</u> subsets of $\{1, 2, 3, \ldots, n\}$, that is, by including the empty set. To include the empty set without affecting the answer we have only to declare that its alternating sum be 0. The subsets of $\{1, 2, 3, \ldots, n\}$ may be divided into two kinds: those that do not contain n and those that do. Moreover, each subset of the first kind may be paired — in a one-to-one correspondence — with a subset of the second kind as follows:

$$\{a_1, a_2, a_3, \ldots, a_i\} \longleftrightarrow \{n, a_1, a_2, a_3, \ldots, a_i\} .$$

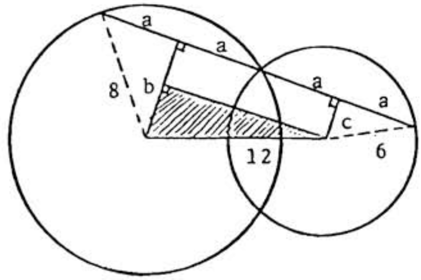
(For the empty set we have the correspondence $\phi \longleftrightarrow \{n\}$.) Then, assuming $n > a_1 > a_2 > \ldots > a_i$, the sum of the alternating sums for each such pair of subsets is given by

$$(a_1 - a_2 + a_3 - \cdots \pm a_i) + (n - a_1 + a_2 - a_3 + \cdots \mp a_i) \;=\; n.$$

And since there are 2^{n-1} such pairs of subsets (why?), the required sum is $n\,2^{n-1}$. Finally, taking $n = 7$. we obtain 448.

14. (130)

We complete the figure as shown. Applying the Pythagorean Theorem to the shaded triangle gives



$$(2a)^2 + (b-c)^2 = 12^2.$$

But $b = \sqrt{64-a^2}$ and $c = \sqrt{36-a^2}$. Thus

$$4a^2 + \left(\sqrt{64-a^2} - \sqrt{36-a^2}\right)^2 = 144.$$

Simplifying gives $a^2 - 22 = \sqrt{(64-a^2)(36-a^2)}$. Squaring and simplifying gives $4a^2 = 130$. Thus $(2a)^2 = 4a^2 = 130$.

15. (175)

Consider the family of all chords emanating from A. Then the locus of the endpoints of these chords is the given circle of radius 5 and the locus of their midpoints is an internally tangent circle (at A) of radius 5/2. (We have simply shrunk the given circle relative to A by a factor of 1/2.) Now BC must be tangent to the smaller circle. For if it cut that circle twice, as in Figure 1, then there would be two chords emanating from A that are bisected by BC.

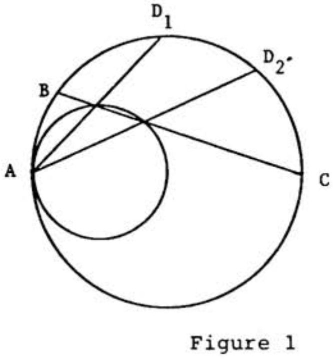


Figure 1

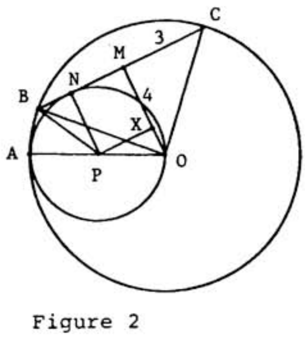


Figure 2

Next consider Figure 2, where O and P are the centers of the two circles and BC is tangent to the smaller circle (at N) as described before. We introduce the following construction lines: OA (through P), OB, OC, $MO \perp BC$, $NP \perp BC$, and $PX \parallel BC$. We seek $\sin \angle BOA$. We will find this by first finding all three sides of $\triangle BOP$. We have

1. $MC = 3$ and $MO = \sqrt{5^2 - 3^2} = 4$.
2. $NP = \frac{1}{2} AO = \frac{5}{2}$.
3. MNPX is a rectangle, so $MX = \frac{5}{2}$ and $OX = \frac{3}{2}$.
4. $\triangle PXO$ is a right triangle with $OP = \frac{5}{2}$ and $OX = \frac{3}{2}$, so $PX = \frac{4}{2} = 2$.
5. $MN = 2$ so $BN = 1$.
6. $\triangle BNP$ is a right triangle, so $BP^2 = 1 + \frac{25}{4} = \frac{29}{4}$.

We now have all three sides of $\triangle BOP$, so by the Law of Cosines applied to $\theta = \angle BOP$ we obtain

$$\frac{29}{4} = 25 + \frac{25}{4} - 2 \cdot 5 \cdot \frac{5}{2} \cos \theta,$$

$$\cos \theta = \frac{24}{25}.$$

Thus $\sin \theta = \sqrt{1 - \left(\frac{24}{25}\right)^2} = \frac{7}{25}$ and the desired answer is 175.

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1. (Answer: 93)

The sum of an arithmetic progression is the product of the number of terms and the arithmetic average of the first and last terms. Therefore, $a_1 + a_2 + \cdots + a_{98} = 98(a_1 + a_{98})/2 = 49(a_1 + a_{98}) = 137$. Similarly, $a_2 + a_4 + \cdots + a_{98} = 49(a_2 + a_{98})/2 = 49(a_1 + 1 + a_{98})/2 = (49(a_1 + a_{98})/2) + (49/2) = (137/2) + (49/2) = 93$.

Alternate Solution. Separating the terms of odd and even subscripts, let $S_o = a_1 + a_3 + \cdots + a_{97}$ and $S_e = a_2 + a_4 + \cdots + a_{98}$. Then each of these series has 49 terms, and since $a_{n+1} = a_n + 1$ for all $n \geq 1$, we have $S_o = S_e - 49$. Furthermore, $S_e + S_o = 137$. Substituting for S_o in the last equation, and solving for S_e, one finds that $S_e = 93$.

2. (592)

Note that n is a common multiple of 5 and 3. As a multiple of 5, n must end in 0, as the digit 5 is not allowed. As a multiple of 3, n must contain a number of 8's equal to a multiple of 3. Hence, in view of the minimality requirement, $n = 8880$, and $8880/15 = 592$ is the answer to the problem.

3. (144)

Let T denote the area of ΔABC, and denote by T_1, T_2 and T_3 the areas of t_1, t_2 and t_3, respectively. Moreover, let c be the length of AB, and let c_1, c_2 and c_3 be the lengths of the bases parallel to AB of t_1, t_2 and t_3, respectively. Then, in view of the similarity of the four triangles, one has

$$\frac{\sqrt{T_1}}{\sqrt{T}} = \frac{c_1}{c}, \quad \frac{\sqrt{T_2}}{\sqrt{T}} = \frac{c_2}{c} \quad \text{and} \quad \frac{\sqrt{T_3}}{\sqrt{T}} = \frac{c_3}{c}.$$

Moreover, since $c_1 + c_2 + c_3 = c$, it follows that

$$\frac{\sqrt{T_1}}{\sqrt{T}} + \frac{\sqrt{T_2}}{\sqrt{T}} + \frac{\sqrt{T_3}}{\sqrt{T}} = \frac{c_1 + c_2 + c_3}{c} = 1,$$

and therefore, $T = (\sqrt{T_1} + \sqrt{T_2} + \sqrt{T_3})^2 = (\sqrt{4} + \sqrt{9} + \sqrt{49})^2 = 144$.

Query: Can you extend the above result to higher dimensions?

4. (649)

Let n denote the number of positive integers in S, and let m denote their sum. Then, on the basis of the information given,

$$\frac{m}{n} = 56 \qquad \text{and} \qquad \frac{m - 68}{n - 1} = 55 .$$

Solving these equations simultaneously yields $n = 13$ and $m = 728$. Now, to maximize the largest element of S, one must minimize the others. To attain this, S must contain eleven 1's, a 68 and a 649, since $728 - 11 - 68 = 649$.

5. (512)

Adding the two equations and using standard log properties yields

$$\log_8 a + \log_8 b + \log_4 a^2 + \log_4 b^2 = \log_8(ab) + 2\log_4(ab) = 12.$$

Moreover, since $\log_8 x = (\log_2 x)/(\log_2 8) = \frac{1}{3}\log_2 x$, and similarly, $\log_4 x = (\log_2 x)/(\log_2 4) = \frac{1}{2}\log_2 x$, the above equation is equivalent to

$$\frac{4}{3}\log_2(ab) = 12.$$

It follows that $\log_2(ab) = 9$, and hence $ab = 2^9 = 512$.

6. (24)

First observe that the three circles are disjoint, i.e. their centers are more than 6 units apart. Next note that any line through (17,76) will divide the area of the circle centered there evenly. Thus the problem reduces to finding the value of m for which the line given by

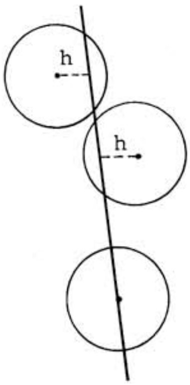
(1) $$y - 76 = m(x - 17)$$

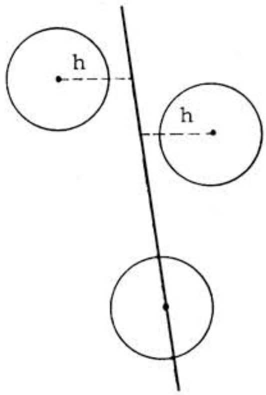
also divides the areas of the other two circles in the desired manner. To this end, note that such a line must pass exactly as far to the right of (14,92) as it passes to the left of (19,84). Denoting this distance by h, it follows that $(14+h,92)$ and $(19-h,84)$ must satisfy Equation (1), i.e.

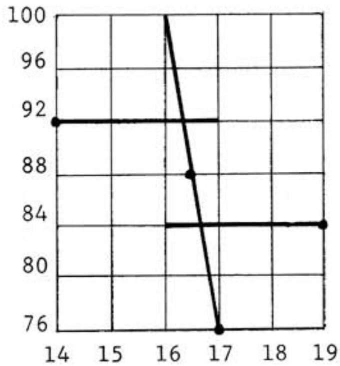
(2) $$\frac{16}{h-3} = m = \frac{8}{2-h},$$

from which $h = \frac{7}{3}$, $m = -24$ and the answer to the problem is 24.

Note. As shown in the first figure below, in this case the uniqueness of the solution is assured by the fact that $h < 3$, the common length of the circles' radii. If the circles were positioned differently with respect to one another, or if the value of h were larger than 3, as exemplified in the second figure below, then the resulting line would not necessarily yield a unique solution to the problem.







Alternate Solution. Note that the lines passing through (16.5 , 88), the center of symmetry of the two circles centered at (14,92) and (19,84), divide the areas of these two circles in the manner desired. Consequently, the line through (16.5 , 88) and (17,76), whose slope is $(88 - 76)/(16.5 - 17) = -24$, provides the answer to the problem. The uniqueness of the solution follows from observing that it intersects the radius from (14,92) to (17,92) of one circle and the radius from (16,84) to (19,84) of the other circle, as shown in the third figure above. (Note the difference in scales.)

7. (997)

Rather than assaulting $f(84)$ directly, it is advantageous to start with values of $f(n)$ near $n = 1000$, and search for a pattern when n is less than 1000, thereby utilizing the recursive definition of f.

Indeed, one finds that

$$(1)\qquad \begin{aligned} f(999) &= f(f(1004)) = f(1001) = 998\,, \\ f(998) &= f(f(1003)) = f(1000) = 997\,, \\ f(997) &= f(f(1002)) = f(999) \ \ = 998\,, \\ f(996) &= f(f(1001)) = f(998) \ \ = 997\,, \\ f(995) &= f(f(1000)) = f(997) \ \ = 998\,, \end{aligned}$$

on the basis of which one may conjecture that

$$(2)\qquad f(n) = \begin{cases} 997\,, & \text{if } n \text{ is even and } n < 1000\,, \\ 998\,, & \text{if } n \text{ is odd and } n < 1000\,. \end{cases}$$

To prove (2), it is convenient to use downward induction. In this, the inductive step is to prove (2) for n, assuming that it is true for all m, $n < m < 1000$. Since the definition relates $f(n)$ to $f(n+5)$, we can do the inductive step only when $n+5 < 1000$, that is, we must verify the truth of (2) for $n = 999, 998, \ldots, 995$ separately. This was done in (1). Now for $n < 995$,

$$f(n) = f(f(n+5)) = \begin{cases} f(997) = 998\,, & \text{if } n+5 \text{ is even}\,, \\ f(998) = 997\,, & \text{if } n+5 \text{ is odd}\,. \end{cases}$$

Noting that n is even when $n+5$ is odd, and that n is odd when $n+5$ is even, completes the proof. In particular, it follows from (2) that $f(84) = 997$.

8. (160)

Let $w = z^3$. Then the given equation reduces to

$$w^2 + w + 1 = 0\,,$$

whose solutions are $(-1+i\sqrt{3})/2$ and $(-1-i\sqrt{3})/2$, with arguments of $120°$ and $240°$, respectively. From these, one finds the following six values for the argument of z:

$$\frac{120°}{3},\ \frac{120°+360°}{3},\ \frac{120°+720°}{3},\ \frac{240°}{3},\ \frac{240°+360°}{3},\ \frac{240°+720°}{3}.$$

Clearly, only the second one of these, $160°$, is between $90°$ and $180°$.

Alternate Solution. Multiplying the given equation by $z^3 - 1 = 0$ yields

$$z^9 - 1 = 0,$$

whose solutions are the ninth roots of unity:

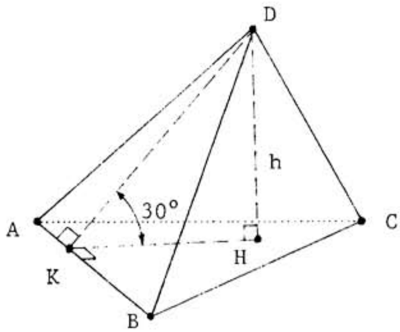
$$z_n = \cos(n \cdot 40^\circ) + i\sin(n \cdot 40^\circ), \quad n = 0, 1, 2, \dots, 8.$$

Of these, only z_3 and z_4 are in the second quadrant. However, since the solutions of $z^9 - 1 = 0$ are distinct, and since z_3 is a solution of $z^3 - 1 = 0$, it cannot be a solution of the original equation. It follows that the desired root is z_4, with degree measure 160.

9. (20)

Let V be the volume of tetrahedron $ABCD$ and h the altitude of the tetrahedron corresponding to D. Then $V = h \cdot \text{area}(\Delta ABC)/3$, which will be determined upon finding h.

By definition, for the angle between faces ABC and ABD to be 30°, planes perpendicular to AB must cut the two faces in rays which form a 30° angle. Choose such a plane through D, let K be its intersection with AB, and let H be the point on the line of intersection of the plane chosen and the plane of ΔABC so that $DH \perp KH$. These are shown in the adjoining figure.



Since $DK \perp AB$, we find that $DK = 2 \cdot \text{area}(\Delta ABD)/AB = 8\text{cm}$. Moreover, since ΔDKH is a 30°-60°-90° triangle, it follows that $h = DH = DK/2 = 4\text{cm}$. Consequently, substituting into the formula given in the first paragraph, we find that $V = 20\text{cm}^3$.

10. (119)

Given that $s = 30 + 4c - w > 80$, the problem calls for finding the lowest value of s, for which the corresponding value of c is unique. To this end, first observe that if $c + w \le 25$, then by increasing the number of correct answers by 1 and the number of wrong answers by 4, one attains the same score. (This is made possible by the equivalence of the above inequality to $(c+1) + (w+4) \le 30$). Consequently, one must have

(1) $$c + w \geq 26 .$$

Next observe that

(2) $$w \leq 3 ,$$

for otherwise one could reduce the number of wrong answers by 4 and the number of correct answers by 1, and still attain the same score. (The latter is possible since $s > 80$ clearly implies that $c \geq 13$.) Now to minimize s, we must minimize c and maximize w, subject to Inequalities (1) and (2) above. This leads to $w = 3$, $c = 23$ and $s = 30 + 4 \cdot 23 - 3 = 119$.

11. (106)

The 12 trees can be planted in $12!$ orders. Let k be the number of orders in which no two birch trees are adjacent to one another. The probability we need is $k/(12!)$. To find k, we will count the number of patterns

$$\underset{1}{\_\_} \ N \ \underset{2}{\_\_} \ N \ \underset{3}{\_\_} \ N \ \underset{4}{\_\_} \ N \ \underset{5}{\_\_} \ N \ \underset{6}{\_\_} \ N \ \underset{7}{\_\_} \ N \ \underset{8}{\_\_}$$

where the 7 N's denote nonbirch (i.e., maple and oak) trees, and slots 1 through 8 are to be occupied by birch trees, at most one in each slot. There are $7!$ orders for the nonbirch trees, and for each ordering of them there are $8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$ ways to place the birch trees. Thus, we find that $k = (7!) \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4$, $\frac{m}{n} = \frac{(7!) \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4}{12!} = \frac{7}{99}$ and $m + n = 106$.

Note. We have assumed in this solution that each tree is distinguishable. The problem can also be interpreted to mean that trees are distinguishable if and only if they are of different species. In that case, the calculations (i.e., the numerator and the denominator) are different, but the probability turns out to be the same.

12. (401)

First we use the given equations to find various numbers in the domain to which f assigns the same values. We find that

(1) $$f(x) = f(2+(x-2)) = f(2-(x-2)) = f(4-x)$$

and that

(2) $$f(4-x) = f(7-(x+3)) = f(7+(x+3)) = f(x+10).$$

From Equations (1) and (2) it follows that

(3) $$f(x+10) = f(x).$$

Replacing x by $x+10$ and then by $x-10$ in Equation (3), we get $f(x+10) = f(x+20)$ and $f(x-10) = f(x)$. Continuing in this way, it follows that

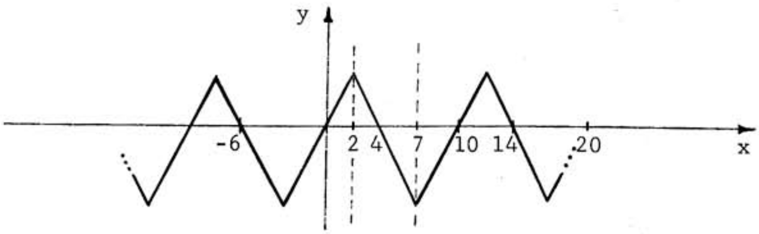
(4) $$f(x+10n) = f(x), \quad \text{for } n = \pm 1, \pm 2, \pm 3, \ldots .$$

Since $f(0) = 0$, Equation (4) implies that

$$f(\pm 10) = f(\pm 20) = \cdots = f(\pm 1000) = 0,$$

necessitating a total of 201 roots for the equation $f(x) = 0$ in the closed interval $[-1000, 1000]$.

Next note that by setting $x = 0$ in Equation (1), $f(4) = f(0) = 0$ follows. Therefore, setting $x = 4$ in Equation (4), we obtain 200 more roots for $f(x) = 0$ at $x = -996, -986, \ldots, -6, 4, 14, \ldots, 994$. Since the zig-zag function pictured below satisfies the given conditions and has precisely these and no other roots, the answer to the problem is 401.



Note. Geometrically, the conditions imply symmetry with respect to the lines $x = 2$ and $x = 7$. The solution above shows how to draw consequences from these symmetries through algebra. More generally, one can prove that if a function is symmetric with respect to $x = a$ and $x = b > a$, then it must also have translational symmetry for every integral multiple of $2(b-a)$. This fact can also be proven geometrically, leading to an alternate formulation of the solution.

13. (15)

In order to simplify the notation, let

(1) $a = \cot^{-1} 3$, $b = \cot^{-1} 7$, $c = \cot^{-1} 13$, $d = \cot^{-1} 21$,

(2) $e = a+b$ and $f = c+d$.

Then, in view of (1) and (2) above, the problem is equivalent to finding the value of $10\cot(e+f)$.

It is easy to show that in general,

(3) $$\cot(x+y) = \frac{(\cot x)(\cot y) - 1}{\cot x + \cot y}.$$

Moreover, even if $\cot^{-1}$ is viewed as a multiple valued function,

(4) $$\cot(\cot^{-1} x) = x, \quad \text{for all real } x.$$

Utilizing Equations (1) through (4) above, we find that

$$\cot e = \cot(a+b) = \frac{3\cdot 7 - 1}{3+7} = 2, \quad \cot f = \cot(c+d) = \frac{13\cdot 21 - 1}{13+21} = 8,$$

$$\cot(e+f) = \frac{2\cdot 8 - 1}{2+8} = \frac{3}{2},$$

and therefore, $10\cot(e+f) = 15$ is the answer to the problem.

Note. More generally, since the numbers 3, 7, 13 and 21 are all of the form $1+n+n^2$, one may attempt to express $\cot^{-1}(1+n+n^2)$ more advantageously. Indeed, it is not difficult to show that

$$\cot^{-1}(1+n+n^2) = \tan^{-1}(n+1) - \tan^{-1} n$$

within an integral multiple of π. Consequently, if k is a positive integer, then

$$\sum_{n=1}^{k} \cot^{-1}(1+n+n^2) = \tan^{-1}(k+1) - \tan^{-1} 1,$$

within an integral multiple of π. From this, one can prove that

$$\cot\left(\sum_{n=1}^{k} \cot^{-1}(1+n+n^2)\right) = \frac{k+2}{k}.$$

14. (38)

We will show that if k is an even integer and if $k \geq 40$, then k is expressible as the sum of two composites. This leaves 38 as the candidate for the largest even integer not expressible in such manner; it is easy to check that 38 indeed satisfies this requirement. The proof of our claim for $k \geq 40$ hinges on the fact that if n is odd and greater than 1, then $5n$ is an odd composite ending in 5. So, to express k as desired, it suffices to find small odd composites ending in 5, 7, 9, 1 and 3, and to add these to numbers of the form $5n$. Indeed, 15, 27, 9, 21 and 33 will satisfy the above condition, and in each of the following cases one can find an odd integer n, $n > 1$, such that

if k ends in 0 (i.e. 40, 50,...), then $k = 15 + 5n$,
if k ends in 2 (i.e. 42, 52,...), then $k = 27 + 5n$,
if k ends in 4 (i.e. 44, 54,...), then $k = 9 + 5n$,
if k ends in 6 (i.e. 46, 56,...), then $k = 21 + 5n$,
if k ends in 8 (i.e. 48, 58,...), then $k = 33 + 5n$.

Alternate Solution. First observe that $6n+9$ is an odd composite number for $n = 0, 1, 2, \ldots$. Now partition the set of even positive integers into three residue classes, modulo 6, and note that in each of the following cases the indicated decompositions are satisfied by some nonnegative integer, n:

if $k \equiv 0 \pmod 6$ and $k \geq 18$, then $k = 9 + (6n+9)$,
if $k \equiv 2 \pmod 6$ and $k \geq 44$, then $k = 35 + (6n+9)$,
if $k \equiv 4 \pmod 6$ and $k \geq 34$, then $k = 25 + (6n+9)$.

This takes care of all even integers greater than 38. Checking again shows that 38 is the answer to the problem.

15. (36)

The claim that the given system of equations is satisfied by x^2, y^2, z^2 and w^2 is equivalent to claiming that

$$(1) \qquad \frac{x^2}{t-1} + \frac{y^2}{t-9} + \frac{z^2}{t-25} + \frac{w^2}{t-49} = 1$$

is satisfied by $t = 4, 16, 36$ and 64. Multiplying to clear fractions, we find that for all values of t for which it is defined (i.e., $t \neq 1$, 9, 25 and 49), Equation (1) is equivalent to the polynomial equation

$$\begin{aligned} &(t-1)(t-9)(t-25)(t-49) \\ (2) \qquad &- x^2(t-9)(t-25)(t-49) - y^2(t-1)(t-25)(t-49) \\ &- z^2(t-1)(t-9)(t-49) - w^2(t-1)(t-9)(t-25) = 0, \end{aligned}$$

where the left member may be viewed as a fourth degree polynomial in t. Since $t = 4$, 16, 36 and 64 are known to be roots, and a 4^{th} degree polynomial can have at most 4 roots, these must be all the roots. It follows that (2) is equivalent to

$$(3) \qquad (t-4)(t-16)(t-36)(t-64) = 0.$$

Since the coefficient of t^4 is 1 in both (2) and (3), one may conclude that the coefficients of the other powers of t must also be the same. In particular, equating the coefficients of t^3, we have

$$1 + 9 + 25 + 49 + x^2 + y^2 + z^2 + w^2 = 4 + 16 + 36 + 64,$$

from which $x^2 + y^2 + z^2 + w^2 = 36$.

Note. By equating the left members of Equations (2) and (3) and letting $t = 1$, one also finds that $x^2 = 11025/1024$. Similarly, letting $t = 9$, 25 and 49 in succession yields $y^2 = 10395/1024$, $z^2 = 9009/1024$ and $w^2 = 6435/1024$. One can show that these values indeed satisfy the given system of equations, and that their sum is 36.

AIME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

3rd ANNUAL AMERICAN INVITATIONAL MATHEMATICS EXAMINATION 1985

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will find the time to share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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1. (Answer: 384)

For each $n > 1$, $x_{n-1}x_n = n$. Using this fact for $n = 2, 4, 6$ and 8, one obtains $x_1x_2\cdots x_8 = (x_1x_2)(x_3x_4)(x_5x_6)(x_7x_8) = 2\cdot4\cdot6\cdot8 = 384$.

Note. Except for the fact that it must be nonzero, the value of x_1 does not affect the solution.

2. (26)

The volume of a cone with a circular base of radius r and height h is given by $\frac{\pi}{3}hr^2$. Denoting the length of the legs of the right triangle by a and b, this implies that

$$\frac{\pi}{3}ba^2 = 800\pi \quad \text{and} \quad \frac{\pi}{3}ab^2 = 1920\pi.$$

Dividing the first of these equations by the second one yields $\frac{a}{b} = \frac{5}{12}$. Hence, $a = \frac{5}{12}b$ and $a^3 = \frac{5}{12}ba^2 = \frac{5}{12}(800)(3) = 1000$. It follows that $a = 10$, $b = 24$ and that the triangle's hypotenuse (by Pythagoras' Theorem) is 26 cm.

3. (198)

First note that one may write c in the form

(1) $$c = a(a^2 - 3b^2) + i[b(3a^2 - b^2) - 107].$$

From this, in view of the fact that c is real, one may conclude that

(2) $$b(3a^2 - b^2) = 107.$$

Since a and b are positive integers, and since 107 is prime, two possible cases arise from (2):

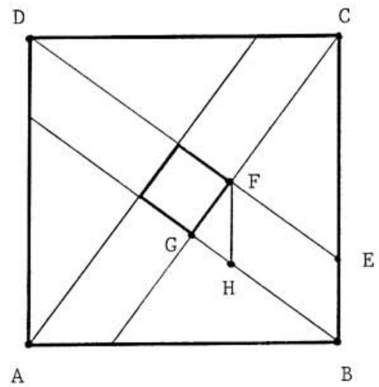
either $b = 107$ and $3a^2 - b^2 = 1$,
or $b = 1$ and $3a^2 - b^2 = 107$.

In the first case, $3a^2 = 107^2 + 1$ would follow. But this is impossible, since $107^2 + 1$ is not a multiple of 3.

In the second case, one finds that $a = 6$ and hence, in view of (1), $c = a(a^2 - 3b^2) = 6(6^2 - 3\cdot1^2) = 198$.

4. (32)

Let A, B, ..., H be the points labeled in the adjoining figure, with FH parallel to EB. Noting that ΔFGH is similar to ΔDCE, it then follows that



$$\frac{FG}{FH} = \frac{DC}{DE},$$

from which

$$FG^2 = \frac{DC^2}{DE^2} \cdot FH^2.$$

In view of the given information, this is equivalent to

$$\frac{1}{1985} = \frac{1}{1+(1-\frac{1}{n})^2} \cdot \frac{1}{n^2},$$

from which

$$2n^2 - 2n + 1 = 1985.$$

Since this last equation is equivalent to $2(n-32)(n+31) = 0$, and since n is positive, it follows that $n = 32$.

5. (986)

Calculating the first eight terms of the sequence, one finds that it cycles in blocks of six terms; i.e., for $n = 1, 2, 3, \ldots$, $a_{n+6} = a_n$. More specifically,

$$a_n = \begin{cases} a_1 & \text{if } n = 1, 7, 13, \ldots, \\ a_2 & \text{if } n = 2, 8, 14, \ldots, \\ a_2 - a_1 & \text{if } n = 3, 9, 15, \ldots, \\ -a_1 & \text{if } n = 4, 10, 16, \ldots, \\ -a_2 & \text{if } n = 5, 11, 17, \ldots, \\ a_1 - a_2 & \text{if } n = 6, 12, 18, \ldots . \end{cases}$$

Since the sum of any six consecutive terms of the sequence is zero, if we let s_n be the sum of the first n terms, then

$$s_n = \begin{cases} a_1 & \text{if } n = 1, 7, 13, \dots, \\ a_1 + a_2 & \text{if } n = 2, 8, 14, \dots, \\ 2a_2 & \text{if } n = 3, 9, 15, \dots, 2001, \dots, \\ 2a_2 - a_1 & \text{if } n = 4, 10, 16, \dots, 1492, \dots, \\ a_2 - a_1 & \text{if } n = 5, 11, 17, \dots, 1985, \dots, \\ 0 & \text{if } n = 6, 12, 18, \dots . \end{cases}$$

Therefore,

$$s_{1985} = a_2 - a_1 = 1492$$

and

$$s_{1492} = 2a_2 - a_1 = 1985,$$

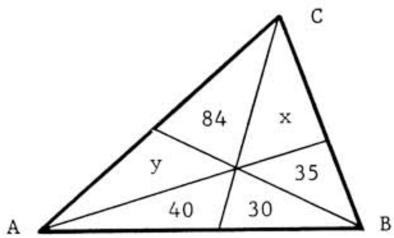
from which $a_2 = 493$ and $s_{2001} = 2a_2 = 986$.

6. (315)

The key to the solution is the fact that if two triangles have the same height, then their areas are proportional to their bases. Therefore, from the figure below, we have the equations

$$\frac{40}{30} = \frac{40 + y + 84}{30 + 35 + x}, \quad \frac{35}{x} = \frac{35 + 30 + 40}{x + 84 + y} \quad \text{and} \quad \frac{84}{y} = \frac{84 + x + 35}{y + 40 + 30}.$$

Solving the first two of these equations simultaneously, one finds that $x = 70$ and $y = 56$. After checking that these values also satisfy the third equation, one may conclude that the area of ΔABC is $30 + 35 + 70 + 84 + 56 + 40$ or 315.



Query. Can you show that such a triangle indeed exists? Can you construct a similar problem with integer areas? For how many of the six small triangles can one choose the area arbitrarily?

7. (757)

Since the prime factorization of positive integers is unique, and since 4 is relatively prime to 5 and 2 is relatively prime to 3, one may conclude that there exist positive integers m and n such that

$$a = m^4, \quad b = m^5, \quad c = n^2 \quad \text{and} \quad d = n^3.$$

Then

$$19 = c - a = n^2 - m^4 = (n - m^2)(n + m^2).$$

Since 19 is a prime, and since $n - m^2 < n + m^2$, it follows that

$$n - m^2 = 1 \quad \text{and} \quad n + m^2 = 19.$$

Therefore, $m = 3$, $n = 10$, $d = 1000$, $b = 243$ and $d - b = 1000 - 243 = 757$.

8. (61)

Two preliminary observations are needed:

(i) Each A_i should be 2 or 3, because for any scheme meeting this condition, $M < 1$; while for any other choice of the A_i's, $M > 1$.

(ii) There is only one way to sum seven integers, each of them 2 or 3, and to obtain 19 : two of them must be 2, while the other five must be 3.

In view of the above, and to make M as small as possible, one must round down (to 2) the two numbers with smallest decimal parts (i.e., a_1 and a_2), and round up (to 3) the other five a_i's. Thereby one finds that

$$M = \max\{|a_2 - 2|, |a_3 - 3|\} = \max\{.61, .35\} = .61,$$

and that $100M = 61$.

Note. More generally, one can show that if any a_i's are listed so that their decimal parts are in increasing order, if the sum of the a_i's is D, and if the sum of their integral parts is I, then the last $D - I$ of the a_i's should be rounded up (to the next integer), while the others should be rounded down.

Such unusual methods of rounding are necessary, for example, in the apportionment of congressional seats to the individual states in the United States. (The U.S. Constitution requires each state to be represented in proportion to its fraction of the U.S. population, but each state must also have an integer number of representatives, and the total size of the House is fixed.) For informative discussions and other solutions to the apportionment problem the reader is advised to read the following article in The American Mathematical Monthly: M.L. Balinski and H.P. Young, The Quota Method of Apportionment, vol.82(1975), pp 701-730.

9. (49)

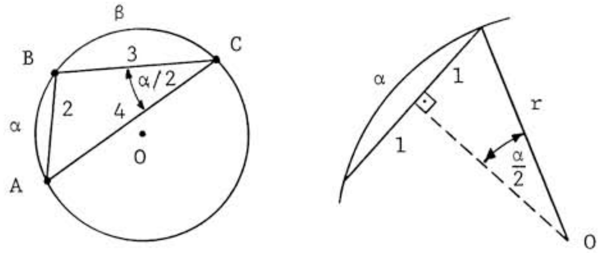
Since any two chords of equal length subtend equal angles, the parallelism of the chords is irrelevant. Therefore, we may choose points A, B and C, as shown in the first accompanying figure, so that $AB = 2$, $BC = 3$ and (since $\widehat{AC} = \alpha + \beta$) $AC = 4$. Since $\measuredangle ACB = \alpha/2$, by the Law of Cosines we have

$$\cos\frac{\alpha}{2} = \frac{AC^2 + BC^2 - AB^2}{2\cdot AC\cdot BC} = \frac{4^2+3^2-2^2}{2\cdot 4\cdot 3} = \frac{7}{8}.$$

Therefore, one finds that

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 = 2\cdot\frac{49}{64} - 1 = \frac{17}{32},$$

and that the desired answer is $17+32$ or 49.



Alternate Solution. Attacking the problem more directly, let r denote the radius of the circle, and note that $\sin\frac{\alpha}{2} = \frac{1}{r}$, as shown in the second figure above. One similarly finds that $\sin\frac{\beta}{2} = \frac{3}{2r}$ and $\sin\frac{\alpha+\beta}{2} = \frac{2}{r}$. Since $\sin(\frac{\alpha}{2}+\frac{\beta}{2}) = \sin\frac{\alpha}{2}\cos\frac{\beta}{2} + \cos\frac{\alpha}{2}\sin\frac{\beta}{2}$, it follows that

$$\frac{2}{r} = \frac{1}{r}\sqrt{1-\frac{9}{4r^2}} + \frac{3}{2r}\sqrt{1-\frac{1}{r^2}}. \tag{1}$$

Solving this for $\frac{1}{r^2}$, one finds that it is equal to $\frac{15}{64}$. Upon checking that this root is not extraneous to (1), it follows that

$$\cos\alpha = 1 - 2\sin^2\frac{\alpha}{2} = 1 - 2\cdot\frac{15}{64} = \frac{17}{32},$$

as in the first solution.

10. (600)

Introduce the notation

(1) $$f(x) = \lfloor 2x \rfloor + \lfloor 4x \rfloor + \lfloor 6x \rfloor + \lfloor 8x \rfloor,$$

and observe that if n is a positive integer, then from (1)

(2) $$f(x+n) = f(x) + 20n$$

follows. In particular, this means that if an integer k can be expressed in the form $f(x_o)$ for some real number x_o, then for $n = 1, 2, 3, \ldots$ one can express $k+20n$ similarly; i.e., $k+20n = f(x_o) + 20n = f(x_o+n)$. In view of this, one may restrict attention to determining which of the first 20 positive integers are generated by $f(x)$ as x ranges through the half-open interval $(0, 1]$.

Next observe that as x increases, the value of $f(x)$ changes only when either $2x$, $4x$, $6x$ or $8x$ attains an integral value, and that the change in $f(x)$ is always to a new, higher value. In the interval $(0, 1]$ such changes occur precisely when x is of the form m/n, where $1 \le m \le n$ and $n = 2, 4, 6$ or 8. There are 12 such fractions; in increasing order they are:

$$\frac{1}{8},\ \frac{1}{6},\ \frac{1}{4},\ \frac{1}{3},\ \frac{3}{8},\ \frac{1}{2},\ \frac{5}{8},\ \frac{2}{3},\ \frac{3}{4},\ \frac{5}{6},\ \frac{7}{8} \text{ and } 1.$$

Therefore, only 12 of the first 20 positive integers can be represented in the desired form. Since $1000 = (50)(20)$, in view of (2), this implies that in each of the 50 sequences,

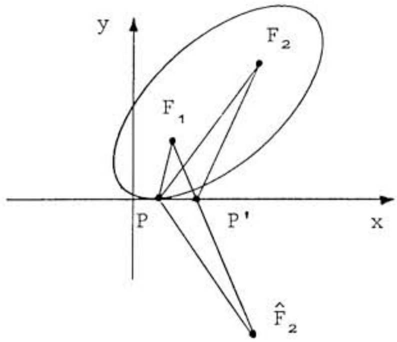
$1, 2, 3, \ldots, 20;\ 21, 22, 23, \ldots, 40;\ \ldots;\ 981, 982, 983, \ldots, 1000,$

of 20 consecutive integers only 12 can be so expressed, leading to a total of $(50)(12)$ or 600 positive integers of the desired form.

11. (85)

Let F_1 and F_2 denote the given foci, $(9, 20)$ and $(49, 55)$, respectively, and let $\hat{F}_2 : (49, -55)$ be the reflection of F_2 in the x-axis. We claim that k, the length of the major axis of the ellipse, is equal to the length of the segment $F_1\hat{F}_2$, which is easily found to be 85 by the distance formula between two points in the plane.

To prove our claim, let P be the point of tangency of the ellipse to the x-axis, and assume that the segment $F_1\hat{F}_2$ intersects the x-axis at the point P', distinct from P, as shown in the adjacent figure, where the location of the foci is purposefully distorted. Then, by the Triangle Inequality, one finds that



(1) $$PF_1 + P\hat{F}_2 > F_1\hat{F}_2 = P'F_1 + P'\hat{F}_2 = P'F_1 + P'F_2.$$

However, by the definition of the ellipse, for every point Q on the ellipse, $QF_1 + QF_2 = k$. Hence, in particular,

(2) $$PF_1 + PF_2 = k.$$

Moreover, for every point Q outside the ellipse, one must have $QF_1 + QF_2 > k$. Since, by the definition of tangency, P' must lie outside the ellipse, in particular it follows that

(3) $$P'F_1 + P'F_2 > k.$$

From (2) and (3) one may therefore conclude that

$$PF_1 + P\hat{F}_2 = PF_1 + PF_2 < P'F_1 + P'F_2,$$

which is contrary to (1). This contradiction establishes the falsity of the assumption that P and P' are distinct, and hence completes the proof of the claim.

12. (182)

For $n = 0, 1, 2, \ldots$, let a_n be the probability that the bug is at vertex A after crawling exactly n meters. Then

(1) $$a_{n+1} = \frac{1}{3}(1 - a_n),$$

because the bug can be at vertex A after crawling $n+1$ meters if and only if

(i) it was not at A following a crawl of n meters (this has probability $1 - a_n$)

and (ii) from one of the other vertices it heads toward A (this has probability $\frac{1}{3}$).

Now since $a_0 = 1$ (i.e., the bug starts at vertex A), from (1) we have

$$a_1 = 0,\quad a_2 = \frac{1}{3},\quad a_3 = \frac{2}{9},\quad a_4 = \frac{7}{27},\quad a_5 = \frac{20}{81},\quad a_6 = \frac{61}{243},$$

and $p = a_7 = \frac{182}{729}$, leading to 182 as the answer to the problem.

Note. The above calculations may be somewhat simplified by observing that for $n \ge 1$ the numerators of a_n and a_{n+1} sum to 3^{n-1}. Alternately, one may write (1) in the form

$$a_n - \frac{1}{4} = -\frac{1}{3}(a_{n-1} - \frac{1}{4}),$$

which (upon successive substitutions, starting with $a_0 - \frac{1}{4} = \frac{3}{4}$) yields the explicit formula

$$a_n = \frac{1}{4} + (-\frac{1}{3})^n(\frac{3}{4}).$$

13. (401)

More generally, we will show that if a is a positive integer and if d_n is the greatest common divisor of $a+n^2$ and $a+(n+1)^2$, then the maximum value of d_n is $4a+1$, attained when $n = 2a$. It will follow that for the sequence under consideration the answer is $4(100)+1$ or 401.

To prove the above, note that if d_n divides $a+(n+1)^2$ and $a+n^2$, then it also divides their difference; i.e.,

(1) $$d_n \mid (2n+1).$$

Now, since $2(a+n^2) = n(2n+1) + (2a-n)$, it follows from (1) that

(2) $$d_n \mid (2a-n).$$

Hence from (1) and (2), $d_n \mid ((2n+1) + 2(2a-n))$, or

(3) $$d_n \mid (4a+1).$$

Consequently, $1 \le d_n \le 4a+1$, so $4a+1$ is the largest possible value of d_n. It is attained, since for $n = 2a$ we have

$$a + n^2 = a + (2a)^2 = a(4a+1)$$

and

$$a + (n+1)^2 = a + (2a+1)^2 = (a+1)(4a+1).$$

14. (25)

Assume that a total of n players participated in the tournament. We will obtain two expressions in n: one by considering the total number of points gathered by all of the players, and one by considering the number of points gathered by the losers (10 lowest scoring contestants) and those gathered by the winners (other $n-10$ contestants) separately. To obtain the desired expressions, we will use the fact that if k players played against one another, then they played a total of $k(k-1)/2$ games, resulting in a total of $k(k-1)/2$ points to be shared among them.

In view of the last observation, the n players gathered a total of $n(n-1)/2$ points in the tournament. Similarly, the losers had $10 \cdot 9/2$ or 45 points in games among themselves; since this accounts for half of their points, they must have had a total of 90 points. In games among themselves the $n-10$ winners similarly gathered $(n-10)(n-11)/2$ points; this also accounts for only half of their total number of points (the other half coming from games against the losers), so their total was $(n-10)(n-11)$ points. Thus we have the equation

$$n(n-1)/2 = 90 + (n-10)(n-11),$$

which is equivalent to

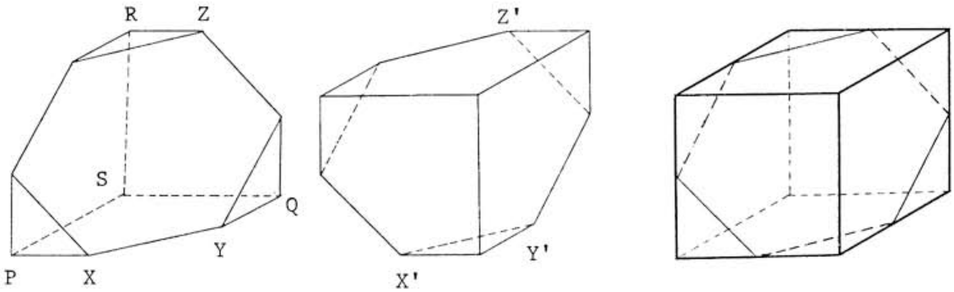
$$n^2 - 41n + 400 = 0.$$

Since the left member of this equation may be factored as $(n-16)(n-25)$, it follows that $n = 16$ or 25. We discard the first of these in view of the following observation: if there were only 16 players in the tournament, then there would have been only 6 winners, and the total of their points would have been 30 points, resulting in an average of 5 points for each of them. This is less than the 90/10 or 9 points gathered, on the average, by each of the losers! Therefore, $n = 25$; i.e., there were 25 players in the tournament.

Note. The AIME participants should recognize that there exists at least one tournament with 25 contestants meeting the conditions of the problem, for otherwise the problem would not have been posed. They are urged to attempt the reconstruction of such a tournament.

15. (864)

At each of the vertices P, Q, R and S, marked in the first figure below, each of the three facial angles measures 90°. Consequently, the polyhedron may be immediately recognized as a portion of a cube. Moreover, two such polyhedra may be fitted together along their hexagonal faces. To accomplish this, flip and rotate by 60° the second polyhedron (as shown in the second figure below), and then slide them together until X and X', Y and Y', as well as Z and Z' coincide. As shown in the third figure below, the result of the procedure is a cube of volume 12^3 or 1728 cm^3. Therefore, the volume of the polyhedron under consideration is 1728/2 or 864 cm^3.



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Readers of this Pamphlet are encouraged to submit alternate (and perhaps more elegant) solutions, insightful remarks, interesting extensions, answers to the queries, as well as other related materials to the Chairman of the AIME for possible inclusion in the journal ARBELOS. Constructive criticism of the AIME problems and solutions will also be most appreciated.

AIME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

4th ANNUAL
AMERICAN INVITATIONAL
MATHEMATICS EXAMINATION
1986

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will find the time to share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

AMERICAN MATHEMATICS COMPETITIONS

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1. (Answer: 337)

   Let $y = \sqrt[4]{x}$. Then the equation may be written in the form

   $$y^2 - 7y + 12 = 0,$$

   whose roots are $y=3$ and $y=4$. Consequently, we obtain the x-values of 3^4 and 4^4, whose sum is 337.

2. (104)

   Repeated use of the identity $(x+y)(x-y) = x^2 - y^2$ leads to

   $(\sqrt{5}+\sqrt{6}+\sqrt{7})(\sqrt{5}+\sqrt{6}-\sqrt{7}) = (\sqrt{5}+\sqrt{6})^2 - (\sqrt{7})^2 = (11+2\sqrt{30}) - 7 = 4+2\sqrt{30}$,

   $(\sqrt{5}-\sqrt{6}+\sqrt{7})(-\sqrt{5}+\sqrt{6}+\sqrt{7}) = (\sqrt{7})^2 - (\sqrt{5}-\sqrt{6})^2 = 7 - (11-2\sqrt{30}) = -4+2\sqrt{30}$

   and $(4+2\sqrt{30})(-4+2\sqrt{30}) = (2\sqrt{30})^2 - 4^2 = 120 - 16 = 104$.

3. (150)

   Multiplying both sides of $\cot x + \cot y = 30$ by $\tan x \tan y$, we find that

   $$\tan y + \tan x = 30 \tan x \tan y.$$

   Since the left side of this equation is equal to 25, it follows that

   $$\tan x \tan y = \frac{25}{30} = \frac{5}{6}.$$

   Therefore,

   $$\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} = \frac{25}{1-\frac{5}{6}} = 150.$$

4. (181)

   Adding the five given equations, and then dividing both sides of the resulting equation by 6, yields

   $$(1) \qquad x_1 + x_2 + x_3 + x_4 + x_5 = 31.$$

   Subtracting (1) from the given fourth and fifth equations, we find that $x_4 = 17$ and $x_5 = 65$. Consequently, $3x_4 + 2x_5 = 51 + 130 = 181$.

5. (890)

By division we find that $n^3 + 100 = (n+10)(n^2 - 10n + 100) - 900$. Thus, if $n+10$ divides $n^3 + 100$, then it must also divide 900. Moreover, since n is maximized whenever $n+10$ is, and since the largest divisor of 900 is 900, we must have $n+10 = 900$. Therefore, $n = 890$.

6. (033)

Let k be the number of the page that was counted twice. Then, $0 < k < n+1$, and $1+2+\cdots+n+k$ is between $1+2+\cdots+n$ and $1+2+\cdots+n+(n+1)$. In other words, $n(n+1)/2 < 1986 < (n+1)(n+2)/2$; i.e.,

$$n(n+1) < 3972 < (n+1)(n+2).$$

By trial and error (clearly, n is a little larger than 60) we find that $n = 62$. Thus $k = 1986 - (62)(63)/2 = 1986 - 1953 = 33$.

7. (981)

If we use only the first six non-negative integral powers of 3, namely 1, 3, 9, 27, 81 and 243, then we can form only 63 terms, since

$$\binom{6}{1} + \binom{6}{2} + \cdots + \binom{6}{6} = 2^6 - 1 = 63.$$

Consequently, the next highest power of 3, namely 729, is also needed.

After the first 63 terms of the sequence the next largest ones will have 729 but not 243 as a summand. There are 32 of these, since $\binom{5}{0} + \binom{5}{1} + \cdots + \binom{5}{5} = 32$, bringing the total number of terms to 95. Since we need the 100^{th} term, we must next include 243 and omit 81. Doing so, we find that the 96^{th}, 97^{th}, ..., 100^{th} terms are: $729+243$, $729+243+1$, $729+243+3$, $729+243+3+1$ and $729+243+9 = 981$.

Alternate Solution. Note that a positive integer is a term of this sequence if and only if its base 3 representation consists only of 0's and 1's. Therefore, we can set up a one-to-one correspondence between the positive integers and the terms of this sequence by representing both with binary digits (0's and 1's), first in base 2 and then in base 3:

$$\begin{array}{rcrcrcr} 1 & = & 1_{(2)} & \longleftrightarrow & 1_{(3)} & = & 1 \\ 2 & = & 10_{(2)} & \longleftrightarrow & 10_{(3)} & = & 3 \\ 3 & = & 11_{(2)} & \longleftrightarrow & 11_{(3)} & = & 4 \\ 4 & = & 100_{(2)} & \longleftrightarrow & 100_{(3)} & = & 9 \\ 5 & = & 101_{(2)} & \longleftrightarrow & 101_{(3)} & = & 10 \\ & & & \vdots & & & \end{array}$$

This is a correspondence between the two sequences in the order given, that is, the k-th positive integer is made to correspond to the k-th sum (in increasing order) of distinct powers of 3. This is because, when the binary numbers are written in increasing order, they are still in increasing order when interpreted in any other base. (If you can explain why this is true when interpreted in base 10, you should be able to explain it in base 3 as well.)

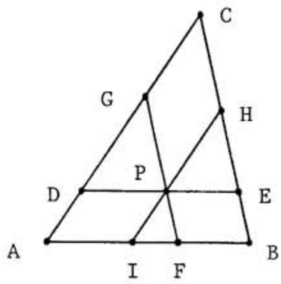
Therefore, to find the 100^{th} term of the sequence, we need only look at the 100^{th} line of the above correspondence:

$$100 = 1100100_{(2)} \longleftrightarrow 1100100_{(3)} = 981.$$

8. (141)

The number $1000000 = (2^6)(5^6)$ has $(6+1)(6+1) = 49$ distinct positive divisors. To see this, observe that they are all of the form $(2^i)(5^j)$; thus there are seven choices for i (0, 1, 2, ..., 6) and, independently, the same seven choices for j. Apart from 1000, the other 48 divisors form 24 pairs such that the product of each pair is 1000000. Since one of these pairs consists of the improper divisors 1 and 1000000, it follows that the product of all proper divisors of 1000000 is $(1000)(1000000)^{23}$ or 10^{141}. Moreover, since the sum, S, of the logarithms is equal to the logarithm of the product of these numbers, $S = 141$. The nearest integer to 141 is clearly 141.

9. (306)



As shown in the figure on the right, $EH = BC - (BE + HC) = BC - (FP + PG) = 450 - d$. In like manner, $GD = 510 - d$. Moreover, from the similarity of ΔDPG and ΔABC we have $DP/GD = AB/CA$. Hence

(1) $$DP = \frac{AB}{CA} \cdot GD = \frac{425}{510}(510 - d) = 425 - \frac{5}{6}d.$$

In like manner, since ΔPEH and ΔABC are similar, $PE/EH = AB/BC$. Hence

(2) $$PE = \frac{AB}{BC} \cdot EH = \frac{425}{450}(450 - d) = 425 - \frac{17}{18}d.$$

Since $d = DP + PE$, adding (1) and (2) we find that $d = 850 - \frac{16}{9}d$, from which $d = 306$.

10. (358)

Adding (abc) to N, and observing that each of the digits a, b and c appears exactly twice in each column, we are led to the equation

(1) $$N + (abc) = 222(a + b + c).$$

In view of this, the problem can be resolved by searching for an integral multiple of 222, say $222k$, which is larger than N (since $(abc) \neq 0$) and less than $N + 1000$ (since (abc) is a three-digit number), so that (1) is satisfied. That is, the sum of the digits of $222k - N$ must be k. If this holds, then $222k - N$ is the answer to the problem.

In our case, since $(14)(222) < 3194$ and $(19)(222) > 3194 + 1000$, the search is limited to $k = 15, 16, 17$ and 18. Of these only 16 works. Specifically, $(16)(222) - 3194 = 358$ and $3 + 5 + 8 = 16$.

Note. By also observing that $2N = 9(27a + 47b + 49c) + (a + b + c)$, one can slightly simplify the above computations. With the help of this observation one can also show that, whatever (abc) is, the magician can determine it uniquely.

11. (816)

Substituting $y - 1$ for x, the given expression becomes

$$1 - (y-1) + (y-1)^2 - (y-1)^3 + \cdots + (y-1)^{16} - (y-1)^{17},$$

which may be written in the form

$$(1) \quad 1 + (1-y) + (1-y)^2 + (1-y)^3 + \cdots + (1-y)^{16} + (1-y)^{17}.$$

Note that each $(1-y)^k$ term in (1) will yield a y^2 term for $2 \le k \le 17$. More specifically, by the Binomial Theorem, each of the summands in (1) contributes $\binom{k}{2}$ or $k(k-1)/2$ to the coefficient of y^2. Therefore, the problem is equivalent to computing the sum

$$\binom{2}{2} + \binom{3}{2} + \binom{4}{2} + \cdots + \binom{17}{2}.$$

To this end, one may proceed directly (i.e., by calculating and adding the sixteen numbers 1, 3, 6, ..., 136); or use the result of the derivation

$$\begin{aligned} \sum_{k=2}^{n} \frac{k(k-1)}{2} = \sum_{k=1}^{n} \frac{k(k-1)}{2} &= \frac{1}{2}\left(\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k \right) \\ &= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} \right) \\ &= \frac{(n+1)\,n\,(n-1)}{6}\ ; \end{aligned}$$

or use the more general formula

$$\sum_{k=m}^{n} \binom{k}{m} = \binom{n+1}{m+1}.$$

In any case, the desired sum is equal to 816.

12. (061)

First we show that S contains at most 5 elements. Suppose otherwise. Then S has at least $\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4}$ or 56 subsets of 4 or fewer members. The sum of each of these subsets is at most 54 (since

$15+14+13+12 = 54$), hence, by the Pigeonhole Principle, at least two of these sums are equal. If the subsets are disjoint, we are done; if not, then the removal of the common element(s) yields the desired contradiction.

Next we attempt to construct such a 5-element set S, by choosing its elements as large as possible. Including 15, 14 and 13 in S leads to no contradiction, but if 12 is also in S, then (in view of $12+15 = 13+14$) the conditions on S would be violated. Hence we must omit 12. No contradiction results from letting 11 be a member of S, but then $10 \notin S$ since $10+15 = 11+14$, and $9 \notin S$ since $9+15 = 11+13$. So we must settle for 8 as the fifth element of S. Indeed, $S = \{8, 11, 13, 14, 15\}$ satisfies the conditions of the problem, yielding $8+11+13+14+15$ or 61 as the candidate for its solution.

Finally, to show that the maximum is indeed 61, suppose that the sum is more for another choice of S. Observe that this set must also contain 15, 14 and 13, for if even the smallest of them (13) is omitted, the maximum possible sum (62) is achievable only by including 10, 11 and 12, but then $15+11 = 14+12$. Having chosen 15, 14 and 13, we must exclude 12, as noted before. If 11 is included, then we are limited to the sum of 61 as above. If 11 is not included, then even by including 10 and 9 (which we can't) we could not surpass 61 since $15+14+13+10+9 = 61$. Consequently, 61 is indeed the maximum sum one can attain.

13. (560)

Think of such sequences of coin tosses as progressions of blocks of T's and H's, to be denoted by (T) and (H), respectively. Next note that each HT and TH subsequence signifies a transition from (H) to (T) and from (T) to (H), respectively. Since there should be three of the first kind and four of the second kind in each of the sequences of 15 coin tosses, one may conclude that each such sequence is of the form

$$(T)(H)(T)(H)(T)(H)(T)(H). \tag{1}$$

Our next concern is the placement of T's and H's in their respective blocks, so as to assure that each sequence will have two HH and five TT subsequences. To this end, we will assume that each block in (1) initially contains only one member. Then, to satisfy the conditions of the problem, it will suffice to place 2 more H's into the (H)'s and 5 more T's into the (T)'s. Thus, to solve the problem, we must count the number of ways this can be accomplished.

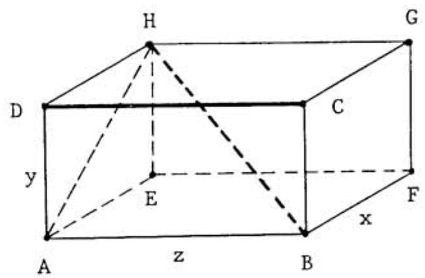
Recall that the number of ways to put p indistinguishable balls (the extra H's and T's in our case) into q distinguishable boxes (the (H)'s and (T)'s, distinguished by their order in the sequence) is given by the formula $\binom{p+q-1}{p}$. (Students who are not familiar with this fact should verify it.) In our case, it implies that the 2 H's can be placed in the 4 (H)'s in $\binom{2+4-1}{2}$ or 10 ways, and the 5 T's can be placed in the 4 (T)'s in $\binom{5+4-1}{5}$ or 56 ways. The desired answer is the product, 560, of these numbers.

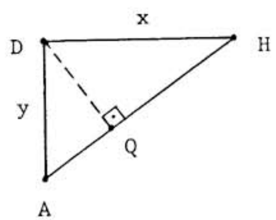
14. (750)

In order to find the volume of P, we will determine its dimensions, x, y and z. To this end, consider the rectangular parallelepiped shown in the first figure below, with vertices and side lengths as indicated. For definiteness, we labeled the box so that

(1) $\quad d(BH, CD) = 2\sqrt{5}, \quad d(BH, AE) = \dfrac{30}{\sqrt{13}}$ and $d(BH, AD) = \dfrac{15}{\sqrt{10}}$,

where, in general, d(RS, TU) denotes the distance between lines RS and TU.





Now the distance from BH to CD is equal to the distance from CD to the plane ABH, which is the same as the length of the perpendicular from D to the diagonal AH of rectangle AEHD. To see this, note that this perpendicular is also perpendicular to the plane ABH and the line CD. If one "slides" it over so that its top moves from D towards C, it eventually intersects line BH. This gives an equally long segment, which is perpendicular to both CD and BH, so its length is indeed the distance d(BH, CD). This distance is found via similar triangles, as shown in the second figure above, in which $DQ/DA = DH/AH$, and hence $DQ = xy/\sqrt{x^2+y^2}$. Treating the other distances similarly, in view of (1), this leads to the equations

$$(2) \qquad \frac{xy}{\sqrt{x^2+y^2}} = 2\sqrt{5}, \qquad \frac{yz}{\sqrt{y^2+z^2}} = \frac{30}{\sqrt{13}} \quad \text{and} \quad \frac{zx}{\sqrt{z^2+x^2}} = \frac{15}{\sqrt{10}}.$$

Upon squaring each equation in (2), taking reciprocals and simplifying, one arrives at the system

$$(3) \qquad \frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{20}, \qquad \frac{1}{y^2} + \frac{1}{z^2} = \frac{13}{900} \quad \text{and} \quad \frac{1}{z^2} + \frac{1}{x^2} = \frac{2}{45}.$$

We solve (3) by first adding the equations therein, and then subtracting from $\frac{1}{2}$ times the resulting equation each of the original equations in (3). Thus we find that $1/x^2 = 1/25$, $1/y^2 = 1/100$ and $1/z^2 = 1/225$. From these, $x=5$, $y=10$ and $z=15$. Consequently, the volume of P is $xyz = 750$.

15. (400)

More generally, we will show that if $\triangle ABC$ is a right triangle with right angle at C, if $AB = 2r$ and if the acute angle between the medians emanating from A and B is θ, then

$$(1) \qquad \text{area}(\triangle ABC) = \frac{4}{3} r^2 \tan\theta.$$

In our case, $r = 60/2 = 30$ and $\tan\theta = 1/3$ (determined either by the standard formula for the tangent of the angle between two given lines, or by glancing at the first accompanying figure), so the answer to the problem is $(4/3)(30)^2(1/3)$ or 400.

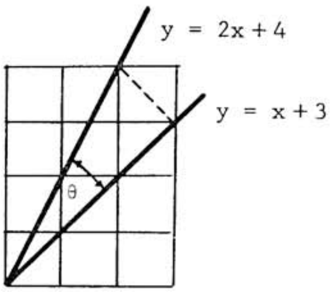
To establish (1), first note that

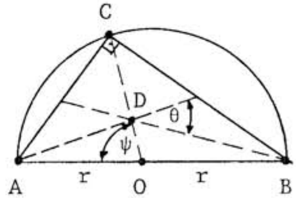
(2) $$\text{area}(\Delta ABC) = r^2 \sin\psi,$$

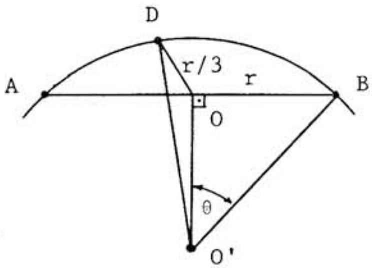
where $\psi = \measuredangle AOC$, as shown in the second figure below, where O is the mid-point of AB, D is the centroid of ΔABC and $CO = r$ (since ΔABC is a right triangle). Consequently, to prove (1), it suffices to verify that $\sin\psi = (4/3)\tan\theta$, which is equivalent to establishing that

(3) $$\cos\phi = -(4/3)\tan\theta,$$

where $\phi = \psi + 90^\circ$. This consideration leads us to the third figure below, where O' is the center of the circle through . . D and B, and $\measuredangle DOO' = \phi$.



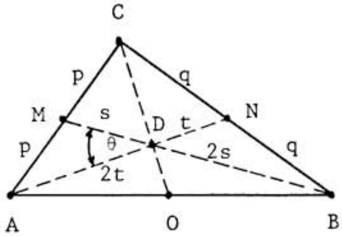




To prove (3), we will apply the Law of Cosines to $\Delta DOO'$. (The fact that $\measuredangle OO'B = \theta$ comes from the observation that $\measuredangle ADB = 180° - \theta$, and hence arc ADB has central angle 2θ; $DO = r/3$ is a well-known fact concerning the centroid.) Noting that $DO' = BO' = r\csc\theta$ and $OO' = r\cot\theta$ (from $\Delta BOO'$), indeed,

$$\cos\phi = \frac{(r/3)^2 + (r\cot\theta)^2 - (r\csc\theta)^2}{2(r/3)(r\cot\theta)} = -\frac{4}{3}\tan\theta,$$

as was to be shown.



Alternate Solution. As in the diagram shown on the right, let M, N and O be the midpoints of the sides of ΔABC, D be its centroid, p, q, s and t be the distances indicated, and $\theta = \measuredangle ADM$. As in the previous solution, also observe that $\tan\theta = 1/3$, and hence $\sin\theta = 1/\sqrt{10}$. Then from $\text{area}(\Delta ABC) = 6\cdot\text{area}(\Delta ADM)$,

(1) $$\text{area}(\Delta ABC) = 6st\sin\theta = \frac{6}{\sqrt{10}}st,$$

and, since ΔABC is a right triangle,

(2) $$\text{area}(\Delta ABC) = 2pq.$$

Moreover, by the Pythagorean Theorem, we find that

$$4p^2 + 4q^2 = AB^2 = 3600,$$
$$p^2 + 4q^2 = 9s^2,$$
$$4p^2 + q^2 = 9t^2.$$

With the help of these expressions from (1) and (2) it follows that

$$\begin{aligned}(\text{area}(\Delta ABC))^2 &= \frac{18}{5}s^2t^2\\ &= \frac{18}{5}\cdot\frac{p^2+4q^2}{9}\cdot\frac{4p^2+q^2}{9}\\ &= \frac{2}{45}((2p^2+2q^2)^2 + 9p^2q^2)\\ &= \frac{2}{45}((3600/2)^2 + \frac{9}{4}(\text{area}(\Delta ABC))^2)\\ &= 144{,}000 + \frac{1}{10}(\text{area}(\Delta ABC))^2.\end{aligned}$$

Consequently, $\frac{9}{10}(\text{area}(\Delta ABC))^2 = 144{,}000$, from which $\text{area}(\Delta ABC) = 400$.

AIME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

5th ANNUAL AMERICAN INVITATIONAL MATHEMATICS EXAMINATION 1987

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AMERICAN MATHEMATICS COMPETITIONS

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1. (Answer: 300)

Since there is no carrying in the addition, the ones column must add to 2, the tens column to 9, the hundreds column to 4 and the thousands column to 1 for each simple ordered pair of non-negative integers summing to 1492. To get a single digit d as the sum of two digits, there are $d+1$ ways:

$$0+d,\ 1+(d-1),\ 2+(d-2),\ \ldots,\ d+0.$$

Thus the number of simple ordered pairs of non-negative integers that sum to 1492 is $(1+1)(4+1)(9+1)(2+1) = 300$.

2. (137)

Let O and $\tilde{O}$ be the centers of the two spheres, and let P and $\tilde{P}$ be the two points where the extensions of the segment $O\tilde{O}$ pierce the two spheres, respectively, so that O is between P and $\tilde{O}$, and $\tilde{O}$ is between O and $\tilde{P}$. Then the desired maximum distance is

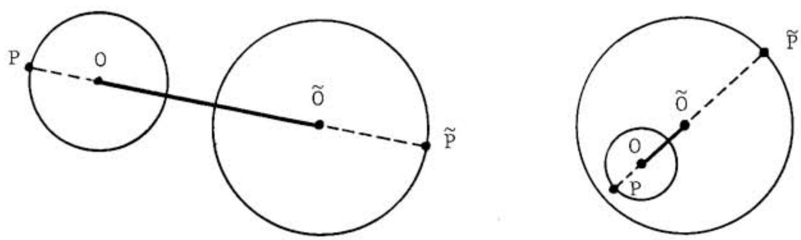
$$P\tilde{P} = PO + O\tilde{O} + \tilde{O}\tilde{P}, \qquad (1)$$

where PO and $\tilde{O}\tilde{P}$ are the given radii and $O\tilde{O}$ is found by the Distance Formula. In our case, $P\tilde{P} = 19 + 31 + 87 = 137$.

To see that (1) indeed yields the maximum distance, note that by the Triangle Inequality, for any points Q and $\tilde{Q}$ on the spheres with centers O and $\tilde{O}$, respectively,

$$Q\tilde{Q} \le QO + O\tilde{Q} \le QO + O\tilde{O} + \tilde{O}\tilde{Q} = PO + O\tilde{O} + \tilde{O}\tilde{P} = P\tilde{P}.$$

Note. The above solution does not depend on the position of the spheres relative to one another, as can be seen in the two configurations below, showing cross sections of the spheres by planes containing their centers.



3. (182)

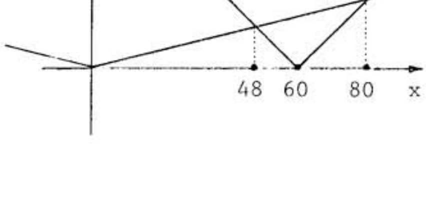
Let k be a positive integer, and let $1, d_1, d_2, \ldots, d_{n-1}, d_n, k$ be its divisors in ascending order. Then $1 \cdot k = d_1 \cdot d_n = d_2 \cdot d_{n-1} = \cdots$. For k to be nice, we must have $n = 2$. Moreover, d_1 must be prime, for otherwise, the proper divisors of d_1 would have appeared in the listing above between 1 and d_1. Similarly, d_2 is either a prime or the square of d_1, for otherwise d_1 could not be the only divisor between 1 and d_2. Therefore, k is either the product of two distinct primes or -- being the product of a prime and its square -- is the cube of a prime.

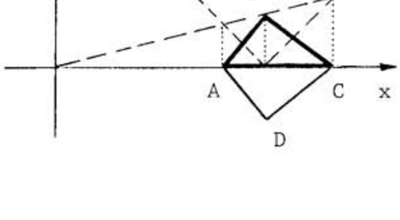
In view of the above, one can easily list the first ten nice numbers. They are: 6, 8, 10, 14, 15, 21, 22, 26, 27 and 33. Their sum is 182.

4. (480)

First we note that the graph of the given equation is symmetric with respect to the x-axis, since the replacement of y by $-y$ does not change the equation. Consequently, it suffices to assume that $y \ge 0$, find the area enclosed above the x-axis, and then double this area to find the answer to the problem.

By sketching both $y = |x/4|$ and $y = |x - 60|$ on the same graph, as shown in the first figure below, we note that $y = |x/4| - |x - 60| \ge 0$ only if the graph of $y = |x/4|$ lies above that of $y = |x - 60|$. This occurs only between $x = 48$ (where $x/4 = 60 - x$) and $x = 80$ (where $x/4 = x - 60$). In this interval, the graph of $y = |x/4| - |x - 60|$ with the x-axis forms ΔABC, as shown in the second figure. The base of this triangle is $80 - 48 = 32$, its altitude (at $x = 60$) is 15, hence its area is 240. Kite $ABCD$ is the graph of $|x - 60| + |y| = |x/4|$; the area enclosed by it is $2 \cdot 240 = 480$.





5. (588)

Rewrite the given equation in the form

$$(y^2 - 10)(3x^2 + 1) = 3\cdot 13^2,$$

and note that, since y is an integer and $3x^2+1$ is a positive integer, $y^2 - 10$ must be a positive integer. Consequently, $y^2 - 10 = 1, 3, 13, 39, 169$ or 507, implying that $y^2 = 11, 13, 23, 49, 179$ or 517. Since the only perfect square in the second list is 49, it follows that $y^2 - 10 = 39$, implying that $3x^2 + 1 = 13$, $x^2 = 4$ and $3x^2y^2 = 12\cdot 49 = 588$.

6. (193)

Since trapezoids XYQP and ZWPQ have the same area, and since their parallel sides are of the same length, their heights must also be equal, each of length $BC/2$. Moreover, since XY is one fourth of the perimeter of rectangle ABCD, it follows that $XY = (AB + BC)/2$, and that the area of trapezoid XYQP is $\frac{PQ + [(AB+BC)/2]}{2}\cdot\frac{BC}{2}$. Since this must be equal to one fourth of the area of rectangle ABCD, one finds that

$$\frac{PQ + [(AB+BC)/2]}{2}\cdot\frac{BC}{2} = \frac{AB\cdot BC}{4},$$

from which $AB = BC + 2\cdot PQ = 19 + 2\cdot 87 = 193$ cm.

Note. This problem arose from the following puzzle: how can one divide a rectangular cake, with a thin layer of icing on the top and sides, into n pieces of equal volume, so that each piece has the same amount of icing? The reader may wish to explore such equal-division problems for $n > 4$.

7. (070)

Since both 1000 and 2000 are of the form 2^m5^n, the numbers a, b and c must also be of this form. More specifically,

(1) $$a = 2^{m_1}5^{n_1}, \qquad b = 2^{m_2}5^{n_2}, \qquad c = 2^{m_3}5^{n_3},$$

where the m_i and n_i are non-negative integers for $i = 1, 2, 3$.

Then, in view of the definition of $[r,s]$, and since

(2) $\quad [a,b] = 2^3 5^3, \qquad [b,c] = 2^4 5^3, \qquad [c,a] = 2^4 5^3,$

the following equalities must hold:

(3) $\quad \max\{m_1, m_2\} = 3, \qquad \max\{m_2, m_3\} = 4, \qquad \max\{m_3, m_1\} = 4$

and

(4) $\quad \max\{n_1, n_2\} = 3, \qquad \max\{n_2, n_3\} = 3, \qquad \max\{n_3, n_1\} = 3.$

To satisfy (3), we must have $m_3 = 4$, and either m_1 or m_2 must be 3, while the other one can take the values of 0, 1, 2 or 3. There are 7 such ordered triples, namely (0,3,4), (1,3,4), (2,3,4), (3,0,4), (3,1,4), (3,2,4) and (3,3,4).

To satisfy (4), two of n_1, n_2 and n_3 must be 3, while the third one ranges through the values of 0, 1, 2 and 3. The number of such ordered triples is 10; they are (3,3,0), (3,3,1), (3,3,2), (3,0,3), (3,1,3), (3,2,3), (0,3,3), (1,3,3), (2,3,3) and (3,3,3).

Since the choice of (m_1, m_2, m_3) is independent of the choice of (n_1, n_2, n_3), they can be chosen in $7 \cdot 10 = 70$ different ways. This is the number of ordered triples (a,b,c) satisfying the given conditions.

8. (112)

By first writing the inequalities in the form $\frac{13}{7} < \frac{n+k}{n} < \frac{15}{8}$, we can see that they are equivalent to

$$48n < 56k < 49n.$$

Consequently, the problem is to find the longest open interval $(48n, 49n)$ that contains exactly one integral multiple of 56.

Since the length of the above interval is n, it contains $n-1$ integers. If $n - 1 \geq 2 \cdot 56$, the interval will contain at least two multiples of 56. Hence, $2 \cdot 56 = 112$ is the largest candidate for n. Indeed, we find that

$$48 \cdot 112 = 56 \cdot 96 < 56 \cdot 97 < 56 \cdot 98 = 49 \cdot 112,$$

also exhibiting that $k = 97$ is the unique positive integer corresponding to $n = 112$.

9. (033)

Noting that $\angle APB = \angle BPC = \angle CPA = 120°$, and applying the Law of Cosines to $\triangle APB$, $\triangle BPC$ and $\triangle CPA$, we find that

(1) $$(AB)^2 = (PA)^2 + (PB)^2 + PA \cdot PB = 100 + 36 + 60 = 196,$$

(2) $$(BC)^2 = (PB)^2 + (PC)^2 + PB \cdot PC = 36 + (PC)^2 + 6\,PC$$

and

(3) $$(CA)^2 = (PC)^2 + (PA)^2 + PC \cdot PA = (PC)^2 + 100 + 10\,PC.$$

Since $(AB)^2 + (BC)^2 = (CA)^2$ by the Pythagorean Theorem, it follows from (1), (2) and (3) that

$$196 + [36 + (PC)^2 + 6\,PC] = (PC)^2 + 100 + 10\,PC,$$

from which $PC = 33$.

10. (120)

Let v_1 denote Al's speed (in steps per unit time) and t_1 his time. Similarly, let v_2 and t_2 denote Bob's speed and time. Moreover, let v be the speed of the escalator, and let x be the number of steps visible at any given time. Then, from the information given,

(1) $$v_1 = 3v_2, \qquad v_1 t_1 = 150, \qquad v_2 t_2 = 75.$$

From (1) it follows that

(2) $$t_2 / t_1 = 3/2.$$

We also know that $x = (v_2 + v)t_2 = (v_1 - v)t_1$, from which we have $v = (x - 75)/t_2 = (150 - x)/t_1$, and hence

(3) $$t_2 / t_1 = (x - 75)/(150 - x).$$

Therefore, from (2) and (3), upon setting their right sides equal, we find that $x = 120$.

Alternate Solution. Assume that Al and Bob start at the same time from their respective ends of the escalator. Then the number of steps initially separating them is the same as the number of visible steps on the escalator.

Hence, to solve the problem, we must find the number of steps each of them takes until they meet, and then add these two numbers.

Since Al can take $3 \cdot 75 = 225$ steps while Bob takes 75 steps, it follows (from $150 = (2/3) \cdot 225$) that Al walks down the escalator in 2/3 of the time it takes Bob to walk up. Therefore, they meet 2/5 of the way from the bottom of the escalator. To that point, Al takes $(3/5) \cdot 150 = 90$ steps, while Bob takes $(2/5) \cdot 75 = 30$ steps. As indicated above, the sum of these, 120, is the number of visible steps of the escalator.

Notes. The second solution illustrates a general way to find the number of exposed steps on a moving escalator: find a friend to start simultaneously at the opposite end, and simply add the number of steps you have each taken before you meet.

Such escalator problems were rather popular at one time. This one was fashioned after a problem of Henry Dudeney (1857-1930), England's most famous creator of puzzles.

11. (486)

To solve the problem, we must find integers n and k such that n is non-negative, k is as large as possible, and

$$3^{11} = (n+1) + (n+2) + \cdots + (n+k). \tag{1}$$

Noting that

$$\begin{aligned}(n+1) + (n+2) + \cdots + (n+k) &= [1+2+\cdots+(n+k)] - [1+2+\cdots+n] \\ &= \frac{(n+k)(n+k+1)}{2} - \frac{n(n+1)}{2} \\ &= k(k+2n+1)/2,\end{aligned}$$

it follows that (1) is equivalent to

$$k(k+2n+1) = 2 \cdot 3^{11}. \tag{2}$$

In solving (2), we must ensure that the smaller factor, k, is as large as possible, and that n is a non-negative integer. These conditions lead to $k = 2 \cdot 3^5 = 486$, $n = 121$ and $3^{11} = 122 + 123 + \cdots + 607$.

Alternate Solution. Let m be the average of the k consecutive integers. If k is odd, then m must be the middle integer, and $km = 3^{11}$. Now $k = 3^5$ and $m = 3^6$ is the best we can do, for if $k = 3^6$ then $m - (k-1)/2$, the smallest summand, is negative. But if k is even, then m lies halfway between the middle two integers in the sum. Thus $(2m)k = 2\cdot 3^{11}$ and now the largest even divisor of $2\cdot 3^{11}$ which does not give rise to a negative first summand is $2\cdot 3^5 = 486$. This is the answer.

12. (019)

We solve the equivalent problem of finding the smallest positive integer n for which

$$(1) \qquad n^3 + 1 < (n + 10^{-3})^3.$$

This is equivalent to the given problem because

$$n < \sqrt[3]{m} < n + 10^{-3} \iff n^3 < m < (n + 10^{-3})^3,$$

and because if some integer m satisfies the double inequality on the right above, then $n^3 + 1$ is the smallest such m.

Rewriting (1) in the form

$$(2) \qquad \frac{1000}{3} < n^2 + \frac{n}{1000} + \frac{1}{3{,}000{,}000},$$

we observe that n^2 must be near $1000/3$, for the contributions of the other two terms on the right side of (2) are relatively small. Consequently, since $18^2 < 1000/3 < 19^2$, we expect that either $n = 18$ or $n = 19$. In the first case, (2) is not satisfied; this can be verified by an easy caculation. It is even easier to show that $n = 19$ satisfies (2), so it is the smallest positive integer with the desired property. The corresponding $m = 19^3 + 1 = 6860$ is the smallest positive integer whose cube root has a positive decimal part which is less than $1/1000$.

13. (931)

The key property of bubble pass is that immediately after r_k is compared with its predecessor and possibly switched, the current r_k is the largest member of the set $\{r_1, r_2, \ldots, r_k\}$. Also, this set is the same at this point as it was originally, though the order of its elements in the sequence may be very different. These assertions can be verified by induction.

For a number m, which is initially before r_{30} in the sequence, to end up as r_{30}, two things must happen: m must move into the 30^{th} position when the current r_{29} $(=m)$ and r_{30} are compared, and it must not move out of that position when compared to r_{31}. Therefore, by the key property, m must be the largest number in the original $\{r_1, r_2, \ldots, r_{30}\}$, but not the largest in the original $\{r_1, r_2, \ldots, r_{31}\}$. In other words, of the first 31 numbers originally, the largest must be r_{31} and the second largest must be m, which in our case is r_{20}.

Whatever the first 31 numbers are, there are $31!$ equally likely orderings of them. Of these, $29!$ have the largest of them in the 31^{st} slot and the second largest in the 20^{th} slot. (The other 29 numbers have $29!$ equally likely orderings.) Thus the desired probability must be $\frac{p}{q} = \frac{29!}{31!} = \frac{1}{930}$, and hence $p+q = 931$.

14. (373)

Since $324 = 18^2 = 4\cdot 3^4$, we may use the factorization

$$\begin{aligned} x^4 + 4y^4 &= x^4 + 4x^2y^2 + 4y^4 - 4x^2y^2 \\ &= (x^2 + 2y^2)^2 - (2xy)^2 \\ &= [(x^2 + 2y^2) - 2xy][(x^2 + 2y^2) + 2xy] \\ &= [(x^2 - 2xy + y^2) + y^2][(x^2 + 2xy + y^2) + y^2] \\ &= [(x-y)^2 + y^2][(x+y)^2 + y^2], \end{aligned}$$

which yields in our case

$$n^4 + 324 = [(n-3)^2 + 9][(n+3)^2 + 9].$$

In view of this, the given fraction can be written as

$$\frac{(7^2+9)(13^2+9)(19^2+9)(25^2+9)\ \cdots\ (55^2+9)(61^2+9)}{(1^2+9)(7^2+9)(13^2+9)(19^2+9)\ \cdots\ (49^2+9)(55^2+9)},$$

which simplifies to $\dfrac{61^2+9}{1^2+9} = \dfrac{3730}{10} = 373.$

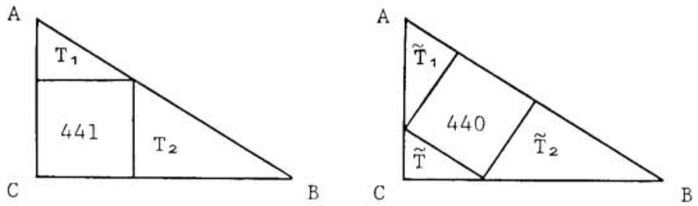
15. (462)

Let $a = BC$, $b = AC$. We will first find the hypotenuse c of ΔABC and the altitude h on c, because these are relatively easy to compute, and because from

$$c^2 = a^2 + b^2 \quad \text{and} \quad ch = ab$$

it is easy to find $a + b$ (without first finding a and b, which is harder!).

Consider the ratios of the areas of the smaller triangles surrounding S_1 and S_2, using the fact that all five of these triangles are similar to one another and to ΔABC. To simplify the notation, let T denote both ΔABC and its area, and similarly, let T_1, T_2, $\tilde{T}_1$, $\tilde{T}_2$ and $\tilde{T}$ denote both the triangles indicated in the figures below and their respective areas.



Since $\dfrac{\tilde{T}_1}{T_1} = \dfrac{\tilde{T}_2}{T_2} = \dfrac{440}{441}$, we find that

$$T = \tilde{T}_1 + \tilde{T}_2 + 440 + \tilde{T} = \frac{440}{441}(T_1 + T_2 + 441) + \tilde{T} = \frac{440}{441}T + \tilde{T}.$$

Therefore, $\tilde{T} = \dfrac{1}{441}T$, and hence the corresponding parts of triangles $\tilde{T}$ and T are in a linear ratio of 1 to 21.

It follows that $c = 21\sqrt{440}$, since the hypotenuse of $\tilde{T}$ is $\sqrt{440}$. Moreover, $h = 21\tilde{h}$, where $\tilde{h}$ denotes the altitude of $\tilde{T}$ on its hypotenuse. Combining the latter equation with the observation $h = \tilde{h} + \sqrt{440}$, we find that $h = \dfrac{21}{20}\sqrt{440}$,

$$ab = ch = (21\sqrt{440})\left(\frac{21}{20}\sqrt{440}\right) = 21^2 \cdot 22,$$

$$(a+b)^2 = c^2 + 2ab = 21^2 \cdot 440 + 2 \cdot 21^2 \cdot 22 = 21^2 \cdot 22^2,$$

and

$$AC + CB = a + b = 21 \cdot 22 = 462.$$

Notes. More generally, if $\text{area}(S_1) = p^2$ and $\text{area}(S_2) = q^2$, one can show that

(1) $$p = ab/(a+b) \quad \text{and} \quad q = ab\sqrt{a^2+b^2}/(a^2+ab+b^2).$$

Then, if p and q are given, either by solving the equations in (1) simultaneously (in the unknowns $a+b$ and ab) or otherwise, one can show that

(2) $$a+b = p+(p^2/\sqrt{p^2-q^2}) \quad \text{and} \quad ab = p(a+b).$$

Knowing the values of $a+b$ and ab, one can also determine the values of a and b explicitly with the help of the Quadratic Formula. In the present problem they turn out to be $21(11 \pm 3\sqrt{11})$. Since these two numbers are in the ratio of $10+3\sqrt{11}$ to 1, the accompanying figures are obviously not drawn to scale.

AIME SOLUTIONS PAMPHLET

FOR STUDENTS AND TEACHERS

6th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
1988

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1. (Answer: 770)

There are 2^{10} configurations of the ten buttons. We are to exclude those in which none or all of the buttons are depressed, as well as those in which exactly five buttons are depressed. Thus the total number of additional combinations is

$$2^{10} - 1 - 1 - \binom{10}{5} = 1024 - 2 - 252 = 770.$$

2. (169)

First we observe that, perhaps after a nonrepeating initial segment, the sequence $f_1(11)$, $f_2(11)$, ... is periodic. To see this, it suffices to note that, for $k < 1000$,

$$f_1(k) \le f_1(999) = (9+9+9)^2 = 729 < 1000.$$

Next we compute $f_n(11)$ for the first few values of n, in the hope that the length of the period is short. This expectation is reasonable since the terms of the sequence are perfect squares, and since there are only 31 perfect squares less than 1000. We find that

$$\begin{aligned}
f_1(11) &= (1+1)^2 = 4, \\
f_2(11) &= f_1(f_1(11)) = f_1(4) = 4^2 = 16, \\
f_3(11) &= f_1(f_2(11)) = f_1(16) = (1+6)^2 = 49, \\
f_4(11) &= f_1(f_3(11)) = f_1(49) = (4+9)^2 = 169, \\
f_5(11) &= f_1(f_4(11)) = f_1(169) = (1+6+9)^2 = 256, \\
f_6(11) &= f_1(f_5(11)) = f_1(256) = (2+5+6)^2 = 169.
\end{aligned}$$

We stop at this point, since $f_n(11)$ depends only on $f_{n-1}(11)$, and hence the numbers 256 and 169 will continue to alternate. More precisely, for $n \ge 4$,

$$f_n(11) = \begin{cases} 169, & \text{if } n \text{ is even}, \\ 256, & \text{if } n \text{ is odd}. \end{cases}$$

Since 1988 is even, it follows that $f_{1988}(11) = 169$.

3. (027)

Since $\log_8 x = \frac{1}{\log_x 8} = \frac{1}{3\log_x 2} = \frac{1}{3}\log_2 x$ and $\log_8(\log_2 x) = \frac{1}{3}\log_2(\log_2 x)$,

the given equation is equivalent to

(1) $$\log_2(y/3) = (1/3)\log_2 y,$$

where $y = \log_2 x$. From (1), $\log_2(y/3)^3 = \log_2 y$, hence $(y/3)^3 = y$; i.e.,

(2) $$y(y^2 - 27) = 0.$$

Since $y \neq 0$ (for otherwise, neither side of (1) would be defined), it follows from (2) that $y^2 = (\log_2 x)^2 = 27$.

4. (020)

If n is a positive integer, then

(1) $$\sum_{k=1}^{n}|x_k| - |\sum_{k=1}^{n}x_k| \le \sum_{k=1}^{n}|x_k| < n,$$

since $|\sum_{k=1}^{n}x_k| \ge 0$ and $|x_k| < 1$ for $1 \le k \le n$. Since it is given that

(2) $$\sum_{k=1}^{n}|x_k| - |\sum_{k=1}^{n}x_k| = 19,$$

it follows that $19 < n$. Thus the answer is 20 if there is a solution to (2) with $n = 20$. One such solution is

$$x_k = \begin{cases} .95 & \text{if } k \text{ is odd,} \\ -.95 & \text{if } k \text{ is even.} \end{cases}$$

5. (634)

The divisors of 10^{99} are of the form $2^a \cdot 5^b$, where a and b are integers with $0 \le a \le 99$ and $0 \le b \le 99$. Since there are 100 choices for both a and b, 10^{99} has $100 \cdot 100$ positive integer divisors. Of these, the multiples of $10^{88} = 2^{88} \cdot 5^{88}$ must satisfy the inequalities $88 \le a \le 99$ and $88 \le b \le 99$. Thus there are 12 choices for both a and b; i.e., $12 \cdot 12$ of the $100 \cdot 100$ divisors of 10^{99} are multiples of 10^{88}. Consequently, the desired probability is $\frac{m}{n} = \frac{12 \cdot 12}{100 \cdot 100} = \frac{9}{625}$ and $m + n = 634$.

6. (142)

Let a and b denote the numbers in two of the squares as shown in the first figure below, and compute the two neighboring entries in terms of them. Then the common difference in the third row is $b - 2a$, while in the fourth row it is $2b - a - 74$. Consequently,

$$2a + 4(b - 2a) = 186 \quad \text{and} \quad a + 2(2b - a - 74) = 103.$$

Solving these equations simultaneously, we find that $a = 13$ and $b = 66$. Therefore, the entries in the third and fourth row, and then in the fourth column may be computed to find that the number in the square marked by the asterisk (*) is 142. For completeness, the second figure shows the rest of the entries as well.

			*	
	74			
2a	b			186
a	2b-74	103		
0				

52	82	112	142	172
39	74	109	144	179
26	66	106	146	186
13	58	103	148	193
0	50	100	150	200

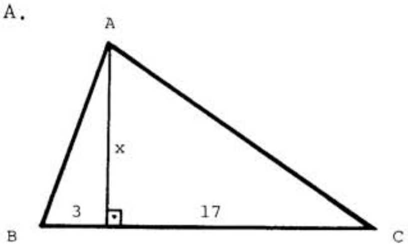
7. (110)

Let x be the length of the altitude from A. Then

$\tan^{-1}(22/7) = \tan^{-1}(3/x) + \tan^{-1}(17/x)$.

Taking tangents of both sides, and using the formula for $\tan(\alpha + \beta)$, we obtain

$$\frac{22}{7} = \frac{(3/x) + (17/x)}{1 - (3/x)(17/x)},$$



which simplifies to

$$11x^2 - 70x - 561 = (x - 11)(11x + 51) = 0.$$

Since x must be positive, we conclude that $x = 11$, and that the area of ΔABC is $(1/2)(3 + 17)(11) = 110$.

8. (364)

The third property is most useful in the form $f(x,z) = \frac{z}{z-x} f(x,z-x)$, which is valid whenever $z > x$. To obtain this, set $z = y + x$, so that $y = z - x$. Now substitute for y in the original third property. Using this new form of the third property and the second given property of f repeatedly, we obtain

$$\begin{aligned} f(14,52) &= \frac{52}{38} f(14,38) = \frac{52}{38}\frac{38}{24} f(14,24) = \frac{52}{24}\frac{24}{10} f(14,10) \\ &= \frac{26}{5} f(10,14) = \frac{26}{5}\frac{14}{4} f(10,4) = \frac{91}{5} f(4,10) = \frac{91}{5}\frac{10}{6} f(4,6) \\ &= \frac{91}{3}\frac{6}{2} f(4,2) = 91\, f(2,4) = 91\,\frac{4}{2} f(2,2) = 364, \end{aligned}$$

where the last equality is a consequence of the first given property of f.

<u>Note</u>. The computations may be somewhat simplified if we introduce a new function g by letting $g(x,y) = \frac{1}{xy} f(x,y)$. The student should verify that $f(x,y)$ is the least common multiple of x and y, while $g(x,y)$ is the reciprocal of the greatest common divisor of x and y. In fact, computing with g is like using Euclid's Algorithm in slow motion!

9. (192)

If the cube of an integer ends in 8, then the integer itself must end in 2; i.e., must be of the form $10k + 2$. Therefore,

(1) $$n^3 = (10k+2)^3 = 1000k^3 + 600k^2 + 120k + 8,$$

where the penultimate term, $120k$, determines the penultimate digit of n^3, which must also be 8. In view of this, $12k$ must also end in 8; i.e., k must end in 4 or 9, and hence be of the form $5m + 4$. Thus

(2) $$\begin{aligned} n^3 &= (10(5m+4) + 2)^3 \\ &= 125000m^3 + 315000m^2 + 264600m + 74088. \end{aligned}$$

Since the first two terms on the right of (2) end in 000, while the last term ends in 088, it follows that $264600m$ must end in 800. The smallest m which will ensure this is $m = 3$, implying that $k = 5\cdot 3 + 4 = 19$, and $n = 10\cdot 19 + 2 = 192$. (Indeed, $192^3 = 7,077,888$.)

10. (840)

Let V, E, D and I denote the number of vertices, edges, facial diagonals and interior diagonals of the polyhedron. We will evaluate them in succession, starting with V.

Since at each vertex of the polyhedron there is exactly one of each kind of face, and since the 4 vertices of each of the square faces must be at different vertices of the polyhedron,

$$V = 4 \times \text{number of square faces} = 4 \times 12 = 48.$$

(Note that the same value results if we consider the hexagonal or octagonal faces.)

Since there are 3 edges emanating from each vertex of the polyhedron, and since 3V counts each edge twice,

$$E = \frac{1}{2} \times 3V = 72.$$

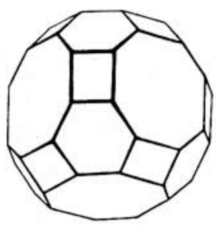
To find D, note that each square face has 2 diagonals, each hexagonal face has 9 diagonals, and each octagonal face has 20 diagonals. Consequently,

$$D = 12 \times 2 + 8 \times 9 + 6 \times 20 = 216.$$

Finally, because the polyhedron is convex, we may find I by subtracting the number of edges and the number of facial diagonals from the total number of ways to connect pairs of vertices for the polyhedron. Thus

$$I = \binom{48}{2} - E - D = 1128 - 72 - 216 = 840.$$

Note. The polyhedron in question is the "great rhombicuboctahedron", also known as the (rhombi)truncated cuboctahedron. It is one of the thirteen semi-regular (Archimedean) solids. See Fejes Tóth's Regular Figures (Pergamon Press, 1964) or Coxeter's Regular Polytopes (Dover, 1973).



11. (163)

Let $y = mx + b$ be a mean line for the complex numbers $w_k = u_k + iv_k$, where u_k and v_k are real, and $k = 1, 2, \ldots, n$. Assume that the complex numbers $z_k = x_k + iy_k$, where x_k and y_k are real, are chosen on the line $y = mx + b$ so that

$$\sum_{k=1}^{n} (z_k - w_k) = 0.$$

Then

$$\sum x_k = \sum u_k, \quad \sum y_k = \sum v_k, \quad \text{and} \quad y_k = mx_k + b \quad (1 \le k \le n),$$

where $\sum$ means summation as k ranges from 1 to n. Consequently,

$$\sum v_k = \sum y_k = \sum (mx_k + b) = m\sum x_k + nb = (\sum u_k)m + nb.$$

Since in our case, $n = 5$, $b = 3$, $\sum u_k = 3$, and $\sum v_k = 504$, it follows that $504 = 3m + 15$ and hence $m = 163$.

Note. We have shown only that $m = 163$ is necessary for $y = mx + 3$ to be a mean line for the given set of points. The reader should find corresponding $z_1, z_2, \ldots, z_5$ to verify the sufficiency of this choice for m.

The definition of mean line given in this problem is not the standard one; i.e., usually it is also required that each of the segments connecting w_k to z_k be perpendicular to the mean line. Can the reader show that the two definitions are equivalent, and that a mean line for a set of complex numbers is nothing more than a line through the centroid of the set?

Query. What if $\sum u_k = 0$?

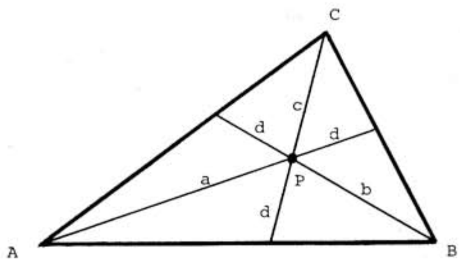
12. (441)

First observe that

(1) $$\frac{d}{d + a} = \frac{\text{area}(\triangle BPC)}{\text{area}(\triangle BAC)},$$

(2) $$\frac{d}{d + b} = \frac{\text{area}(\triangle CPA)}{\text{area}(\triangle CBA)},$$

(3) $$\frac{d}{d + c} = \frac{\text{area}(\triangle APB)}{\text{area}(\triangle ACB)}.$$



Then, since area($\triangle BPC$) + area($\triangle CPA$) + area($\triangle APB$) = area($\triangle ABC$), the sum of (1), (2) and (3) simplifies to

$$\frac{d}{d+a} + \frac{d}{d+b} + \frac{d}{d+c} = 1.$$

Multiplying through by $(d+a)(d+b)(d+c)$, expanding and grouping like terms, we may write the above as

(4) $$2d^3 + (a+b+c)d^2 - abc = 0.$$

From (4), in view of the given data it follows that $abc = 2\cdot 3^3 + 43\cdot 3^2 = 441$.

Note. One such triangle has $a = b = 21$ and $c = 1$. Show that there are infinitely many noncongruent triangles meeting the conditions of the problem. Is it true that if two of the quantities $a+b+c$, abc, d are given, then the third is uniquely determined and can be realized geometrically?

13. (987)

Since the roots of $x^2 - x - 1 = 0$ are $p = \frac{1}{2}(1+\sqrt{5})$ and $q = \frac{1}{2}(1-\sqrt{5})$, these must also be roots of $ax^{17} + bx^{16} + 1 = 0$. Thus

$$ap^{17} + bp^{16} = -1 \qquad \text{and} \qquad aq^{17} + bq^{16} = -1.$$

Multiplying the first of these equations by q^{16}, the second one by p^{16}, and using the fact that $pq = -1$, we find that

$$ap + b = -q^{16} \qquad \text{and} \qquad aq + b = -p^{16}.$$

Thus

(1) $$a = \frac{p^{16} - q^{16}}{p - q} = (p^8 + q^8)(p^4 + q^4)(p^2 + q^2)(p + q).$$

Since

$$\begin{aligned} p + q &= 1, \\ p^2 + q^2 &= (p+q)^2 - 2pq = 1 + 2 = 3, \\ p^4 + q^4 &= (p^2 + q^2)^2 - 2(pq)^2 = 9 - 2 = 7, \\ p^8 + q^8 &= (p^4 + q^4)^2 - 2(pq)^4 = 49 - 2 = 47, \end{aligned}$$

substitution into (1) yields

$$a = (47)(7)(3)(1) = 987.$$

Note. Substituting for p and q into (1) gives

$$a = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{16} - \left(\frac{1-\sqrt{5}}{2}\right)^{16}\right],$$

whose right side may be recognized as (the Binet form of) the 16-th Fibonacci number F_{16}. It is easiest to compute F_{16} by iteration, using the recursive definition: $F_1 = F_2 = 1$, $F_n = F_{n-1} + F_{n-2}$ for $n > 2$.

Alternate Solution. The other factor is of degree 15 and we write (with slight malice aforethought)

$$(-1 - x + x^2)(-c_0 + c_1x - \dots + c_{15}x^{15}) = 1 + bx^{16} + ax^{17}.$$

Comparing coefficients:

$$x^0: \quad c_0 = 1,$$
$$x^1: \quad c_0 - c_1 = 0 \quad \Rightarrow \quad c_1 = 1,$$
$$x^2: \quad -c_0 - c_1 + c_2 = 0 \quad \Rightarrow \quad c_2 = 2,$$

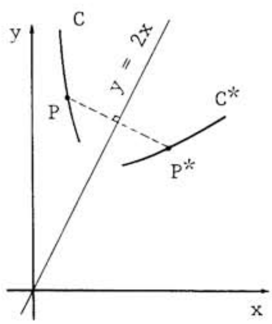
and for $3 \le k \le 15$,

$$x^k: \quad -c_{k-2} - c_{k-1} + c_k = 0.$$

It follows that for $k \le 15$, $c_k = F_{k+1}$, the $(k+1)$-st Fibonacci number. Thus $a = c_{15} = F_{16} = 987$.

14. (084)

Let $P(u,v)$ be any point on C, and let $P^*(x,y)$ be the corresponding point on C^*, that is, the reflection of P in the line $y = 2x$. Connect P and P^* by a straight line segment as shown in the figure. Then the problem is to find an equation relating x and y.



Since PP^* is perpendicular to the line $y = 2x$, its slope is $-\frac{1}{2}$. Thus

(1) $$\frac{y - v}{x - u} = -\frac{1}{2}.$$

Furthermore, since the midpoint of PP^* is on the line $y = 2x$, its coordinates must satisfy the equation of this line; that is,

(2) $$\frac{y + v}{2} = 2\cdot\frac{x + u}{2}.$$

Solving (1) and (2) simultaneously for u and v yields $u = (4y - 3x)/5$ and $v = (4x + 3y)/5$. Substituting these into $uv = 1$ (which holds since P is on C) we find that

$$12x^2 - 7xy - 12y^2 + 25 = 0.$$

This is the equation of C^* in the form desired; in it $bc = (-7)(-12) = 84$.

15. (704)

At any given time, the letters in the box are in decreasing order from top to bottom. Thus the sequence of letters in the box is uniquely determined by the set of letters in the box. We have two cases: letter 9 arrived before lunch or it did not.

Case 1. Since letter 9 arrived before lunch, no further letters will arrive, and the number of possible orders is simply the number of subsets of $T = \{1, 2, \ldots, 6, 7, 9\}$ which might still be in the box. In fact, each subset of T is possible, because the secretary might have typed letters not in the subset as soon as they arrived and not typed any others. Since T has 8 elements, it has $2^8 = 256$ subsets (including the empty set).

Case 2. Since letter 9 didn't arrive before lunch, the question is: where can it be inserted in the typing order? Any position is possible for each subset of $U = \{1, 2, \ldots, 6, 7\}$ which might have been left in the box during lunch (in descending order). For instance, if the letters in the box during lunch are 6,3,2, then the typing order 6,3,9,2 would occur if the boss would deliver letter 9 just after letter 3 was typed. There would seem to be $k+1$ places at which letter 9 could be inserted into a sequence of k letters. However, if letter 9 is inserted at the beginning of the sequence (i.e., at the top of the pile, so it arrives before any after-lunch typing is done), then we are duplicating an ordering from Case 1. Thus if k letters are in the basket after returning from lunch, then there are k places to insert letter 9 (without duplicating any Case 1 orderings). Thus we obtain

$$\sum_{k=0}^{7} k\binom{7}{k} = 7(2^{7-1}) = 448$$

new orderings in Case 2.

Combining these cases gives $256 + 448 = 704$ possible typing orders.

<u>Note</u>. The reasoning in Case 2 can be extended to cover both cases by observing that in any sequence of k letters not including letter 9, there are $k+2$ places to insert letter 9, counting the possibility of not having to insert it (i.e., if it arrived before lunch) as one of the cases. This yields

$$\sum_{k=0}^{7} (k+2)\binom{7}{k} = 704$$

possible orderings, in agreement with the answer found previously.

AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

7th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)

TUESDAY, MARCH 21, 1989

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

It is hoped that teachers will find the time to share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Questions and comments about the problems and solutions (but **not** requests for the Solutions Pamphlet) should be addressed to:

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1. (Answer: 869)
The data

$$\begin{array}{ccccccc} 4{\cdot}3{\cdot}2{\cdot}1 & + & 1 & = & 5^2 & = & (3\cdot 2-1)^2 \\ 5{\cdot}4{\cdot}3{\cdot}2 & + & 1 & = & 11^2 & = & (4\cdot 3-1)^2 \\ 6{\cdot}5{\cdot}4{\cdot}3 & + & 1 & = & 19^2 & = & (5\cdot 4-1)^2 \end{array}$$

suggest that $(k+1)(k)(k-1)(k-2)+1 = [k(k-1)-1]^2 = [(k^2-k)-1]^2$. The calculations

$$\begin{aligned}(k+1)(k)(k-1)(k-2)+1 &= \left[(k+1)(k-2)\right]\left[k(k-1)\right]+1 \\ &= (k^2-k-2)(k^2-k)+1 \\ &= (k^2-k)^2-2(k^2-k)+1 \\ &= \left[(k^2-k)-1\right]^2\end{aligned}$$

show that this is true. Thus $\sqrt{(31)(30)(29)(28)+1} = 30^2-30-1 = 869$.

2. (Answer: 968)
For $3 \le k \le 10$, each choice of k points will yield a convex polygon with k vertices. Because k points can be chosen from 10 in $\binom{10}{k}$ ways, the answer to the problem is

$$\begin{aligned}&\binom{10}{3}+\binom{10}{4}+\cdots+\binom{10}{10} \\ &\qquad = \left[\binom{10}{0}+\binom{10}{1}+\cdots+\binom{10}{10}\right]-\left[\binom{10}{0}+\binom{10}{1}+\binom{10}{2}\right] \\ &\qquad = (1+1)^{10}-(1+10+45) \\ &\qquad = 968.\end{aligned}$$

Query. Where have we used the stipulation that the polygons are convex?

3. (Answer: 750)
Since $\frac{n}{810} = 0.\mathrm{d}25\mathrm{d}25\mathrm{d}25\ldots$, we have $1000\frac{n}{810} = \mathrm{d}25.\mathrm{d}25\mathrm{d}25\ldots$. Subtracting gives

$$\frac{999}{810}n = 1000\frac{n}{810} - \frac{n}{810} = \mathrm{d}25 = 100\mathrm{d}+25.$$

Consequently $999n = 810(100\mathrm{d}+25)$, which leads to $37n = 750(4\mathrm{d}+1)$. Noting that 750 and 37 are relatively prime, we see that $4\mathrm{d}+1$ must be a multiple of 37. Since d is a single digit, $\mathrm{d} = 9$ and hence $n = 750$.

4. (Answer: 675)
Since a, b, c, d and e are consecutive integers, $b+c+d=3c$ and $a+b+c+d+e=5c$. Let m and n be positive integers such that $b+c+d=m^2$ and $a+b+c+d+e=n^3$. Then

$$3c=m^2 \tag{1}$$

and

$$5c=n^3 \tag{2}$$

From (1) we see that $3|m$ and hence $3|c$. (If j and k are integers, $j|k$ is read "j divides k" and means that j is a factor of k.) Therefore, (2) implies that $3|n$ and hence that $3^3|c$. From (2) we also find that $5|n$, which leads to $5^2|c$. Consequently, $25\cdot 27$ divides c. It is easy to verify that $c=25\cdot 27=675$ is the solution we seek.

5. (Answer: 283)
Let r be the probability of getting heads when the coin is tossed once. Then the probability of getting k heads out of n tosses is $\binom{n}{k}r^k(1-r)^{n-k}$. Consequently, the information given in the problem leads to the equation

$$\binom{5}{1}r(1-r)^4=\binom{5}{2}r^2(1-r)^3,$$

from which $r=0, \frac{1}{3}$ or 1. Since the desired probability is not 0, we may conclude that $r=\frac{1}{3}$. Thus the probability of obtaining exactly 3 heads in 5 tosses is

$$\frac{i}{j}=\binom{5}{3}r^3(1-r)^2=\binom{5}{3}\left(\tfrac{1}{3}\right)^3\left(1-\tfrac{1}{3}\right)^2=\frac{40}{243}$$

and $i+j=40+243=283$.

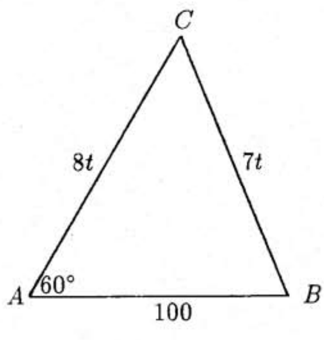
6. (Answer: 160)
Suppose that after t seconds, Allie and Billie meet at point C. Then $\triangle ABC$ has sides of $AB=100$, $AC=8t$ and $BC=7t$, with $\overline{BC}$ opposite the $60°$ angle. Hence, by the Law of Cosines,

$$(7t)^2=100^2+(8t)^2-2(100)(8t)\cos 60°,$$

from which

$$0=3t^2-160t+2000=(3t-100)(t-20).$$

Thus $t=20$ or $t=\dfrac{100}{3}$. Since the meeting takes place at the earliest possible time, we must have $t=20$ and hence $AC=8\cdot 20=160$ meters.



7. (Answer: 925)

We seek the values of k, n and d such that

$$36 + k = (n - d)^2 \tag{1}$$
$$300 + k = n^2 \tag{2}$$
$$596 + k = (n + d)^2. \tag{3}$$

Subtracting (1) from (3), we find that $4nd = 560$, from which

$$nd = 140. \tag{4}$$

Multiplying equation (2) by 2 and subtracting the result from the sum of (1) and (3), we find that $2d^2 = 32$, giving $d = \pm 4$. Combining this with (4), we find $n = \pm 35$. Using (2) again, we have $k = n^2 - 300 = 35^2 - 300 = 925$.

Note. One may also find k by solving the equation

$$\sqrt{36 + k} + \sqrt{596 + k} = 2\sqrt{300 + k}.$$

8. (Answer: 334)

Observe that subtracting 3 times the second equation from the sum of the first equation and 3 times the third equation gives an equation with the desired expression on the left. Thus

$$16x_1 + 25x_2 + 36x_3 + 49x_4 + 64x_5 + 81x_6 + 100x_7 = 1(1) - 3(12) + 3(123) = 334.$$

More formally, we can deduce the relation mentioned above by finding constants a, b and c such that

$$an^2 + b(n + 1)^2 + c(n + 2)^2 = (n + 3)^2 \tag{1}$$

holds for all n. Considering (1) as a polynomial identity in n, we expand and simplify both sides, then equate coefficients of like powers of n. We obtain the equations

$$\begin{aligned} a + b + c &= 1 \\ 2b + 4c &= 6 \\ b + 4c &= 9. \end{aligned}$$

The solution of this system is $a = 1$, $b = -3$ and $c = 3$.

Note. For the sake of completeness, one should check that the system of three equations given in the problem does have a solution. One such solution is $x_1 = 797/4$, $x_2 = -916/4$, $x_3 = 319/4$ and $x_4 = x_5 = x_6 = x_7 = 0$.

It is interesting to note that the number of variables and their values are of little significance. The reader may wish to investigate generalizations of these results to problems in which the coefficients are cubes, fourth powers, etc.

9. (Answer: 144)

It is clear that $n \geq 134$. We can get an upper bound on n by noting that

$$\begin{aligned} n^5 &= 133^5 + 110^5 + 84^5 + 27^5 \\ &< 133^5 + 110^5 + (27+84)^5 \\ &< 3(133)^5 \\ &< \frac{3125}{1024}(133)^5 \\ &= \left(\frac{5}{4}\right)^5 (133)^5. \end{aligned}$$

Thus $n < (\frac{5}{4})(133)$, giving $n \leq 166$. Next note that, when an integer is raised to the fifth power, its units digit is unchanged. It follows that n has the same units digit as the sum $133 + 110 + 84 + 27$; i.e., the units digit of n is 4, and n is one of the four numbers 134, 144, 154, 164. Since $133 \equiv 1 \pmod 3$, $110 \equiv 2 \pmod 3$, $84 \equiv 0 \pmod 3$ and $27 \equiv 0 \pmod 3$, we have

$$n^5 = 133^5 + 110^5 + 84^5 + 27^5 \equiv 1^5 + 2^5 \equiv 0 \pmod 3.$$

This means that n is a multiple of 3, and we conclude that $n = 144$.

Note. Euler's original conjecture was, that for any integer $n \geq 3$, the equation

$$x_1^n + x_2^n + x_3^n + \cdots + x_{n-1}^n = x_n^n$$

has no non-trivial integer solutions. The "spoilers of Euler" were L. J. Lander, T. R. Parkin and J. L. Selfridge. Their work was published in *Mathematics of Computation*, **21**(1967), 446–459. Recently, N. D. Elkies, (a USAMO winner) showed that Euler's conjecture is also false in the case $n = 4$.

10. (Answer: 994)
First note that

$$\frac{\cot\gamma}{\cot\alpha+\cot\beta}=\frac{\dfrac{\cos\gamma}{\sin\gamma}}{\dfrac{\cos\alpha\sin\beta+\sin\alpha\cos\beta}{\sin\alpha\sin\beta}}=\frac{\cos\gamma\sin\alpha\sin\beta}{\sin\gamma\sin(\alpha+\beta)}=\frac{\cos\gamma\sin\alpha\sin\beta}{\sin^2\gamma},$$

and that

$$\frac{\sin\alpha}{a}=\frac{\sin\beta}{b}=\frac{\sin\gamma}{c}$$

implies

$$\frac{\sin\alpha\sin\beta}{\sin^2\gamma}=\frac{ab}{c^2}.$$

Thus

$$\frac{\cot\gamma}{\cot\alpha+\cot\beta}=\frac{ab\cos\gamma}{c^2}.$$

Hence, by the Law of Cosines,

$$\frac{\cot\gamma}{\cot\alpha+\cot\beta}=\frac{a^2+b^2-c^2}{2c^2}=\frac{1989c^2-c^2}{2c^2}=994.$$

11. (Answer: 947)
Let M be the mode and m be the mean. We may assume, without loss of generality, that $M\geq m$. For D to be as large as possible, $M=1000$, since if $M=1000-k$, then increasing M by k increases m by no more than k. As a result, D certainly does not decrease.

Given that $M=1000$, we must make m as small as possible. Now 1000 must occur in the sample at least twice, for otherwise it could not be the unique mode. If 1000 occurs exactly twice, then every other number in the sample must occur once. In this case, m will be smallest if the other $121-2$ values are $1, 2, 3, \ldots, 119$. This leads to a mean of

$$\frac{\frac{(119)(119+1)}{2}+2000}{121}=\frac{9140}{121}.$$

If $M=1000$ occurs exactly 3 times, then every other value can occur at most twice, and m will be smallest if the other 118 sample values are $1, 1, 2, 2, 3, 3, \ldots, 59, 59$. We then have a mean of

$$\frac{\frac{(118)(59+1)}{2}+3000}{121}=\frac{6540}{121}.$$

If the mode occurs 4 times, the smallest possible mean is

$$\frac{\frac{(117)(39+1)}{2}+4000}{121}=\frac{6340}{121}.$$

If the mode occurs 5 times, the smallest mean is

$$\frac{\frac{(116)(29+1)}{2}+5000}{121}=\frac{6740}{121}$$

and if the mode occurs 6 times, the smallest mean is

$$\frac{\frac{(115)(23+1)}{2}+6000}{121}=\frac{7380}{121}.$$

Finally, if $M=1000$ occurs exactly n times, with $n\geq 7$, then

$$m\geq\frac{1000n}{121}\geq\frac{7000}{121}.$$

The above shows that the smallest m occurs when $M=1000$ occurs exactly 4 times, and this m is $\frac{6340}{121}=52+\frac{48}{121}$. Thus the largest value of D is $1000-(52+\frac{48}{121})$, and $\lfloor D\rfloor=947$.

Query. Can you state and solve an analogous problem for real number data?

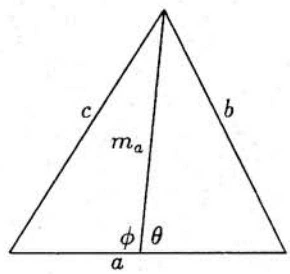
12. (Answer: 137)
We first show that if a, b and c are the three sides of a triangle and m_a is the median to side a, then

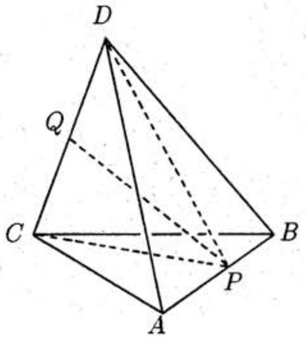
$$m_a^2=\frac{1}{4}(2b^2+2c^2-a^2). \tag{1}$$

We then apply (1) three times in order to find the desired d^2.

To prove (1), consider the figure shown below. Apply the Law of Cosines to each of the two smaller triangles to get

$$\begin{aligned}m_a^2&=\frac{1}{2}(m_a^2+m_a^2)\\&=\frac{1}{2}[(c^2-\frac{1}{4}a^2+am_a\cos\phi)\\&\qquad+(b^2-\frac{1}{4}a^2+am_a\cos\theta)]\\&=\frac{1}{4}(2b^2+2c^2-a^2)\\&\qquad+\frac{1}{2}am_a(\cos\phi+\cos(\pi-\phi))\\&=\frac{1}{4}(2b^2+2c^2-a^2).\end{aligned}$$





Next, in the tetrahedron shown on the left, let P be the midpoint of $\overline{AB}$ and Q be the midpoint of $\overline{CD}$. We apply (1) to find $(PC)^2$ (since $\overline{PC}$ is a median of ΔABC) and $(PD)^2$ (since $\overline{PD}$ is a median of ΔABD). We find

$$(PC)^2 = \frac{1}{4}[2(AC)^2 + 2(BC)^2 - (AB)^2] = \frac{1009}{4}$$

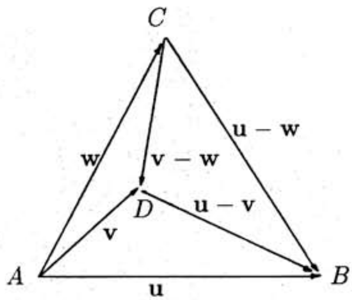
and

$$(PD)^2 = \frac{1}{4}[2(AD)^2 + 2(BD)^2 - (AB)^2] = \frac{425}{4}.$$

Finally, we again use (1) to find $d^2 = (PQ)^2$ from ΔCDP:

$$(PQ)^2 = \frac{1}{4}[2(PC)^2 + 2(PD)^2 - (CD)^2] = 137.$$

Alternate Solution. Introduce the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ to denote the directed edges from A. (See the accompanying figure.) The other three edges, with orientations as indicated, are given by $\mathbf{u}-\mathbf{v}$, $\mathbf{u}-\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$. Moreover, the vector from A to the midpoint of $\overline{AB}$ is $\frac{1}{2}\mathbf{u}$ and the vector from A to the midpoint of $\overline{CD}$ is $\frac{1}{2}(\mathbf{v}+\mathbf{w})$. Consequently, the vector from the midpoint of $\overline{CD}$ to the midpoint of $\overline{AB}$ is $\frac{1}{2}\mathbf{u} - \frac{1}{2}(\mathbf{v}+\mathbf{w})$. We seek the square of the length of this last vector. Recalling that for a vector $\mathbf{x}$, $|\mathbf{x}|^2 = \mathbf{x}\cdot\mathbf{x}$, we have



$$d^2 = \frac{1}{4}(\mathbf{u}-\mathbf{v}-\mathbf{w})\cdot(\mathbf{u}-\mathbf{v}-\mathbf{w}) = \frac{1}{4}(|\mathbf{u}|^2+|\mathbf{v}|^2+|\mathbf{w}|^2-2\mathbf{u}\cdot\mathbf{v}-2\mathbf{u}\cdot\mathbf{w}+2\mathbf{v}\cdot\mathbf{w}). \quad (1)$$

To find $\mathbf{u}\cdot\mathbf{v}$, $\mathbf{u}\cdot\mathbf{w}$ and $\mathbf{v}\cdot\mathbf{w}$, note that $|\mathbf{x}-\mathbf{y}|^2 = |\mathbf{x}|^2+|\mathbf{y}|^2-2\mathbf{x}\cdot\mathbf{y}$, which implies

$$2\mathbf{x}\cdot\mathbf{y} = |\mathbf{x}|^2+|\mathbf{y}|^2-|\mathbf{x}-\mathbf{y}|^2. \quad (2)$$

Applying (2) to (1) yields the general formula

$$d^2 = \frac{1}{4}(|\mathbf{v}|^2+|\mathbf{w}|^2+|\mathbf{u}-\mathbf{v}|^2+|\mathbf{u}-\mathbf{w}|^2-|\mathbf{u}|^2-|\mathbf{v}-\mathbf{w}|^2). \quad (3)$$

The given measurements yield $d^2 = 137$.

Note. For an alternate derivation of (3) the reader may wish to consult Straszewicz's excellent *Mathematical Problems and Puzzles from the Polish Olympiad*, published by Pergamon Press in 1965, but unfortunately long out of print.

13. (Answer: 905)

We first show that, given any set of 11 consecutive integers from $\{1, 2, 3, \ldots, 1989\}$, at most 5 of these 11 can be elements of S. We prove this fact for the set $T = \{1, 2, 3, \ldots, 11\}$, but the same proof works for any set of 11 consecutive integers. Consider the following partition of T, where each subset was formed so that it can contribute at most one element to S:

$$\{1,5\} \qquad \{2,9\} \qquad \{3,7\} \qquad \{4,11\} \qquad \{6,10\} \qquad \{8\}. \tag{1}$$

If it were possible to have 6 elements of T in S, then each of the sets in (1) would have to contribute exactly one element. That this is impossible is shown by the following chain of implications:

$$8 \in S \Longrightarrow 1 \notin S \Longrightarrow 5 \in S \Longrightarrow 9 \notin S \Longrightarrow 2 \in S \Longrightarrow 6 \notin S \Longrightarrow 10 \in S$$
$$\Longrightarrow 3 \notin S \Longrightarrow 7 \in S \Longrightarrow 11 \notin S \Longrightarrow 4 \in S \Longrightarrow 8 \notin S.$$

With the aid of (1), or otherwise, it is easy to find a 5-element subset of T that satisfies the key property of S (i.e., no two numbers differ by 4 or 7). One such set is

$$T' = \{1, 3, 4, 6, 9\}.$$

We also find (perhaps to our surprise) that T' has the remarkable property of allowing for a periodic continuation. That is, if I denotes the set of integers, then

$$S' = \{k + 11n \mid k \in T' \quad \text{and} \quad n \in I\}$$

also has the property that no two elements in the set differ by 4 or 7. Moreover, since $1989 = 180 \cdot 11 + 9$, it is clear that S cannot have more than $181 \cdot 5 = 905$ elements. Because the largest element in T' is 9, it follows that the set

$$S = S' \cap \{1, 2, 3, \ldots, 1989\}$$

has 905 elements and hence shows that the upper bound of 905 on the size of the desired set can be attained. This completes the argument.

Note. The reader may wish to find other 5-element subsets of $\{1, 2, 3, \ldots, 11\}$ that exhibit the key property of S. Which of these subsets can be used, as above, to generate a maximal S?

The reader is also encouraged to explore similar problems with other pairs (triples, etc.) of integers in place of 4 and 7, and to find the appropriate motivations for the choice of 11 as the size of the blocks of integers considered in the above solution.

14. (Answer: 490)

If

$$\begin{aligned} k = (a_3a_2a_1a_0)_{-3+i} &= a_3(-3+i)^3 + a_2(-3+i)^2 + a_1(-3+i) + a_0 \\ &= a_3(-18+26i) + a_2(8-6i) + a_1(-3+i) + a_0 \\ &= (-18a_3 + 8a_2 - 3a_1 + a_0) + (26a_3 - 6a_2 + a_1)i \end{aligned}$$

is a real integer, then its imaginary part must vanish. Thus

$$26a_3 - 6a_2 + a_1 = 0. \tag{1}$$

Since a_3, a_2, $a_1 \in \{0, 1, \ldots, 9\}$ and $a_3 \neq 0$, we see that (1) can hold only if $a_3 = 1$ or $a_3 = 2$.

If $a_3 = 1$, then $6a_2 - a_1 = 26$ and the restrictions on a_2 and a_1 force $a_2 = 5$ and $a_1 = 4$. In this case, we have

$$k = (a_3a_2a_1a_0)_{-3+i} = -18a_3 + 8a_2 - 3a_1 + a_0 = 10 + a_0.$$

Since $a_0 \in \{0, 1, 2, \ldots, 9\}$, we see that k can be any one of 10, 11, ..., 19.

If $a_3 = 2$, then $6a_2 - a_1 = 52$, leading to $a_2 = 9$ and $a_1 = 2$. It follows that $k = 30 + a_0$ and k can be any one of 30, 31, ..., 39. Adding the possibilities from the two cases gives the answer

$$(10 + 11 + \cdots + 19) + (30 + 31 + \cdots + 39) = 490.$$

Note. For more about complex bases, the reader is referred to W. Gilbert's article in the March 1984 issue of *Mathematics Magazine*.

15. (Answer: 108)

We are given length information about the three segments through P. Our strategy is to translate one of these segments to form a new triangle (inside of $\triangle ABC$) for which we know all three sides, and hence the area. We then multiply this area by an appropriate ratio to obtain the area of $\triangle ABC$.

We first find the lengths of $\overline{CP}$ and $\overline{PF}$. To this end, observe that

$$\frac{\text{Area}(\triangle BPC)}{\text{Area}(\triangle BAC)} = \frac{PD}{AD} = \frac{6}{6+6} = \frac{1}{2} \tag{1}$$

and

$$\frac{\text{Area}(\triangle APC)}{\text{Area}(\triangle ABC)} = \frac{PE}{BE} = \frac{3}{3+9} = \frac{1}{4}. \tag{2}$$

Thus

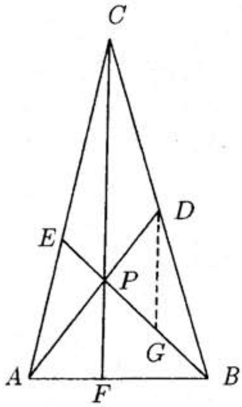
$$\begin{aligned}\frac{PF}{CF} &= \frac{\text{Area}(\triangle APB)}{\text{Area}(\triangle ACB)}\\ &= \frac{\text{Area}(\triangle ACB) - \text{Area}(\triangle APC) - \text{Area}(\triangle BPC)}{\text{Area}(\triangle ACB)} \qquad (3)\\ &= 1 - \frac{1}{4} - \frac{1}{2} = \frac{1}{4}.\end{aligned}$$

Since $CF = 20$, it follows that $PF = 5$ and $CP = 15$. Furthermore,

$$\begin{aligned}\frac{BD}{CD} = \frac{\text{Area}(\triangle ADB)}{\text{Area}(\triangle ACD)} &= \frac{\text{Area}(\triangle PDB)}{\text{Area}(\triangle PDC)}\\ &= \frac{\text{Area}(\triangle ADB) - \text{Area}(\triangle PDB)}{\text{Area}(\triangle ACD) - \text{Area}(\triangle PDC)} = \frac{\text{Area}(\triangle APB)}{\text{Area}(\triangle ACP)} = 1,\end{aligned}$$

where the last equality results from dividing equation (3) by equation (2).

Next construct $\overline{DG}$ parallel to $\overline{CF}$, with G on $\overline{PB}$ as shown. We show that $\triangle GDP$ is a right triangle with sides of $\frac{9}{2}$, 6 and $\frac{15}{2}$ and then show that the area of $\triangle GDP$ is $\frac{1}{8}$ of the area of $\triangle ABC$. It will then follow that the desired answer is $8 \cdot \frac{27}{2} = 108$.



To establish the above claims, first note that $\triangle BDG \sim \triangle BCP$. Since $BD = \frac{1}{2}BC$, the sides of these two triangles are in a ratio of 1:2. It follows that $DG = \frac{1}{2}PC = \frac{15}{2}$ and $PG = GB = \frac{1}{2}PB = \frac{9}{2}$. Since $PD = 6$ was given, we see that $\triangle GDP$ is a right triangle as claimed. Next note that

$$\frac{\text{Area}(\triangle PBC)}{\text{Area}(\triangle GBD)} = \left(\frac{2}{1}\right)^2 = 4 \qquad \text{and} \qquad \frac{\text{Area}(\triangle GBD)}{\text{Area}(\triangle GPD)} = \frac{BG}{PG} = 1.$$

Using these ratios with (1) gives

$$\begin{aligned}\text{Area}(\triangle ABC) &= \frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle PBC)} \cdot \frac{\text{Area}(\triangle PBC)}{\text{Area}(\triangle GBD)} \cdot \frac{\text{Area}(\triangle GBD)}{\text{Area}(\triangle GPD)} \cdot \text{Area}(\triangle GPD)\\ &= 2 \cdot 4 \cdot 1 \cdot \frac{27}{2} = 108.\end{aligned}$$

Note. Implicit in this solution is a method for constructing $\triangle ABC$ with straightedge and compass from the given data. The reader is invited to explore the conditions such data must satisfy in order to ensure that $\triangle ABC$ exists (and is unique).

Alternate Solution. Let $P = (0,0)$, $D = (6,0)$, $A = (-6,0)$, $E = (h,k)$ and $B = (-3h,-3k)$. Solving the equations for $\overline{AE}$ and $\overline{BD}$ simultaneously, we find $C = (3h+12, 3k)$. Next, the coordinates of F can be found by solving the equations of $\overline{CP}$ and $\overline{AB}$ simultaneously; the result is $F = (-4-h, -k)$. Finally, solving the equations $h^2 + k^2 = 9$ and $(4+h)^2 + k^2 = 25$ (arising from $PE = 3$ and $CF = 20$, respectively) one finds that $h = 0$ and $k = 3$. Once we have the coordinates of A, B and C, we can find that the area of $\triangle ABC$ is 108.

AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

8th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
TUESDAY, MARCH 20, 1990

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

Correspondence about the problems and solutions (but **not** requests for the Solutions Pamphlet) should be addressed to:

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1. (Answer: 528)
Between 1 and 500, there are $\lfloor\sqrt{500}\rfloor = 22$ perfect squares and $\lfloor\sqrt[3]{500}\rfloor = 7$ perfect cubes. Among these integers there are $\lfloor\sqrt[6]{500}\rfloor = 2$ of them (1 and 64) that are counted twice. Thus there are $22 + 7 - 2 = 27$ integers between 1 and 500 that are not in the sequence. To get the 500^{th} number, we must append 27 integers to the list $2, 3, 5, \ldots, 500$ of 473 non-squares and non-cubes. Since we cannot use 512, the last number will be 528.

2. (Answer: 828)
We split 52 into two parts to obtain squares in each set of parentheses:

$$\begin{aligned}
&(52+6\sqrt{43})^{3/2}-(52-6\sqrt{43})^{3/2}\\
&\quad=(43+6\sqrt{43}+9)^{3/2}-(43-6\sqrt{43}+9)^{3/2}\\
&\quad=\left[(\sqrt{43}+3)^2\right]^{3/2}-\left[(\sqrt{43}-3)^2\right]^{3/2}\\
&\quad=(\sqrt{43}+3)^3-(\sqrt{43}-3)^3\\
&\quad=(43\sqrt{43}+3\cdot 3\cdot 43+3\cdot 3^2\sqrt{43}+3^3)\\
&\qquad\qquad-(43\sqrt{43}-3\cdot 3\cdot 43+3\cdot 3^2\sqrt{43}-3^3)\\
&\quad=828.
\end{aligned}$$

Alternate Solution. Let $\alpha = (52 + 6\sqrt{43})^{1/2}$ and $\beta = (52 - 6\sqrt{43})^{1/2}$. We wish to find $\alpha^3 - \beta^3 = (\alpha - \beta)(\alpha^2 + \alpha\beta + \beta^2)$. Now

$$\alpha^2+\beta^2=104 \qquad \text{and} \qquad \alpha\beta=(52^2-36\cdot 43)^{1/2}=(1156)^{1/2}=34.$$

Thus $(\alpha - \beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2 = 104 - 68 = 36$, so $\alpha - \beta = 6$ and $\alpha^3 - \beta^3 = 6(104 + 34) = 828$.

3. (Answer: 117)
In a regular n-gon, each interior angle has radian measure $(n-2)\pi/n$. The information in the problem says

$$\frac{59}{58}=\left(\frac{r-2}{r}\pi\right)\Big/\left(\frac{s-2}{s}\pi\right)=\frac{rs-2s}{rs-2r}. \qquad (*)$$

Solving for r gives

$$r=\frac{116s}{118-s}.$$

Since r must be positive, we must have $s \leq 117$. Indeed, if $s = 117$ then we find $r = 116 \cdot 117$ and equation $(*)$ will be satisfied.

4. (Answer: 013)
Let $x^2 - 10x = y$. The equation in the problem then becomes

$$\frac{1}{y-29} + \frac{1}{y-45} - \frac{2}{y-69} = 0,$$

from which

$$\frac{1}{y-29} - \frac{1}{y-69} = \frac{1}{y-69} - \frac{1}{y-45},$$

and

$$\frac{-40}{(y-29)(y-69)} = \frac{24}{(y-45)(y-69)}$$

follows. This equation has $y = 39$ as its only solution. We then note that $x^2 - 10x = 39$ is satisfied by the positive number 13.

5. (Answer: 432)
Suppose the prime factorization of n has the form

$$n = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k},$$

where $p_1, p_2, \ldots, p_k$ are the distinct prime divisors of n and $r_1, r_2, \ldots, r_k$ are positive integers. Then the number of divisors of n is given by

$$(r_1+1)(r_2+1)\cdots(r_k+1).$$

Since this last product must be $75 = 3 \cdot 5 \cdot 5$, we see that n can have at most three distinct prime factors. To ensure that n is divisible by 75 and that the n we obtain is minimal, the prime factors must belong to the set $\{2, 3, 5\}$, with the factor 3 occurring at least once and the factor 5 occurring at least twice. Thus

$$n = 2^{r_1} 3^{r_2} 5^{r_3}$$

with

$$(r_1+1)(r_2+1)(r_3+1) = 75 \qquad r_2 \geq 1, r_3 \geq 2.$$

It is not hard to write down the ordered triples (r_1, r_2, r_3) that satisfy the above conditions:

$$(4,4,2) \qquad (4,2,4) \qquad (2,4,4) \qquad (0,4,14)$$
$$(0,14,4) \qquad (0,2,24) \qquad (0,24,2).$$

Among the above ordered triples, the minimum value for n occurs when $r_1 = r_2 = 4$ and $r_3 = 2$. Thus our answer is $n/75 = 2^4 3^3 = 432$.

6. (Answer: 840)
Let

$$X = \text{the number of fish in the lake on May 1,}$$
$$Y = \text{the number of fish in the lake on September 1.}$$

From the data in the problem we find that $Y = .75X + .40Y$, and that the number of tagged fish in the lake on September 1 is $.75(60) = 45$. Thus, assuming the tagged fish are fairly represented in the September 1 sample, we have

$$\frac{3}{70} = \frac{45}{Y}.$$

Hence $Y = 1050$ and $X = .60Y/.75 = 840$.

7. (Answer: 089)
Extend $\overline{PR}$ through R to T, where T is selected so that $PQ = PT$. Since $PQ = 25$ and $PR = 15$, the point T has coordinates

$$\begin{aligned} P + \frac{25}{15}(R-P) &= (-8,5) + \frac{25}{15}[(1,-7)-(-8,5)] \\ &= (-8,5) + \frac{5}{3}(9,-12) \\ &= (7,-15). \end{aligned}$$

Now $\angle P$ in $\triangle PQR$ and $\angle P$ in $\triangle PQT$ have the same bisector. Since $\triangle PQT$ is isosceles, with $PQ = PT$, this bisector intersects $\overline{QT}$ at its midpoint, $(-4,-17)$. Thus the slope of the bisector is $-\frac{11}{2}$ and its equation can be written in the form $11x + 2y + 78 = 0$. Hence $a + c = 11 + 78 = 89$.

Alternate Solution. Consider the vectors $\overrightarrow{PQ} = -7\vec{\imath} - 24\vec{\jmath}$ and $\overrightarrow{PR} = 9\vec{\imath} - 12\vec{\jmath}$. Let $\vec{v} = a\vec{\imath} + b\vec{\jmath}$ be a vector parallel to the bisector of $\angle P$. Then the angle between vectors $\overrightarrow{PQ}$ and $\vec{v}$ is equal to the angle between vectors $\vec{v}$ and $\overrightarrow{PR}$. Let ϕ be the measure of each of these angles. Then

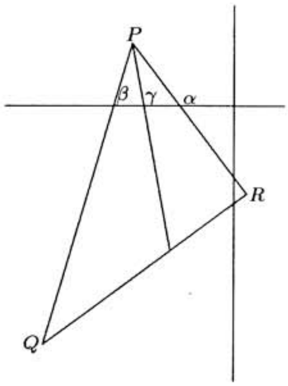
$$\frac{\vec{v}\cdot\overrightarrow{PQ}}{\|\vec{v}\|\,\|\overrightarrow{PQ}\|} = \cos\phi = \frac{\vec{v}\cdot\overrightarrow{PR}}{\|\vec{v}\|\,\|\overrightarrow{PR}\|}$$

giving

$$\frac{-7a-24b}{25} = \frac{9a-12b}{15},$$

which simplifies to $11a + 2b = 0$. Hence $b = -\frac{11}{2}a$ and it follows that the bisector of $\angle P$ has slope $-\frac{11}{2}$. The equation of the bisector can be written as $11x + 2y + 78 = 0$, so $a + c = 11 + 78 = 89$.

Alternate Solution. Let α, β, γ, respectively, be the angles that $\overline{PR}$, $\overline{PQ}$ and the bisector of $\angle P$ make with the x-axis. (These angles are measured counterclockwise from the x-axis.) Let m be the slope of the bisector of $\angle P$. Then the slope of $\overline{PR}$ is $\tan\alpha = -\frac{4}{3}$, the slope of $\overline{PQ}$ is $\tan\beta = \frac{24}{7}$, and the slope of the bisector of $\angle P$ is $\tan\gamma = m$. Since $\alpha - \gamma = \frac{1}{2}\angle P = \gamma - \beta$, we have $\tan(\alpha-\gamma) = \tan(\gamma-\beta)$. Using the formula for the tangent of the difference of two angles gives



$$\frac{\tan\alpha - \tan\gamma}{1+\tan\alpha\tan\gamma} = \frac{\tan\gamma - \tan\beta}{1+\tan\gamma\tan\beta},$$

which leads to

$$\frac{-\frac{4}{3} - m}{1 - \frac{4}{3}m} = \frac{m - \frac{24}{7}}{1 + \frac{24}{7}m}.$$

The last equation has solutions $m = -\frac{11}{2}$ and $m = \frac{2}{11}$. The solution $-\frac{11}{2}$ is the slope of the internal bisector of $\angle P$. (Some other line has slope $\frac{2}{11}$. Which one?) We then find that the equation of the bisector can be written in the form $11x + 2y + 78 = 0$, and $a + c = 11 + 78 = 89$.

8. (Answer: 560)
Consider the eight shots that must be fired to break the eight targets. Of the eight, any subset of three shots may be the shots used to break the targets in the first column (but once these three shots are chosen the rules of the match determine the order in which the targets in the first column will be broken by these shots.) This set of shots for the first column may be chosen in $\binom{8}{3}$ ways. From the remaining five shots, the three used to break the targets in the other column of three may be chosen in $\binom{5}{3}$ ways, while the remaining two shots will be used to break the remaining two targets. Combining, we find that the number of orders in which the targets can be broken is

$$\binom{8}{3}\binom{5}{3}\binom{2}{2} = \binom{8}{3,3,2} = \frac{8!}{3!3!2!} = 560.$$

9. (Answer: 073)
There are 2^n possible sequences of length n that can be formed from the letters **T** and **H**. Let $A(n)$ be the number of these sequences in which there are no adjacent occurrences of **H**. The values $A(1) = 2$, $A(2) = 3$ and $A(3) = 5$ may be found by simply listing all possible outcomes for tosses of 1, 2 and 3 coins respectively. For higher values of n, we may find $A(n)$ by using the recursion relation $A(n+2) = A(n+1) + A(n)$, which holds for any positive integer n. This recursion relation is true because $A(n+2)$ counts two distinct types of sequences: those with no consecutive **H**'s that end with **T** (there are $A(n+1)$ of them) and those with no consecutive **H**'s that end with **TH** (there are $A(n)$ of these).

It follows that the values $A(n)$ are Fibonacci numbers, so $A(10) = 144$. Hence the probability of tossing a coin ten times and never having heads occur on consecutive tosses is $144/1024 = 9/64$ and $i + j = 73$.

10. (Answer: 144)
First observe that if $z \in A$ and $w \in B$, then

$$(zw)^{144} = (z^{18})^8(w^{48})^3 = 1.$$

This shows that the set C is contained in the set of 144^{th} roots of unity. Next we show that any 144^{th} root of unity is in C, thereby showing that C has 144 elements. Let x be a 144^{th} root of unity. Then there is an integer k with

$$x = \cos\left(\frac{2\pi}{144}k\right) + i\sin\left(\frac{2\pi}{144}k\right) = \operatorname{cis}\left(\frac{2\pi}{144}k\right) = \left[\operatorname{cis}\left(\frac{2\pi}{144}\right)\right]^k,$$

where the last equality follows by an application of DeMoivre's formula. We next express the greatest common divisor of 18 and 48 as $6 = 3 \cdot 18 - 48$ and use this in the following:

$$\operatorname{cis}\left(\frac{2\pi}{144}\right) = \operatorname{cis}\left(\frac{2\pi}{864}\cdot 6\right) = \operatorname{cis}\left(\frac{2\pi}{864}(3\cdot 18 - 48)\right) = \operatorname{cis}\left(\frac{2\pi}{48}3\right)\operatorname{cis}\left(\frac{2\pi}{18}(-1)\right).$$

By another application of DeMoivre's formula, we now have

$$x = \left[\operatorname{cis}\left(\frac{2\pi}{48}3\right)\operatorname{cis}\left(\frac{2\pi}{18}(-1)\right)\right]^k = \operatorname{cis}\left(\frac{2\pi}{48}3k\right)\operatorname{cis}\left(\frac{2\pi}{18}(-k)\right),$$

which shows that x is a product of elements from A and B. Hence the set of 144^{th} roots of unity is a subset of C. We may conclude that C is the set of 144^{th} roots of unity, so C has 144 elements.

11. (Answer: 023)
If $n!$ can be expressed as the product of $n-3$ consecutive integers, then there is a positive integer k such that

$$n! = (n+k)(n+k-1)\cdots(k+4) = \frac{(n+k)!}{(k+3)!}.$$

We can express this last relation as

$$\frac{(n+k)!/n!}{(k+3)!} = 1$$

and expand to get

$$\frac{n+k}{k+3}\cdot\frac{n+k-1}{k+2}\cdots\frac{n+2}{5}\cdot\frac{n+1}{4!} = 1.$$

If $n > 23$, then the k factors on the left of the previous equation all exceed 1 and the equation cannot be true. On the other hand, $n = 23$ and $k = 1$ is an obvious solution to (1) and shows that $n = 23$ is the answer to the problem.

Note. The above argument can be generalized to prove the following result: *Let $r \geq 2$ be an integer. The largest integer n for which $n!$ can be written as the product of $n - r$ consecutive positive integers is $n = (r+1)! - 1$.*

12. (Answer: 720)
Position the 12-gon in the Cartesian plane with its center at the origin and one vertex at $(12, 0)$. Compute the sum, S, of the lengths of the eleven segments emanating from this vertex. The coordinates of the other vertices are given by $(12\cos kx, 12\sin kx)$ where $x = 30°$ and $k = 1, 2, \ldots, 11$. The length of the segment joining $(12, 0)$ to $(12\cos kx, 12\sin kx)$ is

$$12\sqrt{(\cos kx - 1)^2 + (\sin kx)^2} = 12\sqrt{2 - 2\cos kx} = 24\sin\frac{kx}{2}.$$

Thus the sum of the lengths of the 11 segments from $(12, 0)$ is

$$S = 24(\sin 15° + \sin 30° + \cdots + \sin 150° + \sin 165°).$$

Since $\sin t = \sin(180° - t)$ we may write

$$S = 48(\sin 15° + \sin 30° + \sin 45° + \sin 60° + \sin 75°) + 24\sin 90°.$$

Now

$$\begin{aligned}\sin 15^\circ + \sin 75^\circ &= \sin(45^\circ - 30^\circ) + \sin(45^\circ + 30^\circ)\\ &= 2\sin 45^\circ \cos 30^\circ\\ &= \frac{1}{2}\sqrt{6}.\end{aligned}$$

Thus

$$\begin{aligned}S &= 48\Big(\frac{1}{2}\sqrt{6} + \frac{1}{2} + \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\Big) + 24\\ &= 48 + 24\sqrt{2} + 24\sqrt{3} + 24\sqrt{6}.\end{aligned}$$

The same value, S, occurs if we add the lengths of all segments emanating from any other vertex of the 12-gon. Since each segment is counted at two vertices (its endpoints) the total length of all such segments is

$$\frac{1}{2}(12S) = 288 + 144\sqrt{2} + 144\sqrt{3} + 144\sqrt{6}.$$

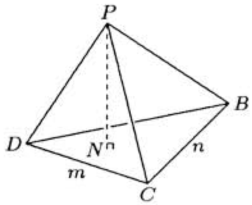
Hence $a + b + c + d = 288 + 144 + 144 + 144 = 5 \cdot 144 = 720$.

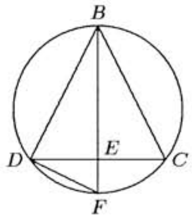
13. (Answer: 184)
Note that 9^k has one more digit than 9^{k-1}, except in the case when 9^k starts with a 9. In the latter case, long division shows that 9^{k-1} starts with a 1 and has the same number of digits as 9^k. Therefore, when the powers of 9 from 9^0 to 9^{4000} are computed there are 3816 increases in the number of digits. Thus there must be $4000 - 3816 = 184$ instances when computing 9^k from 9^{k-1} $(1 \le k \le 4000)$ does not increase the number of digits. Since $9^0 = 1$ does not have leading digit 9 we can conclude that 9^k $(1 \le k \le 4000)$ has a leading digit of 9 exactly when there is no increase in the number of digits in computing 9^k from 9^{k-1}. It follows that 184 of the numbers must start with the digit 9.

Note. We did not need to know that the leading digit of 9^{4000} is 9, but it was important to note that the leading digit of 9^0 is *not* 9.

14. (Answer: 594)

Let m and n denote CD and BC, respectively. Three of the faces are lettered PBC, PCD, and PDB (see diagram). Let N be the point where the altitude from P meets BCD. We first show that N is the circumcenter of $\triangle BCD$. To see this, note that $\triangle PNB$, $\triangle PNC$ and $\triangle PND$ are congruent by the hypotenuse-leg criterion. It follows that $\overline{BN}$, $\overline{CN}$ and $\overline{DN}$ all have the same length r; this r is the radius (and hence N is the center) of the circle that circumscribes $\triangle BCD$. We will find the value of r and use it to find PN. From B draw a diameter of the circumcircle. Let the other end of this diameter be F and let E be the point where the diameter meets $\overline{CD}$. Then $\angle BFD \cong \angle ECB$ since both angles subtend the same arc on the circumcircle. Hence the two right triangles $\triangle BFD$ and $\triangle BCE$ are similar, implying $BF/BD = BC/BE$. Since $BF = 2r$, the last equation gives





$$r = \frac{n^2}{2(BE)} = \frac{n^2}{\sqrt{4n^2 - m^2}}.$$

Now using $PB = \frac{1}{2}\sqrt{m^2+n^2}$ we can find PN, the altitude of the pyramid, by the Pythagorean theorem:

$$PN^2 = PB^2 - BN^2 = \frac{m^2+n^2}{4} - \frac{n^4}{4n^2 - m^2}.$$

Hence

$$PN = \frac{m}{2}\sqrt{\frac{3n^2 - m^2}{4n^2 - m^2}}.$$

Thus, the volume of pyramid $PBCD$ is

$$\frac{1}{3}(PN)(\text{Area}(\triangle BCD)) = \frac{1}{3}\cdot\frac{m}{2}\sqrt{\frac{3n^2-m^2}{4n^2-m^2}}\cdot\frac{m}{2}\sqrt{n^2 - \frac{m^2}{4}} = \frac{m^2}{24}\sqrt{3n^2 - m^2}.$$

Since $m^2 = 432$ and $n^2 = 507$, the volume is 594.

15. (Answer: 020)
For $n = 1$ and $n = 2$, the identity

$$(1) \qquad (ax^{n+1} + by^{n+1})(x + y) - (ax^n + by^n)xy = ax^{n+2} + by^{n+2}$$

yields the equations

$$7(x + y) - 3xy = 16 \qquad \text{and} \qquad 16(x + y) - 7xy = 42.$$

Solving these last two equations simultaneously, one finds that

$$(2) \qquad x + y = -14 \qquad \text{and} \qquad xy = -38.$$

Applying (1) with $n = 3$ then gives

$$ax^5 + by^5 = (42)(-14) - (16)(-38) = -588 + 608 = 20.$$

Note: From (2) we can solve for x and y. We obtain $x = -7 \pm \sqrt{87}$ and $y = -7 \mp \sqrt{87}$, from which $a = \dfrac{49}{76} \pm \dfrac{457}{6612}\sqrt{87}$ and $b = \dfrac{49}{76} \mp \dfrac{457}{6612}\sqrt{87}$.

AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

9th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
TUESDAY, MARCH 19, 1991

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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1. (Answer: 146)

Let $a = x + y$ and $b = xy$, and note that the given equations imply

$$a + b = 71 \qquad \text{and} \qquad ab = 880.$$

Solving simultaneously, one finds that $\{a, b\} = \{16, 55\}$; i.e., either

$$x + y = 55 \qquad \text{and} \qquad xy = 16 \tag{1}$$

or

$$x + y = 16 \qquad \text{and} \qquad xy = 55. \tag{2}$$

It is easy to check that (1) has no solution in integers while in (2) we have $\{x, y\} = \{5, 11\}$. Consequently, $x^2 + y^2 = 5^2 + 11^2 = 146$.

2. (Answer: 840)

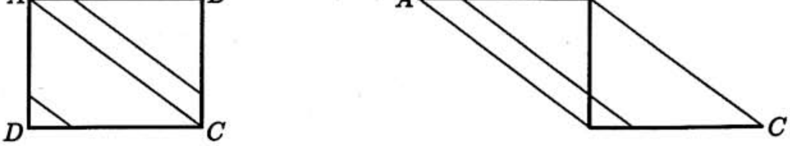
By symmetry, the sum of the lengths of the 335 segments is equal to

$$AC + 2\sum_{k=1}^{167} P_kQ_k.$$

For $1 \le k \le 167$ we have $P_kB = AB(1 - k/168)$ and $BQ_k = BC(1 - k/168)$. It follows that $\triangle P_kBQ_k \sim \triangle ABC$, so $P_kQ_k = AC(1 - k/168)$. Thus the sum of the 335 segments is

$$AC\left(1 + 2\sum_{k=1}^{167}\left(1 - \frac{k}{168}\right)\right) = 5\left(1 + \frac{2}{168}\sum_{j=1}^{167} j\right) = 5\left(1 + \frac{2}{168}\frac{167 \cdot 168}{2}\right) = 840.$$

Alternate Solution. Cut the rectangle along diagonal $\overline{AC}$, then reposition $\triangle ABC$ so vertices B and C of $\triangle ABC$ are coincident with vertices A and D, respectively, of $\triangle ADC$. The resulting figure is a parallelogram. In doing this cutting and repositioning we see that, except for diagonal $\overline{AC}$ which has length 5, all of the other 334 segments can be grouped into pairs whose lengths sum to 5. (See figure.) Hence the desired sum is $5 + 167 \cdot 5 = 840$.



3. (Answer: 166)

For $1 \le k \le 1000$,

$$\frac{A_k}{A_{k-1}} = \frac{\dfrac{1000!}{k!(1000-k)!}(0.2)^k}{\dfrac{1000!}{(k-1)!(1001-k)!}(0.2)^{k-1}} = \frac{1001-k}{k}(0.2).$$

This ratio never equals 1, and exceeds 1 if and only if $1001 - k > 5k$. This last inequality is true just for $k \le 166$. Thus we have

$$A_0 < A_1 < \ldots < A_{166}$$

while

$$A_{166} > A_{167} > \ldots > A_{1000}.$$

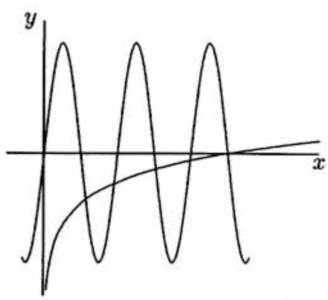
Hence A_k is largest for $k = 166$.

4. (Answer: 159)

Since $|\sin\theta| \le 1$ for all real θ, we need only consider those values of x for which

$$\left|\frac{1}{5}\log_2 x\right| \le 1.$$

This inequality is satisfied by all values of x between $1/32$ and 32, inclusive. We first consider $\frac{1}{32} \le x < 1$. For such x we have $-1 \le \frac{1}{5}\log_2 x < 0$, while $\sin 5\pi x \le 0$ only for $\frac{1}{5} \le x \le \frac{2}{5}$ and $\frac{3}{5} \le x \le \frac{4}{5}$. It follows that the graphs of $y = \frac{1}{5}\log_2 x$ and $y = \sin 5\pi x$ meet at 4 points for $\frac{1}{32} \le x < 1$. (See accompanying figure.) When $1 < x \le 32$, we have $0 < \frac{1}{5}\log_2 x \le 1$ while $\sin 5\pi x \ge 0$ only for $\frac{2k}{5} \le x \le \frac{2k+1}{5}$ $(k = 3, 4, \ldots, 79)$. The graphs of the two functions intersect at 2 points on each of these 77 intervals, giving 154 points of intersection for $1 < x \le 32$. Since both functions take on the value 0 when $x = 1$, we have a total of $4 + 154 + 1 = 159$ solutions to the equation.



5. (Answer: 128)

For a fraction to be in lowest terms, its numerator and denominator must be relatively prime. Thus any prime factor that occurs in the numerator cannot occur in the denominator, and vice-versa. There are eight prime factors of 20!, namely 2, 3, 5, 7, 11, 13, 17, and 19. For each of these prime factors, one must decide only whether it occurs in the numerator or in the denominator. These eight decisions can be made in a total of $2^8 = 256$ ways. However, not all of the 256 resulting fractions will be less than 1. Indeed, they can be grouped into 128 pairs of reciprocals, each of which will have exactly one fraction less than 1. Thus the number of rational numbers with the desired property is 128.

6. (Answer: 743)

The given sum has 73 terms, each of which equals either $\lfloor r \rfloor$ or $\lfloor r \rfloor + 1$. This is because $19/100, 20/100, \ldots, 91/100$ are all less than 1. In order for the sum to be 546, it is necessary that $\lfloor r \rfloor$ be 7, because $73 \cdot 7 < 546 < 73 \cdot 8$. Now suppose that $\lfloor r + \frac{k}{100} \rfloor = 7$ for $19 \le k \le m$ and $\lfloor r + \frac{k}{100} \rfloor = 8$ for $m + 1 \le k \le 91$. Then

$$7(m - 18) + 8(91 - m) = 546,$$

giving $m = 56$. Thus $\lfloor r + \frac{56}{100} \rfloor = 7$ but $\lfloor r + \frac{57}{100} \rfloor = 8$. It follows that $7.43 \le r < 7.44$, and hence that $\lfloor 100r \rfloor = 743$.

7. (Answer: 383)

The fraction on the right side of the equation can be simplified to the form

$$\frac{ax + b}{cx + d},$$

for some real numbers a, b, c, and d. It follows that the given equation is quadratic, and hence has at most 2 solutions. Next, observe that any solution to

$$x = \sqrt{19} + \frac{91}{x} \qquad (*)$$

is also a solution to the original equation. This can be seen by repeatedly replacing each occurrence of x in the right side of $(*)$ by $\sqrt{19} + \dfrac{91}{x}$ until the equation in the problem results. Equation $(*)$ has two solutions,

$$x = \frac{\sqrt{19} + \sqrt{383}}{2} \qquad \text{and} \qquad x = \frac{\sqrt{19} - \sqrt{383}}{2},$$

so these must be the roots of the equation given in the problem. The sum of the absolute values of these roots is $A = \sqrt{383}$, and $A^2 = 383$.

8. (Answer: 010)

Suppose $x^2 + ax + 6a = 0$ has integer roots m and n, with $m \le n$. Since

$$x^2 + ax + 6a = (x - m)(x - n) = x^2 - (m + n)x + mn,$$

we must have $a = -(m + n)$ and $6a = mn$. This implies that a must be an integer and that $-6(m+n) = mn$. This last equation is equivalent to $mn+6m+6n+36 = 36$, or

$$(m + 6)(n + 6) = 36.$$

It is not hard to see that the only integer solutions with $m \le n$ are the ten pairs $(-42, -7)$, $(-24, -8)$, $(-18, -9)$, $(-15, -10)$, $(-12, -12)$, $(-5, 30)$, $(-4, 12)$, $(-3, 6)$, $(-2, 3)$, $(0, 0)$. The corresponding values of $a = -(m + n)$ are 49, 32, 27, 25, 24, -25 -8, -3, -1, and 0. Thus there are 10 values of a for which $x^2 + ax + 6a = 0$ has integer roots.

9. (Answer: 044)

Since $\sec^2 x - \tan^2 x = 1$, we see that $\sec x - \tan x = 1/p$, where p stands for $22/7$. This leads to

$$2 \sec x = p + \frac{1}{p} \qquad \text{and} \qquad 2 \tan x = p - \frac{1}{p},$$

and then to

$$\cos x = \frac{2p}{p^2 + 1} \qquad \text{and} \qquad \sin x = \frac{p^2 - 1}{p^2 + 1}.$$

It is now an easy matter to show that

$$\csc x + \cot x = \frac{p + 1}{p - 1} = \frac{29}{15}.$$

Alternate solution. We apply the half-angle identities

$$\tan(x/2) = \csc x - \cot x \qquad \text{and} \qquad \cot(x/2) = \csc x + \cot x.$$

Since

$$\frac{22}{7} = \sec x + \tan x = \csc\left(\frac{\pi}{2} + x\right) - \cot\left(\frac{\pi}{2} + x\right) = \tan\left(\frac{1}{2}\left(\frac{\pi}{2} + x\right)\right) = \tan\left(\frac{\pi}{4} + \frac{x}{2}\right),$$

we have

$$\begin{aligned} \frac{m}{n} = \csc x + \cot x = \cot\frac{x}{2} &= \tan\left(\frac{\pi}{2} - \frac{x}{2}\right) = \tan\left(\frac{3\pi}{4} - \left(\frac{\pi}{4} + \frac{x}{2}\right)\right) \\ &= \tan\left(\frac{3\pi}{4} - \arctan\frac{22}{7}\right) = \frac{-1 - \frac{22}{7}}{1 - \frac{22}{7}} = \frac{29}{15}. \end{aligned}$$

10. (Answer: 532)

Let S_a, the three-letter string received when aaa is transmitted, be $x_1x_2x_3$ and let S_b be $y_1y_2y_3$, where each of the x_k, y_k is an a or a b. It will be convenient to introduce the symbol $\prec$ to denote that one string of letters precedes another in alphabetical order. (Thus, if S_1 and S_2 are two strings of letters, then $S_1 \prec S_2$ is to be read "S_1 precedes S_2 alphabetically.") We will find the probability that $S_a \prec S_b$. Since the reception of any one letter is independent of that of any of the other letters, we have

$$\begin{aligned}
\operatorname{Prob}(S_a \prec S_b) &= \operatorname{Prob}(x_1x_2x_3 \prec y_1y_2y_3) \\
&= \operatorname{Prob}(x_1 \prec y_1) + \operatorname{Prob}(x_1 = y_1 \text{ and } \mathrm{x}_2 \prec \mathrm{y}_2) \\
&\qquad + \operatorname{Prob}(x_1 = y_1 \text{ and } \mathrm{x}_2 = \mathrm{y}_2 \text{ and } \mathrm{x}_3 \prec \mathrm{y}_3) \\
&= \operatorname{Prob}(x_1 \prec y_1) + \operatorname{Prob}(x_1 = y_1) \cdot \operatorname{Prob}(x_2 \prec y_2) \\
&\qquad + \operatorname{Prob}(x_1 = y_1) \cdot \operatorname{Prob}(x_2 = y_2) \cdot \operatorname{Prob}(x_3 \prec y_3). \qquad (*)
\end{aligned}$$

Now $x_1 \prec y_1$ is true if and only if $x_1 = a$ and $y_1 = b$; that is, if and only if these leading letters were received correctly. Since for each letter there is a $\frac{2}{3}$ probability that it was received correctly, we conclude that

$$\operatorname{Prob}(x_1 \prec y_1) = \frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}.$$

Similarly, $\operatorname{Prob}(x_2 \prec y_2) = \operatorname{Prob}(x_3 \prec y_3) = \frac{4}{9}$. The relation $x_1 = y_1$ is true if and only if one of these letters was received correctly and the other was received incorrectly. Thus

$$\operatorname{Prob}(x_1 = y_1) \;=\; \operatorname{Prob}(x_1 = y_1 = a) + \operatorname{Prob}(x_1 = y_1 = b) \;=\; \frac{2}{3} \cdot \frac{1}{3} + \frac{1}{3} \cdot \frac{2}{3} = \frac{4}{9}.$$

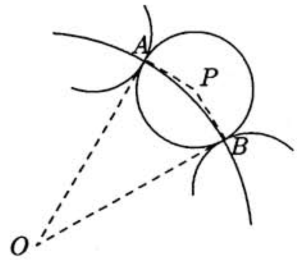
Identical reasoning shows that $\operatorname{Prob}(x_2 = y_2) = \frac{4}{9}$ also. Substituting these probabilities in $(*)$ we have

$$\operatorname{Prob}(S_a \prec S_b) = \frac{4}{9} + \left(\frac{4}{9}\right)^2 + \left(\frac{4}{9}\right)^3 = \frac{532}{729}.$$

The desired numerator is 532.

11. (Answer: 135)

Since the 12 disks cover C and each of the disks is tangent to its two neighbors, C must pass through the 12 points of tangency. The accompanying figure shows one of the covering disks, arcs of the two adjacent disks, and part of C. Let A and B be the points of tangency. By symmetry, the lines mutually tangent to adjacent disks must all pass through the center, O, of C. Let P be the center of the disk shown. Then $\angle PBO$ is a right angle, $\angle BOA = \frac{1}{12}2\pi = \frac{\pi}{6}$ and $\angle POB = \frac{\pi}{12}$. Thus the radius of each of the twelve disks is

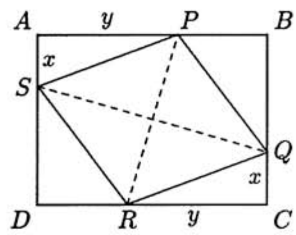


$$PB = BO\tan(\angle POB) = \tan\frac{\pi}{12}$$
$$= \tan\left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \frac{\tan\frac{\pi}{3} - \tan\frac{\pi}{4}}{1 + \tan\frac{\pi}{3}\tan\frac{\pi}{4}} = \frac{\sqrt{3}-1}{1+\sqrt{3}} = 2 - \sqrt{3}.$$

Hence each of the disks has area $\pi(2-\sqrt{3})^2 = \pi(7-4\sqrt{3})$. The sum of the areas of the twelve disks is
$$12\pi(7-4\sqrt{3}) = \pi(84 - 48\sqrt{3}),$$
and we have $a + b + c = 84 + 48 + 3 = 135$.

12. (Answer: 677)

Let x and y stand for QC and RC respectively, and note that these are also the lengths of $\overline{SA}$ and $\overline{PA}$ as well. The diagonals of a rhombus bisect each other at right angles, hence $PQRS$ and its diagonals divide the rectangle into eight right triangles. Six of these triangles have side lengths of 15, 20, and 25, while the other two have sides of length x, y, and 25. Summing the areas of these eight pieces, we find that

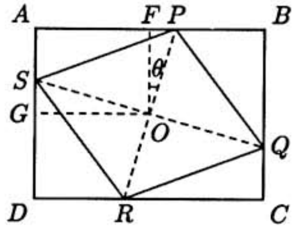


$$6(150) + 2\left(\frac{1}{2}xy\right) = \text{Area(ABCD)}$$
$$= (20+x)(15+y),$$

which leads to $3x+4y = 120$. Combining this with $x^2+y^2 = 625$ leads to the quadratic equation
$$5x^2 - 144x + 880 = 0.$$

Factoring gives $(5x-44)(x-20) = 0$. Note that x cannot be 20 since this would imply $BC = 40$, which is inconsistent with $PR = 30$. Hence $x = 44/5$, $y = 117/5$, and the perimeter of rectangle $ABCD$ is $2(15 + 20 + x + y) = 672/5$.

Alternate Solution. Let x and y stand for AS and AP respectively, and let O denote the intersection of $\overline{PR}$ and $\overline{QS}$. Let F and G, respectively, be the feet of the perpendiculars from O to $\overline{AB}$ and $\overline{AD}$. Then $\angle FOP = \angle GOS$. Let θ be the measure of each of these angles. We then have



$$\cos\theta = \frac{(y+15)/2}{20} = \frac{(x+20)/2}{15}, \qquad (*)$$

from which we obtain $3y-4x=35$. The equation $3x+4y=120$ can be obtained as in the previous solution. Solving the linear equations

$$\begin{aligned} 3x+4y &= 120 \\ -4x+3y &= 35 \end{aligned}$$

simultaneously gives $x = 44/5$ and $y = 117/5$. The perimeter of $ABCD$ is thus $2(15+20+x+y) = 672/5$.

Note. From equation $(*)$ it follows that all rectangles that circumscribe a given rhombus have the same shape.

13. (Answer: 990)

Let R and B, respectively, denote the numbers of red and blue socks in the drawer. Because the probability of obtaining a non-matching pair is 1/2, we have

$$\frac{RB}{\binom{R+B}{2}} = \frac{1}{2}.$$

This leads to $(R+B)(R+B-1) = 4RB$, which can be written as $(R-B)^2 = R+B$. This shows that the total number of socks in the drawer is a perfect square. Let $n = R-B$, so $n^2 = R+B$. Then $R = (n^2+n)/2$. Since $R+B \le 1991$, we must have $|n| \le \sqrt{1991} < 45$. We then see that the largest possible value of R occurs when $n = 44$, and this value of R is 990.

14. (Answer: 384)

Label the remaining four vertices C, D, E, and F, in the natural order. Draw diagonals $\overline{AC}$, $\overline{AD}$, and $\overline{AE}$. and let their lengths be x, y, and z respectively. Draw $\overline{BD}$, $\overline{BE}$, and $\overline{DF}$ and note that $BD = z$, $BE = y$, and $DF = z$, since chords of congruent arcs are congruent. Next we apply Ptolemy's Theorem: *For a quadrilateral inscribed in a circle, the product of the lengths of the diagonals is equal to the sum of the products of the lengths of opposite sides.* Using Ptolemy's Theorem on quadrilaterals $ABCD$, $ABDE$, and $ADEF$ respectively, we obtain

$$
\begin{aligned}
xz &= 31 \cdot 81 + 81y && (1)\\
y^2 &= 31 \cdot 81 + z^2 &&\\
z^2 &= 81^2 + 81y. && (2)
\end{aligned}
$$

Thus $y^2 - 31 \cdot 81 = 81^2 + 81y$, implying

$$y^2 - 81y - 81 \cdot 112 = (y + 63)(y - 144) = 0.$$

Since y cannot be -63, we must have $y = 144$. Substituting in (2) and then in (1) we obtain $z = 135$ and $x = 105$. The sum of the three diagonals is then $x + y + z = 384$.

Alternate Solution. Let R be the radius of the circle and $2x$ the measure of one of the central angles subtended by a side of length 81. Since there are five such sides in the hexagon, we must have $x < 36°$. We now have

$$81 = 2R \sin x \qquad \text{and} \qquad 31 = 2R\sin(\pi - 5x) = 2R \sin 5x, \tag{1}$$

and the sum of the lengths of the three diagonals from A is

$$2R(\sin 2x + \sin 3x + \sin 4x). \tag{2}$$

We will use (1) to find the value of $\sin x$ and then use this to evaluate the sum in (2). By DeMoivre's formula,

$$
\begin{aligned}
\sin 5x = \operatorname{Im}(\cos 5x + i \sin 5x) &= \operatorname{Im}\left[(\cos x + i \sin x)^5\right]\\
&= 5\cos^4 x \sin x - 10 \cos^2 x \sin^3 x + \sin^5 x\\
&= \sin x \left[5(1 - \sin^2 x)^2 - 10 \sin^2 x(1 - \sin^2 x) + \sin^4 x\right]\\
&= \sin x(16 \sin^4 x - 20 \sin^2 x + 5).
\end{aligned}
$$

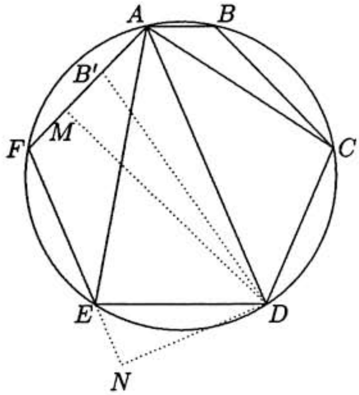
Using (1) we obtain

$$\frac{31}{81} = \frac{\sin 5x}{\sin x} = 16 \sin^4 x - 20 \sin^2 x + 5 = \left(4 \sin^2 x - \frac{5}{2}\right)^2 - \frac{5}{4},$$

from which $\sin x = \pm\sqrt{11}/6$ or $\pm\sqrt{34}/6$. Since $x < 36°$ and $\sin x$ must be positive, we conclude that $\sin x = \sqrt{11}/6$ and $\cos x = 5/6$. We can now evaluate the sum in (2):

$$\begin{aligned}
&2R(\sin 2x + \sin 3x + \sin 4x)\\
&\qquad = \frac{81}{\sin x}\left[2\sin x\cos x + (3\sin x - 4\sin^3 x) + 4\sin x\cos x(1 - 2\sin^2 x)\right]\\
&\qquad = 81\left[2\cos x + (3 - 4\sin^2 x) + 4\cos x(1 - 2\sin^2 x)\right]\\
&\qquad = 81\left[\frac{5}{3} + \left(3 - \frac{11}{9}\right) + \frac{10}{3}\left(1 - \frac{11}{18}\right)\right]\\
&\qquad = 135 + 243 - 99 + 270 - 165\\
&\qquad = 384.
\end{aligned}$$

Alternate Solution Label the remaining vertices C, D, E, F. We shall find and make use of a few pairs of similar right triangles to calculate AD, AE, and AC, in this order.

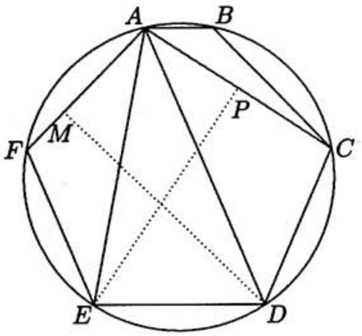


Mark point B' on $\overline{AF}$ so that $AB = AB'$. Since $\angle BAD$ and $\angle FAD$ intercept congruent arcs, these angles are congruent, and it follows that $\triangle BAD$ and $\triangle B'AD$ are congruent. Thus $BD = B'D$. Since $BD = FD$ is also true, we have $B'D = FD$. If we now let M denote the midpoint of $\overline{B'F}$ we see that $\triangle AMD$ is a right triangle with leg $\overline{AM}$ of length 56. Now drop a perpendicular from D to the extension of $\overline{FE}$ and let N be the foot of this perpendicular. Since trapezoid $ADEF$ is isosceles, we deduce that $\angle FAD = \angle NED$, and hence that $\triangle AMD \sim \triangle END$. Thus

$$\frac{56}{AD} = \frac{AM}{AD} = \frac{EN}{ED} = \frac{\frac{AD-81}{2}}{81},$$

from which $AD^2 - 81AD - 2 \cdot 56 \cdot 81 = 0$. The solutions to this quadratic are -63 and 144, hence $AD = 144$.

The preceding argument has also shown that $EN = 63/2$. The Pythagorean theorem now gives us $DN = 45\sqrt{11}/2$, and a second application, to $\triangle FND$, gives $FD = 135$. This is also the length of $\overline{AE}$.



To find AC, consider isosceles $\triangle AEC$. Let P be the midpoint of $\overline{AC}$, so that $\triangle PAE$ is a right triangle. This triangle is similar to $\triangle MAD$ since $\angle PAE = \angle MAD$. We thus obtain

$$\frac{AP}{AM} = \frac{AE}{AD},$$

from which

$$AP = (AM)(AE)/AD = \frac{105}{2},$$

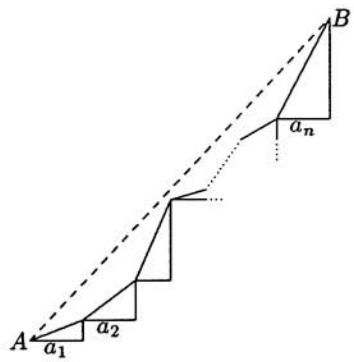
and $AC = 2AP = 105$. The required sum is therefore $AC + AD + AE = 384$.

15. (Answer: 012)

We interpret each term

$$t_k = \sqrt{(2k-1)^2 + a_k^2}$$

as the length of the hypotenuse of a right triangle with legs of length $2k-1$ and a_k. Put the triangles together in a "staircase" arrangement as shown in the diagram, and let A and B be the initial and terminal points of the broken path formed by the hypotenuses. The distance from A to B is



$$\sqrt{\left(\sum_{k=1}^{n} a_k\right)^2 + \left(\sum_{k=1}^{n}(2k-1)\right)^2} = \sqrt{17^2 + n^4},$$

while the sum $\sum_{k=1}^{n} t_k$ is the length of the path from A to B formed by the hypotenuses of the triangles. It follows immediately that $\sum_{k=1}^{n} t_k \geq \sqrt{17^2 + n^4}$, and that equality is obtained by choosing the a_k so that the broken path is actually a straight line. Thus $S_n = \sqrt{17^2 + n^4}$ is the minimum possible value of the given sum. When S_n is an integer, the equation $17^2 = S_n^2 - n^4 = (S_n - n^2)(S_n + n^2)$ implies that

$$S_n + n^2 = 17^2$$

and

$$S_n - n^2 = 1.$$

Solving this system yields $S_n = 145$ and $n = 12$.

AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

10th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)

THURSDAY, APRIL 2, 1992

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers will share these solutions with their students, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

Correspondence about the problems and solutions (but **not** requests for the Solutions Pamphlet) should be addressed to:

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1. (Answer: 400)
A positive rational number that is less than 10 and has denominator 30 can be written in the form
$$\frac{30n+r}{30},$$
where n and r are integers satisfying $0 \le n \le 9$ and $0 \le r < 30$. Furthermore, such a fraction is in lowest terms if and only if r and 30 are relatively prime; i.e., if and only if $r \in \{1, 7, 11, 13, 17, 19, 23, 29\}$. Thus there are 10 choices for n and 8 choices for r, and no two pairs of choices (n, r) give the same value of $(30n + r)/30$. It follows that the desired sum has $8 \cdot 10 = 80$ terms. These may be paired by noting that $k/30$ is one of these fractions if and only if $10 - k/30 = (300 - k)/30$ is as well. Since the sum of each of these pairs is 10, we find that the sum of all such fractions is
$$\left(\frac{1}{30} + \frac{300-1}{30}\right) + \left(\frac{7}{30} + \frac{300-7}{30}\right) + \cdots + \left(\frac{149}{30} + \frac{300-149}{30}\right) = 40 \cdot 10 = 400.$$

2. (Answer: 502)
An ascending positive integer must have distinct, nonzero digits. Thus the digits must be a subset of two or more elements from the set $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Conversely, any subset of S that has two or more elements corresponds to a unique ascending positive integer in which the elements of the subset are arranged in increasing order. It follows that the number of ascending positive integers is equal to the number of subsets of S that have two or more elements. Since a nine-element set has $2^9 = 512$ subsets and ten of these subsets have fewer than two elements, the number of ascending positive integers is $512 - 10 = 502$.

3. (Answer: 164)
Let

$W =$ the player's number of wins at the start of the weekend

and

$M =$ the number of matches played at the start of the weekend.

We are given $W/M = .500$ and $(W+3)/(M+4) > .503$. Thus $M = 2W$ and
$$W + 3 > .503(2W + 4) = 1.006W + 2.012.$$
It follows that $W < (3 - 2.012)/.006 = 164.\overline{6}$. Finally, note that if $W = 164$ and $M = 328$, then $W/M = .500$ and $(W+3)/(M+4) > .503$. Hence, the largest number of matches that the player could have won before the start of the weekend is 164.

4. (Answer: 062)
Row n of Pascal's triangle consists of the binomial coefficients $\binom{n}{k}$, $k = 0, 1, \ldots, n$. If three consecutive entries in row n of Pascal's triangle are in the ratio $3:4:5$, then there is a positive integer k for which

$$\frac{3}{4} = \frac{\binom{n}{k-1}}{\binom{n}{k}} = \frac{\frac{n!}{(k-1)!(n-k+1)!}}{\frac{n!}{k!(n-k)!}} = \frac{k!(n-k)!}{(k-1)!(n-k+1)!} = \frac{k}{n-k+1}$$

and

$$\frac{4}{5} = \frac{\binom{n}{k}}{\binom{n}{k+1}} = \frac{\frac{n!}{k!(n-k)!}}{\frac{n!}{(k+1)!(n-k-1)!}} = \frac{(k+1)!(n-k-1)!}{k!(n-k)!} = \frac{k+1}{n-k}.$$

It follows that

$$3n - 7k = -3 \qquad \text{and} \qquad 4n - 9k = 5.$$

Solving simultaneously gives $k = 27$ and $n = 62$. Thus, the consecutive entries $\binom{62}{26}$, $\binom{62}{27}$, $\binom{62}{28}$ in row 62 of Pascal's triangle are in the ratio $3:4:5$.

5. (Answer: 660)
First note that

$$0.\overline{abc} = \frac{abc}{999},$$

and that $999 = 3^3 37$.
If abc is divisible by neither 3 nor 37, then this fraction is already in lowest terms. By the Inclusion-Exclusion Principle, there are

$$999 - \left(\frac{999}{3} + \frac{999}{37}\right) + \left(\frac{999}{3 \cdot 37}\right) = 999\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{37}\right) = 648$$

such numbers.
Some of the reduced fractions may have numerators that are divisible by 3 or 37. Such fractions must have the form

$$\frac{k}{37}, \text{ where } k \text{ is a multiple of 3 but not a multiple of 37}$$

or

$$\frac{l}{3^m}, \text{ where } l \text{ is a multiple of 37 but not a multiple of 3, and } m = 1, 2, 3.$$

There are no fractions of the second type in S, since any fraction of this form is greater than 1. There are 12 fractions of the first type in S, one for each of $k = 3, 6, \ldots, 36$. Thus the number of distinct numerators in the set of reduced fractions is $648 + 12 = 660$.

6. (Answer: 156)

Let n have decimal representation $1abc$. If one of a, b, or c is 5, 6, 7, or 8, then there will be carrying when n and $n+1$ are added. If $b=9$ and $c \neq 9$, or if $a=9$ and either $b \neq 9$ or $c \neq 9$, there will also be carrying when n and $n+1$ are added.

If n is not one of the integers described above, then n has one of the forms

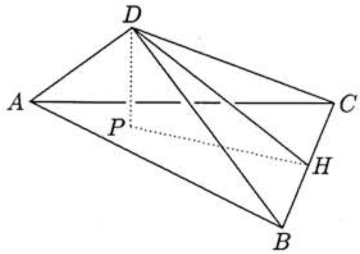
$$1abc \qquad 1ab9 \qquad 1a99 \qquad 1999,$$

where $a,\, b,\, c \in \{0,\, 1,\, 2,\, 3,\, 4\}$. For such n, no carrying will be needed when n and $n+1$ are added. There are $5^3+5^2+5+1=156$ such values of n.

7. (Answer: 320)

Let H be the foot of the perpendicular from D to line BC. Since $BC=10$ and the area of $\triangle BCD$ is 80, $DH=16$. Next, let P be the foot of the perpendicular from D to plane ABC. Then $\triangle HPD$ is a right triangle and $\angle DHP$ measures $30°$. It follows that $DP=\frac{1}{2}DH=8$, and the volume of the tetrahedron is†

$$\frac{1}{3}DP \cdot [ABC] = \frac{1}{3} \cdot 8 \cdot 120 = 320.$$



8. (Answer: 819)

Suppose that the first term of the sequence ΔA is d. Then the sequence ΔA is $(d,\, d+1,\, d+2, \ldots)$ with n^{th} term given by $d+(n-1)$. Hence the sequence A is

$$\big(a_1,\ a_1+d,\ a_1+d+(d+1),\ a_1+d+(d+1)+(d+2),\ \ldots\ \big),$$

with n^{th} term given by

$$a_n = a_1 + (n-1)d + \frac{1}{2}(n-1)(n-2).$$

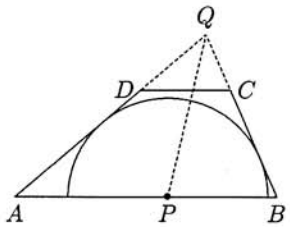
This shows that a_n is a quadratic polynomial in n with leading coefficient $\frac{1}{2}$. Since $a_{19}=a_{92}=0$, we must have

$$a_n = \frac{1}{2}(n-19)(n-92),$$

so $a_1 = \frac{1}{2}(1-19)(1-92) = 819$.

† The area of triangle ABC is denoted $[ABC]$.

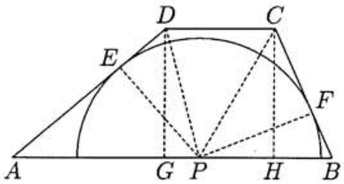
9. (Answer: 164)
Extend $\overline{AD}$ and $\overline{BC}$ until the two segments meet at a point Q. Point P is equidistant from $\overline{AQ}$ and $\overline{BQ}$, so P lies on the bisector of $\angle AQB$. Since the bisector of the angle of a triangle divides the side opposite the angle into segments whose lengths are proportional to the lengths of the adjoining sides, we have



$$\frac{AP}{BP} = \frac{AQ}{BQ} = \frac{AD}{BC} = \frac{7}{5}.$$

Since $AP + PB = 92$, it follows that $AP = 161/3$ and $m + n = 164$.

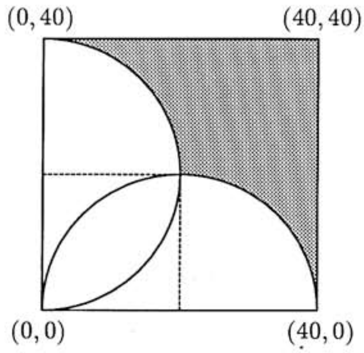
Alternate Solution. Label as E the point at which the circle is tangent to $\overline{AD}$, and as F the point at which the circle is tangent to $\overline{BC}$. Let G be the foot of the perpendicular from D to $\overline{AB}$, and H the foot of the perpendicular from C to $\overline{AB}$. Since $PE = PF$ and $DG = CH$, it follows that



$$\frac{AP}{BP} = \frac{\frac{1}{2}AP \cdot DG}{\frac{1}{2}BP \cdot CH} = \frac{[APD]}{[PBC]} = \frac{\frac{1}{2}AD \cdot EP}{\frac{1}{2}BC \cdot FP} = \frac{AD}{BC} = \frac{7}{5}.$$

Since $AP + PB = 92$, it follows that $AP = 161/3$ and $m + n = 164$.

10. (Answer: 572)
Let $z = x + iy$. For z to be in the region in question, we must have $0 \le x \le 40$ and $0 \le y \le 40$. Hence the region lies in the square with vertices $(0,0)$, $(40,0)$, $(40,40)$, and $(0,40)$. Next note that



$$\frac{40}{\overline{z}} = \frac{40}{x - iy} = \frac{40x}{x^2 + y^2} + i\frac{40y}{x^2 + y^2}.$$

Hence the restrictions on the real and imaginary parts of $40/\overline{z}$ give

$$0 \le \frac{40x}{x^2 + y^2} \le 1 \quad \text{and} \quad 0 \le \frac{40y}{x^2 + y^2} \le 1,$$

from which

$$(x-20)^2 + y^2 \ge 20^2 \quad \text{and} \quad x^2 + (y-20)^2 \ge 20^2.$$

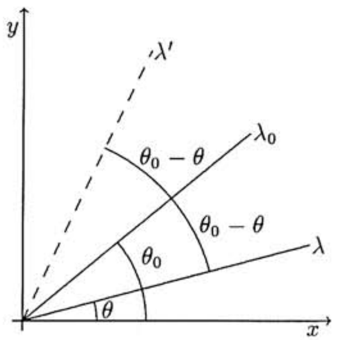
Thus the region in question lies outside the circle with center $(20,0)$ and radius 20 and also outside the circle with center $(0,20)$ and radius 20, as indicated by the shaded portion of the diagram. As suggested by the dashed lines in the diagram, the area of the region is $3/4$ the area of the square minus the area of two quarter-circles. Hence

$$\text{Area}(A) = \frac{3}{4}\cdot 40^2 - \frac{2}{4}(\pi\cdot 20^2) = 200(6-\pi) \approx 571.7,$$

so that the desired number is 572.

11. (Answer: 945)
Let λ_0 and λ be lines through the origin making angles of θ_0 and θ, respectively, with the positive x-axis. When λ is reflected in λ_0, the resulting line λ' makes an angle of

$$\theta_0 + (\theta_0 - \theta) = 2\theta_0 - \theta$$



with the positive x-axis. Thus, if λ is reflected in ℓ_1, then the result is a line λ_1 that passes through the origin and makes an angle of $2\frac{\pi}{70} - \theta$ with the positive x-axis. Reflecting λ_1 in the line ℓ_2 gives a line λ_2 through the origin that makes an angle of

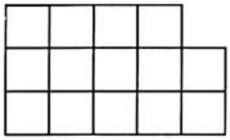
$$2\frac{\pi}{54} - \left(2\frac{\pi}{70} - \theta\right) = -\frac{8\pi}{945} + \theta$$

with the positive x-axis. Thus $R(\lambda)$ is obtained by rotating λ through $-8\pi/945$ radians and $R^{(m)}(\lambda)$ is obtained by rotating λ through $-8m\pi/945$ radians. For $R^{(m)}(\lambda) = \lambda$ to hold, $8m/945$ must be an integer. The smallest positive integer value of m for which this is true is 945.

Queries. If the line ℓ did not pass through the origin, would the above answer be affected? If R were defined by consecutive reflection about an odd number of lines through the origin, then m would be either 1 or 2. Why?

12. (Answer: 792)

At any stage of the game, the uneaten squares will form columns of non-increasing heights as we read from left to right. It is not hard to show that this condition is not only necessary, but is also sufficient for a given configuration of squares to occur in a game. (The reader should prove this fact.) Moreover, any such configuration can be completely described by the twelve-step polygonal path that runs from the upper left to the lower right of the original board, forming the boundary between the eaten and uneaten squares. This polygonal boundary can be described by a twelve-letter sequence of **V**'s and **H**'s . Such a sequence contains seven **H**'s, where each **H** represents the top of an uneaten column (or the bottom of a completely eaten one) and five **V**'s, where each **V** represents a one-unit drop in vertical height in moving from the top of an uneaten column to the top of an adjacent, but shorter column. For example, the state that appears in the diagram accompanying the problem is described by **HHHVHVVHHHVV**, while the state in the diagram above is given by **VVHHHHVHVVHH**. There are $\frac{12!}{7!5!} = 792$ sequences of seven **H**'s and five **V**'s, including the sequences **HHHHHHHVVVVV** and **VVVVVHHHHHHH**, which describe the full board and the empty board, respectively.



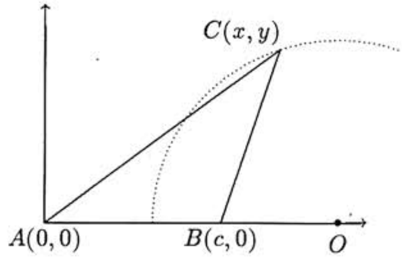
Note. The game of *Chomp* is due to David Gale, and was introduced (and named) by Martin Gardner in his *Scientific American* column "Mathematical Games". The column reappeared in Gardner's collection *Knotted Doughnuts*.

13. (Answer: 820)

Let $AB = c$, $AC = br$, and $BC = ar$, with $a < b$. We shall show that the locus of all such points C is a circle whose center is on line AB and whose radius is $abc/(b^2 - a^2)$. (This circle is called a circle of Apollonius.) The radius of the circle serves as the height of the triangle of maximal area, so the desired area is

$$\frac{1}{2}c\left(\frac{abc}{b^2 - a^2}\right).$$

Taking $a : b = 40 : 41$ and $c = 9$, we get an answer of 820.



One way to proceed is with coordinates: Let $A = (0,0)$, $B = (c, 0)$, and $C = (x, y)$. Then $BC/AC = a/b$ becomes

$$\frac{\sqrt{(x-c)^2 + y^2}}{\sqrt{x^2 + y^2}} = \frac{a}{b}.$$

Squaring both sides and rearranging terms leads to

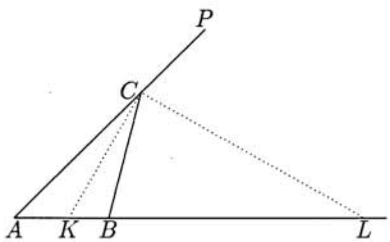
$$(b^2 - a^2)x^2 - 2b^2cx + (b^2 - a^2)y^2 = -b^2c^2.$$

Completing the square then gives

$$\left(x - \frac{b^2c}{b^2 - a^2}\right)^2 + y^2 = \frac{a^2b^2c^2}{(b^2 - a^2)^2}.$$

Hence the set of all vertices C satisfying the conditions of the problem is the circle of center $O = \left(\frac{b^2c}{b^2 - a^2}, 0\right)$ and radius $\frac{abc}{b^2 - a^2}$.

Alternate Solution. Assume $a < b$ and let K and L satisfy $AK : KB = b : a = AL : LB$, with K on $\overline{AB}$ and L on the extension of $\overline{AB}$ through B. Extend $\overline{AC}$ through C to P. Because $AC : CB = b : a$ also, the Angle Bisector Theorem implies that $\overline{CK}$ bisects angle ACB and $\overline{CL}$ bisects the exterior angle BCP. It follows that angle KCL is a right angle, so C lies on the circle that has KL as a diameter. It is straightforward to calculate that $KB = ac/(a + b)$ and that $BL = ac/(b - a)$. Therefore the radius of the circle is $abc/(b^2 - a^2)$, which serves as the altitude of the triangle ABC of maximal area. When $c = 9$ and $b : a = 41 : 40$, this area is 820.



Alternate Solution. Let $AB = c$, $AC = b$, and $BC = a$, with $a < b$. Then

$$[ABC] = \frac{1}{2}ab\sin C = \frac{1}{2}ab\sin C\left(\frac{\frac{c}{\sin C}\frac{c}{\sin C}}{\frac{a}{\sin A}\frac{b}{\sin B}}\right) = \frac{1}{2}c^2\frac{\sin A\sin B}{\sin C},$$

where we have used the Law of Sines for the second equality above. Since $C = \pi - A - B$, this equation can be rewritten as

$$[ABC] = \frac{1}{2}c^2\frac{\sin A\sin B}{\sin(A+B)} \le \frac{1}{2}c^2\frac{\sin A\sin B}{\sin(B+A)\sin(B-A)},$$

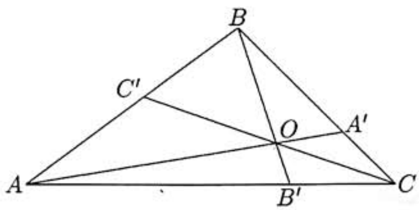
where equality holds if and only if $B = \frac{\pi}{2} + A$. Note that $B > A$ follows from $b > a$. Using the trigonometric identity $\sin(B+A)\sin(B-A) = \sin^2 B - \sin^2 A$ and then the Law of Sines again, we have

$$[ABC] \le \frac{1}{2}c^2\frac{\sin A\sin B}{\sin^2 B - \sin^2 A} = \frac{1}{2}c^2\frac{ab}{b^2-a^2}.$$

If $c = 9$ and $b:a = 41:40$, we see that the maximum possible area is 820, and that this maximum is attained when $B - A = \frac{\pi}{2}$.

14. (Answer: 094)
Since $\triangle AOB$ and $\triangle A'OB$ share an altitude, as do $\triangle AOC$ and $\triangle A'OC$, we have

$$\begin{aligned}\frac{AO}{OA'} = \frac{[AOB]}{[A'OB]} = \frac{[COA]}{[COA']} &= \frac{[AOB]+[COA]}{[A'OB]+[COA']} \\ &= \frac{[AOB]+[COA]}{[BOC]} \\ &= \frac{z+y}{x},\end{aligned}$$



where $x = [BOC]$, $y = [COA]$, and $z = [AOB]$. Similarly,

$$\frac{BO}{OB'} = \frac{x+z}{y} \quad \text{and} \quad \frac{CO}{OC'} = \frac{y+x}{z}.$$

We then have

$$\begin{aligned}\frac{AO}{OA'}\frac{BO}{OB'}\frac{CO}{OC'} &= \frac{(z+y)(x+z)(y+x)}{xyz}\\ &= \frac{yz^2+y^2z+x^2z+xz^2+xy^2+x^2y+2xyz}{xyz}\\ &= \frac{yz(z+y)+xz(x+z)+xy(y+x)}{xyz}+2\\ &= \frac{z+y}{x}+\frac{x+z}{y}+\frac{y+x}{z}+2.\end{aligned}$$

Hence,

$$\frac{AO}{OA'}\frac{BO}{OB'}\frac{CO}{OC'} = \left(\frac{AO}{OA'}+\frac{BO}{OB'}+\frac{CO}{OC'}\right)+2 = 92+2 = 94.$$

15. (Answer: 396)

Let $f(m)$ be the number of ending zeros in the decimal expansion of $m!$. It is clear that $f(m)$ is a nondecreasing function of m. Furthermore, when m is a multiple of 5 we have

$$f(m) = f(m+1) = f(m+2) = f(m+3) = f(m+4) < f(m+5).$$

Thus if we list the numbers $f(k)$ for $k = 0, 1, 2, \ldots$, we obtain

$$0, 0, 0, 0, 0, 1, 1, 1, 1, 1, 2, 2, 2, 2, 2, \ldots, 4, 4, 4, 4, 4, 6, 6, 6, 6, 6, \ldots, \qquad (*)$$

and each number in this list appears 5 times. We would like to know if the number 1991 appears in this list. It is well known (and easy to show) that the number of zeros at the end of $m!$ is

$$f(m) = \sum_{k=1}^{\infty} \left\lfloor \frac{m}{5^k} \right\rfloor.$$

If there is an m for which $f(m) = 1991$, then

$$1991 < \sum_{k=1}^{\infty} \frac{m}{5^k} = m\frac{\frac{1}{5}}{1-\frac{1}{5}} = \frac{m}{4}.$$

Hence $m > 4 \cdot 1991 = 7964$. Using the above formula for $f(m)$ we find that $f(7965) = 1988$, and once this is known we can readily ascertain that $f(7975) = 1991$. Now if the list $(*)$ is carried out to the term $f(7979) = 1991$ we have

$$0, 0, 0, 0, 0, 1, 1, 1, 1, 1, \ldots, 1989, 1991, 1991, 1991, 1991, 1991.$$

This list contains 7980 terms, and each integer in the sequence occurs exactly 5 times. Thus the list has $7980/5 = 1596$ distinct integer values from the set $\{0, 1, 2, \ldots, 1991\}$. Hence $1992 - 1596 = 396$ of these integers do not appear in the list. Consequently there are 396 positive integers less than 1992 that are not factorial tails.

AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

11th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)

THURSDAY, APRIL 1, 1993

Sponsored by

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

Correspondence about the problems and solutions (but **not** requests for the Solutions Pamphlet) should be addressed to:

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1. (Answer: 728)
We first count those integers of the desired type with 4 or 6 as the thousands digit. In this case, the thousands digit can be chosen in 2 ways, and then the units digit (0, 2, 4, 6, or 8) can be chosen in 4 ways. There are then 8 choices for the hundreds digit and then 7 for the tens digit. Thus there are $2 \cdot 4 \cdot 8 \cdot 7 = 448$ integers of the type we seek with 4 or 6 as the thousands digit. Similarly, if the thousands digit is 5, we have $1 \cdot 5 \cdot 8 \cdot 7 = 280$ even integers with four different digits. Thus we have a total of $448 + 280 = 728$ integers.

2. (Answer: 580)
Consider a Cartesian coordinate system with origin at the candidate's starting point, positive x-axis pointing east, and positive y-axis pointing north. At the end of the 40^{th} day the candidate is at the point

$$\begin{aligned}
&\left(\frac{1^2-3^2+5^2-7^2+\cdots+37^2-39^2}{2}, \frac{2^2-4^2+6^2-8^2+\cdots+38^2-40^2}{2}\right)\\
=&\left(\frac{(1-3)(1+3)+(5-7)(5+7)+\cdots+(37-39)(37+39)}{2},\right.\\
&\qquad\left.\frac{(2-4)(2+4)+(6-8)(6+8)+\cdots+(38-40)(38+40)}{2}\right)\\
=&(-4-12-20-\cdots-76, -6-14-22-\cdots-78)\\
=&\left(10\frac{-4-76}{2}, 10\frac{-6-78}{2}\right)\\
=&(-400, -420).
\end{aligned}$$

Thus the candidate's distance from his starting point is $\sqrt{400^2+420^2} = 580$.

3. (Answer: 943)
Let F be the total number of fish caught during the festival and C be the total number of contestants. Then $C-(9+5+7) = C-21$ contestants each caught 3 or more fish, and these contestants caught a total of $F-(0\cdot 9+1\cdot 5+2\cdot 7) = F-19$ fish. Hence

$$\frac{F-19}{C-21} = 6. \tag{1}$$

Similarly, $C-(5+2+1) = C-8$ contestants each caught 12 or fewer fish, and these contestants caught a total of $F-(5\cdot 13+2\cdot 14+1\cdot 15) = F-108$ fish. Thus

$$\frac{F-108}{C-8} = 5. \tag{2}$$

Solving (1) and (2) simultaneously, we find $C = 175$ and $F = 943$.

4. (Answer: 870)
Since $a+d=b+c$, we may take $(a, b, c, d) = (a, a+x, a+y, a+x+y)$, where x and y are integers with $0 < x < y$. Then

$$93 = bc - ad = (a+x)(a+y) - a(a+x+y) = xy,$$

from which either $(x, y) = (1, 93)$ or $(x, y) = (3, 31)$. In the first case,

$$(a, b, c, d) = (a, a+1, a+93, a+94)$$

is in the desired range for $a = 1, 2, \ldots, 405$. In the second case,

$$(a, b, c, d) = (a, a+3, a+31, a+34)$$

is in the desired range for $a = 1, 2, \ldots, 465$. These two sets of four-tuples are disjoint, so a total of $405 + 465 = 870$ four-tuples of integers (a, b, c, d) satisfy the given conditions.

5. (Answer: 763)
For positive integers n, we have

$$\begin{aligned} P_n(x) &= P_{n-1}(x-n) \\ &= P_{n-2}(x-n-(n-1)) \\ &= P_{n-3}(x-n-(n-1)-(n-2)) \\ &\ \ \vdots \\ &= P_0(x-n-(n-1)-\cdots-2-1), \end{aligned}$$

from which

$$P_n(x) = P_0\left(x - \frac{1}{2}n(n+1)\right).$$

Hence

$$P_{20}(x) = P_0\left(x - \frac{1}{2}20\cdot 21\right) = P_0(x-210) = (x-210)^3 + 313(x-210)^2 - 77(x-210) - 8.$$

The coefficient of x in this polynomial is

$$3(210)^2 - 313\cdot 2\cdot 210 - 77 = 210(630-626) - 77 = 763.$$

6. (Answer: 495)
The sum of nine consecutive integers is 9 times the fifth number, the sum of ten consecutive integers is 5 times the sum of the fifth and sixth numbers, and the sum of eleven consecutive integers is 11 times the sixth number. Thus any positive integer that can be written as a sum of nine, ten, and eleven consecutive positive integers must be a multiple of 9, 5, and 11. The smallest such number is 495. It is readily verified that

$$\begin{aligned} 495 &= 51 + 52 + \cdots + 59 \\ &= 45 + 46 + \cdots + 54 \\ &= 40 + 41 + \cdots + 50. \end{aligned}$$

7. (Answer: 005)
Since we may rotate the brick before we attempt to place it in the box, we may assume that $a_1 < a_2 < a_3$ and that $b_1 < b_2 < b_3$. The brick will then fit in the box if and only if $a_1 < b_1$, $a_2 < b_2$, and $a_3 < b_3$. Because each selection of 6 dimensions is equally likely, there is no loss of generality in assuming the brick and box dimensions are selected from the set $\{1, 2, 3, 4, 5, 6\}$. There are $\binom{6}{3} = 20$ ways to select the dimensions of a brick-box pair from $\{1, 2, 3, 4, 5, 6\}$.
If the brick does fit inside the box, then we must have $a_1 = 1$ and $b_3 = 6$. In addition, we must have $b_2 > b_1 > a_1 = 1$ and $b_2 > a_2$, so $6 > b_2 > 3$.
If $b_2 = 4$, then $a_3 = 5$. Taking b_1 to be either 2 or 3 will result in a pair of dimensions for which the brick fits in the box.
If $b_2 = 5$, then taking b_1 to be 2, 3, or 4 will result in a box that can hold the brick. Thus there are 5 ways to select the two sets of dimensions from $\{1, 2, 3, 4, 5, 6\}$ so that the brick fits inside the box. It follows that the probability that the brick will fit inside the box is $5/20 = 1/4$. The sum of the numerator and denominator is 5.

8. (Answer: 365)
In order that $A \cup B = S$, for each element s of S exactly one of the following three statements is true:

$s \in A$ and $s \notin B$ $\qquad$ $s \notin A$ and $s \in B$ $\qquad$ $s \in A$ and $s \in B$.

Hence if S has n elements, there are 3^n ways to choose the sets A and B. Except for pairs with $A = B$, this total counts each pair of sets twice. Since $A \cup B = S$ with $A = B$ occurs if and only if $A = B = S$, the number of pairs of subsets of S whose union is S is

$$\frac{3^n - 1}{2} + 1,$$

which is 365 when $n = 6$.

Alternate Solution. Let S be the set with n elements and A and B be two subsets whose union is S. If $|A| = k$, then B must contain the $n-k$ elements of S not in A. There are $\binom{n}{k}$ such sets A. Each of the k elements in A may or may not be in B, so for each set A there are 2^k sets B that can be paired with A. Note that

$$\sum_{k=0}^{n} \binom{n}{k} 2^k = (1+2)^n = 3^n$$

counts each pair of sets twice, except the pair $A = B = S$, which occurs once. Hence the number of pairs of subsets whose union is S is $(3^n+1)/2$. When $n = 6$, this gives 365.

9. (Answer: 118)

Let A be the point labeled 1. For any positive integer n we can find the point labeled by n by counting $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$ points around the circle in the clockwise direction, with the count starting at A. It follows that two positive integers l and m will label the same point if and only if $\frac{1}{2}l(l+1)$ and $\frac{1}{2}m(m+1)$ have the same remainder when divided by 2000. Thus, if k is a positive integer that labels the same point as 1993, then

$$2\left(\frac{1993(1993+1)}{2} - \frac{k(k+1)}{2}\right) = (1993-k)(1994+k)$$

must be a multiple of $4000 = 2^5 5^3$. It is clear that $k = 1993$ satisfies these conditions; we need to see if there is a positive integer $k < 1993$ that also satisfies these conditions and, if any exist, find the smallest such integer. Since $1993-k$ and $1994+k$ are of different parity and cannot both be multiples of 5, one of these integers must be a multiple of 125 and one must be a multiple of 32. If $k < 1993$, then $1994+k < 32 \cdot 125 = 4000$, so exactly one of $1993-k$ and $1994+k$ is a multiple of 125 and the other is a multiple of 32. We consider these two cases.

Case 1. $125 \mid (1993-k)$ and $32 \mid (1994+k)$

Because $1993 = 15 \cdot 125 + 118$ and $1994 = 62 \cdot 32 + 10 = 63 \cdot 32 - 22$, it follows that $k - 118$ and $k - 22$ are divisible by 125 and 32, respectively. In other words, $k = 118 + 125r$ and $k = 22 + 32s$ for non-negative integers r and s. It is now evident that $k \geq 118$ and that $k = 118$ arises from $r = 0$ and $s = 3$.

Case 2. $125 \mid (1994+k)$ and $32 \mid (1993-k)$

Because $1994 = 15 \cdot 125 + 119$ and $1993 = 62 \cdot 32 + 9$, it follows that $k + 119$ and $k - 9$ are divisible by 125 and 32 respectively. Thus $k = 125r - 119$ and $k = 32s + 9$ for non-negative integers r and s. From this we obtain $125r = 128 + 32s$, so r is a multiple of 32. Thus for some integer t we have $k = 125 \cdot 32t - 119$. It follows that any positive integer k satisfying this case is greater than 1993.

Hence 118 is the smallest positive integer that labels the same point as 1993.

Note. In analyzing the above cases, we are solving systems of congruences. In Case 1 the system is

$$k \equiv 118 \pmod{125}$$
$$k \equiv 22 \pmod{32}.$$

Since 32 and 125 are relatively prime, it follows from the Chinese Remainder Theorem that a solution to the system exists, and that all solutions are congruent modulo $125 \cdot 32 = 4000$.

10. (Answer: 250)
Since $F - E + V = 2$, and $F = 32$ it follows that

$$E = V + 30.$$

Since $T + P$ faces meet at each vertex, there are $T + P$ edges that meet at each vertex. Hence $2E = V(T + P)$, from which $V(T + P) = 2(V + 30)$ and

$$V(T + P - 2) = 60. \tag{1}$$

Each triangular face has three vertices, so the product VT counts each triangular face 3 times. Thus the total number of triangular faces is $VT/3$. Similarly, the total number of pentagonal faces is $VP/5$. Because every face is a triangle or a pentagon,

$$V\left(\frac{T}{3} + \frac{P}{5}\right) = 32. \tag{2}$$

Combining (1) and (2) we have

$$60V\left(\frac{T}{3} + \frac{P}{5}\right) = 32V(T + P - 2),$$

from which

$$3T + 5P = 16.$$

The only non-negative integer solution of this equation is $T = P = 2$. From (1) we find $V = 30$, so $100P + 10T + V = 250$.

Note. The polyhedron described above is called an icosidodecahedron.

11. (Answer: 093)

For any game the probability that the first player wins the game is

$$\frac{1}{2}+\left(\frac{1}{2}\right)^3+\left(\frac{1}{2}\right)^5+\cdots=\frac{\frac{1}{2}}{1-\left(\frac{1}{2}\right)^2}=\frac{2}{3}.$$

Hence the probability that the second player wins the game is $1-\frac{2}{3}=\frac{1}{3}$. Now let P_k denote the probability that Alfred wins the k^{th} game. Then $P_1=\frac{2}{3}$ and for $k\geq 2$ we have

$$P_k=\frac{1}{3}P_{k-1}+\frac{2}{3}(1-P_{k-1})=\frac{2}{3}-\frac{1}{3}P_{k-1},$$

from which

$$P_k-\frac{1}{2}=-\frac{1}{3}\left(P_{k-1}-\frac{1}{2}\right).$$

It follows that

$$P_k=\frac{1}{2}+\frac{(-1)^{k-1}}{3^{k-1}}\left(P_1-\frac{1}{2}\right)=\frac{1}{2}+\frac{(-1)^{k-1}}{2\cdot 3^k}.$$

When $k=6$, this probability is $364/729$. Then $m+n=1093$ and the last three digits are 093.

Note. The probability P_1 that the person going first wins the game can be computed without using geometric series. Note that after each player tosses a tail, the game essentially starts anew. Hence

$$P_1=\frac{1}{2}+\left(\frac{1}{2}\right)^2P_1,$$

from which we find $P_1=2/3$.

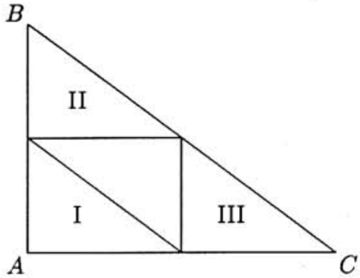
12. (Answer: 344)

First note that, since P_1 is inside $\triangle ABC$, all subsequent points P_k will also be inside the triangle. Furthermore, as will be shown below, once any subsequent P_k is given, then P_1 is uniquely determined. Suppose that $P_k = (x_k, y_k)$ is known. Since P_k is inside $\triangle ABC$, we have

$$0 < x_k < 560, \qquad 0 < y_k < 420, \qquad 0 < 420x_k + 560y_k < 420 \cdot 560.$$

If A is rolled, then

$$(x_{k+1}, y_{k+1}) = P_{k+1} = \frac{1}{2}P_k = \left(\frac{x_k}{2}, \frac{y_k}{2}\right),$$



so the range of possible positions of P_{k+1} is limited to the original triangle contracted by a factor of 1/2 (region I in the diagram). Hence if A is rolled, then P_{k+1} is in the interior of region I, and we may conclude that

$$420x_{k+1} + 560y_{k+1} < \frac{1}{2} \cdot 420 \cdot 560.$$

Similarly, if B is rolled, then P_{k+1} is in the interior of region II, so $y_{k+1} > 210$. If C is rolled, then P_{k+1} is in the interior of region III, so $x_{k+1} > 280$. Thus, for $k \geq 2$, P_k must lie in one of the regions I,II,III, and its predecessor is uniquely determined. For example if $P_k = (x_k, y_k)$ lies in region II, then P_k must be the midpoint of $\overline{BP_{k-1}}$. It follows that $P_{k-1} = 2P_k - B = (2x_k, 2y_k - 420)$. We can now construct a "predecessor function" as follows: if $k \geq 2$ and $P_k = (x_k, y_k)$, then

$$P_{k-1} = \begin{cases} (2x_k, 2y_k - 420) & \text{if } y_k > 210 \\ (2x_k - 560, 2y_k) & \text{if } x_k > 280 \\ (2x_k, 2y_k) & \text{if } 420x_k + 560y_k < \frac{1}{2}420 \cdot 560 \end{cases}$$

It is now easy to trace $P_7 = (14, 92)$ back to P_1:

$$P_7 = (14, 92) \implies P_6 = (28, 184) \implies P_5 = (56, 368) \implies P_4 = (112, 316) \implies$$
$$P_3 = (224, 212) \implies P_2 = (448, 4) \implies P_1 = (336, 8).$$

We then see that $k + m = 336 + 8 = 344$.

Query. What is the set of all points P_7 for which there exists a corresponding P_1 inside the triangle?

Note. The above analysis shows that the point $P_7 = (14, 92)$ is generated from $P_1 = (336, 8)$ by the sequence of die outcomes C, B, B, B, A, A.

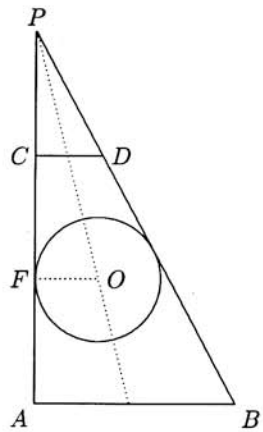
If a fair die is used in generating points $P_2, P_3, \cdots, P_N$ for some large value of N, the graph will almost always resemble the Sierpinski triangle, one of the better known fractals.

13. (Answer: 163)

Let A and C, respectively, be Kenny's and Jenny's positions at the instant when the building first blocks their line of sight, and let B and D be their positions when they can first see each other again. Let P be the point where $\overline{AC}$ extended meets $\overline{BD}$ extended. These two segments are tangent to the building and $\overline{AC} \perp \overline{AB}$. Let O be the center of the building, F the point at which $\overline{AC}$ is tangent to the building, and t the time in seconds that passes as Kenny walks from A to B. Then $AB = 3t$, $CD = t$, $CF = FA = 100$, and $OF = 50$. Since $\triangle PAB \sim \triangle PCD$, we see that

$$3 = \frac{AB}{CD} = \frac{PA}{PC} = \frac{200 + PC}{PC},$$

so $PC = 100$. Let θ be the measure of $\angle FPO$. Since $\overrightarrow{PO}$ is the bisector of $\angle BPA$, it follows that $\angle BPA$ has measure 2θ. Thus



$$t = CD = PC \tan 2\theta = 100\frac{2\tan\theta}{1 - \tan^2\theta}.$$

Using

$$\tan\theta = \frac{OF}{PF} = \frac{50}{200} = \frac{1}{4},$$

we compute

$$t = 100\frac{2 \cdot \frac{1}{4}}{1 - \left(\frac{1}{4}\right)^2} = \frac{160}{3}.$$

The sum of the numerator and denominator is 163.

Alternate Solution. Let Kenny walk to the right along the x-axis, Jenny to the right along the line $y = 200$, and let the building be centered at $(50, 100)$. Then Kenny and Jenny lose sight of each other as they (simultaneously) cross the y-axis. If we assume this happens at time 0, then at time t, Kenny's position will be $(3t, 0)$ and Jenny's position will be $(t, 200)$. An equation for the line determined by Kenny's and Jenny's positions at time t is

$$100x + ty - 300t = 0.$$

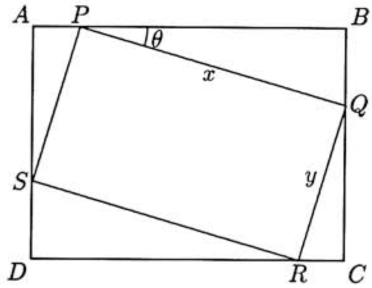
We wish to find the time $t > 0$ when the distance from the center of the building to this line is 50. Using the formula for the distance from a point to a line, we have

$$\frac{|100 \cdot 50 + t \cdot 100 - 300t|}{\sqrt{100^2 + t^2}} = 50,$$

which reduces to $15t^2 - 800t = 0$. This will be the case when $t = 0$ or $t = 160/3$.

14. (Answer: 448)

Let $ABCD$ be the outer rectangle, with $AB = 8$. Let $PQRS$ be an inscribed rectangle, with P, Q, R, S on $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, $\overline{DA}$ respectively. Note that angles QPB, RQC, SRD, and PSA are congruent; let θ denote their common measure. Let $x = PQ$ and $y = QR$. Without loss of generality, we may assume that the figure has been labeled so that $y < x$ and $\theta < 45°$. (Why?) Then



$$BA = BP + PA = x\cos\theta + y\sin\theta = 8$$
$$BC = BQ + QC = x\sin\theta + y\cos\theta = 6,$$

and these equations can be solved simultaneously to give

$$x = \frac{8\cos\theta - 6\sin\theta}{\cos^2\theta - \sin^2\theta} \qquad \text{and} \qquad y = \frac{6\cos\theta - 8\sin\theta}{\cos^2\theta - \sin^2\theta}.$$

Thus the perimeter of $PQRS$ is

$$2x + 2y = \frac{28}{\cos\theta + \sin\theta}.$$

Since $\cos\theta + \sin\theta = \sqrt{2}\sin(\theta + 45°)$ and $0 < \theta < 45°$, it follows that the perimeter of $PQRS$ is a decreasing function of θ. It will be shown below that BQ is an increasing function of θ, and that $PQRS$ is stuck if and only if $3 < BQ$. Therefore, we seek the perimeter of the rectangle that results when Q is the midpoint of $\overline{BC}$ (and S is the midpoint of $\overline{DA}$). Because Q and S are midpoints of opposite sides of $ABCD$, rectangle $PQRS$ has area 24 and diagonal 8, yielding the equations

$$xy = 24$$
$$x^2 + y^2 = 64.$$

Combining these, we find that $(x+y)^2 = 64 + 2\cdot 24 = 112$, so $2(x+y) = 2\sqrt{112} = \sqrt{448}$ is the smallest perimeter for an unstuck inscribed rectangle.

To complete the demonstration, let O be the intersection of $\overline{AC}$ and $\overline{BD}$, and consider circles centered at O and intersecting all four sides of $ABCD$. These circles all have diameters between 8 and 10. Except for the extreme cases, each such circle intersects $ABCD$ at eight points, P, P', Q, Q', R, R', S, S', given in cyclical order so that P and P' are on $\overline{AB}$, Q and Q' are on $\overline{BC}$, etc. Note that $PQRS$ and $PQ'RS'$ are inscribed rectangles.

Rectangle $PQRS$ is unstuck (Figure 1), because its vertices can be moved along the arcs PS', QP', RQ', and SR', which lie inside $ABCD$. Note that $\angle QOP' = 2\theta$. As the diameter decreases, both 2θ and BQ increase, because P' and Q drift away from B. This shows that BQ is an increasing function of θ.

Rectangle $PQ'RS'$ is stuck (Figure 2) when the diameter exceeds 8 (and $3 < BQ'$). This is because arcs $Q'Q$ and RR' are outside $ABCD$, so Q' and R are free to move only along arc $Q'R$, which is impossible.

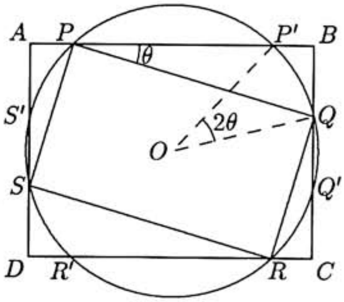


Figure 1 (Unstuck)

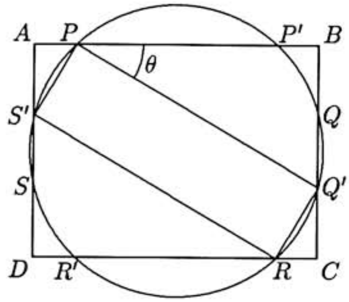
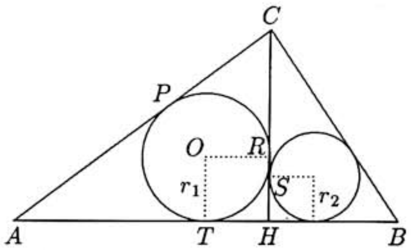


Figure 2 (Stuck)

Query. We have shown that $x+y$ is a decreasing function of θ. Are x and y themselves decreasing functions of θ?

15. (Answer: 997)

Let $a = BC$, $b = AC$, $c = AB$, $h = CH$, $p = AH$, and $q = BH$. Let O be the center of the circle inscribed in $\triangle AHC$, let r_1 be the radius of this circle, and let T and P, respectively, be the points where this circle is tangent to $\overline{AB}$ and $\overline{AC}$. Since $\angle CHA$ is a right angle, we have $\overline{OR} \perp \overline{OT}$, and hence $RH = OT = r_1$. Similarly, $SH = r_2$, where r_2 is the radius of the circle inscribed in $\triangle CHB$. Thus



$$RS = |RH - SH| = |r_1 - r_2|.$$

Next note that

$$b = AC = AP + CP = AT + CR = (p - r_1) + (h - r_1),$$

from which $r_1 = (p + h - b)/2$. Similarly, $r_2 = (q + h - a)/2$.
Thus

$$RS = |r_1 - r_2| = \left|\frac{p+h-b}{2} - \frac{q+h-a}{2}\right| = \frac{1}{2}|(p-q) + (a-b)|. \qquad (*)$$

By the Pythagorean Theorem, $a^2 - q^2 = h^2 = b^2 - p^2$, so $p^2 - q^2 = b^2 - a^2$. From this we have

$$p - q = \frac{(b+a)(b-a)}{p+q} = \frac{(b+a)(b-a)}{c}.$$

Substituting this last expression into (∗) gives

$$RS = \frac{1}{2}\left|\frac{(b+a)(b-a)}{c} + (a-b)\right| = \frac{|b-a|}{2c}|a+b-c|.$$

With $a = 1993$, $b = 1994$, and $c = 1995$, we find

$$RS = \frac{1}{2 \cdot 1995}1992 = \frac{332}{665},$$

so $m + n = 332 + 665 = 997$.

AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

12th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)

THURSDAY, March 31, 1994

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 063)
A positive integer that is one less than a perfect square is of the form

$$n^2 - 1 = (n-1)(n+1),$$

for $n = 2, 3, \ldots$. Such a number is a multiple of 3 if and only if n is not a multiple of 3. Thus the $(2k-1)^{\text{st}}$ and $(2k)^{\text{th}}$ terms of the sequence are $(3k-1)^2 - 1$ and $(3k+1)^2 - 1$ respectively. Therefore, the 1994^{th} term of the sequence is

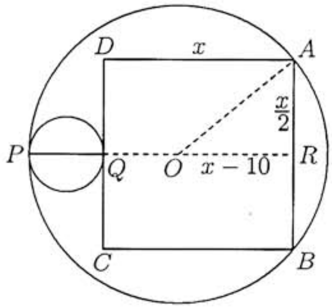
$$(3 \cdot 997 + 1)^2 - 1 = (3000 - 8)^2 - 1 = 3000^2 - 16 \cdot 3000 + 63.$$

When this number is divided by 1000, the remainder is 63.

2. (Answer: 312)
Let O be the center of the large circle. Note that P, Q, and O are collinear since the circles are tangent. Let the line through P, Q, and O intersect $\overline{AB}$ in R and let $x = AB$. Then $RO = RQ - OQ = x - 10$. Because $\overline{CD}$ is tangent at Q to the smaller circle, it follows that $AR = x/2$ and that $\angle ARO$ is a right angle. Hence, by the Pythagorean Theorem,

$$(x-10)^2 + (x/2)^2 = 20^2.$$

Solving for x, we obtain $x = 8 \pm \sqrt{304}$. Since $x > 0$, we have $x = 8 + \sqrt{304}$, and $m + n = 8 + 304 = 312$.



3. (Answer: 561)
Using $f(x) = x^2 - f(x-1)$ repeatedly, we have

$$\begin{aligned}
f(94) &= 94^2 - f(93)\\
&= 94^2 - 93^2 + f(92)\\
&= 94^2 - 93^2 + 92^2 - f(91)\\
&\ \ \vdots\\
&= 94^2 - 93^2 + 92^2 - \cdots + 20^2 - f(19)\\
&= (94+93)(94-93) + (92+91)(92-91) + \cdots\\
&\qquad + (22+21)(22-21) + 20^2 - 94\\
&= (94 + 93 + 92 + \cdots + 21) + 306\\
&= \frac{94+21}{2} \cdot 74 + 306\\
&= 4561.
\end{aligned}$$

Thus, when $f(94)$ is divided by 1000, the remainder is 561.

4. (Answer: 312)
Let
$$S_n = \lfloor \log_2 1 \rfloor + \lfloor \log_2 2 \rfloor + \lfloor \log_2 3 \rfloor + \cdots + \lfloor \log_2 n \rfloor.$$
Note that, for nonnegative integer k, there are 2^k positive integers x for which $\lfloor \log_2 x \rfloor = k$, namely $x = 2^k, 2^k+1, \ldots, 2^{k+1}-1$. Thus, if r is a positive integer,
$$S_{2^r-1} = 0 + (1+1) + (2+2+2+2) + \cdots + \Big(\underbrace{(r-1)+(r-1)+\cdots+(r-1)}_{2^{r-1}\text{ terms}}\Big).$$
The right side of this expression has
$$\begin{aligned} 2^r - 2^1 \text{ terms } &\geq 1 \\ 2^r - 2^2 \text{ terms } &\geq 2 \\ 2^r - 2^3 \text{ terms } &\geq 3 \\ &\vdots \\ 2^r - 2^{r-2} \text{ terms } &\geq r-2 \\ 2^r - 2^{r-1} \text{ terms } &= r-1. \end{aligned}$$
It follows that
$$\begin{aligned} S_{2^r-1} &= (2^r - 2^1) + (2^r - 2^2) + (2^r - 2^3) + \cdots + (2^r - 2^{r-1}) \\ &= (r-1)2^r - (2^r - 2) \\ &= (r-2)2^r + 2. \end{aligned}$$
Taking $r = 8$ in this last equation we obtain $S_{255} = 1538 < 1994$. Setting $r = 9$ we find $S_{511} = 3586 > 1994$. Hence, if $S_n = 1994$, then $255 = 2^8 - 1 < n < 2^9 - 1$. Therefore
$$1994 = S_n = S_{255} + (n - 255)8 = 8n - 502,$$
and this yields $n = 312$.

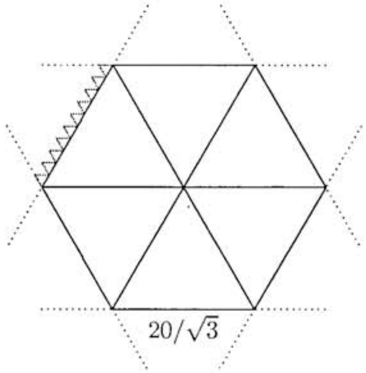
5. (Answer: 103)
Consider each positive integer less than 1000 to be a three-digit number by prefixing 0's to numbers with fewer than three digits. The sum of the products of the digits of all such positive numbers is
$$\begin{aligned} &(0\cdot 0\cdot 0 + 0\cdot 0\cdot 1 + 0\cdot 0\cdot 2 + \cdots + 9\cdot 9\cdot 8 + 9\cdot 9\cdot 9) - 0\cdot 0\cdot 0 \\ &\qquad = (0+1+2+\cdots+9)^3 - 0. \end{aligned} \qquad (*)$$
However, $p(n)$ is the product of the non-zero digits of n. The sum of these products can be found by replacing 0 by 1 in the above expression, since ignoring 0's is equivalent

to thinking of them as 1's in the products. (Note that the final 0 in (*) becomes a 1 and compensates for the contribution of 000 after it is changed to 111.) Hence

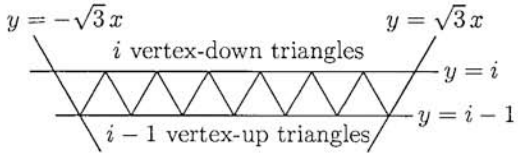
$$\sum_{n=1}^{999} p(n) = (1+1+2+\cdots+9)^3 - 1 = 46^3 - 1 = (46-1)(46^2+46+1) = 3^3 \cdot 5 \cdot 7 \cdot 103,$$

and the largest prime factor is 103.

6. (Answer: 660)
The six outermost lines determine a regular hexagon of side length $20/\sqrt{3}$. The three lines through the origin cut this hexagon into 6 equilateral triangles, each with side length $20/\sqrt{3}$. Since each of these large triangles has side length 10 times the side length of the small triangles, each of the large equilateral triangles is cut into 10^2 small triangles. Hence the hexagon is cut into $6 \cdot 10^2 = 600$ small triangles. In addition there is a row of ten triangles outside, but adjacent to, each side of the hexagon. (Some of these triangles are shown in the figure.) Therefore the total number of small triangles in the configuration is $600 + 6 \cdot 10 = 660$.



Alternate Solution. Let n be a positive integer and consider the figure obtained by drawing the lines for $-n \leq k \leq n$. By symmetry, the number of equilateral triangles of side $2/\sqrt{3}$ obtained is 6 times the number that lie in the upper half plane in the wedge bounded by the lines $y = \sqrt{3}\,x$ and $y = -\sqrt{3}\,x$. Below the line $y = 1$ there is 1 triangle in the wedge. Between $y = 1$ and $y = 2$ there are 3 triangles. For $1 \leq i \leq n$, there are $2i - 1$ triangles between $y = i - 1$ and $y = i$, and in this row there are $i - 1$ triangles oriented vertex-up and i oriented vertex-down. Above the line $y = n$ there is a row of n vertex-up triangles, but because there is no horizontal line above $y = n$, there are none that are vertex-down. Hence the number of equilateral triangles of side length $2/\sqrt{3}$ is

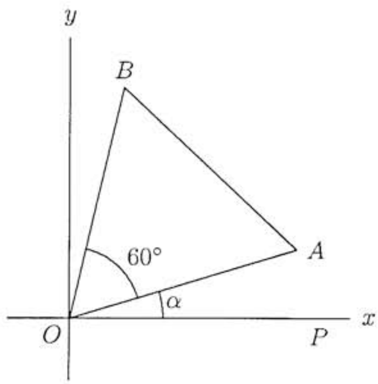


$$6\left((1+3+\cdots+2n-1)+n\right) = 6(n^2+n).$$

When $n = 10$ we have 660 such triangles.

7. (Answer: 072)
The equation $x^2 + y^2 = 50$ is that of the circle with center $(0, 0)$ and radius $5\sqrt{2}$, and $ax + by = 1$ is an equation of a line. The problem statement is equivalent to requiring that the line and the circle intersect and that each intersection be a lattice point. There are 12 lattice points on the circle: $(\pm1, \pm7)$, $(\pm5, \pm5)$, $(\pm7, \pm1)$. Any pair of these points determines a line that intersects the circle in those two points. There are $\binom{12}{2} = 66$ such pairs. Also, at each of the twelve points the tangent line intersects the circle at only that point. Thus, there are 66+12=78 lines that intersect the circle and do so only at lattice points. Any such line can be uniquely written in the form $ax + by = 1$ if and only if the line does not contain the origin. But 6 of the 78 lines *do* contain the origin. These are the lines determined by diametrically opposite points. It follows that there are $78 - 6 = 72$ ordered pairs (a, b) of real numbers for which the given system has at least one solution and has only integer solutions.

8. (Answer: 315)
Let $O = (0, 0)$, $A = (a, 11)$, and $B = (b, 37)$. Note that reflection of the triangle in the y-axis does not change the value of ab. Thus we may assume that the counterclockwise measure of the angle from $\overrightarrow{OA}$ to $\overrightarrow{OB}$ is $60°$.
Let $OB = OA = AB = r$, and let $\angle AOP = \alpha$, where P is a point on the positive x-axis. Then $\angle BOP = \alpha + 60°$. Since



$$\sin(\angle BOP) = \sin(\alpha + 60°)$$
$$= \sin\alpha\cos 60° + \cos\alpha \sin 60°,$$

we have

$$\frac{37}{r} = \frac{11}{r}\cdot\frac{1}{2} + \frac{a}{r}\cdot\frac{\sqrt{3}}{2},$$

from which $a = 21\sqrt{3}$. Similarly

$$\cos(\angle BOP) = \cos(\alpha + 60°) = \cos\alpha\cos 60° - \sin\alpha\sin 60°,$$

which leads to

$$\frac{b}{r} = \frac{a}{r}\cdot\frac{1}{2} - \frac{11}{r}\cdot\frac{\sqrt{3}}{2}.$$

It follows that $b = 5\sqrt{3}$, so $ab = 21\sqrt{3}\cdot 5\sqrt{3} = 315$.

Alternate Solution. Let $O = (0,0)$, $A = (a,11)$, $B = (b,37)$, and assume that the counterclockwise measure of the angle from $\overrightarrow{OA}$ to $\overrightarrow{OB}$ is $60°$. Regard A and B as the complex numbers $a + 11i$ and $b + 37i$, respectively. Since a rotation of $60°$ about the origin is equivalent to multiplication by $\cos 60° + i \sin 60°$, we have

$$(a + 11i)(\cos 60° + i \sin 60°) = b + 37i.$$

Separating the real and imaginary parts yields

$$\begin{aligned} a - 11\sqrt{3} &= 2b \\ 11 + a\sqrt{3} &= 74. \end{aligned}$$

From the second equation we obtain $a = 21\sqrt{3}$, and then the first yields $b = 5\sqrt{3}$. Thus $ab = 315$.

9. (Answer: 394)
Let $n \geq 2$ be an integer and let the bag contain n distinct pairs of tiles. The probability that two of the first three tiles selected make a pair is

$$\frac{\text{\# of ways to select three tiles, two of which match}}{\text{\# of ways to select three tiles}} = \frac{n(2n-2)}{\binom{2n}{3}} = \frac{3}{2n-1}.$$

Now let $P(n)$ be the probability of emptying the bag when the bag initially contains n distinct pairs of tiles. Then $P(2) = 1$ and for $n \geq 3$,

$$P(n) = \frac{3}{2n-1} P(n-1).$$

Using this recursion formula repeatedly, we find that

$$P(n) = \frac{3}{2n-1} \frac{3}{2n-3} \cdots \frac{3}{5} P(2).$$

Setting $n = 6$ we have

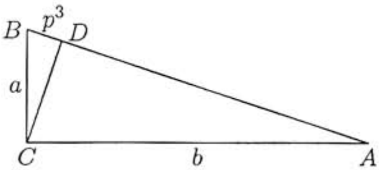
$$P(6) = \frac{3^4}{11 \cdot 9 \cdot 7 \cdot 5} = \frac{9}{385}.$$

The sum of the numerator and denominator is $9 + 385 = 394$.

10. (Answer: 450)

Let a, b, c denote the lengths of $\overline{BC}$, $\overline{AC}$, $\overline{AB}$, respectively. Let $p = 29$, so $BD = p^3$. Since $\triangle BCD \sim \triangle BAC$ we have

$$\frac{p^3}{a} = \frac{a}{c},$$



from which $a^2 = p^3c$. Since p is prime, there is an integer x such that $c = px^2$. It follows that $a = p^2x$. Substituting these expressions for a and c into $b^2 = c^2 - a^2$ we find

$$b^2 = p^2x^4 - p^4x^2 = p^2x^2(x^2 - p^2).$$

Thus, there is a positive integer y with $x^2 - p^2 = y^2$, so $p^2 = x^2 - y^2 = (x-y)(x+y)$. Since p is prime and $x - y < x + y$, we have $x - y = 1$ and $x + y = p^2$, which leads to

$$x = \frac{p^2+1}{2} \quad \text{and} \quad y = \frac{p^2-1}{2}.$$

Since $p = 29$,

$$\cos B = \frac{a}{c} = \frac{p^2x}{px^2} = \frac{p}{(p^2+1)/2} = \frac{29}{(29^2+1)/2}$$

in lowest terms. Hence $m + n = (29^2 + 2 \cdot 29 + 1)/2 = (29+1)^2/2 = 450$.

11. (Answer: 465)

If five bricks in a tower are oriented so that each contributes $10''$ to the height of the tower, then these five bricks contribute $50''$ to the height of the tower. These five bricks can be reoriented so that three contribute $4''$ each and two contribute $19''$ to the height of the tower. Note that with this reorientation, the total height of the tower is unchanged. Thus we may assume that in a tower of bricks there are 0, 1, 2, 3, or 4 bricks oriented so that they each contribute $10''$ to the total height. All other bricks in the tower are oriented to contribute $4''$ or $19''$ to the height.

Suppose that there are y blocks that contribute $10''$ and z blocks that contribute $19''$ to the height of the tower. Then there are $94 - y - z$ bricks that contribute $4''$, and the height of the tower is

$$4(94 - y - z) + 10y + 19z = 376 + 6y + 15z$$

inches. As noted above, we may assume that $0 \le y \le 4$. Because $y + z \le 94$ we must have $0 \le z \le 94 - y$, so there are $95 - y$ possible values for z. In all there are $95 + 94 + 93 + 92 + 91 = 465$ such pairs (y, z), and hence at most 465 different possible tower heights.

We now show that any two of these ordered pairs give different tower heights. Suppose that ordered pairs (y, z) and (u, v) lead to towers of the same height. Then

$$376 + 6y + 15z = 376 + 6u + 15v.$$

It follows that $6(y-u) = 15(v-z)$, which implies that $y-u$ is divisible by 5. Because $|y-u| \leq 4$ we conclude that $y = u$ and then that $z = v$. Thus the 465 ordered pairs correspond to 465 different tower heights.

Alternate Solution. If we have x 4″ bricks, y 10″ bricks and z 19″ bricks in our tower, then $x + y + z = 94$ and the height of the tower is

$$4x + 10y + 19z = 4(94 - y - z) + 10y + 19z = 376 + 3(2y + 5z).$$

There will be a height corresponding to each possible value of $2y + 5z$ where y and z are non-negative integers with $y + z \leq 94$. With these restrictions, we have

$$2y + 5z \leq 5 \cdot 94 = 470.$$

To count the number of possible values of $2y + 5z$ we apply the following theorem:

> If $\mathrm{GCD}(a, b) = 1$, then every integer greater than or equal to $(a-1)(b-1)$ can be written as $ar + bs$ where r and s are nonnegative integers, and the number of nonnegative integers that cannot be written in this way is $(a-1)(b-1)/2$.

Define an integer n between 0 and 470 to be "good" if n can be written as $2y + 5z$ where $y, z \geq 0$ and $y + z \leq 94$, and "bad" otherwise. By the theorem quoted, there are $(2-1)(5-1)/2 = 2$ bad integers between 0 and 3 (namely 1 and 3). Substituting $z = 94 - x - y$, we see that an integer n between 4 and 470 is good if and only if

$$470 - n = 5x + 3y$$

for some nonnegative integers x and y. Thus by the theorem stated above there are $(3-1)(5-1)/2 = 4$ bad integers between $470 - ((3-1)(5-1) - 1) = 463$ and 470 (namely 463, 466, 468, and 469). Hence the number of good integers and the desired number of tower heights is $471 - 6 = 465$.

Note. The number of heights is equal to the number of terms in the expansion of

$$(x^4 + x^{10} + x^{19})^{94}. \qquad (*)$$

In addition, the coefficient of x^n in the expansion of $(*)$ is the number of different ways of obtaining a tower of height n.

12. (Answer: 702)

Suppose that the field is partitioned into squares of side s. Then there are positive integers m, n with

$$\frac{24}{s} = m \qquad \text{and} \qquad \frac{52}{s} = n.$$

Hence

$$\frac{m}{n} = \frac{6}{13},$$

so there is a positive integer k with $m = 6k$ and $n = 13k$. Note that the total number of test plots is a maximum when k is as large as possible. The total length of fence used in partitioning the field into $s \times s$ squares is

$$(m-1)52 + (n-1)24 = k(6 \cdot 52 + 13 \cdot 24) - (52 + 24) = 624k - 76.$$

Since at most 1994 meters of fence can be used, we have

$$624k - 76 \le 1994,$$

so that

$$k \le \frac{1994 + 76}{624} = 3.31\ldots$$

Since k must be an integer, the largest possible value of k is 3. For this value of k we have a total of

$$mn = (6 \cdot 3)(13 \cdot 3) = 702$$

squares, formed by using $624 \cdot 3 - 76 = 1796$ meters of the available 1994 meters of fence.

13. (Answer: 850)

Let $p(x) = x^{10} + (13x-1)^{10}$. If r is a zero of $p(x)$, then

$$-1 = \left(\frac{13r-1}{r}\right)^{10} = \left(\frac{1}{r} - 13\right)^{10}.$$

Thus

$$\left(\frac{1}{r} - 13\right)\left(\frac{1}{\overline{r}} - 13\right) = 1,$$

so that

$$\left(\frac{1}{r_1} - 13\right)\left(\frac{1}{\overline{r_1}} - 13\right) + \cdots + \left(\frac{1}{r_5} - 13\right)\left(\frac{1}{\overline{r_5}} - 13\right) = 5.$$

Expanding and rearranging, we find

$$\left(\frac{1}{r_1\overline{r_1}} + \cdots + \frac{1}{r_5\overline{r_5}}\right) - 13\left(\frac{1}{r_1} + \frac{1}{\overline{r_1}} + \cdots + \frac{1}{r_5} + \frac{1}{\overline{r_5}}\right) + 5 \cdot 169 = 5.$$

Note that $1/r_1$, $1/\overline{r_1}, \ldots, 1/r_5$, $1/\overline{r_5}$ are the zeros of

$$x^{10}p\left(\frac{1}{x}\right) = x^{10} - 130x^9 + \cdots,$$

so

$$\frac{1}{r_1} + \frac{1}{\overline{r_1}} + \cdots + \frac{1}{r_5} + \frac{1}{\overline{r_5}} = 130.$$

Therefore,

$$\frac{1}{r_1\overline{r_1}} + \cdots + \frac{1}{r_5\overline{r_5}} = 13 \cdot 130 - 5 \cdot 169 + 5 = 850.$$

Alternate Solution. Let $p(x) = x^{10} + (13x - 1)^{10}$. If $p(r) = 0$, then

$$\left(13 - \frac{1}{r}\right)^{10} = -1 = \cos 180^\circ + i \sin 180^\circ.$$

It follows that

$$\frac{1}{r} = 13 - (\cos\theta + i\sin\theta)$$

where θ is an odd multiple of 18°. Hence

$$\frac{1}{r\overline{r}} = (13 - (\cos\theta + i\sin\theta))\,(13 - (\cos\theta - i\sin\theta)) = 170 - 26\cos\theta.$$

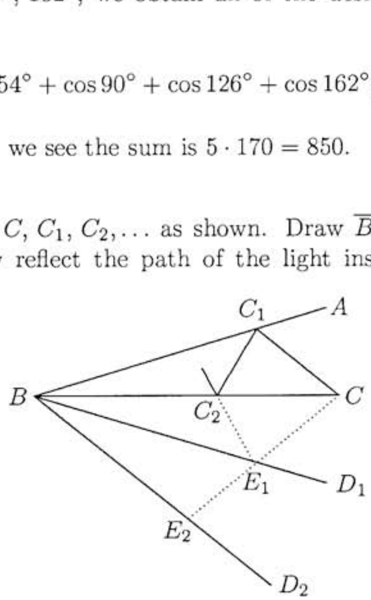
Letting θ take on the values 18°, 54°, 90°, 126°, 162°, we obtain all of the desired products. Thus

$$\frac{1}{r_1\overline{r_1}} + \cdots + \frac{1}{r_5\overline{r_5}} = 5 \cdot 170 - 26(\cos 18^\circ + \cos 54^\circ + \cos 90^\circ + \cos 126^\circ + \cos 162^\circ).$$

Applying the identity $\cos\theta + \cos(180^\circ - \theta) = 0$, we see the sum is $5 \cdot 170 = 850$.

14. (Answer: 071)

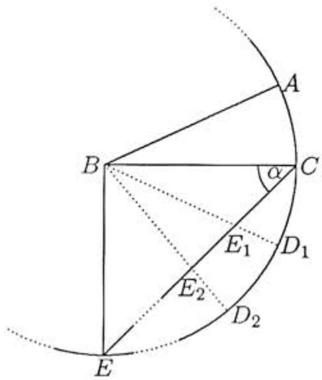
Label the points at which the light reflects as C, C_1, $C_2, \ldots$ as shown. Draw $\overline{BD_1}$ so that $BD_1 = BC$ and $\angle CBD_1 = \beta$. Now reflect the path of the light inside $\angle ABC$ across $\overline{BC}$, and let E_1 be the reflection of C_1. For counting purposes, it doesn't matter whether we look at the real path $CC_1C_2\ldots$ or the reflected path beginning with $\overline{CE_1}$, so assume that the light beam actually begins its path by travelling from C to E_1. Now draw $\overline{BD_2}$ with $\angle D_1BD_2 = \beta$ and $BD_2 = BD_1$. Reflect the new path of the light beam across $\overline{BD_1}$ and let the reflection of C_2 be E_2. Since $\angle CE_1D_1 = \angle C_2E_1B = \angle BE_1E_2$, the path CE_1E_2 must be a straight line.



Repeat the above process by constructing $\overline{BD_3}$, $\overline{BD_4}\ldots$ with $\angle D_iBD_{i+1} = \beta$ and $BD_i = BC$. The result is a new path which follows the ray $\overrightarrow{CE_1}$. Each point where the light beam reflects off $\overline{BA}$ or $\overline{BC}$ will correspond to an intersection point E_i of this ray with some $\overline{BD_i}$. We need to count these intersections. Draw the circle with center B and radius AB as shown. The path that concerns us is the line segment $\overline{CE}$ where E is on the circle and $\angle BCE = \alpha$. (Once the ray leaves the circle, there will be no more reflections in the original path since the light beam will be outside $\triangle ABC$.) To count the number of intersections E_i first find $\angle CBE$. Since $BE = BC$, we have $\angle BEC = \angle BCE$, so $\angle CBE = 180 - 2\alpha$. Hence the number of intersections equals

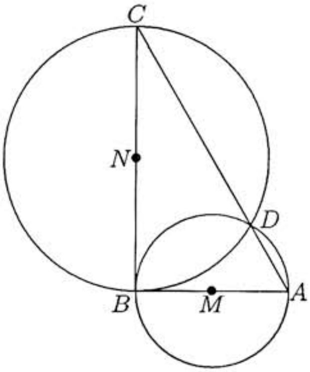
$$\left\lfloor \frac{180 - 2\alpha}{\beta} \right\rfloor + 1,$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to x, and we add 1 to count the first reflection at C. With $\alpha = 19.94$ and $\beta = 1.994$, we obtain 71 reflections.



15. (Answer: 597)

We claim that the set of all fold points of $\triangle ABC$ is the region common to the interiors of the circles that have $\overline{AB}$ and $\overline{BC}$ as their diameters. To establish this, first show that the creases formed by folding vertices A and B onto P are disjoint if and only if P lies inside the circle that has $\overline{AB}$ as diameter. To see this, note that the crease formed by folding any point Q onto P is part of the perpendicular bisector of $\overline{QP}$. If P is outside the circle with diameter $\overline{AB}$, then $\triangle PAB$ is not obtuse, for $\angle P$ is acute, and angles PAB and PBA are at most 60° and 90° respectively. Therefore the circumcenter of $\triangle PAB$ is inside the triangle, requiring that the creases intersect. On the other hand, if P is inside the circle that has diameter $\overline{AB}$, then the perpendicular bisectors of $\overline{PA}$ and $\overline{PB}$ meet at a point that is separated from P by $\overline{AB}$, so the creases do not meet inside $\triangle ABC$. If P is on the circle, then the creases meet on $\overline{AB}$. A similar discussion applies to the circle that has $\overline{BC}$ as diameter and to the circle that has $\overline{AC}$ as diameter. Note, however, that all interior points of $\triangle ABC$ are inside the latter circle. Thus the set of fold points of the triangle is the region common to the interior of the triangle and the interiors of the two circles with diameters $\overline{AB}$ and $\overline{BC}$.



These two circles both intersect $\overline{AC}$ at D, the foot of the perpendicular from B to $\overline{AC}$. The region in question is therefore bounded by two circular arcs. One is a 120° arc of radius 18, centered at M, the midpoint of $\overline{AB}$; the other is a 60° arc of radius $18\sqrt{3}$, centered at N, the midpoint of $\overline{BC}$. The areas of triangles DMB and DNB are $\dfrac{1}{2} \cdot 18 \cdot 18 \cdot \dfrac{\sqrt{3}}{2} = 81\sqrt{3}$ and $\dfrac{1}{2} \cdot 18\sqrt{3} \cdot 18\sqrt{3} \cdot \dfrac{\sqrt{3}}{2} = 243\sqrt{3}$, respectively. The areas of sectors DMB and DNB are $\dfrac{1}{3}\pi \cdot 18^2 = 108\pi$ and $\dfrac{1}{6}\pi(18\sqrt{3})^2 = 162\pi$, respectively. The area that we seek is the sum of the sectors' areas minus the sum of the triangles' areas, which simplifies to $270\pi - 324\sqrt{3}$. The desired sum is $q+r+s = 270+324+3 = 597$.

AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

13th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)

THURSDAY, March 23, 1995

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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1. (Answer: 255)
Observe that the area of S_{i+1} is one fourth that of S_i, and that three fourths of S_{i+1} is not inside S_i. Therefore the area enclosed by at least one of S_1, S_2, S_3, S_4, S_5 is

$$1+\frac{3}{4}\left(\frac{1}{4}+\frac{1}{16}+\frac{1}{64}+\frac{1}{256}\right)=\frac{1279}{1024}.$$

Hence $m-n=1279-1024=255$.

2. (Answer: 025)
Taking logarithms of both sides of the equation, we find that

$$\log\left(\sqrt{1995}x^{\log x}\right)=\log x^2,$$

where all the logarithms are to the base 1995. From this we obtain

$$\frac{1}{2}\log 1995+(\log x)(\log x)=2\log x,$$

which leads to

$$(\log x)^2-2\log x+\frac{1}{2}=0.$$

Solving this last equation gives

$$\log x=1\pm\sqrt{\frac{1}{2}}.$$

Since these values for $\log x$ are both real, the original equation has two positive roots; call them r_1 and r_2. Since $\log r_1 r_2=\log r_1+\log r_2=2$, the product of these roots is

$$r_1 r_2=1995^2=(2000-5)^2=2000^2-10\cdot 2000+25.$$

The last three digits of this number are 025.

3. (Answer: 067)
Since the net movement must be two steps right (R) and two steps up (U), there must be at least four steps. The point $(2,2)$ can be reached in exactly four steps if the sequence is some permutation of R, R, U, U. These four steps can be permuted in

$$\frac{4!}{2!2!}=6$$

ways. Each of these sequences has probability $(1/4)^4$ of occurring. Thus the probability of reaching $(2,2)$ in exactly 4 steps is $6/4^4$.

In moving to $(2,2)$, the total number of steps must be even, since an odd number of steps would reach a lattice point with one even coordinate and one odd coordinate. Next consider the possibility of reaching $(2,2)$ in six steps. A six-step sequence must include the steps R, R, U, U in some order, as well as a pair consisting of R, L (left) or U, D (down), in some order. The steps R, R, U, U, U, D can be permuted in

$$\frac{6!}{3!2!1!} = 60$$

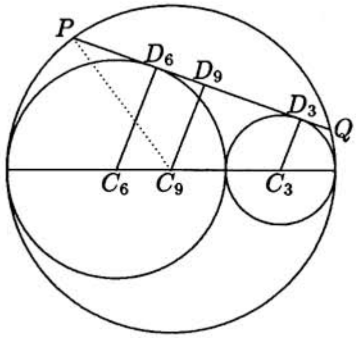
ways, but for 12 of these sequences — namely those that start with some permutation of R, R, U, U — the object actually reaches $(2,2)$ in four steps. A similar analysis holds for the steps R, R, U, U, R, L. Thus there are $2(60-12) = 96$ six-step sequences that reach $(2,2)$, but that do not do so until the sixth step. Each of these 96 sequences occurs with probability $1/4^6$.

Considering the four- and six-step possibilities, we find that the probability of reaching $(2,2)$ in six or fewer steps is

$$\frac{6}{4^4} + \frac{96}{4^6} = \frac{3}{64}.$$

Thus $m + n = 67$.

4. (Answer: 224)
Let $\overline{PQ}$ be the tangent chord, and let C_3, C_6, and C_9 be the centers of the circles of radii 3, 6, and 9, respectively. We see that $C_6C_9 = 3$ and $C_9C_3 = 6$. Let D_3, D_6, and D_9, respectively, be the feet of the perpendiculars from C_3, C_6, and C_9 to $\overline{PQ}$. Then D_3 and D_6 are points of tangency, and $\overline{C_3D_3}$, $\overline{C_6D_6}$, and $\overline{C_9D_9}$ are parallel. It follows that



$$C_9D_9 = \frac{1 \cdot C_3D_3 + 2 \cdot C_6D_6}{3} = 5.$$

Now apply the Pythagorean Theorem to right triangle C_9PD_9 to find that

$$(PQ)^2 = 4(D_9P)^2 = 4\left[(C_9P)^2 - (C_9D_9)^2\right] = 4(9^2 - 5^2) = 224.$$

5. (Answer: 051)
Let the roots be r_1, r_2, r_3, r_4, where $r_1r_2 = 13+i$ and $r_3+r_4 = 3+4i$. Because the polynomial has real coefficients and none of the roots is real, the roots occur in conjugate pairs, say $r_3 = \overline{r_1}$ and $r_4 = \overline{r_2}$. It follows that $r_3r_4 = \overline{r_1r_2} = 13-i$ and $r_1+r_2 = \overline{r_3+r_4} = 3-4i$. The polynomial is therefore

$$[x^2-(3-4i)x+(13+i)][x^2-(3+4i)x+(13-i)]$$

$$= x^4-6x^3+51x^2-70x+170.$$

In particular, $b = 51 = (13+i)+(3-4i)(3+4i)+(13-i)$.

Query. Because the roots occur in conjugate pairs, the given polynomial can be factored as the product of two quadratics that have *real* coefficients. What are these factors?

6. (Answer: 589)
Let $n = p^r q^s$, where p and q are distinct primes. Then $n^2 = p^{2r}q^{2s}$, so n^2 has

$$(2r+1)(2s+1)$$

factors. For each factor less than n, there is a corresponding factor greater than n. By excluding the factor n, we see that there must be

$$\frac{(2r+1)(2s+1)-1}{2} = 2rs+r+s$$

factors of n^2 that are less than n. Because n has $(r+1)(s+1)$ factors (including n itself), and because every factor of n is also a factor of n^2, there are

$$2rs+r+s-[(r+1)(s+1)-1] = rs$$

factors of n^2 that are less than n but not factors of n. When $r=31$ and $s=19$, there are $rs=589$ such factors.

7. (Answer: 027)

Let $s = \sin t + \cos t$ and $p = \sin t \cos t$. It is given that $1 + s + p = \frac{5}{4}$; thus $p = \frac{1}{4} - s$. It then follows that

$$1 = \cos^2 t + \sin^2 t = s^2 - 2p = s^2 + 2s - \frac{1}{2},$$

which leads to $s = -1 \pm \sqrt{10}/2$. Because $-2 < s < 2$, however, the only possible value for s is $-1 + \sqrt{10}/2$. Then

$$(1 - \sin t)(1 - \cos t) = 1 - s + p = \frac{5}{4} - 2s = \frac{13}{4} - \sqrt{10},$$

so $k + m + n = 27$.

8. (Answer: 085)

Suppose that $\dfrac{x+1}{y+1} = m$ holds for some integer $m > 1$, so that

$$x = my + (m - 1). \tag{$*$}$$

Because y is a factor of x, it must also be a factor of $m - 1$. Hence there is a positive integer k with $m - 1 = ky$. Substitute $m = ky + 1$ into $(*)$ to find that

$$x = (ky + 1)y + ky = ky(y + 1) + y. \tag{$\dagger$}$$

It follows from $x \le 100$ that

$$k \le \frac{100 - y}{y(y + 1)}.$$

There are $\left\lfloor \dfrac{100 - y}{y(y+1)} \right\rfloor$ positive integers k that satisfy this inequality. For each positive integer y, equation $(\dagger)$ shows that there is a one-one correspondence between such k and ordered pairs (x, y) with the desired property. Hence the number of pairs is

$$\sum_{y=1}^{99} \left\lfloor \frac{100 - y}{y(y+1)} \right\rfloor = \sum_{y=1}^{9} \left\lfloor \frac{100 - y}{y(y+1)} \right\rfloor$$

$$= \left\lfloor \frac{99}{2} \right\rfloor + \left\lfloor \frac{98}{6} \right\rfloor + \left\lfloor \frac{97}{12} \right\rfloor + \left\lfloor \frac{96}{20} \right\rfloor + \left\lfloor \frac{95}{30} \right\rfloor + \left\lfloor \frac{94}{42} \right\rfloor + \left\lfloor \frac{93}{56} \right\rfloor + \left\lfloor \frac{92}{72} \right\rfloor + \left\lfloor \frac{91}{90} \right\rfloor$$

$$= 49 + 16 + 8 + 4 + 3 + 2 + 1 + 1 + 1$$

$$= 85.$$

9. (Answer: 616)

Let $\theta = \angle BAD$, so that $\angle BDM = 3\theta$ and $\angle ABD = 2\theta$. Combine the Law of Sines and the double-angle formula for the sine function to find that

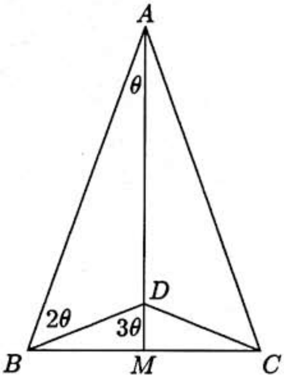
$$\frac{BD}{\sin\theta} = \frac{AD}{\sin 2\theta} = \frac{10}{2\sin\theta\cos\theta},$$

from which $\cos\theta = \dfrac{5}{BD}$ follows. Hence

$$AB = \frac{AM}{\cos\theta} = \frac{11}{5}BD.$$

Apply the Pythagorean Theorem to obtain

$$\left(\frac{11}{5}BD\right)^2 - 11^2 = BM^2 = BD^2 - 1^2.$$



It follows that $BD = \dfrac{5\sqrt{5}}{2}$, hence that $BM = \dfrac{11}{2}$ and $AB = \dfrac{11}{2}\sqrt{5}$. Thus the perimeter of $\triangle ABC$ is $2(AB + BM) = 11\sqrt{5} + 11 = \sqrt{605} + 11$, and $a + b = 616$.

Alternate Solution. Let the bisector of $\angle ABD$ intersect $\overline{AD}$ at E, and let $x = BE = AE$. By the Pythagorean Theorem,

$$BM = \sqrt{BE^2 - EM^2} = \sqrt{x^2 - (11-x)^2} = \sqrt{22x - 121}.$$

By applying the Pythagorean Theorem two more times, we find that

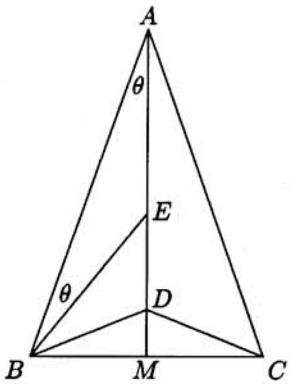
$$AB = \sqrt{BM^2 + AM^2} = \sqrt{22x} \quad \text{and}$$
$$BD = \sqrt{BM^2 + DM^2} = \sqrt{22x - 120}.$$

By the angle-bisector theorem, we have that

$$\frac{AB}{BD} = \frac{AE}{DE},$$

from which

$$\frac{\sqrt{22x}}{\sqrt{22x - 120}} = \frac{x}{10 - x}.$$



By squaring both sides of this equation and solving for x, we find that $x = 55/8$. Hence $BM = 11/2$ and $AB = (11/2)\sqrt{5}$. The perimeter of the triangle is $2(AB + BM) = 11\sqrt{5} + 11 = \sqrt{605} + 11$, so $a + b = 616$.

10. (Answer: 215)

Suppose that $N > 42$ is not the sum of a positive multiple of 42 and a positive composite integer. Let M be the smallest positive integer that makes $N - M$ divisible by 42. Because N is not divisible by 42, it follows that $M < 42$, and the conditions on N imply that no term of the arithmetic progression

$$M \ , M + 42 \ , M + 84 \ , \ \ldots \ , N - 42$$

is composite. If there are at most four terms in this progression, then

$$N \leq (41 + 3 \cdot 42) + 42 = 209.$$

On the other hand, if there are more than four terms, then $M = 5$ is required, for there must be a multiple of 5 among any five consecutive terms of an arithmetic progression whose constant difference is not divisible by 5, and no term after M is composite. Thus the only progression with at least five terms begins

$$5 \ , 47 \ , 89 \ , 131 \ , 173 \ , 215 \ , \ \ldots \ ,$$

which has 215 as its first composite term. Thus 215 is the largest integer that is not the sum of a positive multiple of 42 and a composite positive integer.

Alternate Solution. Because 42 and 5 are relatively prime, every integer can be expressed in the form $42x + 5y$, for some integers x and y. Moreover, (x, y) is a solution of

$$42x + 5y = n$$

if and only if $(x - 5, y + 42)$ is also a solution. Therefore, there is one solution for which $1 \leq x \leq 5$. It follows that the largest integer that cannot be written in the form $42x + 5y$ with $x \geq 1$ and $y \geq 2$ is $42 \cdot 5 + 5 \cdot 1 = 215$. In other words, every integer larger than 215 is the sum of a multiple of 42 and a composite number — a multiple of 5, in fact. Now check that $215 - 42 = 173$, $215 - 2 \cdot 42 = 131$, $215 - 3 \cdot 42 = 89$, $215 - 4 \cdot 42 = 47$, and $215 - 5 \cdot 42 = 5$ are prime, thereby showing that 215 is the largest integer that is not the sum of a positive multiple of 42 and a composite positive integer.

11. (Answer: 040)

Let x, y, z with $x \leq y \leq z$ be the sides of Q, the rectangular parallelepiped that is similar to P. Since Q is cut from P by a plane parallel to one of the faces of P, two of the numbers x, y, z must equal two of the numbers a, b, c. Furthermore

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} < 1, \tag{$*$}$$

so it follows that $y = a$ and $z = b$. Thus

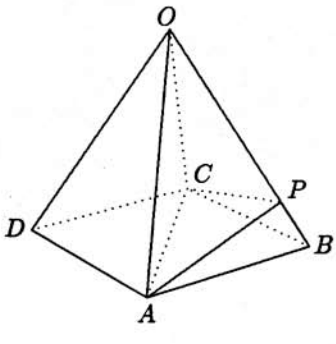
$$\frac{a}{b} = \frac{b}{c},$$

so $ac = b^2 = 1995^2 = 3^2 5^2 7^2 19^2$. Now 1995^2 has $(2+1)^4 = 81$ factors, and $a < c$ follows by substituting $y = a$ and $z = b$ into $(*)$. Thus the triple (a, b, c) can be selected in $(81-1)/2 = 40$ ways.

Each of these choices for a and c results in a rectangular parallelepiped of the type desired. Indeed, if $a < b = 1995 < c$ and $ac = 1995^2$, then cutting P by a plane parallel to the $a \times b$ face and at distance $x = a^2/b$ from that face produces an $x \times a \times b$ parallelepiped similar to P.

12. (Answer: 005)

Let P be the foot of the perpendicular from A to $\overline{OB}$. The pyramid's symmetry implies that P is also the foot of the perpendicular from C to $\overline{OB}$. Without loss of generality, we may assume $OP = 1$, from which $AP = PC = 1$, $OB = OA = \sqrt{2}$, and $BP = \sqrt{2} - 1$ follow. Two applications of the Pythagorean Theorem now give



$$AB^2 = AP^2 + BP^2 = 4 - 2\sqrt{2}$$

and

$$AC^2 = 2 \cdot AB^2 = 8 - 4\sqrt{2}.$$

The measure of the dihedral angle determined by faces OAB and OBC is the same as that of $\angle APC$. Use the Law of Cosines to obtain

$$\cos \angle APC = \frac{AP^2 + CP^2 - AC^2}{2 \cdot AP \cdot CP} = \frac{2 - 8 + 4\sqrt{2}}{2} = -3 + \sqrt{8}.$$

We then have $m + n = -3 + 8 = 5$.

Query. Without loss of generality, one may instead assume that $AB = 1$, so that $AC = \sqrt{2}$. Can you complete the calculation of $\cos \angle APC$?

13. (Answer: 400)

Let m be a positive integer. The largest integer n for which $f(n) = m$ is

$$\begin{aligned}\left\lfloor\left(m+\frac{1}{2}\right)^4\right\rfloor &= \left\lfloor m^4+2m^3+\frac{3}{2}m^2+\frac{1}{2}m+\frac{1}{16}\right\rfloor \\ &= \left\lfloor m^4+2m^3+\frac{1}{2}(3m^2+m)+\frac{1}{16}\right\rfloor \\ &= \left(m+\frac{1}{2}\right)^4-\frac{1}{16},\end{aligned}$$

the last line following because $3m^2 + m$ is even. Therefore the number of integers n with $f(n) = m$ is

$$\left\lfloor\left(m+\frac{1}{2}\right)^4\right\rfloor - \left\lfloor\left((m-1)+\frac{1}{2}\right)^4\right\rfloor = \left(m+\frac{1}{2}\right)^4 - \left(m-\frac{1}{2}\right)^4 = 4m^3+m.$$

Thus $f(n) = m$ for $4m^3 + m$ consecutive positive integers n. Now observe that $6^4 < 1995 < 7^4$, so that either $f(1995) = 6$ or $f(1995) = 7$. Because

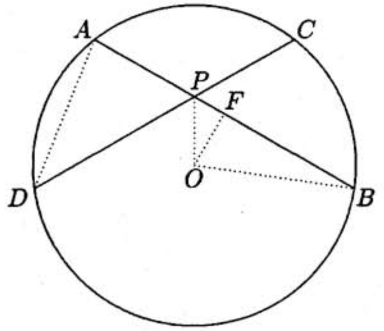
$$\sum_{m=1}^{6}(4m^3+m) = 1785,$$

it follows that $f(1786) = f(1787) = \cdots = f(1995) = 7$, hence that

$$\begin{aligned}\sum_{k=1}^{1995}\frac{1}{f(k)} &= \sum_{k=1}^{1785}\frac{1}{f(k)} + \frac{1995-1785}{7} \\ &= \sum_{m=1}^{6}\frac{4m^3+m}{m}+30 \\ &= \sum_{m=1}^{6}(4m^2+1)+30 = 400.\end{aligned}$$

14. (Answer: 378)
Let the chords be denoted by $\overline{AB}$ and $\overline{CD}$, and the intersection of the chords by P, where $AP \le BP$ and $CP \le DP$. Let O be the center of the circle, and F be the foot of the perpendicular from O to $\overline{AB}$. By the Pythagorean Theorem,

$$\begin{aligned} OF &= \sqrt{OB^2 - FB^2} = \sqrt{42^2 - 39^2} \\ &= \sqrt{(42+39)(42-39)} = 9\sqrt{3}. \end{aligned}$$



Because $OP = 18$, it follows that

$$FP = 9 \text{ and } \angle OPB = 60°,$$

hence that

$$BP = BF + FP = 39 + 9 = 48$$

and $AP = 30$. A similar argument shows that $\angle OPD = 60°$, $DP = 48$, and $CP = 30$. Because $\triangle DPB$ is isosceles and $\angle DPB = 120°$, it follows that inscribed angle ABD is $30°$, and that central angle AOD is $60°$. Thus $\triangle AOD$ is equilateral, with $AD = 42$. The desired area is the sum of the area of $\triangle APD$ and the area of the segment of the circle bounded by $\overline{AD}$ and minor arc AD. This is

$$\begin{aligned} &\text{Area}(\triangle APD) + [\text{Area}(\text{Sector } AOD) - \text{Area}(\triangle AOD)] \\ =&\frac{1}{2}(30)(48)\sin 60° + \left[\frac{1}{6}\pi(42)^2 - \frac{1}{2}(42)^2 \sin 60°\right] \\ =&294\pi - 81\sqrt{3}. \end{aligned}$$

Therefore, $m + n + d = 294 + 81 + 3 = 378$.

15. (Answer: 037)

A *successful string* is a sequence of H's and T's in which $HHHHH$ appears before TT does. Each successful string must belong to one of the following three types:

(i) those that begin with T, followed by a successful string that begins with H;

(ii) those that begin with H, HH, HHH, or $HHHH$, followed by a successful string that begins with T;

(iii) the string $HHHHH$.

Let p_H denote the probability of obtaining a successful string that begins with H, and let p_T denote the probability of obtaining a successful string that begins with T. It follows that

$$p_T = \frac{1}{2}p_H \qquad \text{and} \qquad p_H = \left(\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}\right)p_T + \frac{1}{32}.$$

Solving these equations simultaneously, we find that

$$p_H = \frac{1}{17} \qquad \text{and} \qquad p_T = \frac{1}{34}.$$

Thus the probability of obtaining five heads before obtaining 2 tails is

$$p = p_H + p_T = \frac{3}{34}$$

and $m + n = 37$.

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FOR STUDENTS AND TEACHERS

14th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
THURSDAY, March 28, 1996

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Professor Walter E Mientka, AMC Executive Director
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1. (Answer: 200)
The sum of the entries in the first row, which is $x+115$, equals the sum of the entries in the first column, hence the lower-left entry is 114. Because the sum of the entries in the diagonal that includes the upper-right corner is $x+115$, the central entry must be $x-95$. Because the sum of the entries in the second row is $x+115$, the last entry in that row must be 209. Because the sum of the entries in the third column is $x+115$, the last entry in that column must be $x-190$. This puts x, $x-95$, and $x-190$ on a diagonal. It follows that

$$x+(x-95)+(x-190)=x+115,$$

hence $x=200$.

2. (Answer: 340)
Because

$$\lfloor \log_2 n \rfloor = k \iff 2^k \le n < 2^{k+1},$$

in order for the integer k to be positive and even,

$$n \in \{\underbrace{4,5,6,7}_{4},\underbrace{16,17,\ldots,31}_{16},\underbrace{64,65,\ldots,127}_{64},\underbrace{256,257,\ldots,511}_{256}\},$$

so there are $4+16+64+256=340$ possible choices for n.

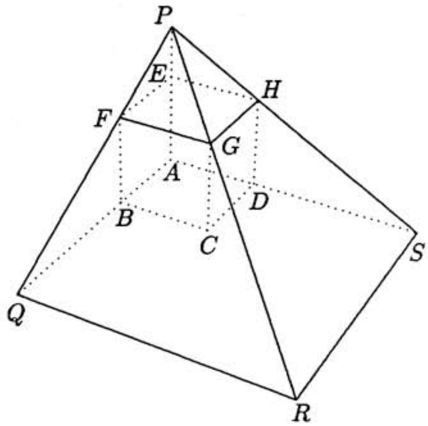
3. (Answer: 044)
Notice that

$$(xy-3x+7y-21)^n=(x+7)^n(y-3)^n.$$

The simplified expansions of $(x+7)^n$ and $(y-3)^n$ have $n+1$ terms each. When these two expansions are multiplied together, $(n+1)^2$ terms of the form cx^jy^k are produced. No two of these terms are like terms because they differ in at least one exponent. Hence the expansion of $(xy-3y+7y-21)^n$ has $(n+1)^2$ terms. To make $(n+1)^2 \ge 1996$, we need $n \ge \sqrt{1996}-1$. The smallest such integer n is 44.

4. (Answer: 166)

Let $ABCDEFGH$ be the cube, P be the point source of light, $PE = x$, and $EA = 1$. In the diagram at right, P, E, and A are collinear, and Q, R, and S are the intersections of the extensions of $\overline{PF}$, $\overline{PG}$, and $\overline{PH}$, respectively, with the plane of $ABCD$. Because the squares $EFGH$ and $ABCD$ are in parallel planes, it follows that pyramids $PEFGH$ and $PAQRS$ are similar. Therefore $AQRS$ is a square, and



$$\frac{AQ}{EF} = \frac{AP}{EP}.$$

Solve this equation to obtain $AQ = \dfrac{x+1}{x}$. The area of the shadow is 48, hence

$$\left(\frac{x+1}{x}\right)^2 - 1 = 48.$$

Thus $x = 1/6$ and $1000\,x = 166\frac{2}{3}$.

5. (Answer: 023)

The first equation implies that $a + b + c = -3$. The second equation implies that $t = -(a+b)(b+c)(c+a)$. It follows that $t = -(-3-c)(-3-a)(-3-b)$, which expands to $t = 27 + 9(a+b+c) + 3(ab+bc+ca) + abc$. The first equation implies that $ab + bc + ca = 4$ and $abc = 11$, hence that $t = 27 - 27 + 12 + 11 = 23$.

OR

The first equation implies that $a + b + c = -3$. It follows that the roots of the second equation are $-3-c$, $-3-a$, and $-3-b$. These are also the roots of the equation $(-x-3)^3 + 3(-x-3)^2 + 4(-x-3) - 11 = 0$, obtained by replacing x by $-x-3$ in the first equation. The leading coefficient of this equation is -1 and the constant term is $(-3)^3 + 3(-3)^2 + 4(-3) - 11 = -23$; thus $t = 23$.

6. (Answer: 049)
The five teams must play a total of $5 \cdot 4/2 = 10$ games, so there are $2^{10} = 1024$ possible outcomes for the tournament. Team A wins all four of its games in $2^{10-4} = 64$ of these outcomes. Because at most one team can be undefeated, there are $5 \cdot 64 = 320$ tournaments that produce an undefeated team. A similar argument shows that 320 of the 1024 possible tournaments produce a winless team. These possibilities are not mutually exclusive, however. In $2^{10-7} = 8$ of the tournaments, team A is undefeated and team B is winless, and there are $5 \cdot 4 = 20$ such two-team permutations. In other words, $8 \cdot 20 = 160$ of the 1024 tournaments have both an undefeated team and a winless team. Thus, according to the inclusion-exclusion principle, there are $1024 - 320 - 320 + 160 = 544$ tournament outcomes in which there is neither an undefeated nor a winless team. All outcomes are equally likely, hence the required probability is $544/1024 = 17/32$, and $17 + 32 = 49$.

7. (Answer: 300)
There are $\binom{49}{2} = 1176$ ways to select the positions of the yellow squares. Because quarter-turns can be applied to the board, however, there are fewer than 1176 inequivalent color schemes. Color schemes in which the two yellow squares are *not* diametrically opposed appear in four equivalent forms. Color schemes in which the two yellow squares *are* diametrically opposed appear in two equivalent forms, and there are $(49 - 1)/2 = 24$ such pairs of yellow squares. Thus the number of inequivalent color schemes is
$$\frac{1176 - 24}{4} + \frac{24}{2} = 300.$$

8. (Answer: 799)
Let $n = 6^{20}$. Suppose that x and y are positive integers for which
$$n = \frac{1}{\frac{1}{2}\left(\frac{1}{x} + \frac{1}{y}\right)} = \frac{2xy}{x + y}.$$
It follows that $xy - \frac{n}{2}x - \frac{n}{2}y = 0$, hence that
$$\left(x - \frac{n}{2}\right)\left(y - \frac{n}{2}\right) = \frac{n^2}{4} = \frac{6^{40}}{4} = 2^{38}3^{40}.$$
Because $2^{38}3^{40}$ has $39 \cdot 41 = 1599$ positive divisors, there are $\frac{1598}{2} = 799$ pairs of unequal positive integers whose product is $2^{38}3^{40}$, and therefore 799 ordered pairs (x, y) of the required type.

9. (Answer: 342)

Suppose that there are 2^k lockers in the row, and let L_k be the number of the last locker opened, once all the lockers are open. After the student makes his first pass along the row, there are 2^{k-1} closed lockers left. These closed lockers all have even numbers and are in descending order from where the student is standing. Now renumber the closed lockers from 1 to 2^{k-1}, starting from the end where the student is standing. Notice that the locker originally numbered n (where n is even) is now numbered $2^{k-1}+1-n/2$. Thus, because L_{k-1} is the number of the last locker opened with this new numbering, we have

$$L_{k-1} = 2^{k-1} + 1 - \frac{L_k}{2}.$$

Solving for L_k we find

$$L_k = 2^k + 2 - 2L_{k-1}.$$

Iterate this recursion once to obtain

$$L_k = 2^k + 2 - 2(2^{k-1} + 2 - 2L_{k-2}) = 4L_{k-2} - 2. \tag{1}$$

When there are $1024 = 2^{10}$ lockers to start with, the last locker to be opened is numbered L_{10}. Apply (1) repeatedly to $L_0 = 1$ to find that $L_2 = 4L_0 - 2 = 2$, $L_4 = 6$, $L_6 = 22$, $L_8 = 86$, and $L_{10} = 342$.

OR

Follow the given solution to the recursion (1), which can be written in the form

$$L_k - \frac{2}{3} = 4\left(L_{k-2} - \frac{2}{3}\right).$$

Because $L_0 = 1$ and $L_1 = 2$, it follows that

$$L_k - \frac{2}{3} = \begin{cases} \left(1 - \dfrac{2}{3}\right) 4^{k/2}, & \text{if } k \text{ is even,} \\ \left(2 - \dfrac{2}{3}\right) 4^{(k-1)/2}, & \text{if } k \text{ is odd.} \end{cases}$$

These formulas may be combined to yield

$$L_k = \frac{1}{3}\left(4^{\lfloor (k+1)/2 \rfloor} + 2\right)$$

for all nonnegative k. In particular, $L_{10} = 342$.

Query: How would the solution change if there were 1000 lockers in the hall?

10. (Answer: 159)
The identity

$$\frac{\cos A + \sin A}{\cos A - \sin A} = \frac{1 + \tan A}{1 - \tan A} = \frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \tan A} = \tan(45^\circ + A)$$

implies that the given equation is equivalent to $\tan 19x^\circ = \tan(45^\circ + 96^\circ) = \tan 141^\circ$. It follows that $19x$ must differ from 141 by a multiple of 180; that is,

$$19x = 141 + 180y = 19(7 + 9y) + (8 + 9y),$$

for some integer y. The smallest positive x corresponds to the smallest nonnegative y for which $8 + 9y = 19z$ for some positive integer z. Solve for y to obtain $y = 2z + \frac{z-8}{9}$, from which it follows that the minimum value for z is 8. Hence $y = 16$ and $x = 159$.

OR

Because $\sin 96^\circ = \cos 6^\circ$, the given equation is equivalent to

$$\tan 19x^\circ = \frac{\cos 96^\circ + \cos 6^\circ}{\cos 96^\circ - \cos 6^\circ}\ .$$

The identities

$$\cos(A + B) + \cos(A - B) = 2\cos A \cos B$$

and

$$\cos(A + B) - \cos(A - B) = -2\sin A \sin B$$

imply that

$$\frac{\cos 96^\circ + \cos 6^\circ}{\cos 96^\circ - \cos 6^\circ} = \frac{2\cos 51^\circ \cos 45^\circ}{-2\sin 51^\circ \sin 45^\circ} = -\frac{\sin 39^\circ}{\cos 39^\circ}\ .$$

It follows that the given equation is equivalent to

$$\tan 19x^\circ = -\tan 39^\circ = \tan 141^\circ.$$

The solution continues as above.

OR

Notice that $A\cos x + B\sin x$ is equivalent to $C\cos(x-\phi)$, where $C^2 = A^2 + B^2$, $A = C\cos\phi$, and $B = C\sin\phi$. Hence the given equation is equivalent to

$$\tan 19x^\circ = \frac{\sqrt{2}\cos(96^\circ - 45^\circ)}{\sqrt{2}\cos(96^\circ + 45^\circ)} = \frac{\cos 51^\circ}{\cos 141^\circ} = \frac{\sin 141^\circ}{\cos 141^\circ} = \tan 141^\circ.$$

The solution continues as before.

11. (Answer: 276)
Divide both sides of the given equation by z^3, which gives $z^3 + z + 1 + z^{-1} + z^{-3} = 0$. This takes the form $w^3 - 2w + 1 = 0$, where $w = z + z^{-1}$. Factor the cubic polynomial to obtain $(w-1)(w^2+w-1) = 0$. Now replace w by $z + z^{-1}$ and multiply both sides of the equation by z^3. This yields

$$0 = (z^2 - z + 1)(z^4 + z^3 + z^2 + z + 1) = \frac{z^3+1}{z+1} \cdot \frac{z^5-1}{z-1}.$$

It follows that the six values for z are the fifth roots of 1 and the cube roots of -1, with the exception of 1 and -1. These roots may be written in polar form $\cos\phi^\circ + i\sin\phi^\circ$, where ϕ takes on the following values: $72, 144, 216, 288, 60, 300$. The roots with positive imaginary part have ϕ-values 72, 144, and 60. The product of these roots is $\cos\theta^\circ + i\sin\theta^\circ$, where $\theta = 72 + 144 + 60 = 276$.

Note: This solution illustrates a general method for solving symmetric equations of degree $2n$, by reducing them to equations of degree n. In this example, it is even possible to find non-trigonometric formulas for the roots, by repeated use of the quadratic formula. In particular, the roots of $w^2 + w - 1 = 0$ are $w = \frac{1}{2}(-1 \pm \sqrt{5})$, and the four z-values from the ensuing equation $z + z^{-1} = w$ are fifth roots of 1. They are $z = \frac{1}{4}\left(-1 - \sqrt{5} \pm i\sqrt{10 - 2\sqrt{5}}\right)$ and $z = \frac{1}{4}\left(-1 + \sqrt{5} \pm i\sqrt{10 + 2\sqrt{5}}\right)$. These formulas for fifth roots imply the ruler-and-compass constructibility of a regular pentagon.

OR

Observe that

$$\begin{aligned} z^6 + z^4 + z^3 + z^2 + 1 &= z^6 - z + (z^4 + z^3 + z^2 + z + 1) \\ &= z(z^5 - 1) + \frac{z^5-1}{z-1} \\ &= (z^5-1)\frac{z^2 - z + 1}{z-1} \\ &= \frac{z^5-1}{z-1} \cdot \frac{z^3+1}{z+1}. \end{aligned}$$

The solution continues as above.

12. (Answer: 058)
Consider the average of all sums of the form

$$|a_1 - a_2| + |a_3 - a_4| + \cdots + |a_{n-1} - a_n|,$$

where n is even and $(a_1, a_2, a_3, \ldots, a_n)$ is a permutation of $(1, 2, 3, \ldots, n)$. Each of the $n!$ sums contains $n/2$ differences of pairs of integers. There are $\binom{n}{2}$ such pairs. For each $k = 1, 2, \ldots, n-1$, there are $n-k$ of these $\binom{n}{2}$ pairs with difference k. Because each of these pairs occurs the same number of times in the $n!$ sums, the average of the differences of all $\frac{n}{2}n!$ pairs is

$$\frac{1}{\binom{n}{2}} \sum_{k=1}^{n-1} k(n-k).$$

Because $k(n-k)$ is the number of subsets $\{a, k+1, b\}$ of $\{1, 2, \ldots, n+1\}$ that have $a < k+1 < b$, it follows that

$$\sum_{k=1}^{n-1} k(n-k) = \binom{n+1}{3}.$$

The average difference is therefore $\dfrac{\binom{n+1}{3}}{\binom{n}{2}} = \dfrac{n+1}{3}$. The average sum of $n/2$ differences is $\dfrac{n(n+1)}{6}$, which equals $55/3$ when $n = 10$. Thus $p + q = 58$.

Note: When $n = 10$, it is easy to calculate the value of $\sum_{k=1}^{n-1} k(n-k)$ directly.

OR

The average is just 5 times the average value of $|a_1 - a_2|$, because the average value of $|a_{2i-1} - a_{2i}|$ is the same for $i = 1, 2, 3, 4, 5$. When $a_1 = k$, the average value of $|a_1 - a_2|$ is

$$\frac{(k-1) + (k-2) + \cdots + 1 + 1 + 2 + \cdots + (10-k)}{9}$$
$$= \frac{1}{9}\left[\frac{k(k-1)}{2} + \frac{(10-k)(11-k)}{2}\right] = \frac{k^2 - 11k + 55}{9}.$$

Thus the average value of the sum is

$$5 \cdot \frac{1}{10} \sum_{k=1}^{10} \frac{k^2 - 11k + 55}{9} = \frac{55}{3},$$

and so $p + q = 58$.

13. (Answer: 065)
Let $AB = c$, $AC = b$, $BC = a$, and notice that $a^2 + b^2 < c^2$. It follows that $\angle C$ is obtuse and that D lies outside $\triangle ABC$. It is given that line AD intersects $\overline{BC}$ at E, the midpoint of $\overline{BC}$. Notice that $\overline{BD}$ is the altitude from B in $\triangle ABE$. Thus

$$\frac{\text{Area}(\triangle ADB)}{\text{Area}(\triangle ABC)} = \frac{\text{Area}(\triangle ADB)}{2\,\text{Area}(\triangle ABE)} = \frac{\frac{1}{2}(BD)(AD)}{2\cdot\frac{1}{2}(BD)(AE)} = \frac{AD}{2\,AE}. \tag{1}$$

To find AE, apply the Law of Cosines twice to obtain

$$b^2 = (AE)^2 + \left(\frac{a}{2}\right)^2 - 2(AE)\left(\frac{a}{2}\right)\cos\angle CEA$$

and

$$c^2 = (AE)^2 + \left(\frac{a}{2}\right)^2 - 2(AE)\left(\frac{a}{2}\right)\cos\angle AEB.$$

Now add these two equations, using the fact that $\cos\angle CEA + \cos\angle AEB = 0$, and solve for AE. The result is

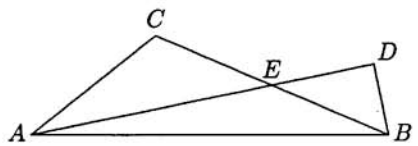
$$AE = \frac{1}{2}\sqrt{2b^2 + 2c^2 - a^2}.$$

Apply the Pythagorean Theorem to find that

$$(AD)^2 + (BD)^2 = c^2$$

and

$$(DE)^2 + (BD)^2 = \tfrac{1}{4}a^2.$$



Substitute $AD = AE + ED$ and subtract to find $(AE)^2 + 2(AE)(ED) = c^2 - \frac{1}{4}a^2$.

Thus

$$\frac{ED}{2\,AE} = \frac{c^2 - \frac{1}{4}a^2 - (AE)^2}{4(AE)^2} = \frac{c^2 - b^2}{2(2b^2 + 2c^2 - a^2)}.$$

Return to (1) to find that

$$\frac{\text{Area}(\triangle ADB)}{\text{Area}(\triangle ABC)} = \frac{AE + ED}{2\,AE} = \frac{1}{2} + \frac{ED}{2\,AE}.$$

When $a = \sqrt{15}$, $b = \sqrt{6}$, and $c = \sqrt{30}$, the desired ratio of areas is $\dfrac{1}{2} + \dfrac{4}{19} = \dfrac{27}{38}$. Hence $m + n = 65$.

14. (Answer: 768)

Let the rectangular solid have width w, length l, and height h, where w, l, and h are positive integers. We will show that the diagonal passes through the interiors of

$$w + l + h - \gcd(w, l) - \gcd(l, h) - \gcd(h, w) + \gcd(w, l, h)$$

of the $1 \times 1 \times 1$ cubes.

Orient the solid in 3-space so that one vertex is at $O = (0, 0, 0)$ and another is at $A = (w, l, h)$. Then $\overline{OA}$ is a diagonal of the solid. Let $P = (x, y, z)$ be a point on this diagonal. Exactly *one* of x, y, z is an integer if and only if P is interior to a face of one of the small cubes. Exactly *two* of x, y, z are integers if and only if P is interior to an edge of one of the small cubes. All *three* of x, y, z are integers if and only if P is a vertex of one of the small cubes. As P moves along the diagonal from O to A, it leaves the interior of a small cube precisely when *at least one* of the coordinates of P is a positive integer. Thus the number of interiors of small cubes through which the diagonal passes is equal to the number of points on the diagonal with at least one positive integer coordinate. Points with positive coordinates on the diagonal $\overline{OA}$ have the form

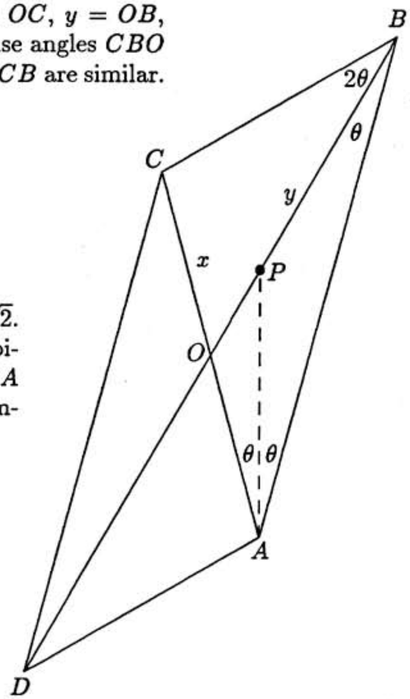
$$P = (wt, lt, ht) \qquad \text{with } 0 < t \leq 1.$$

The first coordinate, wt, will be a positive integer for w values of t, namely for the values $t = 1/w, 2/w, 3/w, \ldots, w/w$. The second coordinate will be an integer for l values of t, and the third coordinate will be an integer for h values of t. The sum $w + l + h$ doubly counts the points with two integer coordinates, however, and it triply counts the points with three integer coordinates. The first two coordinates will be positive integers precisely when t has the form $k/\gcd(w, l)$, for some positive integer k between 1 and $\gcd(w, l)$, inclusive. A similar argument shows that the second and third coordinates will be positive integers for $\gcd(l, h)$ values of t, the third and first coordinates will be positive integers for $\gcd(h, w)$ values of t, and all three will be positive integers for $\gcd(w, l, h)$ values of t. By the inclusion-exclusion principle, P will have one or more positive integer coordinates

$$w + l + h - \gcd(w, l) - \gcd(l, h) - \gcd(h, w) + \gcd(w, l, h)$$

times, which gives 768 when $\{w, l, h\} = \{150, 324, 375\}$.

15. (Answer: 777)



The given data allows us to label $x = OA = OC$, $y = OB$, $\theta = \angle OBA$, and $2\theta = \angle OAB = \angle OBC$. Because angles CBO and CAB are congruent, triangles BCO and ACB are similar. Thus

$$\frac{CB}{CA} = \frac{CO}{CB} = \frac{OB}{BA},$$

or

$$\frac{CB}{2x} = \frac{x}{CB} = \frac{y}{BA}.$$

It follows that $CB = x\sqrt{2}$, and that $BA = y\sqrt{2}$. Now let P be the intersection of $\overline{OB}$ with the bisector of $\angle OAB$. Because angles OAP and OBA are congruent, triangles OPA and OAB are similar. Thus

$$\frac{AP}{BA} = \frac{OA}{OB} = \frac{OP}{OA}.$$

The first equation yields $AP = x\sqrt{2}$. Because $AP = PB$, the second equation yields

(1) $$y^2 - x^2 = xy\sqrt{2}.$$

Apply the Law of Cosines to triangle COB to find that

$$\cos 3\theta = \frac{x^2 + y^2 - (x\sqrt{2})^2}{2xy} = \frac{y^2 - x^2}{2xy},$$

which is $\sqrt{2}/2$, by equation (1). In other words, $3\theta = 45°$, so $\theta = 15°$. It follows that $\angle ACB = 105°$ and $\angle AOB = 135°$, so $r = 105/135 = 7/9 = 0.\overline{7}$ and $1000\,r = 777.\overline{7}$.

OR

Apply the Law of Sines to triangles BOC and ABC to find that

$$BC = \frac{OC \sin 3\theta}{\sin 2\theta} \quad \text{and} \quad BC = \frac{AC \sin 2\theta}{\sin 3\theta},$$

respectively. Because $2 \cdot OC = AC$, it follows that $\sin^2 3\theta = 2\sin^2 2\theta$. Now use the identities $\sin 2\theta = 2\sin\theta\cos\theta$ and $\sin 3\theta = \sin\theta(4\cos^2\theta - 1)$ to produce the equation $(4\cos^2\theta - 1)^2 = 8\cos^2\theta$, then use the identity $2\cos^2\theta = 1 + \cos 2\theta$ to reduce it to $(1 + 2\cos 2\theta)^2 = 4 + 4\cos 2\theta$. This is equivalent to $4\cos^2 2\theta = 3$, hence $\cos 2\theta = \pm\frac{1}{2}\sqrt{3}$. Because it is clear that 2θ is acute, only $\theta = 15°$ is a possibility. Thus $r = 105/135 = 7/9$ and $1000\,r = 777.\overline{7}$.

AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

15th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)

THURSDAY, March 20, 1997

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs. geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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1. (Answer: 750)
A difference of integer squares $a^2 - b^2$ can be factored as $(a+b)(a-b)$, the product of two integers of the same parity. Thus, for an even integer to be a difference of integer squares, it must be a product of two even integers, and therefore must be divisible by 4. Conversely, any integer multiple of 4 may be expressed as the difference of two integer squares, because $4n = (n+1)^2 - (n-1)^2$. Any odd number may be expressed as the difference of two integer squares, because $2n+1 = (n+1)^2 - n^2$. In all, exactly 750 of the integers from 1 to 1000 are differences of integer squares.

2. (Answer: 125)
Notice that rectangles must be formed by choosing two distinct vertical lines and two distinct horizontal lines. Because there are nine vertical lines and nine horizontal lines, the total number of rectangles is $\binom{9}{2}^2 = 1296$. Of these, 64 are 1×1 squares, 49 are 2×2 squares, and, in general, $(9-j)^2$ are $j \times j$ squares, because each square is determined by its size and the position of its upper-left corner. Hence the total number of squares on the 8×8 checkerboard is $64+49+36+25+16+9+4+1 = 204$. Thus
$$\frac{s}{r} = \frac{204}{1296} = \frac{17}{108},$$
and $m+n = 125$.

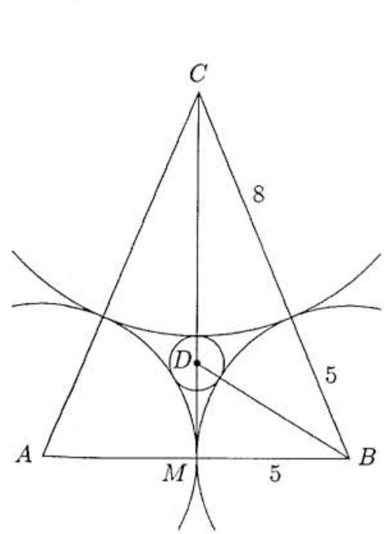
3. (Answer: 126)
Let x and y be the two- and three-digit numbers, respectively. It is given that the juxtaposition is nine times as large as the intended product, hence
$$1000x + y = 9xy, \text{ or}$$
$$\frac{y}{9y - 1000} = x.$$
Because $9y - 1000$ must be positive, the smallest possible value for y is 112. Because x must be at least 10, it follows that $10(9y-1000) \le y$, hence that $y \le \frac{10000}{89} < 113$. Thus y can only be 112, and x must be 14, so $x + y = 126$.

4. (Answer: 017)

Let A, B, C, and D be the centers of the circles of radii 5, 5, 8, and $r = m/n$, respectively. It is clear that C lies outside the strip bounded by the parallel lines that are tangent to both of the circles of radius 5. Thus D lies inside triangle ABC, as in the figure at right. Notice that, because the centers of two tangent circles are collinear with their point of tangency, $AB = 10$, $CA = CB = 13$, and $DB = 5 + r$. Because $DA = DB$, D is on the perpendicular bisector $\overline{MC}$ of $\overline{AB}$, where M is the midpoint of $\overline{AB}$. Apply the Pythagorean Theorem to triangle CMB to find that $CM = 12$, hence that $DM = CM - CD = 12 - (8 + r) = 4 - r$. Now apply the Pythagorean Theorem again, this time to triangle DMB, to find that

$$(5 + r)^2 = 5^2 + (4 - r)^2.$$

The solution to this equation is $r = \dfrac{8}{9}$, hence $m + n = 8 + 9 = 17$.



Query. Given three mutually externally tangent circles, whose radii are $r_1 \le r_2 \le r_3$, under what conditions will there be more than one circle that is externally tangent to all three circles?

5. (Answer: 417)

Let S be the set of real numbers that can be written as fractions whose numerators are 1 or 2 and whose denominators are integers. Because $\frac{1}{4}$ is the greatest element of S that is less than $\frac{2}{7}$, and $\frac{1}{3}$ is the least element of S that is greater than $\frac{2}{7}$, we must find the number of values for r that are closer to $\frac{2}{7}$ than to $\frac{1}{3}$ or $\frac{1}{4}$. Because r can be expressed as a four-place decimal, the inequality

$$\frac{\frac{1}{4} + \frac{2}{7}}{2} < r < \frac{\frac{2}{7} + \frac{1}{3}}{2}$$

implies that $0.2679 \le r \le 0.3095$. Thus there are $3095 - 2679 + 1 = 417$ possible values for r.

Query. The length of the interval $\frac{15}{56} < r < \frac{13}{42}$ is $\frac{1}{24}$. How many four-place decimals might there be in an arbitrary interval of length $\frac{1}{24}$?

6. (Answer: 042)

Let m be the number of sides of the polygon determined by A_n, A_1, and B. The degree measures of the interior angles of the three polygons are $180 - \frac{360}{n}$, 60, and $180 - \frac{360}{m}$. If $6 < n$, the polygons fit together at their common vertex A_1, thus

$$360 = 180 - \frac{360}{n} + 60 + 180 - \frac{360}{m}.$$

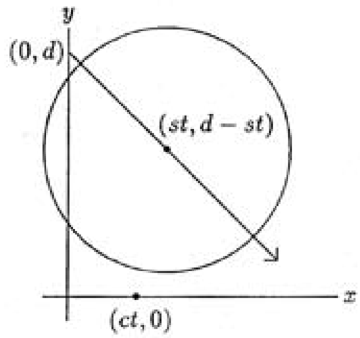
This can be rewritten in the form

$$n = \frac{6m}{m-6} = 6 + \frac{36}{m-6}.$$

It is clear that $m > 6$, so that n is a decreasing function of m. The largest value of n is 42, obtained when $m = 7$.

7. (Answer: 198)

Let r be the radius of the storm, d be the distance from the center of the storm to the car at time $t = 0$, c be the speed of the car, and $s\sqrt{2}$ be the speed of the storm. Set up a coordinate system so that, at $t = 0$, the storm center is at $(0, d)$, and the car is at the origin, moving along the positive x-axis. At any time t, the car is at $(ct, 0)$ and the storm center is at $(st, d - st)$. When the car is entering or leaving the storm circle, the distance between these two points is r. In other words, t_1 and t_2 are the solutions to



$$(ct - st)^2 + (d - st)^2 = r^2,$$

which can be rewritten as $((c-s)^2 + s^2)t^2 - (2ds)t + (d^2 - r^2) = 0$. The sum of the two roots of this quadratic equation is

$$t_1 + t_2 = \frac{2ds}{(c-s)^2 + s^2}.$$

Now use the given data $c = \frac{2}{3}$, $d = 110$, and $s = \frac{1}{2}$ to find that

$$\frac{t_1 + t_2}{2} = \frac{ds}{(c-s)^2 + s^2} = \frac{110 \cdot \frac{1}{2}}{\left(\frac{2}{3} - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2} = 198.$$

The discriminant $4d^2s^2 - 4(d^2 - r^2)((c-s)^2 + s^2)$ of the quadratic equation is nonnegative in this case, therefore t_1 and t_2 are real. The answer is otherwise independent of r.

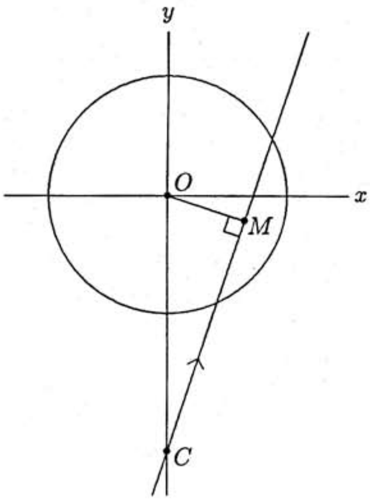
OR

Make the storm center the origin O of a coordinate system that has its positive x-axis pointing east. At time $t = 0$, the car is at $C = (0, -110)$. Each minute thereafter, its position relative to the storm center is shifted by the velocity vector

$$(2/3, 0) - (1/2, -1/2) = (1/6, 1/2),$$

due to the combined motion of the car and storm, respectively. The resulting linear path of the car intersects the storm along a chord of the circle, whose midpoint M is reached at time $t = \frac{1}{2}(t_1 + t_2)$. Because angle CMO is right, and $\tan \angle OCM$ is $\frac{1}{3}$, it follows that $CM = CO \cos \angle OCM = 110 \cdot \frac{3}{\sqrt{10}} = 33\sqrt{10}$. To find the time it takes for the car to reach M, divide this distance by the relative speed of the car, which is $\frac{1}{6}\sqrt{10}$. It follows that

$$\frac{1}{2}(t_1 + t_2) = \frac{33\sqrt{10}}{\frac{1}{6}\sqrt{10}} = 198.$$



8. (Answer: 090)
Each row and each column must contain two 1's and two -1's, so there are $\binom{4}{2} = 6$ ways to fill the first row. There are also six ways to fill the second row. Of these, one way has four matches with the first row, four ways have two matches with the first row, and one way has no matches with the first row. The first case allows one way to fill the third row, the second case allows two ways to fill the third row, and the third case allows six ways to fill the third row. Once the first three rows are filled, the fourth row can be filled in only one way. There are thus $6(1 \cdot 1 + 4 \cdot 2 + 1 \cdot 6) = 90$ ways to fill the array to satisfy the conditions.

9. (Answer: 233)

Notice first that the given data imply that $\langle a^{-1} \rangle = a^{-1}$ and $\langle a^2 \rangle = a^2 - 2$. Hence a must satisfy the equation $a^{-1} = a^2 - 2$, or $a^3 - 2a - 1 = 0$. This factors as

$$(a+1)(a^2 - a - 1) = 0,$$

whose only positive root is $a = \frac{1}{2}(1 + \sqrt{5})$. Now use the relations $a^2 = a + 1$ and $a^3 = 2a + 1$ to calculate

$$\begin{aligned} a^6 &= 8a + 5, \\ a^{12} &= 144a + 89, \\ a^{13} &= 233a + 144, \end{aligned}$$

from which it follows that $a^{12} - 144a^{-1} = \dfrac{a^{13} - 144}{a} = 233.$

OR

As above, show that $a = \frac{1}{2}(1 + \sqrt{5})$. The cubic equation yields $a^3 = 2a + 1 = 2 + \sqrt{5}$. It follows that $a^6 = 9 + 4\sqrt{5}$ and $a^{12} = 161 + 72\sqrt{5}$. The cubic equation also implies that $144a^{-1} = 144(a^2 - 2) = -72 + 72\sqrt{5}$, hence $a^{12} - 144a^{-1} = 233$.

OR

As above, $a = (1 + \sqrt{5})/2$. Let $b = -a^{-1}$ and use Binet's formula for Fibonacci numbers to calculate

$$F_n = \frac{a^n - b^n}{\sqrt{5}} = \frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + \cdots + ab^{n-2} + b^{n-1}.$$

Thus

$$F_n = a^{n-1} + F_{n-1}b = a^{n-1} - F_{n-1}a^{-1}.$$

In particular,

$$233 = F_{13} = a^{12} - F_{12}a^{-1} = a^{12} - 144a^{-1}.$$

10. (Answer: 117)
Consider any pair of cards from the deck. We show that there is exactly one card that can be added to this pair to make a complementary set. If the cards in the pair have the same shape, then the third card must also have this shape, while if the cards have different shapes, then the third card must have the one shape that differs from them. In either case, the shape on the third card is uniquely determined. Similar reasoning shows that the color and the shade on the third card are also uniquely determined. Thus we can count the number of complementary sets by counting the number of pairs of cards and then dividing by 3, because each complementary set is counted three times by this procedure. The number of complementary sets is

$$\frac{1}{3}\binom{27}{2} = \frac{1}{3} \cdot \frac{27 \cdot 26}{2} = 117.$$

11. (Answer: 241)
Because $\sin n^\circ + \cos n^\circ = \sqrt{2}\cos(45-n)^\circ$, it follows that

$$\sum_{n=1}^{44} \sin n^\circ + \sum_{n=1}^{44} \cos n^\circ = \sum_{n=1}^{44} \sqrt{2}\cos(45-n)^\circ = \sum_{n=1}^{44} \sqrt{2}\cos n^\circ.$$

Thus

$$\sum_{n=1}^{44} \sin n^\circ = (\sqrt{2}-1)\sum_{n=1}^{44} \cos n^\circ,$$

which yields $x = 1+\sqrt{2}$ and $\lfloor 100x \rfloor = 241$.

12. (Answer: 058)

The statement implies that f is its own inverse. The inverse may be found by solving $x = \dfrac{ay+b}{cy+d}$ for y. This yields $f^{-1}(x) = \dfrac{dx-b}{-cx+a}$. Because the nonzero numbers a, b, c, and d must therefore be proportional to $-d$, b, c, and $-a$, respectively, it follows that $a = -d$, hence that $f(x) = \dfrac{ax+b}{cx-a}$. The conditions $f(19) = 19$ and $f(97) = 97$ lead to the equations

$$19^2 c = 2 \cdot 19a + b$$
$$97^2 c = 2 \cdot 97a + b.$$

Thus $(97^2 - 19^2)c = 2(97-19)a$, from which follows $a = 58c$, which in turn leads to $b = -1843c$. This determines

$$f(x) = \frac{58x - 1843}{x - 58} = 58 + \frac{1521}{x-58},$$

which never has the value 58.

OR

The number that is not in the domain of f is $x = -d/c$, and the number that is not in the range of f is $y = a/c$. Because $f = f^{-1}$, it follows that $d = -a$. The fixed points of f satisfy

$$ax + b = x(cx+d), \qquad \text{or} \qquad cx^2 - (a-d)x - b = 0.$$

It follows that the sum of the two fixed points is $(a-d)/c = 2a/c$, which is twice the number omitted from the range. Because the fixed points are given as 19 and 97, the number omitted from the range is $(19+97)/2=58$.

OR

The equation $y = \dfrac{ax+b}{cx+d}$ is equivalent to the equation $xy - \dfrac{a}{c}x + \dfrac{d}{c}y = \dfrac{b}{c}$, hence to the equation $\left(x + \dfrac{d}{c}\right)\left(y - \dfrac{a}{c}\right) = \dfrac{bc-ad}{c^2}$. Notice that $bc - ad$ cannot be zero, for this would imply that $f(x) = \dfrac{a}{c}$ for all values of x except $-\dfrac{d}{c}$, contrary to the problem statement. The graph of $y = f(x)$ is therefore a hyperbola, whose center is $\left(\dfrac{-d}{c}, \dfrac{a}{c}\right)$. The statement implies that the line $y = x$ is an axis of symmetry. Because the hyperbola intersects this axis at (19,19) and at (97,97), these points are the vertices of the hyperbola, and its center is midway between them. It follows that $\dfrac{-d}{c} = \dfrac{a}{c} = \dfrac{19+97}{2} = 58$, which is the value omitted from the range of f (and also from the domain of f).

13. (Answer: 066)

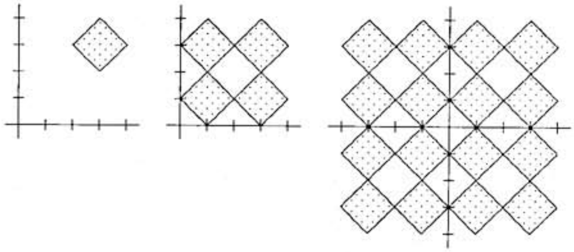
Making use of symmetry, graph the part in the first quadrant and then reflect this in the coordinate axes. In the first quadrant, the defining equation simplifies to

$$\Big||x-2|-1\Big| + \Big||y-2|-1\Big| = 1.$$

Again making use of symmetry, graph the part in the region $2 \le x$, $2 \le y$ and then reflect this in the lines $x = 2$ and $y = 2$. In this region, the equation simplifies further to

$$|x-3| + |y-3| = 1,$$

the graph of which is a square, whose vertices are (3,2), (4,3), (3,4), and (2,3), and whose perimeter is $4\sqrt{2}$.



Reflection in the line $x = 2$ and then in the line $y = 2$ produces a set of squares for which the required length of wire is $4 \cdot 4\sqrt{2} = 16\sqrt{2}$, as shown in the middle figure. Reflection in the coordinate axes then produces a set of squares for which the required length of wire is $4 \cdot 16\sqrt{2} = 64\sqrt{2}$, as shown in the third figure. Thus $a + b = 66$.

14. (Answer: 582)

Because the 1997 roots of the equation are symmetrically distributed in the complex plane, it is no loss of generality to assume that $v = 1$. Let $w = \cos\theta + i\sin\theta$, with $-180° < \theta < 180°$. It is required to find the probability that

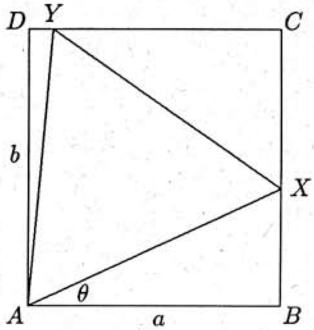
$$|1+w|^2 = |(1+\cos\theta) + i\sin\theta|^2 = 2 + 2\cos\theta \ge 2 + \sqrt{3},$$

which is equivalent to $\cos\theta \ge \frac{1}{2}\sqrt{3}$. Thus $|\theta| \le 30°$. Because $w \ne 1$, the only possible values of θ are

$$\theta = \pm\frac{360°}{1997}, \ \pm\frac{720°}{1997}, \ \pm\frac{1080°}{1997}, \ldots, \ \pm\frac{360k°}{1997},$$

where $k = \lfloor 1997/12 \rfloor = 166$. Hence the probability is $2 \cdot 166/1996 = 83/499$, and $m + n = 83 + 499 = 582$.

15. (Answer: 554)



Let us generalize the problem slightly. Label the rectangle $\mathcal{R} = ABCD$ so that $AB = CD = a$ and $BC = DA = b$. We first show that, given any equilateral triangle $\mathcal{T}$ in $\mathcal{R}$, there is an equilateral triangle in $\mathcal{R}$ that has the same area as $\mathcal{T}$ and that has one vertex at a vertex of $\mathcal{R}$. Given a side of $\mathcal{R}$, find the vertex of $\mathcal{T}$ that is closest to the side, then draw a line through the vertex parallel to that side. When this is done for all four sides of $\mathcal{R}$, we have a rectangle $\mathcal{U}$ that encloses $\mathcal{T}$. One vertex of $\mathcal{T}$ must coincide with a vertex of $\mathcal{U}$, because each side of $\mathcal{U}$ passes through a vertex of $\mathcal{T}$. Keeping rectangle $\mathcal{U}$ inside rectangle $\mathcal{R}$, we can slide $\mathcal{U}$ and $\mathcal{T}$ so that this vertex of $\mathcal{T}$ coincides with a vertex of $\mathcal{R}$. It is therefore no loss of generality to assume that $\mathcal{T}$ has a vertex at A.

If an equilateral triangle AXY exists that has X on $\overline{BC}$ and Y on $\overline{CD}$, then AXY has maximal area among all equilateral triangles that lie inside $\mathcal{R}$. The reason is that any other equilateral triangle with a vertex at A either must have a vertex that lies in or on triangle ADY or must have a vertex that lies in or on triangle ABX; this implies a smaller area.

To see that such a triangle exists, and to find its area, let X and Y be points on $\overline{BC}$ and $\overline{CD}$, respectively, such that $\angle XAY$ is a 60-degree angle. Let $\theta = \angle BAX$, so that $AX = a\sec\theta$ and $AY = b\sec(30^\circ - \theta)$. Triangle AXY is equilateral if and only if $AX = AY$, which is equivalent to $a\cos(30^\circ - \theta) = b\cos\theta$, which, according to the subtraction law for cosines, is equivalent to $a\sqrt{3}\cos\theta + a\sin\theta = 2b\cos\theta$. That is, $\tan\theta = \dfrac{2b}{a} - \sqrt{3}$, which lies between $\tan 0^\circ$ and $\tan 30^\circ = \dfrac{1}{\sqrt{3}}$ if and only if $\dfrac{a}{b}$ lies between $\dfrac{\sqrt{3}}{2}$ and $\dfrac{2}{\sqrt{3}}$. Because $\{a, b\} = \{10, 11\}$, equilateral triangle AXY exists. Furthermore, the area of triangle AXY is

$$\frac{\sqrt{3}}{4}AX^2 = \frac{\sqrt{3}}{4}a^2\sec^2\theta = \frac{\sqrt{3}}{4}a^2\left(1 + \tan^2\theta\right) = \frac{\sqrt{3}}{4}a^2\left[1 + \left(\frac{2b}{a} - \sqrt{3}\right)^2\right],$$

which simplifies to $(a^2 + b^2)\sqrt{3} - 3ab = 221\sqrt{3} - 330$. Hence $p + q + r = 554$.

AMERICAN MATHEMATICS COMPETITIONS

AIME SOLUTIONS PAMPHLET
FOR STUDENTS AND TEACHERS

16th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)

TUESDAY, March 17, 1998

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1. (Answer: 025)
The prime factorizations of the given integers are

$$6^6 = 2^6 3^6, \quad 8^8 = 2^{24}, \quad \text{and} \quad 12^{12} = 2^{24} 3^{12}.$$

Because 2 and 3 are the only prime factors of the least common multiple of the three numbers, $k = 2^m 3^n$ for some nonnegative integers m and n. In order for the least common multiple of $2^6 3^6$, 2^{24}, and $2^m 3^n$ to be $2^{24} 3^{12}$, n must be 12, and m can be any integer from 0 to 24, inclusive. Thus there are 25 acceptable values of k.

2. (Answer: 480)
Points that have integer coordinates are called *lattice points*. The lattice points in question lie within the square defined by $1 \le x \le 30$ and $1 \le y \le 30$. The only lattice points that are *not* included are those for which $2y < x$ or $2x < y$. For positive x and y, these conditions are mutually exclusive. Within the square, the inequality $2y < x$ cannot hold for $y \ge 15$. For each integer y between 1 and 14, inclusive, there are $30 - 2y$ such points that satisfy $2y < x$, namely those for which $2y + 1 \le x \le 30$. The number of points that satisfy $2x < y$ is the same as the number of points that satisfy $2y < x$. The total number of omitted points is therefore

$$2(2 + 4 + \cdots + 28) = 4(1 + 2 + 3 + \cdots + 14) = 420,$$

making the answer $900 - 420 = 480$.

OR

The conditions in the problem can be expressed as $1 \le x \le 30$, $y/2 \le x \le 2y$, and $1 \le y \le 30$. For each value of y from 1 to 15, x must be between $\lceil \frac{y}{2} \rceil$ and $2y$, inclusive, so there are $2y - \lceil \frac{y}{2} \rceil + 1$ values of x. (The value $\lceil r \rceil$ of the *ceiling function* is the smallest integer that is not less than r.) For each value of y from 16 to 30, similar reasoning shows that there are $30 - \lceil \frac{y}{2} \rceil + 1$ values of x. The number of ordered pairs is thus

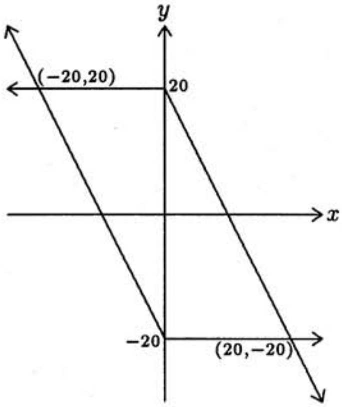
$$\sum_{y=1}^{15} \left(2y - \left\lceil \frac{y}{2} \right\rceil + 1\right) + \sum_{y=16}^{30} \left(30 - \left\lceil \frac{y}{2} \right\rceil + 1\right)$$

$$= \left(\sum_{y=1}^{15} 2y\right) + 15 + 15 \cdot 31 - \sum_{y=1}^{30} \left\lceil \frac{y}{2} \right\rceil$$

$$= 2\sum_{y=1}^{15} y + 480 - 2\sum_{y=1}^{15} y$$

$$= 480.$$

3. (Answer: 800)
For nonnegative values of x, the equation can be written $y^2 - 400 + 2x(y+20) = 0$, or
$$(y - 20 + 2x)(y + 20) = 0.$$
For nonnegative x, the graph thus consists of two rays. For nonpositive values of x, the equation can be written $y^2 - 400 + 2x(y-20) = 0$, or
$$(y + 20 + 2x)(y - 20) = 0.$$
This gives two more rays. The bounded region is therefore enclosed by a parallelogram whose height is 40, whose base is 20, and whose area is 800.

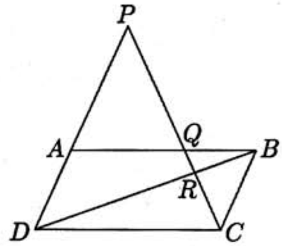


Note: Because $(-x,-y)$ satisfies the given equation if and only if (x,y) does, the graph is symmetric with respect to the origin. The requested area is thus twice the area of the triangle enclosed by the y-axis and the two rays found for nonnegative x.

4. (Answer: 017)
Each player must select an odd number of odd-numbered tiles. Because there are five odd-numbered tiles available, one player must select three of them, and the other two players must select one each. The probability that the first player selects three odd-numbered tiles is $10/84$, for there are $\binom{9}{3} = 84$ ways to select three tiles from the nine available, and there are $\binom{5}{3} = 10$ ways to select three odd-numbered tiles from the five available. Given that this event has occurred, the probability that the second player will choose exactly one odd-numbered tile is $12/20$, for there are $\binom{6}{3} = 20$ ways to select three tiles from the six that remain, and there are $\binom{2}{1}\binom{4}{2} = 12$ ways to select one odd-numbered and two even-numbered tiles. Given that the first two players have each selected an odd number of odd-numbered tiles, the third is sure to do the same. Because any player can be the one who selects three odd-numbered tiles, the desired probability is $3(10/84)(12/20) = 3/14$, so $m + n = 17$.

5. (Answer: 040)
Notice that $\dfrac{k(k-1)}{2}$ is even when $k = 4m$ or $k = 4m+1$, and odd otherwise. It follows that
$$A_{4m-1} + A_{4m} = -\frac{(4m-1)(4m-2)}{2} + \frac{4m(4m-1)}{2} = 4m - 1$$
and
$$A_{4m+1} + A_{4m+2} = \frac{(4m+1)4m}{2} - \frac{(4m+2)(4m+1)}{2} = -4m - 1,$$
hence $A_{4m-1} + A_{4m} + A_{4m+1} + A_{4m+2} = -2$. Thus $A_{19} + A_{20} + A_{21} + \cdots + A_{98}$, which is a sum of eighty terms, equals $20(-2) = -40$.

6. (Answer: 308)
The similarity of triangles RBC and RDP implies that $\dfrac{RC}{RP} = \dfrac{RB}{RD}$, and the similarity of triangles RBQ and RDC implies that $\dfrac{RB}{RD} = \dfrac{RQ}{RC}$. Thus $\dfrac{RC}{RP} = \dfrac{RQ}{RC}$, or $RC^2 = RQ \cdot RP = 112 \cdot 847 = 16 \cdot 7 \cdot 7 \cdot 121$. Hence $RC = 4 \cdot 7 \cdot 11 = 308$.



7. (Answer: 196)
Each x_i can be replaced by $2y_i - 1$, where y_i is a positive integer. Because

$$98 = \sum_{i=1}^{4} x_i = \sum_{i=1}^{4}(2y_i - 1) = 2\left(\sum_{i=1}^{4} y_i\right) - 4$$

it follows that $51 = \sum_{i=1}^{4} y_i$. Each such quadruple (y_1, y_2, y_3, y_4) corresponds in a one-to-one fashion to a row of 51 ones that has been separated into four groups by the insertion of three zeros. For example, $(17, 5, 11, 18)$ corresponds to

$$11111111111111111\,0\,11111\,0\,11111111111\,0\,111111111111111111.$$

There are $\binom{50}{3} = 19600$ ways to insert three zeros into the fifty spaces between adjacent ones. Thus there are $n = 19600$ of the requested sums, and $\dfrac{n}{100} = 196$.

8. (Answer: 618)
Let $m = 1000$. The given sequence is

$$m,\ x,\ m-x,\ 2x-m,\ 2m-3x,\ 5x-3m,\ 5m-8x,\ \ldots .$$

Except for alternating signs, the coefficients of m and x in this sequence appear to belong to the Fibonacci-type sequence 1, 0, 1, 1, 2, 3, 5, 8, 13, 21, …, in which each term is the sum of its two predecessors. Because the goal is to avoid negative terms in the given sequence, an optimal x satisfies as many of the following inequalities as possible before failing:

$$x < m\ ,\ \tfrac{1}{2}m < x\ ,\ x < \tfrac{2}{3}m\ ,\ \tfrac{3}{5}m < x\ ,\ x < \tfrac{5}{8}m\ ,\ \ldots .$$

Each inequality involves a ratio of two successive terms of the Fibonacci sequence. Beginning with the fourth inequality, an optimal x must satisfy

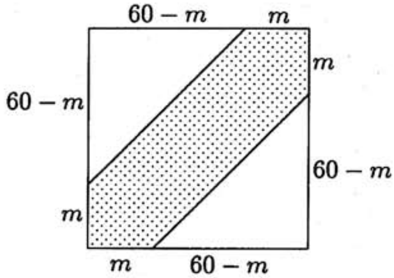
$$600 < x\ ,\ x < 625\ ,\ 615 < x\ ,\ x < 620\ ,\ 617 < x\ ,\ \text{and } x < 619.$$

It follows that $x = 618$, which produces the fourteen-term sequence 1000, 618, 382, 236, 146, 90, 56, 34, 22, 12, 10, 2, 8, -6.

Challenge: Prove that the coefficients of m and x do appear unsigned in the Fibonacci sequence.

9. (Answer: 087)

In the figure below, points in the square correspond to ordered pairs (x, y) of arrival times, with $0 \le x \le 60$ and $0 \le y \le 60$. The shaded points correspond to meetings, which occur if and only if $|x-y| \le m$. The probability of no meeting is $3/5$, which is the ratio of the unshaded region to the area of the whole square. Thus $(60-m)^2 = \frac{3}{5} \cdot 60^2$, whose solutions are $m = 60 \pm 12\sqrt{15}$. Because m must be smaller than 60, it follows that $m = 60 - 12\sqrt{15}$ (about 13.5 minutes), and $a + b + c = 60 + 12 + 15 = 87$.

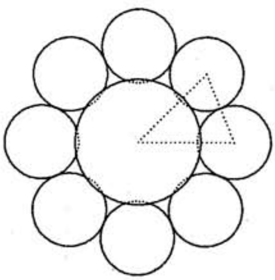


10. (Answer: 152)

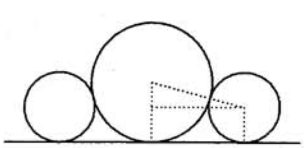
Let d be the distance between the center of one of the eight congruent spheres and the center of the regular octagon. Apply the Law of Cosines to the isosceles triangle formed by the center of the octagon and the centers of two congruent tangent spheres (shown below in the top view). This yields $200^2 = d^2 + d^2 - 2d^2 \cos 45° = d^2(2 - \sqrt{2})$, from which follows

$$d^2 = \frac{40000}{2 - \sqrt{2}} = 40000 + 20000\sqrt{2}.$$

Let r be the radius of the ninth sphere. The center of one of the eight congruent spheres, the center of the octagon, and the center of the ninth sphere form a right triangle (shown below in the side view). Apply the Pythagorean Theorem to obtain $d^2 + (r - 100)^2 = (r + 100)^2$, which is equivalent to $d^2 = 400r$. It follows that $r = 100 + 50\sqrt{2}$, and thus $a + b + c = 152$.



top view

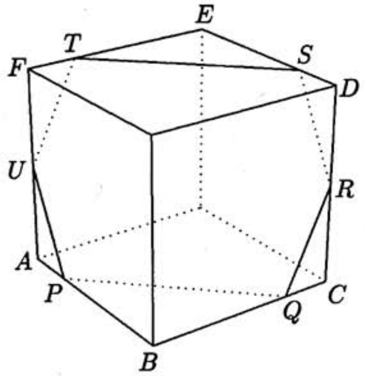


side view

11. (Answer: 525)

Because $BP = BQ$, $\overline{PQ}$ is parallel to $\overline{AC}$. Thus the line through R that is parallel to PQ will intersect $\overline{AF}$ at U so that $AU = CR$. Because the line through R that is parallel to $\overline{PQ}$ is in plane PQR, U is a vertex of the intersection polygon. The midpoint of $\overline{RU}$ is also the center of the cube, so the intersection polygon has point symmetry with respect to the center of the cube. Hence its area is twice the area of isosceles trapezoid $PQRU$, whose altitude is

$$\sqrt{UP^2 - \left(\frac{UR - PQ}{2}\right)^2}.$$



Given the lengths $UR = 20\sqrt{2}$, $PQ = 15\sqrt{2}$, and $UP = \sqrt{125}$, the desired area is found to be

$$(UR + PQ)\sqrt{UP^2 - \left(\frac{UR - PQ}{2}\right)^2} = 35\sqrt{2}\sqrt{\frac{225}{2}} = 525.$$

OR

Set up a coordinate system so that $A = (0,0,0)$, $B = (20,0,0)$, $C = (20,20,0)$, and $D = (20,20,20)$. It follows that $P = (5,0,0)$, $Q = (20,15,0)$, and $R = (20,20,10)$. Plane PQR can be described by an equation $ax + by + cz = d$. Substitute the coordinates of P, Q, and R into this equation to find that

$$d = 5a = 20a + 15b = 20a + 20b + 10c,$$

hence that $a = -b = 2c$. Thus plane PQR is described by $2x - 2y + z = 10$. To find coordinates for the other points where the plane intersects the edges of the cube, replace two of the unknowns by 0 or 20, and solve for the third, which must also be between 0 and 20. This yields the three additional points $S = (15,20,20)$, $T = (0,5,20)$, and $U = (0,0,10)$. The area of hexagon $PQRSTU$ may now be found as above.

12. (Answer: 083)

To see first that there is at most one set of points with the given property, suppose that P', Q', and R' also have the given property. Notice that P' is on $\overline{PE}$ if and only if Q' is on $\overline{QE}$, which is true if and only if R' is on $\overline{RD}$, which is true if and only if P' is on $\overline{PD}$. Conclude that $P' = P$. It follows that $Q' = Q$, because P determines the position of Q on $\overline{EF}$, and $R' = R$, because Q determines the position of R on $\overline{DF}$. Without loss of generality, let $DC = DE = 1$ and $DP = x$. Triangles ABC and DEF have 120-degree rotational symmetry, hence triangle PQR must also. (If this were not true, then a 120-degree rotation would produce another set of points with the given property.) It follows that $EQ = x$ and $PE = 1 - x$. The similarity of triangles DCP and EQP implies that $\dfrac{DP}{EP} = \dfrac{DC}{EQ}$, or $\dfrac{x}{1-x} = \dfrac{1}{x}$. Thus $x^2 = 1 - x$, whose positive solution is $x = \dfrac{1}{2}(\sqrt{5} - 1)$. Apply the Law of Cosines to triangle PQE to obtain

$$\begin{aligned} PQ^2 &= PE^2 + EQ^2 - 2 \cdot PE \cdot EQ \cdot \cos 60^\circ \\ &= (1-x)^2 + x^2 - x(1-x) \\ &= 3x^2 - 3x + 1 \\ &= 3(1-x) - 3x + 1 \\ &= 4 - 6x \\ &= 7 - 3\sqrt{5}. \end{aligned}$$

Therefore

$$\frac{\text{Area}(\triangle ABC)}{\text{Area}(\triangle PQR)} = \frac{AB^2}{PQ^2} = \frac{4}{7 - 3\sqrt{5}} = 7 + 3\sqrt{5},$$

and $a^2 + b^2 + c^2 = 49 + 9 + 25 = 83$.

13. (Answer: 368)

It suffices to express S_{n+1} in terms of S_n and n. The subsets of $\{1,2,3,\ldots,n+1\}$ that do not contain $n+1$ are just the subsets of $\{1,2,3,\ldots,n\}$, and thus the sum of their complex power sums is just S_n. The subsets of $\{1,2,3,\ldots,n+1\}$ that do contain $n+1$ are the subsets of $\{1,2,3,\ldots,n\}$ with $n+1$ adjoined as the greatest element. Notice that $n+1$ will be the only member in one of these subsets, the second member in n of these subsets, and in general will be the k^{th} member in $\binom{n}{k-1}$ of these subsets. The sum of the complex power sums of all subsets of $\{1,2,3,\ldots,n+1\}$ that contain $n+1$ is therefore

$$\begin{aligned} S_n &+ \binom{n}{0}(n+1)i + \binom{n}{1}(n+1)i^2 + \binom{n}{2}(n+1)i^3 + \cdots + \binom{n}{n}(n+1)i^{n+1} \\ &= S_n + i(n+1)\sum_{k=0}^{n}\binom{n}{k}i^k, \end{aligned}$$

which by the Binomial Theorem is equal to $S_n + i(n+1)(1+i)^n$. The desired recursion is therefore $S_{n+1} = 2S_n + i(n+1)(1+i)^n$. Thus

$$S_9 = 2S_8 + 9i(1+i)^8 = -352 - 128i + 9i\cdot 16 = -352 + 16i,$$

so $|p| + |q| = 368$.

Note: To solve the recursion, sum the equations

$$\begin{aligned} S_n &= 2S_{n-1} + in(1+i)^{n-1}, \\ 2S_{n-1} &= 4S_{n-2} + 2i(n-1)(1+i)^{n-2}, \\ 4S_{n-2} &= 8S_{n-3} + 4i(n-2)(1+i)^{n-3}, \\ &\ \vdots \\ 2^{n-2}S_2 &= 2^{n-1}S_1 + 2^{n-2}\,i\,2(i+1), \text{ and} \\ 2^{n-1}S_1 &= 2^{n-1}\,i \end{aligned}$$

to obtain

$$\begin{aligned} S_n &= i\left(2^{n-1} + 2^{n-2}2(i+1) + \cdots + n(1+i)^{n-1}\right) \\ &= i\,2^{n-1}\left(1 + 2z + 3z^2 + \cdots + n\,z^{n-1}\right), \end{aligned}$$

where $z = \frac{1}{2}(1+i)$. Multiply both sides by $1-z$ to obtain

$$\begin{aligned} (1-z)S_n &= i\,2^{n-1}\left(1 + z + z^2 + \cdots + z^{n-1} - n\,z^n\right) \\ &= i\,2^{n-1}\left[\frac{1-z^n}{1-z} - nz^n\right]. \end{aligned}$$

Then multiply both sides by $(1-z)^{-1} = 1+i$ to obtain

$$S_n = i\,2^{n-1}\left[2i\left(1-\left(\frac{1+i}{2}\right)^n\right) - n(1+i)\left(\frac{1+i}{2}\right)^n\right],$$

which simplifies to $S_n = (n+1+i)(1+i)^{n-1} - 2^n$.

Here is a combinatorial approach to evaluating S_n: Observe that, for $1 \le k \le n$ and $1 \le r \le k$, the term ki^r occurs in the sum of complex power sums for each choice of $1 \le a_1 < a_2 \cdots \le n$ in which $a_r = k$. Thus $S_n = \sum_{k=1}^{n}\sum_{r=1}^{k} W(k,r)ki^r$, where $W(k,r)$ is the number of subsets $\{a_1, a_2, \ldots\}$ of $\{1, 2, \ldots, n\}$ in which the r^{th} smallest element is k. Notice that $W(k,r) = \binom{k-1}{r-1}2^{n-k}$, because there are $\binom{k-1}{r-1}$ ways to choose $1 \le a_1 < a_2 < \cdots < a_{r-1} \le k-1$ and then 2^{n-k} ways to choose $\{a_{r+1}, a_{r+2}, \ldots\} \subseteq \{k+1, \ldots, n\}$. Hence, by the Binomial Theorem,

$$\begin{aligned} S_n &= \sum_{k=1}^{n}\sum_{r=1}^{k}\binom{k-1}{r-1}2^{n-k}k\,i^r \\ &= i\,2^{n-1}\sum_{k=1}^{n} k2^{-(k-1)}\sum_{r=1}^{k}\binom{k-1}{r-1}i^{r-1} \\ &= i\,2^{n-1}\sum_{k=1}^{n} kz^{k-1}, \end{aligned}$$

where $z = \frac{1}{2}(1+i)$. Continue as above.

14. (Answer: 130)

First notice that $2mnp = (m+2)(n+2)(p+2)$ implies $\dfrac{2m}{m+2} = \dfrac{(n+2)(p+2)}{np} > 1$, which shows that $m \ge 3$. Next rewrite the equation as

$$\begin{aligned} &2mnp = mnp + 2(mn+np+mp) + 4(m+n+p) + 8, \\ &mnp - 2(mn+np+mp) + 4(m+n+p) - 8 = 8(m+n+p), \\ &(m-2)(n-2)(p-2) = 8(m+n+p), \end{aligned}$$

which suggests replacing $m-2$, $n-2$, and $p-2$ by the positive integers a, b, and c, respectively. Notice that $1 \le a$. The problem is now to find the largest c that satisfies the equation $abc = 8(a+b+c+6)$, which can be rewritten $\dfrac{c}{8} = \dfrac{a+b+6}{ab-8}$. Because $(a-1)(b-1)$ is nonnegative, it follows that $a+b \le ab+1$, hence that

$$\frac{c}{8} = \frac{a+b+6}{ab-8} \le \frac{ab+1+6}{ab-8} = \frac{ab-8+15}{ab-8} = 1 + \frac{15}{ab-8}\,.$$

This shows that c can be no larger than $8 \cdot 16 = 128$, and that c attains this value if $ab = 9$ and $a+b = ab+1 = 10$. Thus $c = 128$ when $a = 1$ and $b = 9$, and $m = 3$, $n = 11$, and $p = 130$ are the dimensions of a possible box.

15. (Answer: 761)

Let $A_n = \{1, 2, 3, \ldots, n\}$ and D_n be the set of dominos that can be formed using integers in A_n. Each k in A_n appears in $2(n-1)$ dominos in D_n, hence appears at most $n-1$ times in a proper sequence from D_n. Except possibly for the integers i and j that begin and end a proper sequence, every integer appears an even number of times in the sequence. Thus, if n is even, each integer different from i and j appears on at most $n-2$ dominos in the sequence, because $n-2$ is even, and i and j themselves appear on at most $n-1$ dominos each. This gives an upper bound of

$$\frac{1}{2}\left[(n-2)^2 + 2(n-1)\right] = \frac{n^2-2n+2}{2}$$

dominos in the longest proper sequence in D_n. This bound is in fact attained for every even n. It is easy to verify this for $n = 2$, so assume inductively that a sequence of this length has been found for a particular value of n. Without loss of generality, assume $i = 1$ and $j = 2$, and let ${}_pX_{p+2}$ denote a four-domino sequence of the form $(p, n+1)(n+1, p+1)(p+1, n+2)(n+2, p+2)$. By appending

$${}_2X_4\ ,\ {}_4X_6\ ,\ \ldots\ ,\ {}_{n-2}X_n\ ,\ (n, n+1)(n+1, 1)(1, n+2)(n+2, 2)$$

to the given proper sequence, a proper sequence of length

$$\frac{n^2-2n+2}{2} + 4\cdot\frac{n-2}{2} + 4 = \frac{n^2+2n+2}{2} = \frac{(n+2)^2 - 2(n+2) + 2}{2}$$

is obtained that starts at $i = 1$ and ends at $j = 2$. This completes the inductive proof. In particular, the longest proper sequence when $n = 40$ is 761.

OR

A proper sequence can be represented by writing the common coordinates of adjacent ordered pairs once. For example, represent (4,7), (7,3), (3,5) as 4,7,3,5. Label the vertices of a regular n-gon 1, 2, 3, …, n. Each domino is thereby represented by a directed segment from one vertex of the n-gon to another, and a proper sequence is represented as a path that retraces no segment. Each time that such a path reaches a non-terminal vertex, it must leave it. Thus, when n is even, it is not possible for such a path to trace every segment, for an odd number of segments emanate from each vertex. By removing $\frac{1}{2}(n-2)$ suitable segments, however, it can be arranged that $n-2$ segments will emanate from $n-2$ of the vertices, and that an odd number of segments will emanate from exactly two of the vertices. In this situation, a path can be found that traces every remaining segment exactly once, starting at one of the two exceptional vertices and finishing at the other. This path will have length $\binom{n}{2} - \frac{1}{2}(n-2)$, which is 761 when $n = 40$.

Note: When n is odd, a proper sequence of length $\binom{n}{2}$ can be found using the dominos of D_n. In this case, the second coordinate of the final domino equals the first coordinate of the first domino. In the language of graph theory, this is an example of an *Eulerian circuit*.

AMERICAN MATHEMATICS COMPETITIONS

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FOR STUDENTS AND TEACHERS

17th ANNUAL
AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)
TUESDAY, March 16, 1999

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This examination was prepared during the tenure of American Mathematics Competitions Executive Director, Dr. Walter E. Mientka.

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1. (Answer: 029)
The common difference of such a sequence must be even, for otherwise at least one of the first five terms would be even and greater than 2. The common difference must also be divisible by 3, for otherwise at least one of the first five terms would be divisible by 3 and greater than 3. Therefore the common difference must be divisible by 6, and the first term of the sequence must be relatively prime to 6. Because the sequence 5, 11, 17, 23, 29 consists exclusively of primes, it follows that the desired prime is 29.

2. (Answer: 118)
The line must go through the point where the diagonals of the parallelogram bisect each other, namely $\left(\dfrac{10+28}{2}, \dfrac{45+153}{2}\right) = (19, 99)$. Thus the slope of the line is $99/19$, and $m + n = 118$.

3. (Answer: 038)
If $n^2 - 19n + 99 = m^2$ for positive integers m and n, then $4m^2 = 4n^2 - 76n + 396 = (2n-19)^2 + 35$. Thus $4m^2 - (2n-19)^2 = 35$, or $(2m+2n-19)(2m-2n+19) = 35$. The sum of the two factors is $4m$, a positive integer, so the pair $(2m+2n-19, 2m-2n+19)$ can only be $(1, 35)$, $(5, 7)$, $(7, 5)$, or $(35, 1)$. Subtract the second factor from the first to discover that $4n - 38$ can be only -34, -2, 2, or 34, from which it follows that n can only be 1, 9, 10, or 18. The sum of these integers is 38.

4. (Answer: 185)
   Notice that O is the center of the circle in which both squares are inscribed. The reflection of either square across the diameter determined by $\overline{OB}$ is another square inscribed in the same circle. Because the circle has only two chords of length 1 that go through B, the squares must be reflected images of each other. In particular, $AB = BC$. Similar reasoning shows that any two adjacent sides of $ABCDEFGH$ have the same length, so the octagon is equilateral. Because the distance from O to all eight sides of the squares is $1/2$, the area of the octagon is eight times the area of triangle AOB; i.e., $8(\frac{1}{2})(\frac{1}{2})AB = 2AB = 86/99$. Thus $m + n = 86 + 99 = 185$.

   OR

   Let J be that vertex of one of the squares for which angle AJB is right, and let $x = AJ$ and $y = BJ$. Then

   $$x + y + \frac{43}{99} = 1, \qquad \text{so} \qquad x + y = \frac{56}{99}, \qquad \text{and} \qquad x^2 + y^2 = \left(\frac{43}{99}\right)^2.$$

   Hence

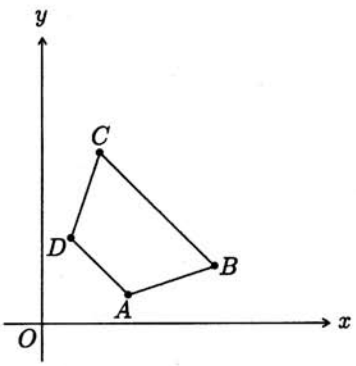
   $$2xy = (x+y)^2 - (x^2 + y^2) = \frac{56^2 - 43^2}{99^2} = \frac{13}{99}.$$

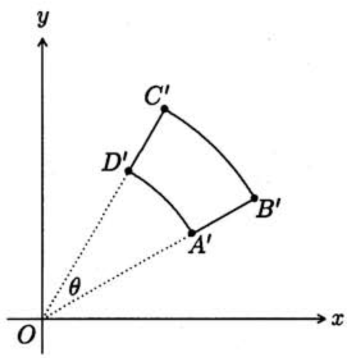
   Because $2xy$ is the combined area of the four corner triangles, the area of the octagon is $1 - \dfrac{13}{99} = \dfrac{86}{99}$, and $m + n = 185$.

   Query: Could this problem have been posed with the hypothesis $AB = 19/99$?

5. (Answer: 223)
   If the final digit of x is less than 8, then $S(x+2) = 2 + S(x)$, so $T(x) = 2$. When the last digit of x is 8, then x has the form $A|B|8$, where B is a block of k nines, k is nonnegative, and the final digit of the block A is not 9. Because $x + 2$ has the form $(A+1)|Z$, where Z is a block of $k+1$ zeros, it follows that $T(x) = S(x) - S(x+2) = S(A) + 9k + 8 - S(A+1) = S(A) + 9k + 8 - S(A) - 1 = 9k + 7$. When the last digit of x is 9, then x has the form $A|B$, where B is a block of k nines, k is positive, and the final digit of A is not 9. Because $x + 2$ has the form $(A+1)|Z|1$, where Z is a block of $k-1$ zeros, it follows that $T(x) = S(x) - S(x+2) = S(A) + 9k - S(A+1) - 1 = 9k - 2$. Notice that the sequence $9k + 7$ for nonnegative k coincides with the sequence $9k - 2$ for positive k. Thus T can have the values $7, 16, 25, \ldots, 1996$, and 2. There are $\frac{1}{9}(1996 - (-2)) + 1 = 223$ values in all.

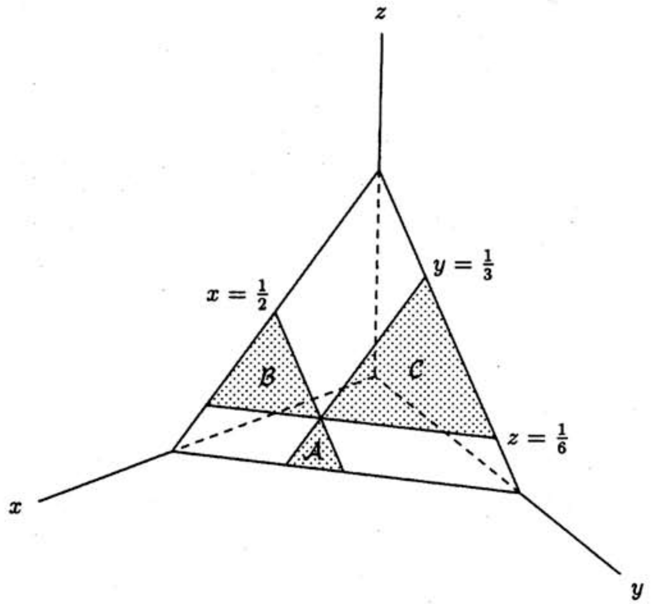
6. (Answer: 314)
Notice that $\overline{AB}$ is contained in the line whose equation is $3y = x$. The image of a point on $\overline{AB}$ must therefore satisfy $3y^2 = x^2$. Because the coordinates of the image points must be positive, the image $\overline{A'B'}$ of $\overline{AB}$ is contained in the line $y\sqrt{3} = x$. In a similar fashion, it follows that the image $\overline{C'D'}$ of $\overline{CD}$ is contained in the line $y = x\sqrt{3}$. An equation for line AD is $x + y = 1200$, so the image of $\overline{AD}$ is contained in the first-quadrant part of the circle $x^2 + y^2 = 1200$. In a similar fashion, it follows that the image of $\overline{BC}$ is contained in the first-quadrant part of the circle $x^2 + y^2 = 2400$. Thus the area of the region enclosed by the image of quadrilateral $ABCD$ is $\frac{\theta}{360}\left(\pi(OB')^2 - \pi(OA')^2\right)$, where O is the origin and θ is the degree measure of the angle formed by $\overline{OA'}$ and $\overline{OD'}$. Notice that $\theta = \tan^{-1}\sqrt{3} - \tan^{-1}\frac{1}{\sqrt{3}} = 30$, because the slope of $\overline{OD'}$ is $\sqrt{3}$ and the slope of $\overline{OA'}$ is $\frac{1}{\sqrt{3}}$. Hence the area of the region enclosed by the image of $ABCD$ is $k = \frac{1}{12}(2400 - 1200)\pi = 100\pi$, and the greatest integer that does not exceed k is 314.





7. (Answer: 650)
A switch will finish in position A if and only if it has been advanced $4k$ times for some integer k. Each advance of a given switch corresponds to a multiple of its label. Let S be the set of integers $2^x3^y5^z$, where x, y, and z take on the values 0, 1, ..., 9. Notice that $2^x3^y5^z$ has $(10 - x)(10 - y)(10 - z)$ multiples in S. Thus the answer to the problem is the number of triples (x, y, z) for which $(10 - x)(10 - y)(10 - z)$ is divisible by 4. There are two cases in which $(10-x)(10-y)(10-z)$ is *not* divisible by 4. If all three factors of $(10-x)(10-y)(10-z)$ are odd, the product will also be odd; this occurs $5 \cdot 5 \cdot 5 = 125$ times. If two of the factors are odd and the third is 2, 6, or 10, the product will be even but not divisible by 4; this occurs $3 \cdot 5 \cdot 5 \cdot 3 = 225$ times. In all, there are $125 + 225 = 350$ triples (x, y, z) for which $(10 - x)(10 - y)(10 - z)$ is not divisible by 4. Therefore after step 1000, the number of switches in position A will be $1000 - 350 = 650$.

8. (Answer: 025)
Notice that $\mathcal{T}$ is a first-octant equilateral triangle, whose vertices are $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. The planes $x=\frac{1}{2}$, $y=\frac{1}{3}$, and $z=\frac{1}{6}$ intersect $\mathcal{T}$ along line segments that are parallel to the sides of $\mathcal{T}$. Let $\mathcal{A}$ consist of those points of $\mathcal{T}$ that satisfy $x \geq \frac{1}{2}$ and $y \geq \frac{1}{3}$. Notice that $z \leq \frac{1}{6}$ for any point in $\mathcal{A}$, so the points of $\mathcal{A}$ support $\left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right)$, with the exception of $\left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right)$ itself. In a similar fashion, let $\mathcal{B}$ consist of those points of $\mathcal{T}$ that satisfy $x \geq \frac{1}{2}$ and $z \geq \frac{1}{6}$, and let $\mathcal{C}$ consist of those points of $\mathcal{T}$ that satisfy $y \geq \frac{1}{3}$ and $z \geq \frac{1}{6}$. Except for the point $\left(\frac{1}{2},\frac{1}{3},\frac{1}{6}\right)$, $\mathcal{S}$ is the union of the equilateral triangles $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, whose sides are $\frac{1}{6}$, $\frac{1}{3}$, and $\frac{1}{2}$ times as long as the sides of $\mathcal{T}$, and whose areas are $\frac{1}{36}$, $\frac{1}{9}$, and $\frac{1}{4}$ times the area of $\mathcal{T}$, respectively. It follows that the area of $\mathcal{S}$ divided by the area of $\mathcal{T}$ is $\frac{1}{36}+\frac{1}{9}+\frac{1}{4}=\frac{7}{18}$. Thus $m+n=25$.



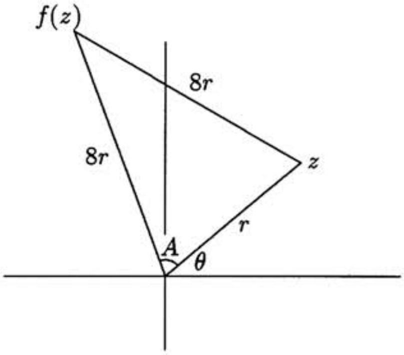
9. (Answer: 259)
Because $(a+bi)z$ is equidistant from z and 0, $|(a+bi)z-z|=|(a+bi)z|$. Thus $|a-1+bi|=|a+bi|$, or $(a-1)^2+b^2=a^2+b^2$, so $a=\dfrac{1}{2}$. Now use the information $|a+bi|=8$ to deduce that $b^2=64-\dfrac{1}{4}=\dfrac{255}{4}$, so that $m+n=259$.

OR

Let $z=r\operatorname{cis}\theta=r(\cos\theta+i\sin\theta)$, where r is positive and $0\le\theta<2\pi$, and let $a+bi=8\operatorname{cis}A$, where $0<A<\pi/2$. Thus $f(z)=8r\operatorname{cis}(\theta+A)$. Notice that $\cos A=\dfrac{r/2}{8r}=\dfrac{1}{16}$, because the triangle in the figure is isosceles. It follows that

$$b^2=8^2\sin^2 A=64(1-\cos^2 A)=\frac{255}{4}.$$

Thus $m+n=259$.



10. (Answer: 489)
Begin generally with k points in the plane, no three of which are collinear. There are $\dbinom{k}{2}$ segments joining the points, and $\dbinom{\binom{k}{2}}{4}$ ways to choose four segments. Because two triangles can share at most one side, four segments cannot form two triangles. Therefore, it suffices to count the ways of choosing a triangle and one additional segment. There are $\dbinom{k}{3}$ ways to choose the vertices of a triangle, and then $\dbinom{k}{2}-3$ ways to choose an additional segment. Hence the probability of obtaining a triangle when $k=10$ is

$$\frac{\dbinom{k}{3}\left(\dbinom{k}{2}-3\right)}{\dbinom{\binom{k}{2}}{4}}=\frac{\dbinom{10}{3}\left(\dbinom{10}{2}-3\right)}{\dbinom{45}{4}}=\frac{16}{473},$$

so that $m+n=489$.

11. (Answer: 177)

$$\begin{aligned}\sum_{k=1}^{35}\sin 5k &= \frac{1}{\sin 5}\sum_{k=1}^{35}\sin 5\sin 5k\\ &= \frac{1}{\sin 5}\sum_{k=1}^{35}\frac{\cos(5k-5)-\cos(5k+5)}{2}\\ &= \frac{1}{\sin 5}\cdot\frac{\cos 0+\cos 5-\cos 175-\cos 180}{2}\\ &= \frac{1-\cos 175}{\sin 175}=\tan\frac{175}{2},\end{aligned}$$

so $m+n=177$.

OR

Let $\operatorname{cis} t=\cos t+i\sin t$. Because $\operatorname{cis} 0=1$, the given series is the imaginary part of the complex series

$$\operatorname{cis} 0+\operatorname{cis} 5+\operatorname{cis} 10+\cdots+\operatorname{cis} 175=\sum_{k=0}^{35}(\operatorname{cis} 5)^k,$$

by the theorem of DeMoivre. This is a geometric series, whose sum is

$$\frac{\operatorname{cis} 0-\operatorname{cis} 180}{1-\operatorname{cis} 5}=\frac{2}{1-\operatorname{cis} 5}=\frac{2}{1-\cos 5-i\sin 5}=\frac{2(1-\cos 5+i\sin 5)}{(1-\cos 5)^2+(\sin 5)^2}.$$

The imaginary part of this sum is

$$\frac{\sin 5}{1-\cos 5}=\frac{\sin 175}{1+\cos 175}=\tan\frac{175}{2}.$$

Thus $m+n=177$.

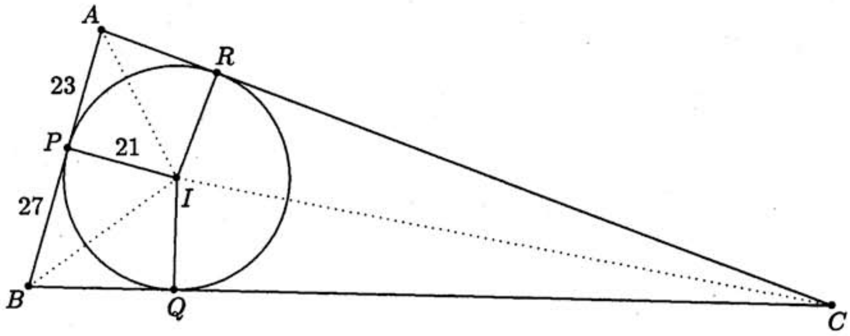
Query: Both solutions make use of the identities $\tan\frac{1}{2}x=\dfrac{\sin x}{1+\cos x}=\dfrac{1-\cos x}{\sin x}$. Can you prove them?

12. (Answer: 345)

Let I be the center of the inscribed circle, and R be the point where the circle is tangent to $\overline{CA}$. Because I is the intersection of the angle bisectors of ABC, it follows that $\angle IAB + \angle IBC + \angle ICA = 90°$. Notice that $\tan \angle IAB = 21/23$, $\tan \angle IBC = 21/27$, and $\tan \angle ICA = 21/CR$. Thus

$$\frac{CR}{21} = \tan(90° - \angle ICA) = \tan(\angle IAB + \angle IBC) = \frac{\frac{21}{23} + \frac{21}{27}}{1 - \frac{21}{23} \cdot \frac{21}{27}} = \frac{35}{6}.$$

Therefore $CR = 245/2$ and the perimeter of triangle ABC is $2\left(23 + 27 + \frac{245}{2}\right) = 345$.



OR

Use the same figure. Let $a = BC$, $b = CA$, $c = AB$, $x = CR$, and $s = \frac{1}{2}(a + b + c)$. Then $s = x + 50$, $s - a = 23$, $s - b = 27$, and $s - c = x$. The area of any triangle is the product of its inradius and its semiperimeter, so $21s = \sqrt{s(s-a)(s-b)(s-c)}$, by Heron's formula. It follows that $21^2(x + 50) = 23 \cdot 27 \cdot x$. Solve this equation to obtain $x = 245/2$ and $2s = 2(x + 50) = 345$.

OR

As in the preceding, let s be the semiperimeter of triangle ABC, and notice that $BC = s - 23$. Because $\overline{IB}$ bisects $\angle ABC$, it follows that

$$\sin \angle ABC = 2 \sin \angle PBI \cos \angle PBI = 2 \cdot \frac{7}{\sqrt{130}} \cdot \frac{9}{\sqrt{130}} = \frac{63}{65}.$$

Hence the area of triangle ABC is $\frac{1}{2} AB \cdot BC \sin \angle ABC = 25(s - 23)\frac{63}{65}$. The area of a triangle is also equal to the product of its inradius and its semiperimeter, so $25(s-23)\frac{63}{65} = 21s$. Solve this equation to find that $2s = 345$, which is the perimeter.

13. (Answer: 742)

Suppose that no two teams win the same number of games. Then, for any k between 0 and 39, exactly one team wins k games. Moreover, a team that wins 38 games can lose only to the team that wins all of its games. An inductive argument shows that in fact each team loses to any team that wins more games, but to no other teams. Thus a tournament in which no two teams win the same number of games is uniquely determined by listing the teams in order of their wins. Given any listing, the probability that each team beat all the teams below it and lost to all the teams above it is $\left(\frac{1}{2}\right)^C$, where $C = \binom{40}{2} = 780$ is the total number of games. Because there are 40! ways to list the teams, the requested probability is $\dfrac{40!}{2^{780}}$. The fraction $\dfrac{40!}{2^{780}}$ is not in lowest terms, however. The number of factors of 2 in 40! is

$$\left\lfloor \frac{40}{2} \right\rfloor + \left\lfloor \frac{40}{2^2} \right\rfloor + \left\lfloor \frac{40}{2^3} \right\rfloor + \cdots = 20 + 10 + 5 + 2 + 1 = 38.$$

Thus the requested probability is $\dfrac{m}{2^{742}}$, where m is an odd integer, so $\log_2 n = 742$.

14. (Answer: 463)

Let ω denote the common measure of angles PAB, PBC, and PCA; let a, b, and c denote BC, CA, and AB; and let x, y, and z denote PA, PB, and PC. Apply the Law of Cosines to triangles PCA, PAB, and PBC to obtain

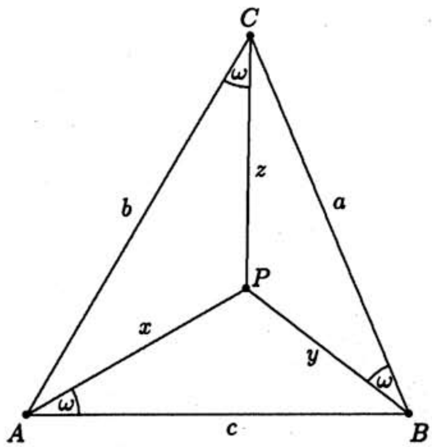
$$\begin{aligned} x^2 &= z^2 + b^2 - 2bz\cos\omega \\ y^2 &= x^2 + c^2 - 2cx\cos\omega \\ z^2 &= y^2 + a^2 - 2ay\cos\omega. \end{aligned}$$

Sum these three equations to obtain $2(cx + ay + bz)\cos\omega = a^2 + b^2 + c^2$. Because the combined area of triangles PAB, PBC, and PCA is $\frac{1}{2}(cx + ay + bz)\sin\omega$, the preceding equation can be rewritten as

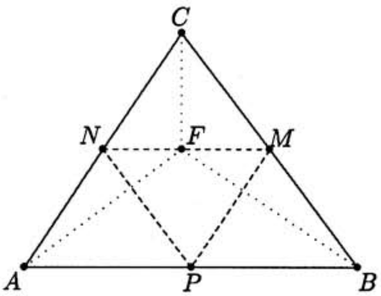
$$\tan\omega = \frac{4[ABC]}{a^2 + b^2 + c^2},$$

where $[ABC]$ denotes the area of triangle ABC. With $a = 14$, $b = 15$, and $c = 13$, use Heron's formula to find that $[ABC] = 84$. It follows that $\tan\omega = 168/295$, so $m + n = 463$.

Query: Triangle ABC has two *Brocard points*, and P is one of them. The other one is the point Q for which angles QBA, QCB, and QAC are equal. What is the common measure of these three angles?



15. (Answer: 408)

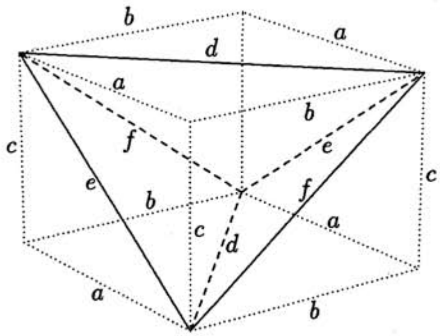


Assign coordinate triples to the vertices, so that $A = (0,0,0)$, $B = (34,0,0)$, $C = (16,24,0)$, and midpoints $M = (25,12,0)$, $N = (8,12,0)$, and $P = (17,0,0)$. Without loss of generality, assume that triangle MNP remains fixed when triangle ABC is folded. Vertex C must then stay in the plane $x = 16$ (which is perpendicular to midline $\overline{MN}$), vertex A must stay in the plane $4y = 3x$ (which is perpendicular to midline $\overline{NP}$), and vertex B must stay in the plane $2x+3y = 68$ (which is perpendicular to midline $\overline{PM}$). The intersection of these three planes includes $F = (16,12,0)$, which happens to be on $\overline{MN}$. The planes also intersect at V, the fourth vertex of the pyramid. Notice that $VF = CF = 12$. Because the planes are all perpendicular to triangle MNP, the altitude drawn to the base MNP of pyramid $VMNP$ is $\overline{VF}$. The volume of the pyramid is therefore

$$\frac{1}{3} \cdot \frac{1}{2} \cdot 17 \cdot 12 \cdot 12 = 408.$$

OR

The pyramid is a tetrahedron with four congruent acute faces. Hence its edges may be regarded as diagonals of the faces of a rectangular parallelepiped, as shown below. The edges are $d = \frac{1}{2}AB = 17$, $e = \frac{1}{2}BC = 15$, and $f = \frac{1}{2}CA = 4\sqrt{13}$. The edges of the parallelepiped are a, b, and c, where $a^2 + b^2 = d^2$, $a^2 + c^2 = e^2$, and $b^2 + c^2 = f^2$. Solve these equations simultaneously to find that $a^2 = \frac{1}{2}(d^2 + e^2 - f^2) = 153$, $b^2 = \frac{1}{2}(f^2 + d^2 - e^2) = 136$, and $c^2 = \frac{1}{2}(e^2 + f^2 - d^2) = 72$. The parallelepiped consists of four congruent pyramids, each of volume $\frac{1}{6}abc$, as well as the given pyramid. Thus the volume of the given pyramid is $\frac{1}{3}abc = \frac{1}{3}\sqrt{153 \cdot 136 \cdot 72} = 408$.



AMERICAN MATHEMATICS COMPETITIONS

18^{th} ANNUAL AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

(AIME)

Tuesday, March 28, 2000

SOLUTIONS PAMPHLET

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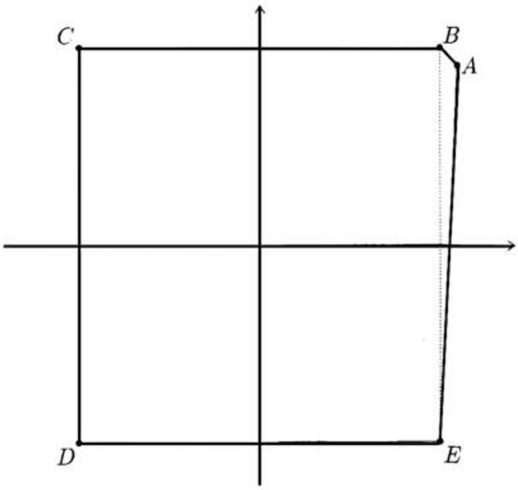
1. (Answer: 008)
The number 10^n can be expressed as the product of 2^n and 5^n, neither of which contains the digit 0 for $n \leq 7$. Since $5^8 = 380625$, and all other pairs of positive integers whose product is 10^8 contain at least one trailing 0, the requested value of n is 8.

2. (Answer: 021)
It follows from the problem statement that the coordinates of A, B, C, D, and E are $A = (u, v), B = (v, u), C = (-v, u), D = (-v, -u)$, and $E = (v, -u)$. The pentagon can be partitioned into rectangle $BCDE$ and triangle ABE, whose areas are $2u \cdot 2v = 4uv$ and $\frac{1}{2}2u(u-v) = u^2 - uv$, so

$$\text{Area}(ABCDE) = 4uv + u^2 - uv = u(3v + u).$$

Thus $u(3v + u) = 451 = 11 \cdot 41$. Because $0 < v < u$, it follows that $u = 11$, $3v + 11 = 41$, $v = 10$, and $u + v = 21$.

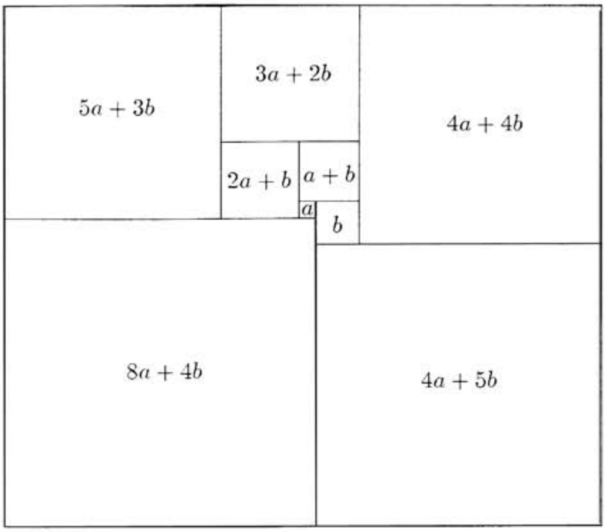


3. (Answer: 667)
The problem statement implies that

$$\frac{2000 \cdot 1999}{2 \cdot 1} a^2 b^{1998} = \frac{2000 \cdot 1999 \cdot 1998}{3 \cdot 2 \cdot 1} a^3 b^{1997}.$$

This is equivalent to $b = 666a$. Because a and b have no common divisors greater than 1, it follows that $a = 1$, $b = 666$, and $a + b = 667$.

4. (Answer: 260)
Let a and b be the lengths of the sides of the two smallest squares, with $a < b$. In the diagram below, each square has been labeled with the length of its sides.



The length $13a + 7b$ of the left side of the rectangle is equal to the length $8a + 9b$ of the right side, so $5a = 2b$. This implies that there is a positive number t for which $a = 2t$ and $b = 5t$. Thus the width-to-height ratio for the rectangle is $\dfrac{12a + 9b}{8a + 9b} = \dfrac{69t}{61t} = \dfrac{69}{61}$. Because the dimensions of the rectangle are relatively prime positive integers, they must be 69 and 61. Therefore the perimeter is $2(69 + 61) = 260$.

5. (Answer: 026)
Suppose that the first box has k_1 marbles, b_1 of which are black, and that the second box has k_2 marbles, b_2 of which are black. Without loss of generality, assume that $k_1 < k_2$. Thus $k_1 + k_2 = 25$ and $(b_1/k_1) \cdot (b_2/k_2) = 27/50$, so $50b_1b_2 = 27k_1k_2$. The latter equation implies that 5 divides either k_1 or k_2. Because $k_1 + k_2 = 25$, both k_1 and k_2 must be divisible by 5. One possibility is that $k_1 = 5$ and $k_2 = 20$, in which case $b_1b_2 = 54$, so $b_1 = 3$ and $b_2 = 18$, because $b_1 < k_1$ and $b_2 < k_2$. In this case, the probability of obtaining two white marbles is $(2/5) \cdot (2/20) = 1/25$. The only other possibility is that $k_1 = 10$ and $k_2 = 15$, in which case $b_1b_2 = 81$, so b_1 and b_2 must each be 9. The probability of obtaining two white marbles is $(1/10) \cdot (6/15) = 1/25$ in this case too. Hence $m + n = 26$.

6. (Answer: 997)
From
$$\frac{x+y}{2} = 2 + \sqrt{xy},$$
it follows that
$$\begin{aligned} x + y - 2\sqrt{xy} &= 4, \\ (\sqrt{y} - \sqrt{x}\,)^2 &= 4, \text{ and} \\ \sqrt{y} - \sqrt{x} &= 2. \end{aligned}$$
Because $y = (2 + \sqrt{x}\,)^2 = x + 4 + 4\sqrt{x}$ is an integer, it follows that $4\sqrt{x}$ must be an integer. Consequently $16x$ is a perfect square, and $\sqrt{x}$ is an integer. From $(2+\sqrt{x}\,)^2 < 10^6$, it follows that $\sqrt{x} < 998$. Thus the 997 solutions are $(x, y) = \left(n^2, (n+2)^2\right)$, for $n = 1, 2, \ldots, 997$.

7. (Answer: 005)
Notice that
$$\begin{aligned} 5 \cdot 29 \cdot \frac{m}{n} &= \left(x + \frac{1}{z}\right)\left(y + \frac{1}{x}\right)\left(z + \frac{1}{y}\right) \\ &= xyz + x + \frac{1}{z} + y + \frac{1}{x} + z + \frac{1}{y} + \frac{1}{xyz} \\ &= 1 + 5 + 29 + \frac{m}{n} + 1. \end{aligned}$$
Thus $144 \cdot \dfrac{m}{n} = 36$, so that $\dfrac{m}{n} = \dfrac{1}{4}$ and $m + n = 5$.

OR

Because
$$5 = x + \frac{1}{z} = x + xy = x + x\left(29 - \frac{1}{x}\right) = 30x - 1,$$
it follows that $x = \dfrac{1}{5}$, $y = 24$, and $z = \dfrac{5}{24}$. Thus $z + \dfrac{1}{y} = \dfrac{5}{24} + \dfrac{1}{24} = \dfrac{1}{4}$.

8. (Answer: 052)
When the cone is held point down, the liquid in the container forms a cone that is similar to the container, the ratio of similarity being $\dfrac{3}{4}$. Thus the volume of the liquid is $\left(\dfrac{3}{4}\right)^3$ times the volume of the container. When the cone is held point up, the air in the container forms a cone whose height is h and whose volume is $1 - \left(\dfrac{3}{4}\right)^3 = \dfrac{37}{64}$ times the volume of the container. Because the cone of air is similar to the container, $\dfrac{37}{4^3} = \dfrac{h^3}{12^3}$, so $h^3 = 3^3 \cdot 37$. It follows that the depth of the liquid is $12 - h = 12 - 3\sqrt[3]{37}$. Thus $m + n + p = 12 + 3 + 37 = 52$.

9. (Answer: 025)
Let $u = \log_{10} x$, $v = \log_{10} y$, and $w = \log_{10} z$. The given equations can be rewritten as

$$\begin{aligned} uv - u - v + 1 &= \log_{10} 2 \\ vw - v - w + 1 &= \log_{10} 2 \\ wu - w - u + 1 &= 1, \end{aligned}$$

and then as

$$\begin{aligned} (u-1)(v-1) &= \log_{10} 2 \\ (v-1)(w-1) &= \log_{10} 2 \\ (w-1)(u-1) &= 1. \end{aligned}$$

It follows from the first two equations that $u = w$, and the third equation then implies that $u = w = 2$ or $u = w = 0$. In the first case, $v = \log_{10} 20$, so $x_1 = 100$, $y_1 = 20$, and $z_1 = 100$. In the second case, $v = \log_{10} 5$, so $x_2 = 1$, $y_2 = 5$, and $z_2 = 1$. Thus $y_1 + y_2 = 25$.

10. (Answer: 173)
Let $S = x_1 + x_2 + x_3 + \cdots + x_{100}$, so $x_k = (S - x_k) - k$ for all integers k between 1 and 100, inclusive. Thus $k + 2x_k = S$ for all such k. Summing these equations for k=1, 2, 3, …, 100 yields

$$\frac{100 \cdot 101}{2} + 2S = 100S,$$

from which $S = \dfrac{2525}{49}$ follows. Thus $x_{50} = \dfrac{S - 50}{2} = \dfrac{75}{98}$, and $m + n = 173$.

11. (Answer: 248)
Because $1000 = 2^3 5^3$, each a/b may be written in the form $2^m 5^n$, where $-3 \le m \le 3$ and $-3 \le n \le 3$. It follows that each a/b appears exactly once in the expansion of

$$\left(2^{-3} + 2^{-2} + \cdots + 2^2 + 2^3\right)\left(5^{-3} + 5^{-2} + \cdots + 5^2 + 5^3\right).$$

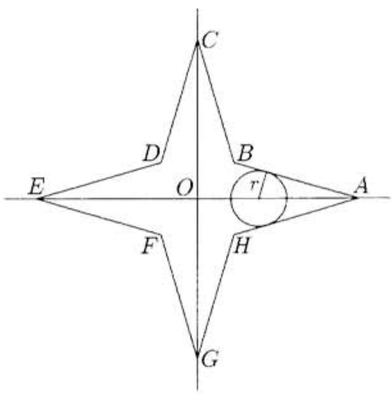
Thus $S = \dfrac{2^4 - 2^{-3}}{2-1} \cdot \dfrac{5^4 - 5^{-3}}{5-1} = \dfrac{127}{8} \cdot \dfrac{19531}{125} = 2480 + \dfrac{437}{1000}$, so $\dfrac{S}{10} = 248 + \dfrac{437}{10000}$.

12. (Answer: 177)
From the given identities $f(x) = f(398 - x)$, $f(x) = f(2158 - x)$, and $f(x) = f(3214 - x)$, derive the following identities:

$$\begin{aligned}
f(x) &= f(2158 - x) = f(3214 - (2158 - x)) = f(1056 + x),\\
f(x) &= f(1056 + x) = f(2158 - (1056 + x)) = f(1102 - x),\\
f(x) &= f(1056 + x) = f(1102 - (1056 + x)) = f(46 - x), \text{ and}\\
f(x) &= f(46 - x) = f(398 - (46 - x)) = f(352 + x).
\end{aligned}$$

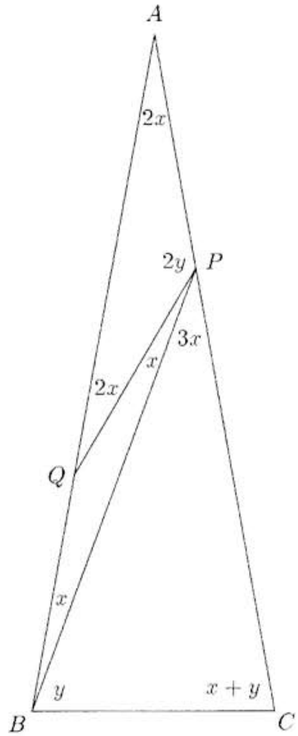
It follows from the last identity that f is periodic, and that the period of f divides 352. Thus every value in the list is found among $f(0)$, $f(1)$, $\ldots$, $f(351)$. The identity $f(x) = f(398 - x)$ implies that $f(200)$, $f(201)$, $\ldots$, $f(351)$ are found among $f(0)$, $f(1)$, $\ldots$, $f(199)$, and the identity $f(x) = f(46 - x)$ implies that $f(0)$, $f(1)$, $\ldots$, $f(22)$ are found among $f(23)$, $f(24)$, $\ldots$, $f(199)$. Thus there can be at most 177 different values in the list. To see that the values $f(23)$, $f(24)$, $\ldots$, $f(199)$ can be distinct, consider the function $f(x) = \cos\left(\frac{360}{352}(x - 23)\right)$, whose argument is interpreted in degrees. It is routine to verify the required identities $f(x) = f(398 - x)$, $f(x) = f(2158 - x)$, and $f(x) = f(3214 - x)$, and to see that $1 = f(23) > f(24) > f(25) > \cdots > f(199) = -1$.

13. (Answer: 731)
Set up a coordinate system in which the axes coincide with the highways. The points that can be reached within six minutes lie on or within circles of radius $14\left(\frac{1}{10} - t\right) = \frac{7}{5} - 14t$ centered at points $(\pm 50t, 0)$ or $(0, \pm 50t)$ on the axes, where $0 \le t \le \frac{1}{10}$. The radius r of each circle is thus linearly related to c, the distance from its center to the origin $O = (0,0)$: namely $r = \frac{7}{5} - \frac{7}{25}c$, or $r = \frac{7}{25}(5 - c)$. Given any one of the circles centered on the positive x-axis, $\frac{r}{5-c} = \frac{7}{25}$ is the sine of the angle formed by the x-axis and a tangent drawn from $A = (5,0)$. It follows that these circles share two common external tangent lines, and that the region in question is a nonconvex octagon $ABCDEFGH$, where B is in the first quadrant. Notice that B is on the line $y = x$, which is one of the four axes of symmetry of the region. An equation for the common external tangent $\overline{AB}$ is $7x + 24y = 35$, because $\tan \angle OAB = \frac{7}{24}$. Set $x = y$ to find that $B = \left(\frac{35}{31}, \frac{35}{31}\right)$. The area of $ABCDEFGH$ is 8 times the area of OAB, or $8 \cdot \frac{1}{2} \cdot 5 \cdot \frac{35}{31} = \frac{700}{31}$. Thus $m + n = 731$.



14. (Answer: 571)

Let x denote the degree measure of angle QPB. Because angle AQP is exterior to isosceles triangle BQP, its measure is $2x$, and angle PAQ has the same measure. Because angle BPC is exterior to triangle BPA, its measure is $3x$. Let y denote the measure of angle PBC. It follows that the measure of angle ACB is $x + y$, and that $4x + 2y = 180$. Two of the angles of triangle APQ have measure $2x$, and thus the measure of angle APQ is $2y$. It follows that $AQ = 2 \cdot AP \cdot \sin y$. Because $AB = AC$ and $AP = QB$, it also follows that $AQ = PC$. Now apply the Law of Sines to triangle PBC to find that



$$\frac{\sin 3x}{BC} = \frac{\sin y}{PC} = \frac{\sin y}{2 \cdot AP \cdot \sin y} = \frac{1}{2 \cdot BC},$$

because $AP = BC$. Hence $\sin 3x = \frac{1}{2}$. This and $4x < 180$ imply that $3x = 30$ and $x = 10$. Thus $y = 70$ and $r = \dfrac{10+70}{2 \cdot 70} = \dfrac{4}{7}$, so $1000r = 571 + \dfrac{3}{7}$.

OR

Let $u = \angle ACB = \angle ABC$. Then $\angle A = \angle AQP = 180 - 2u$, $\angle APQ = 4u - 180$, $\angle QBP = \angle QPB = 90 - u$, $\angle BQC = \angle BCQ = 90 - \frac{1}{2}u$, $\angle CQP = \frac{5}{2}u - 90$, and $\angle QCP = \frac{3}{2}u - 90$. Apply the Law of Sines to triangles APQ and CPQ to obtain

$$\frac{\sin(4u - 180)}{\sin(180 - 2u)} = \frac{AQ}{AP} = \frac{PC}{PQ} = \frac{\sin(\frac{5}{2}u - 90)}{\sin(\frac{3}{2}u - 90)}.$$

This is equivalent to

$$\frac{-\sin 4u}{\sin 2u} = \frac{\cos \frac{5}{2}u}{\cos \frac{3}{2}u},$$

or $-2\cos 2u \cos \frac{3}{2}u = \cos \frac{5}{2}u$. Use the identity $2\cos\alpha\cos\beta = \cos(\alpha - \beta) + \cos(\alpha + \beta)$ to obtain $\cos\frac{7}{2}u + \cos\frac{5}{2}u + \cos\frac{1}{2}u = 0$, and then again to obtain $2\cos 3u \cos\frac{1}{2}u + \cos\frac{1}{2}u = 0$. This implies that $\cos 3u = -\frac{1}{2}$, so that $u = 40$ or $u = 80$. Because $4u$ must be greater than 180, it follows that $u = 80$. Thus $\angle APQ = 4u - 180 = 140$ and $r = 4/7$, as above.

15. (Answer: 927)

Run the process backwards. Start by picking up the card labeled 2000. Next pick up the card labeled 1999, place it on the top of the stack, and bring the bottom card to the top of the stack. Next pick up the card labeled 1998, place it on top of the stack, and bring the bottom card to the top of the stack. The card labeled 1999 is now at the top of a three-card stack. Notice that the top card of an m-card stack will become the top card of a $2m$-card stack after m more cards have been picked up (and m cards have been moved from the bottom of the stack to the top). It follows by induction that the card labeled 1999 is the top card when the number of cards in the stack is $3 \cdot 2^k$ for any nonnegative integer k that satisfies $3 \cdot 2^k < 2000$. In particular, the last time that this happens is just after $3 \cdot 2^9 = 1536$ cards have been picked up. The cards remaining on the table are labeled 1 through 464. After each of the cards labeled 464, 463, $\ldots$, 2 is picked up and placed on top of the stack, another card is brought from the bottom of the stack to the top. Finally, the card labeled 1 is placed on top of the stack and the stack is in its original state. This puts $2 \cdot 463 + 1 = 927$ cards on top of the card labeled 1999.

OR

Because the process causes the cards on the table to appear in ascending order, the card labeled 1999 is the next-to-last card placed on the table. To keep track of that card, first notice that, when a stack of 2^m cards is dealt in this way, the next-to-last card placed on the table begins at position 2^{m-1} in the stack; then apply the process to a stack of $2^{11} = 2048$ cards. After 48 of the cards have been placed on the table and 48 more cards have been moved from the top of the stack to the bottom, a 2000-card stack remains. Remove the cards that are on the table. The next-to-last card that will be placed on the table from the 2000-card stack is the card that began at position 1024 in the 2048-card stack. The position of that card in the 2000-card stack is $1024 - (48 + 48) = 928$, so the number of cards above it is 927.

MATHEMATICAL ASSOCIATION OF AMERICA

AMERICAN MATHEMATICS COMPETITIONS

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19th Annual

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Tuesday, March 27, 2001

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 630)

   Let a represent the tens digit and b the units digit of an integer with the required property. Then $10a+b$ must be divisible by both a and b. It follows that b must be divisible by a, and that $10a$ must be divisible by b. The former condition requires that $b = ka$ for some positive integer k, and the latter condition implies that $k = 1$ or $k = 2$ or $k = 5$. Thus the requested two-digit numbers are 11, 22, 33, ..., 99, 12, 24, 36, 48, and 15. Their sum is $11 \cdot 45 + 12 \cdot 10 + 15 = 630$.

2. (Answer: 651)

   Let $\mathcal{S}$ have n elements with mean x. Then

   $$\frac{nx+1}{n+1} = x - 13 \qquad \text{and} \qquad \frac{nx+2001}{n+1} = x + 27,$$

   or

   $$nx + 1 = (n+1)x - 13(n+1) \qquad \text{and} \qquad nx + 2001 = (n+1)x + 27(n+1).$$

   Subtract the third equation from the fourth to obtain $2000 = 40(n+1)$, from which $n = 49$ follows. Thus $x = 651$.

3. (Answer: 500)

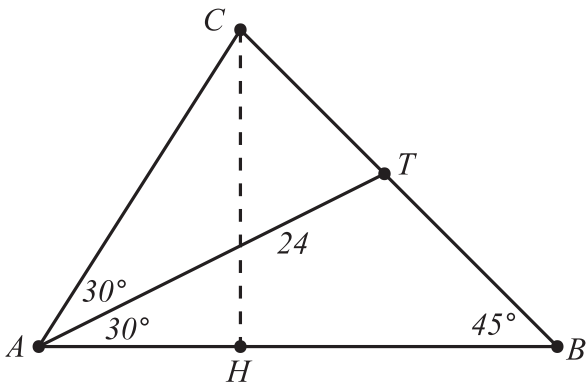
   Apply the binomial theorem to write

   $$\begin{aligned} 0 = x^{2001} + \left(\frac{1}{2} - x\right)^{2001} &= x^{2001} - \left(x - \frac{1}{2}\right)^{2001} \\ &= x^{2001} - x^{2001} + 2001 \cdot x^{2000}\left(\frac{1}{2}\right) - \frac{2001 \cdot 2000}{2} x^{1999}\left(\frac{1}{2}\right)^2 + \cdots \\ &= \frac{2001}{2} x^{2000} - 2001 \cdot 250 x^{1999} + \cdots . \end{aligned}$$

   The formula for the sum of the roots yields $2001 \cdot 250 \cdot \frac{2}{2001} = 500$.

4. (Answer: 291)

   Note that angles C and ATC each measure $75°$, so $AC = AT = 24$. Draw altitude $\overline{CH}$ of triangle ABC. Then triangle ACH is $30° - 60° - 90°$ and triangle BHC is $45° - 45° - 90°$. Now $AH = 12$ and $BH = CH = 12\sqrt{3}$. The area of triangle ABC is thus $(1/2)12\sqrt{3}(12 + 12\sqrt{3}) = 216 + 72\sqrt{3}$.

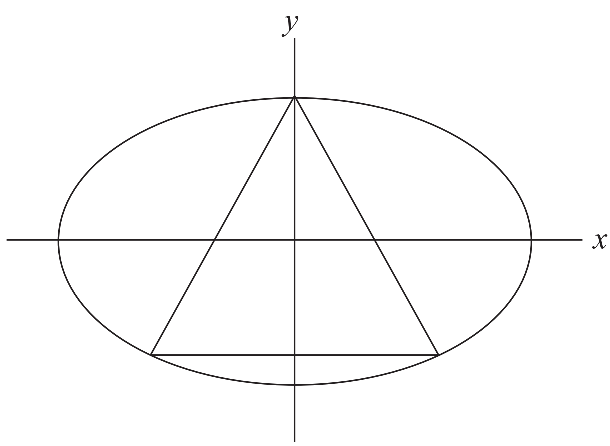


5. (Answer: 937)

   Let the other two vertices of the triangle be (x, y) and $(-x, y)$, with $x > 0$. Then the line through $(0, 1)$ and (x, y) forms a 120-degree angle with the positive x-axis, and its slope is $\tan(120°) = -\sqrt{3}$. Therefore, the line's equation is $y = -\sqrt{3}x + 1$. Substituting this into the equation of the ellipse and simplifying yields

$$13x^2 - 8\sqrt{3}x = 0 \qquad \text{or} \qquad x = \frac{8\sqrt{3}}{13}.$$

   The triangle has sides of length $2x = (16\sqrt{3})/13 = \sqrt{768/169}$, and $m + n = 937$.



OR

Let the other two vertices of the triangle be (x, y) and $(-x, y)$, with $x > 0$. Equating the squares of the distances from $(0, 1)$ to (x, y) and from $(-x, y)$ to (x, y) yields

$$x^2 + (y-1)^2 = 4x^2, \qquad \text{or} \qquad (y-1)^2 = 3x^2.$$

Substituting from the equation of the ellipse, it follows that $13y^2 - 2y - 11 = 0$. The roots of this quadratic are 1 and -11/13. If $y = 1$, then $x = 0$, so $y = -11/13$. Solving for x yields $x = \sqrt{192/169}$, so that the triangle has sides of length $2x = \sqrt{768/169}$, and $m + n = 937$.

Query: There are two other equilateral triangles with one vertex at $(0, 1)$ that are inscribed in the ellipse $x^2 + 4y^2 = 4$. Can you find the lengths of their sides?

6. (Answer: 079)

Any particular outcome of the four rolls has probability $1/6^4$. Given the values of four rolls, there is exactly one order that satisfies the requirement. It therefore suffices to count all the sets of values that could be produced by four rolls, allowing duplicate values. This is equivalent to counting the number of ways to put four balls into six boxes labeled 1 through 6. By thinking of 4 balls and 5 dividers to separate the six boxes, this can be seen to be $\binom{9}{4} = 126$. The requested probability is thus $126/6^4 = 7/72$, so $m + n = 79$.

OR

Let a_1, a_2, a_3, and a_4 be the sequence of values rolled, and consider the difference between the last and the first: If $a_4 - a_1 = 0$, then there is 1 possibility for a_2 and a_3, and 6 possibilities for a_1 and a_4. If $a_4 - a_1 = 1$, then there are 3 possibilities for a_2 and a_3, and 5 possibilities for a_1 and a_4. In general, if $a_4 - a_1 = k$, then there are $6 - k$ possibilities for a_1 and a_4, while the number of possibilities for a_2 and a_3 is the same as the number of sets of 2 elements, with repetition allowed, that can be chosen from a set of $k + 1$ elements. This is equal to the number of ways to put 2 balls in $k + 1$ boxes, or $\binom{k+2}{2}$. Thus there are $\sum_{k=0}^{5} \binom{k+2}{2}(6-k) = 126$ sequences of the type requested, so the probability is $126/6^4 = 7/72$, and $m + n = 79$.

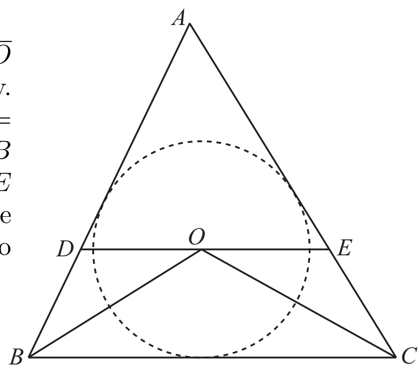
OR

Define an *acceptable* sequence to be one in which each element is between 1 and 6 and is at least as large as the preceding element. Let $A(x, n)$ be the number of acceptable sequences of length n beginning with x. Then, for $1 \le x \le 6$, $A(x, 1) = 1$, and $A(x, n)$ is equal to the number of acceptable sequences of length $n-1$ that begin with a value at least as large as x. That is, $A(x, n) = \sum_{i=x}^{6} A(i, n-1)$. Use this relationship to produce the table shown below. The requested probability is $\frac{56+35+20+10+4+1}{6^4}$ or 7/72.

x	A(x, 1)	A(x, 2)	A(x, 3)	A(x, 4)
1	1	6	21	56
2	1	5	15	35
3	1	4	10	20
4	1	3	6	10
5	1	2	3	4
6	1	1	1	1

7. (Answer: 923)

Let O be the incenter of triangle ABC, so that $\overline{BO}$ and $\overline{CO}$ bisect angles ABC and ACB, respectively. Because $\overline{DE}$ is parallel to $\overline{BC}$, it follows that $\angle DOB = \angle DBO$ and $\angle EOC = \angle ECO$, hence that $DO = DB$ and $EO = EC$. Thus the perimeter if triangle ADE is $AB + AC$. Triangle ADE is similar to triangle ABC, with the ratio of similarity equal to the ratio of perimeters. Therefore



$$\frac{DE}{BC} = \frac{AB + AC}{AB + AC + BC}.$$

Substituting the given values yields $DE = 860/63$, and $m + n = 923$.

OR

Let r be the radius of the inscribed circle, and h be the length of the altitude from A to $\overline{BC}$. Then the area of triangle ABC may be computed in two ways as

$$\frac{1}{2}BC \cdot h = \frac{1}{2}(AB + AC + BC)r, \quad \text{so that} \quad \frac{r}{h} = \frac{BC}{AB + AC + BC}.$$

Triangles ADE and ABC are similar, their ratio of similarity equal to the ratio of any pair of corresponding altitudes. Therefore

$$DE = \frac{h - r}{h}BC = \frac{(AB + AC)BC}{AB + AC + BC}.$$

As above, $m + n = 923$.

8. (Answer: 315)

Suppose that $a_k 7^k + a_{k-1}7^{k-1} + \cdots + a_2 7^2 + a_1 7 + a_0$ is a 7-10 double, with $a_k \neq 0$. In other words, $a_k 10^k + a_{k-1}10^{k-1} + \cdots + a_2 10^2 + a_1 7 + a_0$ is twice as large, so that

$$a_k(10^k - 2\cdot 7^k) + a_{k-1}(10^{k-1} - 2\cdot 7^{k-1}) + \cdots + a_2(10^2 - 2\cdot 7^2) + a_1(10 - 2\cdot 7) + a_0(1-2) = 0.$$

Since the coefficient of a_i in this equation is negative only when $i = 0$ and $i = 1$, and no a_i is negative, it follows that k is at least 2. Because the coefficient of a_i is at least 314 when $i > 2$, and because no a_i exceeds 6 it follows that $k = 2$ and $2a_2 = 4a_1 + a_0$. To obtain the largest possible 7-10 double, first try $a_2 = 6$. Then the equation $12 = 4a_1 + a_0$ has $a_1 = 3$ and $a_0 = 0$ as the solution with the greatest possible value of a_1. The largest 7-10 double is therefore $6 \cdot 49 + 3 \cdot 7 = 315$.

9. (Answer: 061)

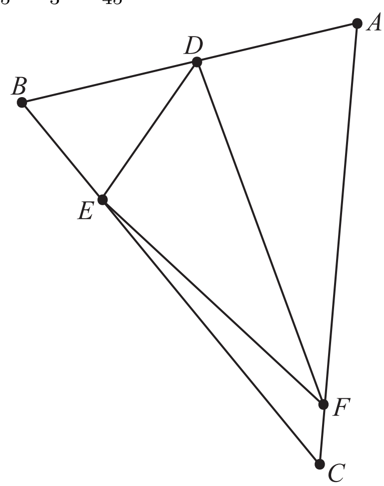
Let $[XYZ]$ denote the area of triangle XYZ. Because p, q, and r are all smaller than 1, it follows that

$$\begin{aligned}
[BDE] &= q(1-p)[ABC],\\
[EFC] &= r(1-q)[ABC],\\
[ADF] &= p(1-r)[ABC],\\
[ABC] &= [DEF] + [BDE] + [EFC] + [ADF]\\
&= [DEF] + \left((p+q+r) - (pq+qr+rp)\right)[ABC], \text{ and}\\
\frac{[DEF]}{[ABC]} &= 1 + pq + qr + rp - (p+q+r).
\end{aligned}$$

Note that

$$pq + qr + rp = \frac{1}{2}[(p+q+r)^2 - (p^2+q^2+r^2)] = \frac{1}{2}\left(\frac{4}{9} - \frac{2}{5}\right) = \frac{1}{45}.$$

Thus the desired ratio is $1 + \frac{1}{45} - \frac{2}{3} = \frac{16}{45}$ and $m + n = 61$.



10. (Answer: 200)

Because the points of S have integer coordinates, they are called *lattice points*. There are $60 \cdot 59 = 3540$ ways to choose a first lattice point and then a distinct second. In order for their midpoint to be a lattice point, it is necessary and sufficient that corresponding coordinates have the same parity. There are $2^2 + 1^2 = 5$ ways for the first coordinates to have the same parity, including 3 ways in which the coordinates are the same. There are $2^2 + 2^2 = 8$ ways for the second coordinates to have that same parity, including 4 ways in which the coordinates are the same. There are $3^2 + 2^2 = 13$ ways for the third coordinates to have the same parity, including 5 in which the coordinates are the same. It follows that there are $5 \cdot 8 \cdot 13 - 3 \cdot 4 \cdot 5 = 460$ ways to choose two distinct lattice points, so that the midpoint of the resulting segment is also a lattice point. The requested probability is $\frac{460}{3540} = \frac{23}{177}$, so $m + n = 200$.

OR

Because there are $3 \cdot 4 \cdot 5 = 60$ points to choose from, there are $\binom{60}{2} = 1770$ ways to choose the two points. In order that the midpoint of the segment joining the two chosen points also be a lattice point, it is necessary and sufficient that corresponding coordinates have the same parity. Notice that there are

$2 \cdot 2 \cdot 3 = 12$ points whose coordinates are all even,
$1 \cdot 2 \cdot 2 = 4$ points whose coordinates are all odd,
$1 \cdot 2 \cdot 3 = 6$ points whose only odd coordinate is x,
$2 \cdot 2 \cdot 3 = 12$ points whose only odd coordinate is y,
$2 \cdot 2 \cdot 2 = 8$ points whose only odd coordinate is z,
$2 \cdot 2 \cdot 2 = 8$ points whose only even coordinate is x,
$1 \cdot 2 \cdot 2 = 4$ points whose only even coordinate is y, and
$1 \cdot 2 \cdot 3 = 6$ points whose only even coordinate is z.

Thus the desired number of segments is

$$\frac{1}{2}(12 \cdot 11 + 4 \cdot 3 + 6 \cdot 5 + 12 \cdot 11 + 8 \cdot 7 + 8 \cdot 7 + 4 \cdot 3 + 6 \cdot 5) = 230,$$

so that the requested probability is $\frac{230}{1770} = \frac{23}{177}$.

11. (Answer: 149)

Suppose that P_i is in row i and column c_i. It follows that

$$x_1 = c_1, x_2 = N + c_2, x_3 = 2N + c_3, x_4 = 3N + c_4, x_5 = 4N + c_5$$

and

$$y_1 = 5c_1 - 4, y_2 = 5c_2 - 3, y_3 = 5c_3 - 2, y_4 = 5c_4 - 1, y_5 = 5c_5.$$

The P_i have been chosen so that

$$\begin{aligned} c_1 &= 5c_2 - 3 \\ N + c_2 &= 5c_1 - 4 \\ 2N + c_3 &= 5c_4 - 1 \\ 3N + c_4 &= 5c_5 \\ 4N + c_5 &= 5c_3 - 2 \end{aligned}$$

Use the first two equations to eliminate c_1, obtaining $24c_2 = N + 19$. Thus $N = 24k + 5$, where $k = c_2 - 1$. Next use the remaining equations to eliminate c_3 and c_4, obtaining $124c_5 = 89N + 7$. Substitute for N to find that $124c_5 = 2136k + 452$, and hence $31c_5 = 534k + 113 = 31(17k + 3) + 7k + 20$. In other words, $7k + 20 = 31m$ for some positive integer m. Now $7k = 31m - 20 = 7(4m - 2) + 3m - 6$. Since 7 must divide $3m - 6$, the minimum value for m is 2, and the smallest possible value of k is therefore 6, which leads to $N = 24 \cdot 6 + 5 = 149$. It is not difficult to check that $c_2 = 7$, $c_1 = 32$, $c_5 = 107$, $c_4 = 5c_5 - 3N = 88$, and $c_3 = 5c_4 - 1 - 2N = 141$ define an acceptable placement of points P_i. The numbers associated with the points are $x_1 = 32$, $x_2 = 156$, $x_3 = 439$, $x_4 = 535$, and $x_5 = 703$.

Note: Modular arithmetic can be used to simplify this solution.

12. (Answer: 005)

Let r be the radius of the inscribed sphere. Because $ABCD$ can be dissected into four tetrahedra, all of which meet at the incenter, have a height of length r, and have a face of the large tetrahedron as a base, it follows that r times the surface area of $ABCD$ equals three times the volume of $ABCD$. To find the area $[ABC]$ of triangular face ABC, first calculate $AB = \sqrt{52}$, $BC = \sqrt{20}$, and $CA = \sqrt{40}$. Then apply the Law of Cosines to find that $\cos\angle CAB = 9/\sqrt{130}$. It follows that $\sin\angle CAB = 7/\sqrt{130}$, so that $[ABC] = \frac{1}{2} \cdot AB \cdot AC \cdot \sin\angle CAB = 14$. The surface area of $ABCD$ is

$$[ABC] + [ABD] + [ACD] + [BCD] = 14 + 12 + 6 + 4 = 36.$$

The volume of tetrahedron $ABCD$ is $\frac{1}{3} \cdot 2 \cdot \frac{1}{2} \cdot 4 \cdot 6 = 8$. Thus $r = 24/36 = 2/3$ and $m + n = 5$.

OR

Because the sphere is tangent to the xy-plane, the yz-plane, and the xz-plane, its center is (r,r,r), where r is the radius of the sphere. An equation for the plane of triangle ABC is $2x+3y+6z=12$, so the sphere is tangent to this plane at $(r+2t, r+3t, r+6t)$, for some positive number t. Thus $2(r+2t)+3(r+3t)+6(r+6t)=12$ and $(2t)^2+(3t)^2+(6t)^2=r^2$, from which follow $11r+49t=12$ and $7t=r$, respectively. Combine these equations to discover that $r=2/3$ and $m+n=5$.

OR

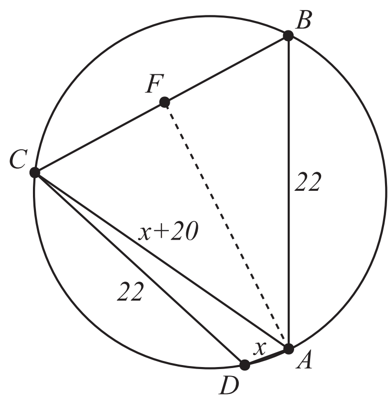
An equation of the plane of triangle ABC is $2x+3y+6z=12$. The distance from the plane to (r,r,r) is

$$\frac{|2r+3r+6r-12|}{\sqrt{2^2+3^2+6^2}}.$$

This leads to $\frac{|11r-12|}{7}=r$, which is satisfied by $r=3$ and $r=2/3$. Since $(3,3,3)$ is outside the tetrahedron, $r=2/3$ and $m+n=5$.

Query: The sphere determined by $r=3$ is outside the tetrahedron and tangent to the planes containing its faces. Can you find the radii of the other three spheres with this property?

13. (Answer: 174)



In the figure, points A, B, C, and D are concyclic, the degree sizes of arcs AB, BC, and CD are all d, and $AB=BC=CD=22$. Note that $\overline{AD}$ is the chord of a $3d$-degree arc. Let $AD=x$. Then $AC=x+20$, because $\overline{AC}$ is the chord of a $2d$-degree arc. In isosceles trapezoid $ABCD$, draw the altitude $\overline{AF}$ from A to $\overline{BC}$, and notice that F divides BC into $BF=11-\frac{x}{2}$ and $CF=11+\frac{x}{2}$. Because the right triangles AFC and AFB share the leg $\overline{AF}$, it follows that

$$(x+20)^2-(11+\frac{x}{2})^2=22^2-(11-\frac{x}{2})^2,$$

which simplifies to $x^2+18x-84=0$. Thus $x=-9+\sqrt{165}$ and $m+n=174$.

OR

Noting that $ABCD$ is a cyclic isosceles trapezoid, apply Ptolemy's Theorem to obtain $AB\cdot BD=BC\cdot AD+CD\cdot AB$, or $(x+20)^2=22x+22^2$. Solve the equation to find that $x=-9+\sqrt{165}$.

Query: If the restriction $d<120$ were removed, then the problem would have an additional solution. Can you find it?

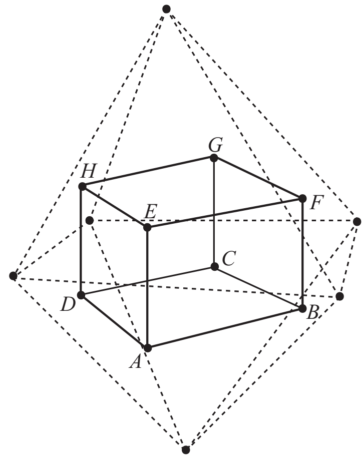
14. (Answer: 351)

The first condition implies that at most ten houses get mail in one day, while the second condition implies that at least six houses get mail. If six houses get mail, they must be separated from each other by a total of at least five houses that do not get mail. The other eight houses that do not get mail must be distributed in the seven spaces on the sides of the six houses that do get mail. This can be done in 7 ways: put two at each end of the street and distribute the other four in $\binom{5}{4} = 5$ ways, or put one in each of the seven spaces and an extra one at one end of the street or the other. If seven houses get mail, they create eight spaces, six of which must contain at least one house that does not get mail. The remaining six houses that do not get mail can be distributed among these eight spaces in 113 ways: six of the eight spaces can be selected to receive a single house in $\binom{8}{6} = 28$ ways; two houses can be placed at each end of the street and two intermediate spaces be selected in $\binom{6}{2} = 15$ ways; and two houses can be placed at one end of the street and four spaces selected for a single house in $2\binom{7}{4} = 70$ ways. Similar reasoning shows that there are $\binom{9}{1} + 1 + 2\binom{8}{2} = 183$ patterns when eight houses get mail, and $2 + \binom{10}{2} = 47$ patterns when nine houses get mail. When ten houses get mail, there is only one pattern, and thus the total number of patterns is $7 + 113 + 183 + 47 + 1 = 351$.

OR

Consider n-digit strings of zeros and ones, which represent no mail and mail, respectively. Such a sequence is called *acceptable* if it contains no occurrences of 11 or 000. Let f_n be the number of acceptable n-digit strings, let a_n be the number of acceptable n-digit strings in which 00 follows the leftmost 1, and let b_n be the number of acceptable n-digit strings in which 01 follows the leftmost 1. Notice that $f_n = a_n + b_n$ for $n \geq 5$. Deleting the leftmost occurrence of 100 shows that $a_n = f_{n-3}$, and deleting 10 from the leftmost occurrence of 101 shows that $b_n = f_{n-2}$. It follows that $f_n = f_{n-2} + f_{n-3}$ for $n \geq 5$. It is straightforward to verify the values of $f_1 = 2$, $f_2 = 3$, $f_3 = 4$, and $f_5 = 7$. Then the recursion can be used to find that $f_{19} = 351$.

15. (Answer: 085)



It is helpful to consider the cube $ABCDEFGH$ shown in the figure. The vertices of the cube represent the faces of the dotted octahedron, and the edges of the cube represent adjacent octahedral faces. Each assignment of the numbers 1,2,3,4,5,6,7, and 8 to the faces of the octahedron corresponds to a permutation of $ABCDEFGH$, and thus to an octagonal circuit of these vertices. The cube has 16 diagonal segments that join nonadjacent vertices. In effect, the problem asks one to count octagonal circuits that can be formed by eight of these diagonals. Six of the diagonals are edges of tetrahedron $ACFH$, six are edges of tetrahedron $DBEG$, and four are *long*, joining a vertex of one tetrahedron to the diagonally opposite point from the other. Notice that each vertex belongs to exactly one long diagonal. It follows that an octagon cannot have two successive long diagonals. Also notice that an octagonal path can jump from one tetrahedron to the other only along one of the long diagonals. it follows that an octagon must contain either 2 long diagonals separated by 3 tetrahedron edges or 4 long diagonals alternating with tetrahedron edges. To form an octagon that contains four long diagonals, choose two opposite edges from tetrahedron $ACFH$ and two opposite edges from tetrahedron $DBEG$. For each of the three ways to choose a pair of opposite edges from tetrahedron $ACFH$, there are two possible ways to choose a pair of opposite edges from tetrahedron $DBEG$. There are 6 distinct octagons of this type and $8 \cdot 2$ ways to describe each of them, making 96 permutations. To form an octagon that contains exactly two of the long diagonals, choose a three-edge path along tetrahedron $ACFH$, which can e done in $4! = 24$ ways. Then choose a three-edge path along tetrahedron $DBEG$ which, because it must start and finish at specified vertices, can be done in only 2 ways. Since this counting method treats each path as different from its reverse, there are $8 \cdot 24 \cdot 2 = 384$ permutations of this type. In all, there are $96 + 384 = 480$ permutations that correspond to octagonal circuits formed exclusively from cube diagonals.
The probability of randomly choosing such a permutation is $\frac{480}{8!} = \frac{1}{84}$, $m + n = 85$.
Note: The cube is called the emphdual of the octahedron.

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1. (Answer: 816)

The possible pairs of consecutive digits are 01, 04, 09, 16, 25, 36, 49, 64, and 81. Choosing 16 as the leftmost pair of digits would yield 1649 as the greatest number with the requested property. Similarly, 25 would yield 25, 36 would yield 3649, 49 would yield 49, 64 would yield 649, and 81 would yield 81649. Of these, 81649 is the largest, and the leftmost three digits are 816.

2. (Answer: 298)

Let s be the number of students who study Spanish but not French, let f be the number of students who study French but not Spanish, and let b be the number of students who study both languages. It is given that $1600.8 < s+b < 1700.85$ and $600.3 < f+b < 800.4$; thus $1601 \leq s+b \leq 1700$ and $601 \leq f+b \leq 800$. Add the last pair of inequalities to obtain $2202 \leq s+f+2b \leq 2500$. Because $s+f+b = 2001$, it follows that $201 \leq b \leq 499$. The triples $(s,f,b) = (1400, 400, 201)$ and $(s,f,b) = (1201, 301, 499)$ show that $m = 201$ and $M = 499$, so $M - m = 298$.

OR

By the Inclusion-Exclusion Principle,

$$M = \lfloor 0.85(2001) \rfloor + \lfloor 0.40(2001) \rfloor - 2001 = 1700 + 800 - 2001 = 499$$

and

$$m = \lceil 0.80(2001) \rceil + \lceil 0.30(2001) \rceil - 2001 = 1601 + 601 - 2001 = 201,$$

so $M - m = 499 - 201 = 298$.

3. (Answer: 898)
For $n > 5$,

$$\begin{aligned} x_n &= x_{n-1} - x_{n-2} + x_{n-3} - x_{n-4} \\ &= (x_{n-2} - x_{n-3} + x_{n-4} - x_{n-5}) - x_{n-2} + x_{n-3} - x_{n-4} \\ &= -x_{n-5}. \end{aligned}$$

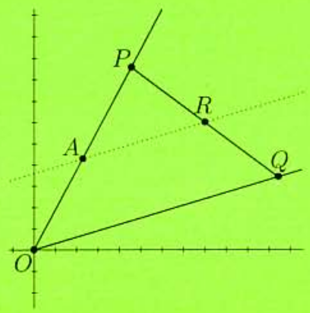
It follows that the sequence repeats in a cycle ten terms long. Hence

$$\begin{aligned} x_{531} + x_{753} + x_{975} &= x_1 + x_3 + x_5 \\ &= x_1 + x_3 + x_4 - x_3 + x_2 - x_1 \\ &= x_4 + x_2 \\ &= 523 + 375 = 898. \end{aligned}$$

OR

Using the theory of difference equations, a characteristic equation for the sequence is $x^4 = x^3 - x^2 + x - 1$ or $x^4 - x^3 + x^2 - x + 1 = 0$. Since $x^5 + 1 = (x+1)(x^4 - x^3 + x^2 - x + 1)$, we can conclude $x_n + x_{n-5} = 0$ and proceed as above.

4. (Answer: 067)
Let $O = (0, 0)$. The line through R that is parallel to $\overline{OQ}$ has equation $10y = 3x + 36$. This line meets $\overline{OP}$ at $A = \left(\frac{16}{7}, \frac{30}{7}\right)$. Because R is the midpoint of $\overline{PQ}$, it follows that A is the midpoint of $\overline{OP}$. Then $P = \left(\frac{32}{7}, \frac{60}{7}\right)$, and $PQ = 2PR = 2\sqrt{\left(\frac{24}{7}\right)^2 + \left(\frac{18}{7}\right)^2} = 2 \cdot \frac{6}{7} \cdot \sqrt{4^2 + 3^2} = \frac{60}{7}$. Thus $m + n = 67$.



OR

Let $P = (8t, 15t)$ and $Q = (10u, 3u)$. Because R is the midpoint of $\overline{PQ}$, it follows that

$$\begin{aligned} 8t + 10u &= 16 \text{ and} \\ 15t + 3u &= 12. \end{aligned}$$

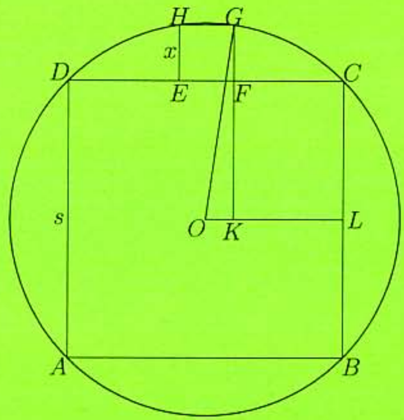
The solution to this system is $t = \dfrac{4}{7}$ and $u = \dfrac{8}{7}$, so $P = \left(\dfrac{32}{7}, \dfrac{60}{7}\right)$, $Q = \left(\dfrac{80}{7}, \dfrac{24}{7}\right)$, and $PQ = \dfrac{1}{7}\sqrt{48^2 + 36^2} = \dfrac{12}{7}\sqrt{4^2 + 3^2} = \dfrac{60}{7}$. Thus $m + n = 67$.

5. (Answer: 253)
The set $\{4, 5, 9, 14, 23, 37, 60, 97, 157, 254\}$ is a ten-element subset of $\{4, 5, 6, \ldots, 254\}$ that does not have the triangle property. Let N be the smallest integer for which $\{4, 5, 6, \ldots, N\}$ has a ten-element subset that lacks the triangle property. Let $\{a_1, a_2, a_3, \ldots, a_{10}\}$ be such a subset, with $a_1 < a_2 < a_3 < \cdots < a_{10}$. Because none of its three-element subsets define triangles, the following must be true:

$$\begin{aligned} N &\geq a_{10} \\ &\geq a_9 + a_8 \\ &\geq (a_8 + a_7) + a_8 = 2a_8 + a_7 \\ &\geq 2(a_7 + a_6) + a_7 = 3a_7 + 2a_6 \\ &\geq 3(a_6 + a_5) + 2a_6 = 5a_6 + 3a_5 \\ &\geq 8a_5 + 5a_4 \\ &\geq 13a_4 + 8a_3 \\ &\geq 21a_3 + 13a_2 \\ &\geq 34a_2 + 21a_1 \\ &\geq 34 \cdot 5 + 21 \cdot 4 = 254 \end{aligned}$$

Thus the largest possible value of n is $N - 1 = 253$.

6. (Answer: 251)
Let O be the center of the circle, and represent the lengths of each side of the small square and the large square by x and s, respectively . Draw $\overline{OL}$ perpendicular to $\overline{BC}$ at L and $\overline{FK}$ perpendicular to $\overline{OL}$ at K. Then $GK = GF + FK = GF + CL = x + \frac{s}{2}$, $OK = x/2$, and the circle's radius is $(1/2)s\sqrt{2}$. Applying the Pythagorean Theorem to triangle OKG, we obtain $(x + \frac{s}{2})^2 + (\frac{x}{2})^2 = (\frac{s\sqrt{2}}{2})^2$. Expanding yields $x^2 + sx + \frac{s^2}{4} + \frac{x^2}{4} = \frac{s^2}{2}$ which leads to $5x^2 + 4sx - s^2 = 0$, or $(5x - s)(x + s) = 0$, so $x = s/5$. The ratio of the squares' areas is thus $1/25$, and $10n + m = 251$.



7. (Answer: 725)

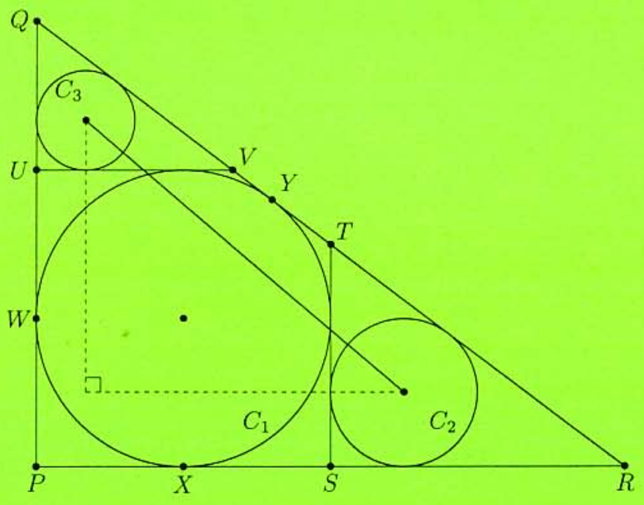
Let r_1, r_2, and r_3 be the radii of circles C_1, C_2, and C_3, respectively. The inradius of any triangle is twice the area divided by the perimeter, so $r_1 = 90 \cdot 120/(90+120+150) = 30$. Because $\triangle RST$ is similar to $\triangle RPQ$ and $RS = PR - 2r_1 = 60$, the similarity constant is $1/2$. Thus $r_2 = 15$. Similarly, $r_3 = 10$. If d is the distance between the centers of C_2 and C_3, then

$$\begin{aligned} d^2 &= (2r_1 + r_2 - r_3)^2 + (2r_1 + r_3 - r_2)^2 \\ &= 65^2 + 55^2 \\ &= 4225 + 3025 \\ &= 7250. \end{aligned}$$

Hence $n = 725$.

OR

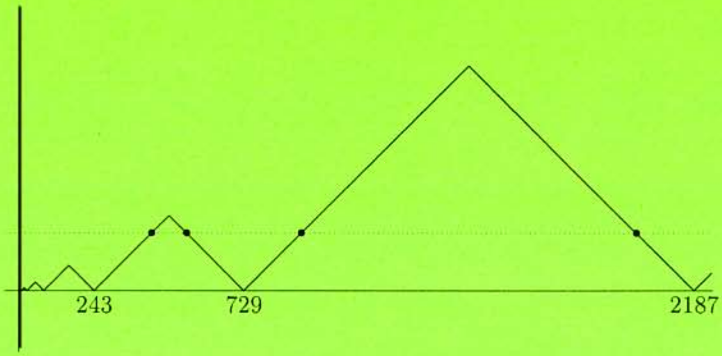
Let W, X, and Y be the points of tangency of circle C_1 to $\overline{PQ}$, $\overline{PR}$, and $\overline{RQ}$, respectively. Note that $PW = PX = r_1$. Then $RY = RX = 120 - r_1$, and $QY = QW = 90 - r_1$, from which we obtain $120 - r_1 + 90 - r_1 = 150$, and $r_1 = 30$. Assign coordinates so that $P = (0,0)$, $Q = (0,90)$, and $R = (120,0)$. Now $U = (0,60)$ and $S = (60,0)$. Because triangles UQV and STR are similar to triangle PQR with similarity constants $1/3$ and $1/2$, respectively, conclude that $r_3 = 10$ and $r_2 = 15$. Thus the centers of circles C_2 and C_3 have coordinates $(75,15)$ and $(10,70)$, respectively. Use the distance formula to find that $d^2 = 65^2 + 55^2 = 7250$.



8. (Answer: 429)
First calculate

$$f(2001) = 3f\left(\frac{2001}{3}\right) = 9f\left(\frac{2001}{9}\right) = \cdots = 729f\left(\frac{2001}{729}\right) = 729\left(1 - \frac{543}{729}\right) = 186.$$

For $1 \le x \le 3$, the graph of $y = f(x)$ consists of segments that join $(2,1)$ to $(1,0)$ and to $(3,0)$. The definition of f implies that $(3a, 3b)$ is on the graph of f whenever (a,b) is, so the positive x-axis and the graph of f form a sequence of isosceles right triangles, each a threefold magnification of its predecessor. Notice that 3^n is the altitude of the triangle whose left vertex is $(3^n, 0)$ and whose right vertex is $(3^{n+1}, 0)$. Because the line $y = 186$ intersects only those triangles whose altitudes are at least 186, the leftmost intersection point is found, as shown, in the triangle whose left vertex is $(243, 0)$ and whose right vertex is $(729, 0)$. The desired point is found on a segment of slope 1, so $x = 243 + 186 = 429$.



9. (Answer: 929)
Number the squares as shown. For i=1, 2, 4, and 5, let Q_i be the event that i is the upper left corner of a 2-by-2 red square, and let $p(E)$ be the probability that event E will occur. By the Inclusion-Exclusion Principle, the probability that the grid *does* have at least one 2-by-2 red square is

1	2	3
4	5	6
7	8	9

$$\begin{aligned}
&p(Q_1) + p(Q_2) + p(Q_4) + p(Q_5) \\
&-p(Q_1 \cap Q_2) - p(Q_1 \cap Q_4) - p(Q_2 \cap Q_5) - p(Q_4 \cap Q_5) - p(Q_1 \cap Q_5) - p(Q_2 \cap Q_4) \\
&+p(Q_1 \cap Q_2 \cap Q_5) + p(Q_1 \cap Q_2 \cap Q_4) + p(Q_1 \cap Q_4 \cap Q_5) + p(Q_2 \cap Q_4 \cap Q_5) \\
&-p(Q_1 \cap Q_2 \cap Q_4 \cap Q_5)
\end{aligned}$$

or

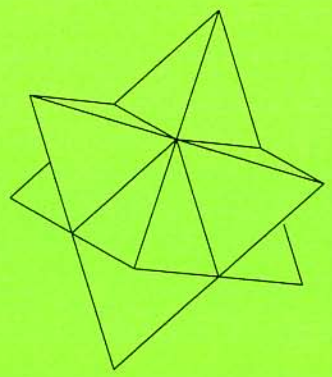
$$4\left(\frac{1}{2}\right)^4 - \left[4\left(\frac{1}{2}\right)^6 + 2\left(\frac{1}{2}\right)^7\right] + 4\left(\frac{1}{2}\right)^8 - \left(\frac{1}{2}\right)^9 = \frac{95}{512}.$$

The probability that the grid does *not* have at least one 2-by-2 red square is therefore $1 - 95/512 = 417/512$, so $m + n = 929$.

10. (Answer: 784)
Because
$$10^j - 10^i = 10^i(10^{j-i} - 1),$$
and $1001 = 7 \cdot 11 \cdot 13$ is relatively prime to 10^i, it is necessary to find i and j so that $10^{j-i} - 1$ is divisible by the primes 7, 11, and 13. Notice that 10^6 is the smallest power of 10 that leaves a remainder of 1 when divided by 7 or 13, and that 10^2 is the smallest power of 10 that leaves a remainder of 1 when divided by 11. Hence $10^i(10^{j-i} - 1)$ is divisible by 1001 if and only if $j - i = 6n$ for some positive integer n. Thus it is necessary to count the number of integer solutions to
$$i + 6n = j, \text{ where } j \le 99, i \ge 0, n > 0.$$
For each $n = 1, 2, 3, \ldots, 16$, there are $100 - 6n$ suitable values of i (and j), so the number of solutions is
$$94 + 88 + 82 + \cdots + 4 = 784.$$

11. (Answer: 341)
The probability P that the team has more wins than losses is the same as the probability that the team has more losses than wins, and hence $P = \frac{1}{2}(1-S)$, where S is the probability that Truncator has the same number of wins as losses. The probability of three wins and three losses is $\binom{6}{3}\left(\frac{1}{3}\right)^6$, the probability of two wins and two losses is $\binom{6}{2}\binom{4}{2}\left(\frac{1}{3}\right)^6$, the probability of one win and one loss is $\binom{6}{1}\binom{5}{1}\left(\frac{1}{3}\right)^6$, and the probability of no wins and no losses is $\left(\frac{1}{3}\right)^6$. Therefore $S = (20 + 90 + 30 + 1)\left(\frac{1}{3}\right)^6 = \frac{141}{729} = \frac{47}{243}$, and $P = \frac{1}{2} \cdot \left(1 - \frac{47}{243}\right) = \frac{98}{243}$. Thus $m + n = 98 + 243 = 341$.

12. (Answer: 101)
The diagram shows $\mathcal{P}_1$. Notice that $\mathcal{P}_0$ has 4 triangular faces, $\mathcal{P}_1$ has 24, and, inductively, $\mathcal{P}_i$ has $4 \cdot 6^i$. This expression therefore counts the small tetrahedra that are attached to $\mathcal{P}_i$ to form $\mathcal{P}_{i+1}$. The volume of each of these small tetrahedra is $(\frac{1}{8})^{i+1}$, and hence the volume of $\mathcal{P}_{i+1}$ is $4 \cdot 6^i(\frac{1}{8})^{i+1} = (\frac{1}{2})(\frac{3}{4})^i$ more than the volume of $\mathcal{P}_i$. In particular, the volume of $\mathcal{P}_3$ is



$$1 + \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)^2 = \frac{69}{32}.$$
Thus $m + n = 101$.

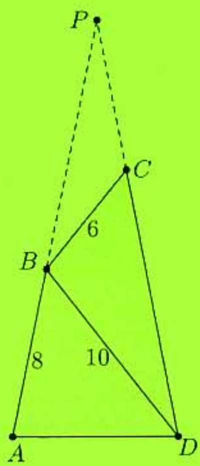
3. (Answer: 069)

Let P be the point where $\overline{AB}$ and $\overline{DC}$ intersect when extended. Since angles PCB and PBD are supplements of angles BCD and ABD, respectively, angles PCB and PBD are congruent, which implies that triangles PCB and PBD are similar. Now

$$\frac{PD-8}{PD}=\frac{PB}{PD}=\frac{CB}{BD}=\frac{6}{10},$$

so $PA = PD = 20$ and $PB = 12$. Then $\dfrac{PC}{12}=\dfrac{6}{10}$, so $PC = 7.2$ and $CD = 12.8 = 64/5$. Thus $m+n = 69$.

Query: Can you prove that B is between A and P and that C is between D and P?



4. (Answer: 840)

Because $|z| = 1, z = \cos\theta + i\sin\theta$, with $0 \le \theta < 360$. Now $\cos 28\theta - \cos 8\theta = 1$ and $\sin 28\theta - \sin 8\theta = 0$. From the latter equation, conclude that $28\theta + 8\theta = 180 + 360k$ for some integer k, so $\theta = 10k + 5$. It follows from the former equation that

$$-2\sin\frac{28\theta+8\theta}{2}\sin\frac{28\theta-8\theta}{2}=1,$$

which is equivalent to $\sin 18\theta \sin 10\theta = -\frac{1}{2}$. Substitute $\theta = 10k+5$ to obtain $\sin(180k + 90)\sin(100k+50) = -\frac{1}{2}$, or $\sin(100k+50) = (-1)^{k+1}\cdot\frac{1}{2}$. When $k = 2m-1$ (that is, when k is odd) $\sin(200m-50) = \frac{1}{2}$. Then $200m - 50 \equiv 30$ or $150 \pmod{360}$ yields $m \equiv 4$ or $1 \pmod 9$, $k \equiv 7$ or $1 \pmod{18}$, and $\theta \equiv 75$ or $15 \pmod{180}$. When $k = 2m$, similar reasoning leads to $\theta \equiv 165$ or $105 \pmod{180}$. Thus $\theta \equiv \pm 15 \pmod{90}$ and $\theta_2 + \theta_4 + \theta_6 + \theta_8 = \Sigma_{k=1}^{4}(90k - 15) = 840$.

OR

Let $\operatorname{cis}\theta$ denote $\cos\theta + i\sin\theta$. If z satisfies $z^{28} - z^8 - 1 = 0$, then $z^8(z^{20}-1) = 1$. Because $|z| = 1$, it follows that $|z^{20} - 1| = 1$ and $|z^{20}| = 1$. This can happen only if $z^{20} = \operatorname{cis}(\pm 60)$, in which case $z^{20} - 1 = \operatorname{cis}(\pm 120)$. Therefore $z^8 = \operatorname{cis}(\mp 120)$ and $z^4 = z^{20}/(z^8)^2 = \operatorname{cis}(\pm 300) = \operatorname{cis}(\mp 60)$, which implies that $z = \operatorname{cis}(90k \mp 15)$ for some integer k. Such z satisfy $z^{28} = \operatorname{cis}(\mp 60) = 1 + \operatorname{cis}(\mp 120) = 1 + z^8$. Thus the equation $z^{28} - z^8 - 1 = 0$ has eight solutions on the unit circle, namely $\theta_1 = 15$, $\theta_2 = 75$, $\theta_3 = 105$, $\theta_4 = 165$, $\theta_5 = 195$, $\theta_6 = 255$, $\theta_7 = 285$, and $\theta_8 = 345$. It follows that $\theta_2+\theta_4+\theta_6+\theta_8 = 840$.

OR

With $z = \cos\theta + i\sin\theta$, write the equation as $z^{18}(z^{10} - z^{-10}) = 1$, and notice that $z^{10} - z^{-10} = 2i\sin 10\theta$. Then, taking the absolute value of each side of the original equation, $2\sin 10\theta = \pm 1$, and thus z is a solution with $|z| = 1$ if and only if

$$z^{18} = \mp i \text{ and } \sin 10\theta = \pm 1/2.$$

Hence the desired values of θ satisfy $18\theta \equiv 270$ or $90 \pmod{360}$ with $10\theta \equiv 30$ or $150 \pmod{360}$ in the first case and $10\theta \equiv 210$ or $330 \pmod{360}$ in the second. Thus

$$\theta \equiv 15 \pmod{20} \text{ and } \theta \equiv 3 \text{ or } 15 \pmod{36}$$

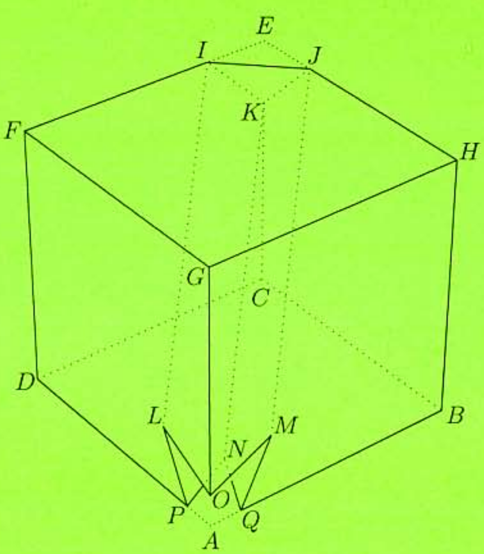
or

$$\theta \equiv 5 \pmod{20} \text{ and } \theta \equiv 21 \text{ or } 33 \pmod{36}.$$

The smallest positive solutions are 75 and 15 in the first case, and 165 and 105 in the second. Solutions are congruent modulo 180, so the solutions between 0 and 360 are 15, 75, 105, 165, 195, 255, 285, 345. Thus $\theta_2 + \theta_4 + \theta_6 + \theta_8 = 840$.

15. (Answer: 417)

Assign coordinates $A = (0,0,0)$, $B = (8,0,0)$, $C = (8,8,0)$, $D = (0,8,0)$, $I = (6,8,8)$, $J = (8,6,8)$, and $K = (8,8,6)$. The line through I that is parallel to $\overline{AE}$ can be described by $(x,y,z) = (6-t, 8-t, 8-t)$, so this line meets the cube again when $x = 0$, at $L = (0,2,2)$. By symmetry, the lines through J and K that are parallel to $\overline{AE}$ intersect the cube again at $M = (2,0,2)$ and $N = (2,2,0)$, respectively. It is straightforward to show that the plane determined by I, J, and L is described by the equation $2z = 2 + x + y$, so that the plane meets the z-axis at $O = (0,0,1)$. By symmetry, the tunnel intersects the x-axis at $Q = (1,0,0)$ and the y-axis at $P = (0,1,0)$. As the diagram shows, one end of the tunnel has a triangular opening IJK, while the other has a non-planar hexagonal opening $LOMQNP$. The surface of $\mathcal{S}$ consists of nine polygonal faces, three of each of three types. It is straightforward to show that the area of pentagon $IFGHJ$ and the area of hexagon $CDPNQB$ are both $8^2 - 2$. To find the area of pentagon $ILOMJ$, first obtain $IJ = 2\sqrt{2}$, $IL = JM = 6\sqrt{3}$, and $LO = OM = \sqrt{5}$. Then calculate the area of rectangle $ILMJ$ to be $12\sqrt{6}$ and the area of isosceles triangle LOM to be $\sqrt{6}$. Thus the total surface area of $\mathcal{S}$ is $3(62 + 62 + 13\sqrt{6}) = 372 + 39\sqrt{6}$, and $m + n + p = 417$.



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SOLUTIONS PAMPHLET

Tuesday, March 26, 2002

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 059)
The probability that a license plate will have a three-letter palindrome is $\frac{(26^2)}{(26^3)} = \frac{1}{26}$ because there are 26 possibilities for each of the first two letters, and the third letter must be the same as the first. Similarly, the probability that a license plate will have a three-digit palindrome is $(10^2)/(10^3) = 1/10$. The probability that a license plate will have both a three-letter palindrome and a three-digit palindrome is $(1/26)(1/10) = 1/260$. Apply the Inclusion-Exclusion Principle to conclude that the probability that a license plate will have at least one palindrome is

$$\frac{1}{26} + \frac{1}{10} - \frac{1}{260} = \frac{35}{260} = \frac{7}{52}\,.$$

Thus $m + n = 59$.

2. (Answer: 154)
Let r be the radius of each circle, and let l and w be the dimensions of the rectangle with $l > w$. Then $14r = l$. Now consider the equilateral triangle whose vertices are the centers of any three mutually tangent circles. The height of such a triangle is $r\sqrt{3}$, so $w = 2r + 2r\sqrt{3}$. It follows that

$$\frac{l}{w} = \frac{14r}{2r(1+\sqrt{3})} = \frac{7(\sqrt{3}-1)}{2} = \frac{\sqrt{147}-7}{2},$$

so $p + q = 154$.

3. (Answer: 025)
Let Dick's age and Jane's age in n years be $10x + y$ and $10y + x$, respectively. At that time, Dick will be $9(x - y)$ years older than Jane, and the sum of their ages will be $11(x + y)$. Dick's age must always exceed Jane's by a multiple of 9; thus Dick's current age must be 34, 43, 52, 61, 70, 79, 88, or 97. Suppose that Dick is 34, so that the sum of their ages is 59. Their age-sum is therefore always odd, and it is not a multiple of 11 until it reaches 77. This takes $n = \frac{1}{2}(77 - 59) = 9$ years. Every 11 years thereafter, as long as Dick has a two-digit age, their ages will be reversals of each other; Dick's ages at those times are 43, 54, 65, 76, 87, and 98. Similar reasoning applies if Dick's current age is 43, and their age-sum is 68: the next age-sum that has the same parity and is divisible by 11 is 88, when Dick is 53 and $n = 10$. Every 11 years thereafter, until Dick is 97, their ages are reversals of each other — five examples in all. Similarly, there are four examples if Dick's current age is 52, four examples if his current age is 61, three examples if his current age is 70, two examples if his current age is 79, one example if his current age is 88, and none if his current age is 97. The total number of ordered pairs is thus $6 + 5 + 4 + 4 + 3 + 2 + 1 = 25$.

OR

Since Dick must always be older than Jane, in n years Jane may be 26, 27, 28, 29, 34,…, 39, 45, …, 49, 56, …, 59, 67, 68, 69, 78, 79 or 89. Dick's age will be the result of reversing the digits of Jane's age. The total number of ordered pairs is thus $4+6+5+4+3+2+1=25$.

4. (Answer: 840)
Because
$$\frac{1}{k^2+k}=\frac{1}{k(k+1)}=\frac{1}{k}-\frac{1}{k+1}\,,$$
the series telescopes, that is,
$$\begin{aligned}1/29 &= a_m+a_{m+1}+\cdots+a_{n-1}\\ &=\left(\frac{1}{m}-\frac{1}{m+1}\right)+\left(\frac{1}{m+1}-\frac{1}{m+2}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right),\end{aligned}$$
so $(1/m)-(1/n)=1/29$. Since neither m nor n is 0, this is equivalent to $mn+29m-29n=0$, from which we obtain $(m-29)(n+29)=-29^2$, or $(29-m)(29+n)=29^2$. Since 29 is a prime and $29+n>29-m$, it follows that $29-m=1$ and $29+n=29^2$. Thus $m=28$, $n=29^2-29$, and $m+n=29^2-1=30\cdot 28=840$.

5. (Answer: 183)
Each pair $\{A_i,A_j\}$ of the $\binom{12}{2}=66$ pairs of vertices generates three squares, one having diagonal A_iA_j, and the other two having A_iA_j as a side. However, each of the three squares $A_1A_4A_7A_{10}$, $A_2A_5A_8A_{11}$, and $A_3A_6A_9A_{12}$ is counted six times. The total number of squares is therefore $3\cdot 66-15=183$.

6. (Answer: 012)
Let $p=\log_{225}x=1/\log_x 225$ and $q=\log_{64}y=1/\log_y 64$. The given equations then take the form $p+q=4$ and $\dfrac{1}{p}-\dfrac{1}{q}=1$, whose solutions are $(p_1,q_1)=(3+\sqrt{5},1-\sqrt{5})$ and $(p_2,q_2)=(3-\sqrt{5},1+\sqrt{5})$. Thus $x_1x_2=225^{p_1}225^{p_2}=225^{p_1+p_2}=225^6$, $y_1y_2=64^{q_1+q_2}=$
64^2, and $\log_{30}(x_1y_1x_2y_2)=\log_{30}\left(225^6 64^2\right)=\log_{30}\left(15^{12}2^{12}\right)=\log_{30}30^{12}=12$.

7. (Answer: 428)
Apply the Binomial Expansion to obtain
$$(10^{2002}+1)^{10/7}=10^{2860}+\frac{10}{7}\cdot 10^{3\cdot 286}+\frac{\frac{10}{7}\cdot\frac{3}{7}}{2}\cdot 10^{-4\cdot 286}+\ldots$$
Thus, only the second term affects the requested digits. Since $1/7=.\overline{142857}$ and 6 is a divisor of $3\cdot 286$, conclude that
$$\frac{10}{7}\cdot 10^{3\cdot 286}=1428571\ldots 571.\overline{428571},$$
so the answer is 428.

8. (Answer: 748)
Suppose $a_1 = x_i$ and $a_2 = x_i + h_i$ for $i = 1, 2$, with $x_2 > x_1 > 0$ and $h_1 > h_2 \geq 0$, so

$$a_9 = 34x_1 + 21h_1 = k = 34x_2 + 21h_2.$$

If h_2 were greater than zero, then k would not be the smallest integer for which the equation $34x + 21h = k$ has a non-unique solution, since $34x_1 + 21(h_1 - h_2) = 34x_2$ would yield a smaller k. Thus,

$$34x_1 + 21h_1 = 34x_2, \quad \text{that is,} \quad 21h_1 = 34(x_2 - x_1),$$

so h_1 must be a positive multiple of 34, and x_2 and x_1 must differ by a multiple of 21. The smallest possible values of h_1, h_2, x_1, x_2, and a_9 that satisfy these conditions and those of the problem are thus $h_1 = 34$, $h_2 = 0$, $x_1 = 1$, $x_2 = 22$, and $a_9 = 34 \cdot 22 + 21 \cdot 0 = 748$. Note that the sequences

$$1, 35, 36, 71, 107, 178, 285, 463, 748, \ldots$$
$$22, 22, 44, 66, 110, 176, 286, 462, 748, \ldots$$

both have $k = 748$ as their ninth term.

OR

Note that $a_9 = 13a_1 + 21a_2$, so the requested value of k is the least positive integer k such that $13x + 21y = k$ has more than one solution (x, y) with $0 < x \leq y$ and x and y integers. If k has this property, then there are integers x, y, u and v with $0 < x < u \leq v < y$ and

$$13x + 21y = k = 13u + 21v.$$

Then $21(y - v) = 13(u - x)$ which implies that $u - x$ is divisible by 21. Thus $u - x \geq 21$ and $v \geq u \geq 22$. Now

$$k = 13u + 21v \geq 13 \cdot 22 + 21 \cdot 22 = 748.$$

To demonstrate that $13x + 21y = 748$ has more than one solution, rewrite the equation as $13(x + y) + 8y = 57 \cdot 13 + 7$, and conclude that 13 must be a divisor of $(8y - 7)$. A few trials reveal that $y = 9$ satisfies this condition. Thus $(43, 9), (43 - 21, 9 + 13) = (22, 22)$, and $(43 - 2 \cdot 21, 9 + 2 \cdot 13) = (1, 35)$ are solutions. Note that $(22, 22)$ and $(1, 35)$ yield the previously mentioned sequences, and $(43, 9)$ yields a sequence that satisfies conditions (2) and (3), but not (1).

9. (Answer: 757)
Let the pickets be numbered consecutively $1, 2, 3, \ldots$. Let H, T, and U be the sets of numbers assigned to the pickets painted by Harold, Tanya, and Ulysses, respectively. Then

$$H = \{1, 1+h, 1+2h, 1+3h, \ldots\}$$
$$T = \{2, 2+t, 2+2t, 2+3t, \ldots\}$$
$$U = \{3, 3+u, 3+2u, 3+3u, \ldots\}.$$

Each picket will be painted exactly once if and only if H, T, and U partition the set of positive integers into mutually disjoint subsets. Clearly, h, t and u are each greater than 1. In fact, $h \geq 3$, since if $h = 2$, then 3 is in H. Also $h < 5$, because if $h \geq 5$, then 4 cannot be in H; since 4 cannot be in U, 4 would have to be in T, making $T = \{2, 4, 6, \ldots\}$, which would make $U = \{3, 5, 7, \ldots\}$, since 5 is not in H. But this leaves no possible value for h. Thus $h = 3$ or $h = 4$. When $h = 3$, $H = \{1, 4, 7, \ldots\}$. Now 5 cannot be in U because 7 would be too, but 7 is in H. So 5 is in T, and $T = \{2, 5, 8, \ldots\}$, which means that $U = \{3, 6, 9, \ldots\}$. When $h = 4$, $H = \{1, 5, 9, \ldots\}$. Since 4 cannot be in U, $T = \{2, 4, 6, \ldots\}$, so $U = \{3, 7, 11, \ldots\}$. The two paintable integers are 333 and 424, whose sum is 757.

10. (Answer: 148)
By the Angle-Bisector Theorem, $BD : DC = AB : AC = 12 : 37$, and thus the area of triangle ADC is 37/49 of the area of triangle ABC. By the Angle-Bisector Theorem, $EG : GF = AE : AF = 3 : 10$, and thus the area of triangle AGF is 10/13 of the area of triangle AEF. The area of triangle AEF is 3/12 of the area of triangle AFB, which is in turn 10/37 of the area of triangle ABC. Since $BC = \sqrt{37^2 - 12^2} = 35$, the area of triangle ABC is 210. It follows that the area of quadrilateral $DCFG$ is

$$\left(\frac{37}{49} - \frac{10}{13} \cdot \frac{3}{12} \cdot \frac{10}{37}\right) 210 = \frac{1110}{7} - \frac{5250}{481} = 158\frac{4}{7} - 10\frac{440}{481},$$

so the requested integer is 148.

11. (Answer: 230)
Place a coordinate system on the cube so that $A = (0,0,0)$, $B = (12,0,0)$, $C = (12,12,0)$, $D = (0,12,0)$, and $P = (12,7,5)$. Point P is the first point where the light hits a face of the cube. Let P_2 be the second point at which the light hits a face, and consider the reflection of the cube in face $BCFG$. Then P_2 is the image of the point at which ray AP next intersects a face of the reflected cube. Continue this process so that the $(k+1)^{\text{st}}$ cube is obtained by reflecting the k^{th} cube in the face containing P_k for $k \geq 2$. Therefore, each intersection of ray AP and a plane with equation $x = 12n$, $y = 12n$, or $z = 12n$, where n is a positive integer, corresponds to a point where the light beam hits a face of the cube. Thus the path will first return to a vertex of the cube when ray AP reaches a point whose coordinates are all multiples of 12. The points on ray AP have coordinates of the form $(12t, 7t, 5t)$, where t is nonnegative, and they will all be multiples of 12 if and only if t is a multiple of 12. This first happens when $t = 12$, which yields the point $(144, 84, 60)$. The requested distance is the same as the distance from this point to A, namely, $12\sqrt{12^2 + 7^2 + 5^2} = 12\sqrt{218}$, so $m + n = 230$.

12. (Answer: 275)
Calculate

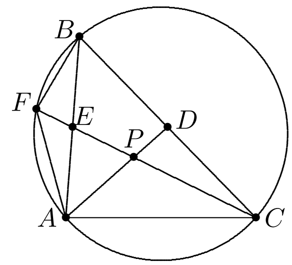
$$F(F(z)) = \frac{\dfrac{z+i}{z-i} + i}{\dfrac{z+i}{z-i} - i} = \frac{z+i+iz+1}{z+i-iz-1} = \frac{1+i}{1-i} \cdot \frac{z+1}{z-1} = i \cdot \frac{z+1}{z-1}$$

and

$$F(F(F(z))) = F\left(i \cdot \frac{z+1}{z-1}\right) = \frac{i \cdot \dfrac{z+1}{z-1} + i}{i \cdot \dfrac{z+1}{z-1} - i} = \frac{z+1+z-1}{z+1-(z-1)} = z,$$

which shows that $z_n = z_{n-3}$ for all $n \geq 3$. In particular, $z_{2002} = z_{2002-667\cdot 3} = z_1 = \dfrac{\frac{1}{137} + 2i}{\frac{1}{137}} = 1 + \dfrac{2}{\frac{1}{137}} i = 1 + 274i$, and $a + b = 275$.

13. (Answer: 063)



Let P be the intersection of $\overline{AD}$ and $\overline{CE}$. Since angles ABF and ACF intercept the same arc, they are congruent, and therefore triangles ACE and FBE are similar. Thus $EF/12 = 12/27$, yielding $EF = 16/3$. The area of triangle AFB is twice that of triangle AEF, and the ratio of the area of triangle AEF to that of triangle AEP is $\frac{16/3}{9}$, since the medians of a triangle trisect each other. Triangle AEP is isosceles, so the altitude to base $\overline{PE}$ has length $\sqrt{12^2 - (9/2)^2} = (1/2)\sqrt{24^2 - 9^2} = (3/2)\sqrt{8^2 - 3^2} = (3/2)\sqrt{55}$, and the area of triangle AEP is $(27/4)\sqrt{55}$. Therefore, $[AFB] = 2[AFE] = 2(16/27) \cdot [AEP] = 2(16/27) \cdot (27/4)\sqrt{55} = 8\sqrt{55}$, and $m + n = 63$.

14. (Answer: 030)

Let x_1, x_2, x_3, $\ldots$, x_n be the members of $\mathcal{S}$, and let

$$s_j = \frac{x_1 + x_2 + x_3 + \cdots + x_n - x_j}{n-1} .$$

It is given that s_j is an integer for any integer j between 1 and n, inclusive. Note that, for any integers i and j between 1 and n, inclusive,

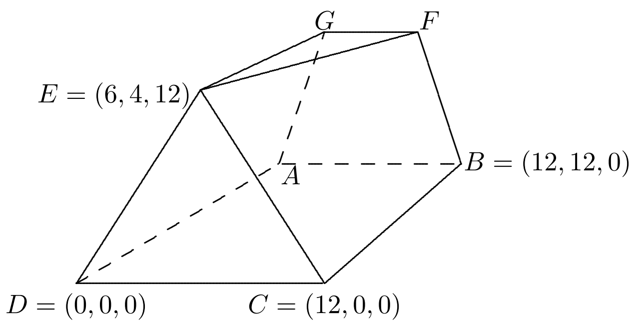
$$s_i - s_j = \frac{x_j - x_i}{n-1},$$

which must be an integer. Also, $x_j = (s_i - s_j)(n-1) + x_i$, and when $x_i = 1$, this implies that each element of $\mathcal{S}$ is 1 more than a multiple of $n-1$. It follows that $(n-1)^2 + 1 \le 2002$, implying that $n \le 45$. Since $n-1$ is a divisor of $2002 - 1$, conclude that $n = 2$ or $n = 4$ or $n = 24$ or $n = 30$, so n is at most 30. A thirty-element set $\mathcal{S}$ with the requested property is obtained by setting $x_j = 29j - 28$ for $1 \le j \le 29$ and $x_{30} = 2002$.

15. (Answer: 163)
Place a coordinate system on the figure so that square $ABCD$ is in the xy-plane, as shown in the diagram. Let $E = (x_1, y_1, 12)$. Because $DE = CE$, it follows that $x_1 = 6$. Because $DE = 14$, it follows that $14^2 = 6^2 + (y_1)^2 + 12^2$, so that $y_1 = 4$. Let $\overline{GK}$ be an altitude of isosceles trapezoid $ABFG$, and notice that the x-coordinates of both G and K are equal to $\frac{1}{2}(AB - GF) = 3$. To find the y-coordinate of G, let $ax + by + cz = d$ be an equation of the plane determined by A, D, and E. Substitute the coordinates of these three points to find that $12b = d$, $0 = d$, and $6a + 4b + 12c = d$, respectively, from which it follows that $b = d = 0$ and $a + 2c = 0$. Thus $G = (3, y_2, z_2)$ lies on the plane $z = 2x$, so $z_2 = 6$. Because $GA = 8$, it follows that $8^2 = 3^2 + (y_2 - 12)^2 + 6^2$, so $y_2 = 12 \pm \sqrt{19}$. Thus

$$EG^2 = (6-3)^2 + (4 - (12 \pm \sqrt{19}))^2 + (12-6)^2 = 128 \pm 16\sqrt{19},$$

and $p + q + r = 163$.



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AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

(AIME)

SOLUTIONS PAMPHLET

Tuesday, April 9, 2002

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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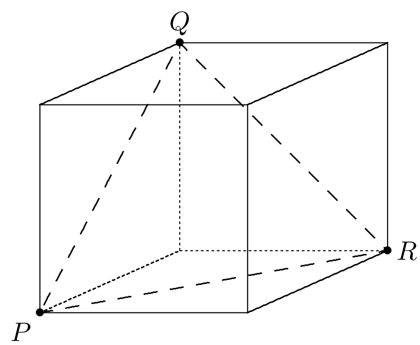
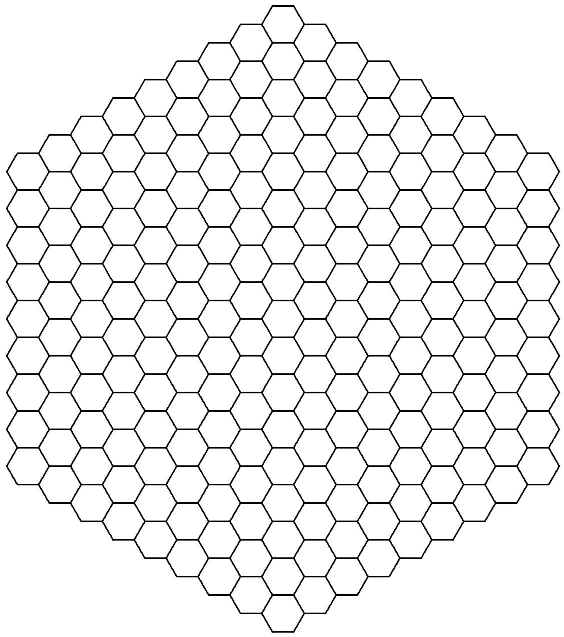
1. (Answer: 009)
Let the hundreds, tens, and units digits of x be h, t, and u, respectively. Then $x = 100h + 10t + u$, $y = 100u + 10t + h$, and $z = |99(h - u)| = 99|h - u|$. Since h and u are between 1 and 9, inclusive, $|h - u|$ must be between 0 and 8, inclusive. Thus there are 9 possible values for z.

2. (Answer: 294)
Notice that PQR is an equilateral triangle, because $PQ = QR = RP = 7\sqrt{2}$. This implies that each edge of the cube is 7 units long. Hence the surface area of the cube is $6(7^2) = 294$.



3. (Answer: 111)
Note that $6 = \log_6 a + \log_6 b + \log_6 c = \log_6 abc$. Then $6^6 = abc = b^3$, so $b = 6^2$ and $ac = 6^4$. Because $b \neq a$ and $b - a$ is the square of an integer, the only possibilities for a are 11, 20, 27, 32, and 35. Of these, only 27 is a divisor of 6^4. Thus $a + b + c = 27 + 36 + 48 = 111$.

4. (Answer: 803)
The garden can be partitioned into regular hexagons congruent to the blocks, and the hexagons on the boundary of the garden form a figure like the path, but with only $n-1$ hexagons on a side.



The figure shows the garden (not including the walk) when $n = 10$. Note that for any $n > 1$, counting the hexagons by columns starting at the left, each column contains one more hexagon than the column adjacent to it on the left, until the center column is reached. Since the leftmost column contains $n-1$ hexagons and the longest column is the $(n-2)^{\text{nd}}$ to the right of it, the longest column contains $(n-1)+(n-2) = 2n-3$ hexagons. The number of hexagons strictly to the left of the center vertical line is therefore

$$(n-1)+n+(n+1)+(n+2)+\cdots+[(n-1)+(n-3)] = \frac{(3n-5)(n-2)}{2},$$

and there are the same number to the right. Since the area of each of the hexagons is six times the area of an equilateral triangle of side 1, the area of the garden is

$$[(2n-3)+(3n-5)(n-2)]\left[\frac{6\sqrt{3}}{4}\right].$$

When $n = 202$ this is $361803\sqrt{3}/2$ square units, so the desired remainder is 803.

OR

For $n > 1$, note that when there are n hexagons per side of the bounding path, there is a total of $6(n-1)$ hexagons on the path. The number of hexagons in the interior of the path is

$$1 + 6 + 12 + 18 + \cdots + 6(n-2) = 1 + 3(n-2)(n-1),$$

so when $n = 202$, the garden can be partitioned into $1 + 3(200)(201) = 120601$ hexagons. Since the area of each hexagon is $6\sqrt{3}/4$, the area of the garden is $361803\sqrt{3}/2$.

5. (Answer: 042)
Note that

$$\frac{6^a}{a^6} = \frac{2^a 3^a}{2^{6n} 3^{6m}}$$

is not an integer if and only if $6n > a$ or $6m > a$, that is, if and only if

$$6 \cdot \max(n, m) > 2^n 3^m. \qquad (\star)$$

When both $n \geq 1$ and $m \geq 1$, there are no values of m and n that satisfy $(\star)$. When $m = 0$, $(\star)$ reduces to $2^n < 6n$, which is satisfied only by $n = 1, 2, 3, 4$. When $n = 0$, $(\star)$ reduces to $3^m < 6m$, which is satisfied only by $m = 1, 2$. Thus there are six values of a for which a^6 not a divisor of 6^a, namely, $a = 2, 4, 8, 16, 3,$ and 9, and their sum is 42.

6. (Answer: 521)
Because $\dfrac{1}{n^2-4} = \dfrac{1}{4}\left(\dfrac{1}{n-2} - \dfrac{1}{n+2}\right)$, the series telescopes, and it follows that

$$\begin{aligned}
1000\sum_{n=3}^{10000} \frac{1}{n^2-4} &= 250\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} - \frac{1}{9999} - \frac{1}{10000} - \frac{1}{10001} - \frac{1}{10002}\right) \\
&= 250 + 125 + \frac{250}{3} + \frac{250}{4} - \frac{250}{9999} - \frac{250}{10000} - \frac{250}{10001} - \frac{250}{10002} \\
&= 520 + \frac{5}{6} - r,
\end{aligned}$$

where the positive number r is less than $1/3$. Thus the requested integer is 521.

7. (Answer: 112)
The sum is a multiple of 200 if and only if $k(k+1)(2k+1) = 6 \cdot 200N = 2^4 \cdot 3 \cdot 5^2 N$ for some positive integer N. Because $2k+1$ is odd and k and $k+1$ cannot both be

even, it follows that either k or $k+1$ is a multiple of 16. Furthermore, the product is divisible by 3 for all integer values of k. (Why?) Substitute $k = 15, 16, 31, 32, \ldots$, and check whether $k(k+1)(2k+1)$ is divisible by 25 to see that $k = 112$ is the smallest positive integer for which $k(k+1)(2k+1)$ is a multiple of 1200.

8. (Answer: 049)
The equation $\left\lfloor \dfrac{2002}{n} \right\rfloor = k$ is equivalent to

$$k \le \frac{2002}{n} < k+1, \quad \text{or} \quad \frac{2002}{k+1} < n \le \frac{2002}{k}\,.$$

In order that there be no solutions, there can be no integer in this interval, that is, $\dfrac{2002}{k}$ and $\dfrac{2002}{k+1}$ must have the same integer part. The length of the interval must be less than 1, so

$$\frac{2002}{k} - \frac{2002}{k+1} < 1$$

which yields $k(k+1) > 2002$, and thus $k \ge 45$. For $k = 45, 46, 47, 48, 49, 50$, the integer part of $2002/k$ is $44, 43, 42, 41, 40, 40$, respectively. Thus $2002/49$ and $2002/50$ have the same integer part, so the least positive integer value of k is 49.

9. (Answer: 501)
Let k be the number of elements of $\mathcal{S}$, and let A and B be two empty jars into which elements of $\mathcal{S}$ will be placed to create two disjoint subsets. For each element x in $\mathcal{S}$, there are three possibilities: place x in A, place x in B, or place x in neither A nor B. Thus the number of ordered pairs of disjoint subsets (A, B) is 3^k. However, this counts the pairs where A or B is empty. Note that for A to be empty, there are two possibilities for each element x in $\mathcal{S}$: place x in B, or do not place x in B. The number of pairs for which A or B is empty is thus $2^k + 2^k - 1 = 2^{k+1} - 1$. Since interchanging A and B does not yield a different set of subsets, there are $\frac{1}{2}(3^k - 2^{k+1} + 1) = \frac{1}{2}(3^k + 1) - 2^k$ sets. When $k = 10$, $n = \frac{3^{10}+1}{2} - 2^{10} = 28501$, and the desired remainder is 501.

OR

For each non-empty subset X of $\mathcal{S}$ with k elements, $k = 1, 2, 3, \ldots, 9$, there are $2^{10-k} - 1$ non-empty subsets of $\mathcal{S}$ that are disjoint with X . Let N be the total number of ordered pairs of non-empty disjoint subsets of $\mathcal{S}$. Then

$$N = \sum_{k=1}^{9} \binom{10}{k} (2^{10-k} - 1) = \sum_{k=1}^{9} \binom{10}{k} (2^k - 1) = \sum_{k=1}^{9} \binom{10}{k} 2^k - \sum_{k=1}^{9} \binom{10}{k}.$$

Note that
$$\sum_{k=0}^{10} \binom{10}{k} = 2^{10},$$
and, from the Binomial Expansion,
$$3^{10} = (1+2)^{10} = \sum_{k=0}^{10} \binom{10}{k} 2^k.$$
Thus $N = \left[3^{10} - (1+2^{10})\right] - \left[2^{10} - (1+1)\right] = 3^{10} - 2^{11} + 1$, and the number of sets of two non-empty disjoint subsets of $\mathcal{S}$ is $\frac{1}{2}(3^{10} - 2^{11} + 1) = 28501$.

10. (Answer: 900)
Because x radians is equivalent to $\dfrac{180x}{\pi}$ degrees, the requested special values of x satisfy $\sin x^\circ = \sin \dfrac{180x}{\pi}^\circ$. It follows from properties of the sine function that either
$$\frac{180x}{\pi} = x + 360j \quad \text{or} \quad 180 - \frac{180x}{\pi} = x - 360k$$
for some integers j and k. Thus either $x = \dfrac{360j\pi}{180-\pi}$ or $x = \dfrac{180(2k+1)\pi}{180+\pi}$, and the least positive values of x are $\dfrac{360\pi}{180-\pi}$ and $\dfrac{180\pi}{180+\pi}$, so $m+n+p+q = 900$.

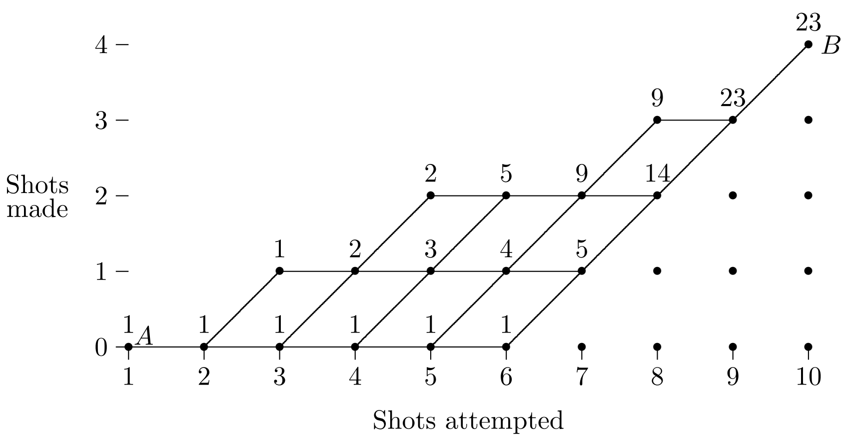
11. (Answer: 518)
Let the two series be
$$\sum_{k=0}^{\infty} a \cdot r^k \quad \text{and} \quad \sum_{k=0}^{\infty} b \cdot s^k.$$
The given conditions imply that $a = 1-r$, $b = 1-s$, and $ar = bs$. It follows that $r(1-r) = s(1-s)$, that is $r - s = r^2 - s^2$. Because the series are not identical, $r \neq s$, leaving $r = 1-s$ as the only possibility, and the series may be written as
$$\sum_{k=0}^{\infty} (1-r) \cdot r^k \quad \text{and} \quad \sum_{k=0}^{\infty} r \cdot (1-r)^k.$$
As we may pick either series as the one whose third term is $1/8$, set $(1-r)r^2 = 1/8$, from which we obtain $8r^3 - 8r^2 + 1 = 0$. The substitution $t = 2r$ yields $t^3 - 2t^2 + 1 = 0$, for which 1 is a root. Factoring gives $(t-1)(t^2 - t - 1) = 0$, so the other two roots are $(1 \pm \sqrt{5})/2$, which implies that $r = 1/2$ or $r = (1 \pm \sqrt{5})/4$. However, if r were $1/2$, the two series would be equal; and if r were $(1-\sqrt{5})/4$, then s would

be $(3+\sqrt{5})/4$, but convergence requires that $|s|<1$. Thus $r=(1+\sqrt{5})/4$, and $-1<s=(3-\sqrt{5})/4<1$. The second term of the series is therefore equal to

$$r(1-r)=\left(\frac{1+\sqrt{5}}{4}\right)\left(\frac{3-\sqrt{5}}{4}\right)=\frac{\sqrt{5}-1}{8},$$

and $100m+10n+p=518$.

12. (Answer: 660)
Let x be the number of attempts and y the number of shots made. The maximum values of y for $x=1,2,\ldots,10$ are 0, 0, 1, 1, 2, 2, 2, 3, 3, and 4, respectively. Since $y=4$ when $x=10$, the minimum values of y when $x=9,8,7,6,5,4,3,2,1$ are 3, 2, 1, 0, 0, 0, 0, 0, 0, respectively. We can represent the possible sequences of made and missed shots in the diagram below.



The possible sequences of made and attempted shots correspond to sequences of ordered pairs (x,y), where x is the number of shots attempted and y is the number of shots made, beginning at $(1,0)$ and ending at $(10,4)$. Each sequence corresponds to a path from A to B that moves right and/or up on the lines in the diagram. The number of paths from A to any point P on the diagram is the sum of the number of paths from A to the points directly before P. Each point is labeled with the number of possible paths from A to that point. Thus, the number of paths from A to B is 23. Each path represents a sequence of 4 made shots and 6 misses, so the requested probability is

$$23(.4)^4(.6)^6=\frac{2^4 3^6 23}{5^{10}}\,,$$

and $(p+q+r+s)(a+b+c)=(2+3+23+5)(4+6+10)=660$.

13. (Answer: 901)

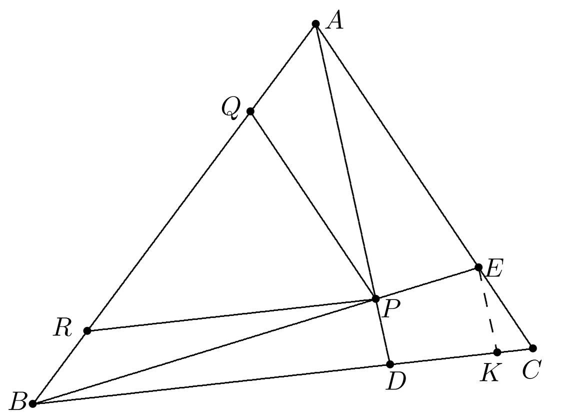
Draw the line through E that is parallel to $\overline{AD}$, and let K be its intersection with $\overline{BC}$. Because $CD = 2$ and $KC : KD = EC : EA = 1 : 3$, it follows that $KD = 3/2$. Therefore,

$$\frac{QP}{AE} = \frac{BP}{BE} = \frac{BD}{BK} = \frac{5}{5 + (3/2)} = \frac{10}{13}.$$

Thus

$$\frac{QP}{AC} = \frac{3}{4} \cdot \frac{10}{13} = \frac{15}{26}.$$

Since triangles PQR and CAB are similar, the ratio of their areas is $(15/26)^2 = 225/676$. Thus $m + n = 901$.



OR

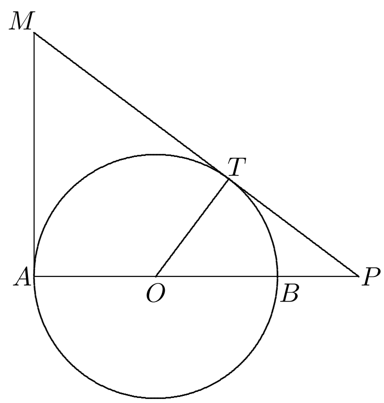
Use *mass points.* Assign a mass of 15 to C. Since $AE = 3 \cdot EC$, the mass at C must be 3 times the mass at A, so the mass at A is 5, and the mass at E is $15 + 5 = 20$. Similarly, the mass at B is $(2/5) \cdot 15 = 6$, so the mass at D is $15 + 6 = 21$, and the mass at P is $6 + 20 = 26$. Draw $\overrightarrow{CP}$ and let it intersect $\overline{AB}$ at F. The mass at F is $26 - 15 = 11$, so $PF/CF = 15/26$, and the ratio of the areas is $(15/26)^2 = 225/676$.

Remark: Research *mass points* to find out more about this powerful method of solving problems involving geometric ratios.

14. (Answer: 098)
Let T and B be the points where the circle meets $\overline{PM}$ and $\overline{AP}$, respectively, with $\overline{ABP}$. Triangles POT and PAM are right triangles that share angle MPA, so they are similar. Let p_1 and p_2 be their respective perimeters. Then $OT/AM = p_1/p_2$. Because $AM = TM$, it follows that $p_1 = p_2 - (AM + TM) = 152 - 2AM$. Thus $19/AM = (152 - 2AM)/152$, so that $AM = 38$ and $p_1 = 76$. It is also true that $OP/PM = p_1/p_2$, so

$$\frac{1}{2} = \frac{OP}{PM} = \frac{OP}{152 - (38 + 19 + OP)}$$

It follows that $OP = 95/3$, and $m + n = 98$.



15. (Answer: 282)
Let r_1 and r_2 be the radii and A_1 and A_2 be the centers of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, and let $P = (u, v)$ belong to both circles. Because the circles have common external tangents that meet at the origin O, it follows that the first-quadrant angle formed by the lines $y = 0$ and $y = mx$ is bisected by the ray through O, A_1, and A_2. Therefore, $A_1 = (x_1, kx_1)$ and $A_2 = (x_2, kx_2)$, where k is the tangent of the angle formed by the positive x-axis and the ray OA_1. Notice that $r_1 = kx_1$ and $r_2 = kx_2$. It follows from the identity $\tan 2\alpha = \dfrac{2\tan\alpha}{1 - \tan^2\alpha}$ that $m = \dfrac{2k}{1 - k^2}$. Now $(PA_1)^2 = (kx_1)^2$, or

$$(u - x_1)^2 + (v - kx_1)^2 = k^2 x_1^2, \text{ so}$$
$$(x_1)^2 - 2(u + kv)x_1 + u^2 + v^2 = 0.$$

In similar fashion, it follows that

$$(x_2)^2 - 2(u + kv)x_2 + u^2 + v^2 = 0.$$

Thus x_1 and x_2 are the roots of the equation

$$x^2 - 2(u + kv)x + u^2 + v^2 = 0,$$

which implies that $x_1x_2 = u^2 + v^2$, and that $r_1r_2 = k^2x_1x_2 = k^2(u^2 + v^2)$. Thus

$$k = \sqrt{\frac{r_1r_2}{u^2 + v^2}}$$

and

$$m = \frac{2k}{1 - k^2} = \frac{2\sqrt{r_1r_2(u^2 + v^2)}}{u^2 + v^2 - r_1r_2}.$$

When $u = 9$, $v = 6$, and $r_1r_2 = 68$, this gives $m = \dfrac{12\sqrt{221}}{49}$, so $a + b + c = 282$.

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AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME)

SOLUTIONS PAMPHLET

Tuesday, March 25, 2003

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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Order prior year Exams, Solutions Pamphlets or Problem Books from:
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1. (Answer: 839)

   Note that
   $$\frac{((3!)!)!}{3!} = \frac{(6!)!}{6} = \frac{720!}{6} = \frac{720 \cdot 719!}{6} = 120 \cdot 719!.$$
   Because $120 \cdot 719! < 720!$, conclude that n must be less than 720, so the maximum value of n is 719. The requested value of $k+n$ is therefore $120+719 = 839$.

2. (Answer: 301)

   The sum of the areas of the green regions is
   $$\begin{aligned} &\left[(2^2-1^2)+(4^2-3^2)+(6^2-5^2)+\cdots+(100^2-99^2)\right]\pi \\ =\ &\left[(2+1)+(4+3)+(6+5)+\cdots+(100+99)\right]\pi \\ =\ &\frac{1}{2}\cdot 100 \cdot 101\pi. \end{aligned}$$
   Thus the desired ratio is
   $$\frac{1}{2}\cdot\frac{100\cdot 101\pi}{100^2\pi} = \frac{101}{200},$$
   and $m+n = 301$.

3. (Answer: 484)

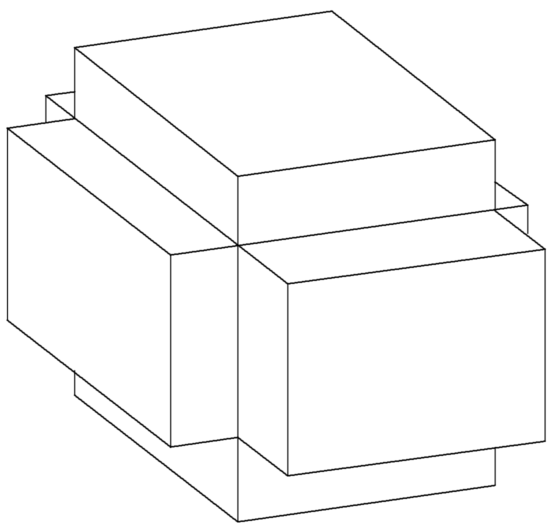
   Since each element x of $\mathcal{S}$ is paired exactly once with every other element in the set, the number of times x contributes to the sum is the number of other elements in the set that are smaller than x. For example, the first number, 8, will contribute four times to the sum because the greater elements of the subsets $\{8,5\}$, $\{8,1\}$, $\{8,3\}$, and $\{8,2\}$ are all 8. Since the order of listing the elements in the set is not significant, it is helpful to first sort the elements of the set in increasing order. Thus, since $\mathcal{S} = \{1,2,3,5,8,13,21,34\}$, the sum of the numbers on the list is $0(1)+1(2)+2(3)+3(5)+4(8)+5(13)+6(21)+7(34) = 484$.

4. (Answer: 012)

   Use logarithm properties to obtain $\log(\sin x\cos x) = -1$, and then $\sin x \cos x = 1/10$. Note that
   $$(\sin x+\cos x)^2 = \sin^2 x + \cos^2 x + 2\sin x\cos x = 1 + \frac{2}{10} = \frac{12}{10}.$$
   Thus
   $$2\log(\sin x + \cos x) = \log\frac{12}{10} = \log 12 - 1,$$
   so
   $$\log(\sin x+\cos x) = \frac{1}{2}(\log 12 - 1),$$
   and $n = 12$.

5. (Answer: 505)

First consider the points in the six parallelepipeds projecting 1 unit outward from the original parallelepiped. Two of these six parallelepipeds are 1 by 3 by 4, two are 1 by 3 by 5, and two are 1 by 4 by 5. The sum of their volumes is $2(1\cdot3\cdot4+1\cdot3\cdot5+1\cdot4\cdot5)=94$. Next consider the points in the twelve quarter-cylinders of radius 1 whose heights are the edges of the original parallelepiped. The sum of their volumes is $4\cdot\frac{1}{4}\pi\cdot1^2(3+4+5)=12\pi$. Finally, consider the points in the eight octants of a sphere of radius 1 at the eight vertices of the original parallelepiped. The sum of their volumes is $8\cdot\frac{1}{8}\cdot\frac{4}{3}\pi\cdot1^3=\frac{4\pi}{3}$. Because the volume of the original parallelepiped is $3\cdot4\cdot5=60$, the requested volume is $60+94+12\pi+4\pi/3=\dfrac{462+40\pi}{3}$, so $m+n+p=462+40+3=505$.



6. (Answer: 348)

The sides of the triangles may be cube edges, face-diagonals of length $\sqrt{2}$, or space-diagonals of length $\sqrt{3}$. A triangle can consist of two adjacent edges and a face-diagonal; three face-diagonals; or an edge, a face-diagonal, and a space-diagonal. The first type of triangle is right with area $1/2$, and there are 24 of them, 4 on each face. The second type of triangle is equilateral with area $\sqrt{3}/2$. There are 8 of these because each of these triangles is uniquely determined by the three vertices adjacent to one of the 8 vertices of the cube. The third type of triangle is right with area $\sqrt{2}/2$. There are 24 of these because there are four space-diagonals and each determines six triangles, one with each cube vertex that is not an endpoint of the diagonal. (Note that there is a total of $\binom{8}{3}=56$ triangles with the desired vertices, which is consistent with the above results.)

The desired sum is thus $24(1/2) + 8(\sqrt{3}/2) + 24(\sqrt{2}/2) = 12 + 4\sqrt{3} + 12\sqrt{2} = 12 + \sqrt{48} + \sqrt{288}$, and $m + n + p = 348$.

7. (Answer: 380)

Let $AD = CD = a$, let $BD = b$, and let E be the projection of D on $\overline{AC}$. It follows that $a^2 - 15^2 = DE^2 = b^2 - 6^2$, or $a^2 - b^2 = 225 - 36 = 189$. Then $(a + b, a - b) = (189, 1)$, $(63, 3)$, $(27, 7)$, or $(21, 9)$, from which $(a, b) = (95, 94)$, $(33, 30)$, $(17, 10)$, or $(15, 6)$. The last pair is rejected since b must be greater than 6. Because each possible triangle has a perimeter of $2a+30$, it follows that $s = 190 + 66 + 34 + 3 \cdot 30 = 380$.

OR

Let (a_k, b_k) be the possible values for a and b, and let n be the number of possible perimeters of $\triangle ACD$. Then $s = \sum_{k=1}^{n}(30+2a_k) = 30n+\sum_{k=1}^{n}[(a_k+b_k)+(a_k-b_k)]$. But $(a_k + b_k)(a_k - b_k) = a_k^2 - b_k^2 = 189 = 3^3 \cdot 7$ which has 4 pairs of factors. Thus $n = 4$. Therefore the sum of the perimeters of the triangles is $30 \cdot 4$ more than the sum of the divisors of 189, that is, $120 + (1 + 3 + 3^2 + 3^3)(1 + 7) = 440$. However, this includes the case where $D = E$, the projection of D on $\overline{AC}$, so $s = 440 - 60 = 380$.

8. (Answer: 129)

Let a, $a + d$, $a + 2d$, and $\dfrac{(a + 2d)^2}{a + d}$ be the terms of the sequence, with a and d positive integers. Then $(a + 30)(a + d) = (a + 2d)^2$, which yields $3a(10 - d) = 2d(2d - 15)$. It follows that either $10 - d > 0$ and $2d - 15 > 0$ or $10 - d < 0$ and $2d - 15 < 0$. In the first case, d is 8 or 9, and the second case has no solutions. When $d = 8$, $a = 8/3$, and when $d = 9$, $a = 18$. Thus, the only acceptable sequence is 18, 27, 36, 48, and the sum is 129.

9. (Answer: 615)

For a balanced four-digit integer, the sum of the leftmost two digits must be at least 1 and at most 18. Let $f(n)$ be the number of ways to write n as a sum of two digits where the first is at least 1, and let $g(n)$ be the number of ways to write n as a sum of two digits. For example, $f(3) = 3$, since $3 = 1+2 = 2+1 = 3+0$, and $g(3) = 4$. Then

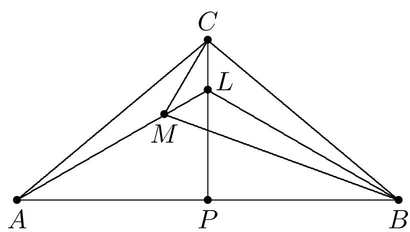
$$f(n) = \begin{cases} n & \text{for} 1 \le n \le 9, \\ 19 - n & \text{for} 10 \le n \le 18, \end{cases} \quad \text{and} \quad g(n) = \begin{cases} n+1 & \text{for} 1 \le n \le 9, \\ 19 - n & \text{for} 10 \le n \le 18. \end{cases}$$

For any balanced four-digit integer whose leftmost and rightmost digit pairs both have sum n, the number of possible leftmost digit pairs is $f(n)$ because the leftmost digit must be at least 1, and the number of possible rightmost digit pairs is $g(n)$. Thus there are $f(n)\cdot g(n)$ four-digit balanced integers whose leftmost and rightmost digit pairs both have sum n. The total number of balanced four-digit integers is then equal to

$$\begin{aligned}\sum_{n=1}^{18} f(n)\cdot g(n) &= \sum_{n=1}^{9} n(n+1) + \sum_{n=10}^{18}(19-n)^2 = \sum_{n=1}^{9}(n^2+n) + \sum_{n=1}^{9} n^2 \\ &= 2\sum_{n=1}^{9} n^2 + \sum_{n=1}^{9} n = 2(1^2+2^2+\cdots+9^2) + (1+2+\cdots+9) \\ &= 615.\end{aligned}$$

10. (Answer: 083)

Let $\overline{CP}$ be the altitude to side $\overline{AB}$. Extend $\overline{AM}$ to meet $\overline{CP}$ at point L, as shown. Since $\angle ACL = 53^\circ$, conclude that $\angle MCL = 30^\circ$. Also, $\angle LMC = \angle MAC + \angle ACM = 30^\circ$. Thus $\triangle MLC$ is isosceles with $LM = LC$ and $\angle MLC = 120^\circ$. Because L is on the perpendicular bisector of $\overline{AB}$, $\angle LBA = \angle LAB = 30^\circ$ and $\angle MLB = 120^\circ$. It follows that $\angle BLC = 120^\circ$. Now consider $\triangle BLM$ and $\triangle BLC$. They share $\overline{BL}$, $ML = LC$, and $\angle MLB = \angle CLB = 120^\circ$. Therefore they are congruent, and $\angle LMB = \angle LCB = 53^\circ$. Hence $\angle CMB = \angle CML + \angle LMB = 30^\circ + 53^\circ = 83^\circ$.



OR

Without loss of generality, assume that $AC = BC = 1$. Apply the Law of Sines in $\triangle AMC$ to obtain

$$\frac{\sin 150^\circ}{1} = \frac{\sin 7^\circ}{CM},$$

from which $CM = 2\sin 7^\circ$. Apply the Law of Cosines in $\triangle BMC$ to obtain $MB^2 = 4\sin^2 7^\circ + 1 - 2\cdot 2\sin 7^\circ \cdot \cos 83^\circ = 4\sin^2 7^\circ + 1 - 4\sin^2 7^\circ = 1$. Thus $CB = MB$, and $\angle CMB = 83^\circ$.

11. (Answer: 092)

Because $\cos(90^\circ - x) = \sin x$ and $\sin(90^\circ - x) = \cos x$, it suffices to consider x in the interval $0^\circ < x \le 45^\circ$. For such x,

$$\cos^2 x \ge \sin x \cos x \ge \sin^2 x,$$

so the three numbers are not the lengths of the sides of a triangle if and only if

$$\cos^2 x \ge \sin^2 x + \sin x \cos x,$$

which is equivalent to $\cos 2x \ge \frac{1}{2} \sin 2x$, or $\tan 2x \le 2$. Because the tangent function is increasing in the interval $0^\circ \le x \le 45^\circ$, this inequality is equivalent to $x \le \frac{1}{2}(\arctan 2)^\circ$. It follows that

$$p = \frac{\frac{1}{2}(\arctan 2)^\circ}{45^\circ} = \frac{(\arctan 2)^\circ}{90^\circ},$$

so $m + n = 92$.

12. (Answer: 777)

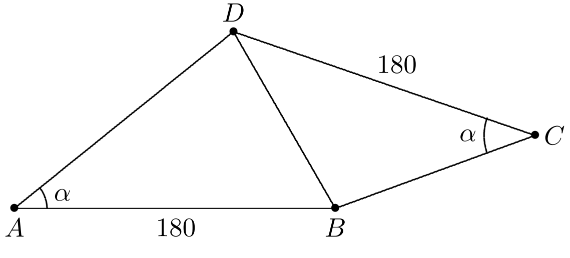
Let $\angle A = \angle C = \alpha$, $AD = x$, and $BC = y$. Apply the Law of Cosines in triangles ABD and CDB to obtain

$$BD^2 = x^2 + 180^2 - 2 \cdot 180x \cos\alpha = y^2 + 180^2 - 2 \cdot 180y \cos\alpha.$$

Because $x \ne y$, this yields

$$\cos\alpha = \frac{x^2 - y^2}{2 \cdot 180(x - y)} = \frac{x + y}{360} = \frac{280}{360} = \frac{7}{9}.$$

Thus $\lfloor 1000 \cos A \rfloor = 777$.



OR

Assume without loss of generality that AD is the greater of AD and BC. Then there is a point P on $\overline{AD}$ with $AP = BC$. Because $\triangle BAP \cong \triangle DCB$, conclude that $BP = BD$, and altitude $\overline{BH}$ of isosceles $\triangle BPD$ bisects $\overline{PD}$. Now $\cos A = AH/180$, and because $AH = AP + (PD/2) = AD - (PD/2)$,

$$AH = \frac{AP + AD}{2} = \frac{BC + AD}{2} = \frac{640 - 2 \cdot 180}{2} = 140.$$

Thus $\cos A = 140/180 = 7/9$, and $\lfloor 1000 \cos A \rfloor = 777$.

13. (Answer: 155)

Since $2003 < 2047 = 2^{11} - 1$, the integers in question have at most 11 digits in base 2. Since the base-2 representation of a positive integer must begin with 1, the number of $(d+1)$-digit numbers with exactly $(k+1)$ 1's is $\binom{d}{k}$. The number of 1's exceeds the number of 0's if and only if $k+1 > (d+1)/2$, or $k \geq d/2$. Thus the number of integers whose base-2 representation consists of at most 11 digits and that have more 1's than 0's is the sum of the entries in rows 0 through 10 in Pascal's Triangle that are on or to the right of the vertical symmetry line. The sum of all entries in these rows is $1+2+4+\cdots+1024 = 2047$, and the sum of the center elements is $\sum_{i=0}^{5} \binom{2i}{i} = 1+2+6+20+70+252 = 351$, so the sum of the entries on or to the right of the line is $(2047+351)/2 = 1199$. The 44 integers less than 2048 and greater than 2003 all have at least six 1's, because they are all greater than 1984, which is 11111000000 in base 2, so they have all been included in the total. Thus the required number is $1199 - 44 = 1155$, whose remainder upon division by 1000 is 155.

14. (Answer: 127)

To find the smallest n, it is sufficient to consider the case in which the string 251 occurs immediately after the decimal point. To show this, suppose that in the decimal representation of m/n, the string 251 does not occur immediately after the decimal point. Then $m/n = .A251\ldots$, where A represents a block of k digits, $k \geq 1$. This implies that $10^k m/n - A = .251\ldots$, but $10^k m/n - A$, which is between 0 and 1, can then be expressed in the form a/b, where a and b are relatively prime positive integers and $b \leq n$. Now

$$\frac{251}{1000} \leq \frac{m}{n} < \frac{252}{1000}.$$

It follows that $251n \leq 1000m < 252n = 251n + n$. The remainder when $1000m$ is divided by 251 must therefore be less than n, so it is sensible to investigate multiples of 251 that are close to and less than a multiple of 1000. When $m = 1$, $n = 3$ yields $3 \cdot 251 = 753$ as the multiple of 251 that is closest to and less than 1000; but the remainder is greater than 3. When $m = 2$, $n = 7$ yields $3 \cdot 251 + 4 \cdot 251 = 1757$ as the multiple of 251 that is closest to and less than 2000; but the remainder is greater than 7. More generally, $(4m-1)251$ is less than $1000m$ when $m \leq 62$, and the remainder is $1000m - (4m-1)251 = 251 - 4m$. The remainder is less than n when $251 - 4m < 4m - 1$, that is, when $m > 31$. Thus the minimum value of m is 32, and the minimum value of n is $4 \cdot 32 - 1 = 127$.

15. (Answer: 289)

Let $AB = c$, $BC = a$, and $CA = b$. Since $a > c$, F is on $\overline{BC}$. Let ℓ be the line passing through A and parallel to $\overline{DF}$, and let ℓ meet $\overline{BD}$, $\overline{BE}$, and $\overline{BC}$ at D', E', and F' respectively. Since $\overline{AF'}$ is parallel to $\overline{DF}$,

$$\frac{m}{n} = \frac{DE}{EF} = \frac{D'E'}{E'F'}.$$

In $\triangle ABF'$, $\overline{BD'}$ is both an altitude and an angle-bisector, so $\triangle ABF'$ is isosceles with $BF' = BA = c$. Hence $AD' = D'F'$, and

$$\frac{AE'}{E'F'} = \frac{AD' + D'E'}{E'F'} = \frac{D'F' + D'E'}{E'F'} = \frac{E'F' + 2D'E'}{E'F'} = 1 + \frac{2m}{n}.$$

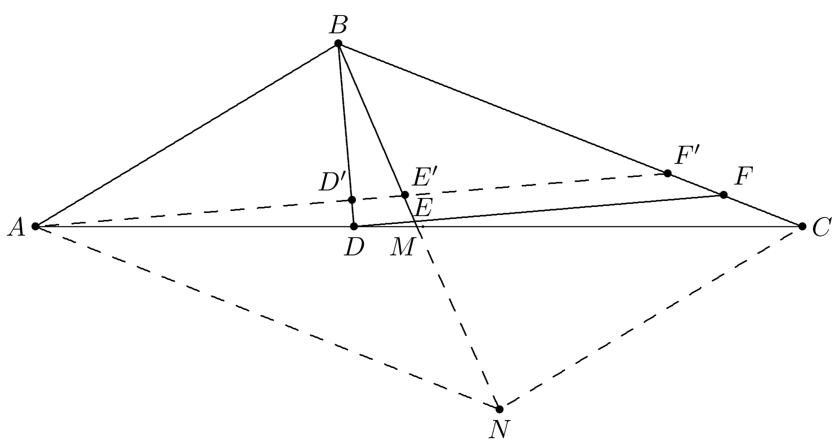
Extend $\overline{BM}$ through M to N so that $BM = MN$, and draw $\overline{AN}$ and $\overline{CN}$. Quadrilateral $ABCN$ is a parallelogram because diagonals $\overline{AC}$ and $\overline{BN}$ bisect each other. Hence $AN = BC = a$ and triangles $AE'N$ and $F'E'B$ are similar. Therefore

$$1 + \frac{2m}{n} = \frac{AE'}{E'F'} = \frac{AN}{F'B} = \frac{a}{c},$$

and

$$\frac{m}{n} = \frac{a - c}{2c} = \frac{507 - 360}{720} = \frac{49}{240},$$

so $m + n = 289$.



OR

Let $AB = c$, $BC = a$, and $CA = b$. Let D' and D'' be the points where the lines parallel to line DF and containing A and C, respectively, intersect $\overrightarrow{BD}$, and let E' and F', be the points where $\overrightarrow{AD'}$ meets $\overline{BM}$ and $\overline{BC}$, respectively. Let G be the point on $\overline{BM}$ so that lines FG and AC are parallel. Note that

$$\frac{c}{a} = \frac{AD}{DC} = \frac{DD'}{DD''} = \frac{FF'}{FC} = \frac{BF - BF'}{BC - BF} = \frac{BF - c}{a - BF},$$

which yields $BF = \dfrac{2ac}{a+c}$.

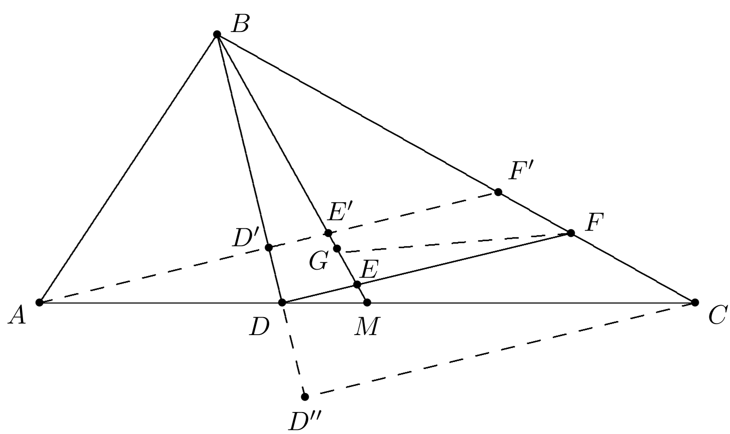
Also, $\dfrac{GF}{MC} = \dfrac{BF}{BC}$, so $GF = \dfrac{b}{2} \cdot \dfrac{BF}{a} = \dfrac{b}{2} \cdot \dfrac{2c}{a+c} = \dfrac{bc}{a+c}$.

But $\dfrac{c}{a} = \dfrac{AD}{DC} = \dfrac{AD}{b - AD}$ yields $AD = \dfrac{bc}{a+c}$.

Therefore $AD = GF$, which implies that $ADFG$ is a parallelogram, so $\overline{AG}$ is parallel to $\overline{DF}$. Thus $G = E'$, and then

$$\frac{DE}{EF} = \frac{DM}{GF} = \frac{AM - AD}{AD} = \frac{\frac{1}{2}(AD + CD) - AD}{AD} = \frac{\frac{1}{2}(CD - AD)}{AD} = \frac{1}{2}\left(\frac{CD}{AD} - 1\right),$$

$$\text{so} \quad \frac{DE}{EF} = \frac{1}{2}\left(\frac{a}{c} - 1\right) = \frac{a-c}{2c}.$$



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The

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MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

21st Annual *(Alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Tuesday, April 8, 2003

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 336)

   Call the three integers a, b, and c, and, without loss of generality, assume $a \leq b \leq c$. Then $abc = 6(a+b+c)$, and $c = a+b$. Thus $abc = 12c$, and $ab = 12$, so $(a,b,c) = (1,12,13)$, $(2,6,8)$, or $(3,4,7)$, and $N = 156$, 96, or 84. The sum of the possible values of N is 336.

2. (Answer: 120)

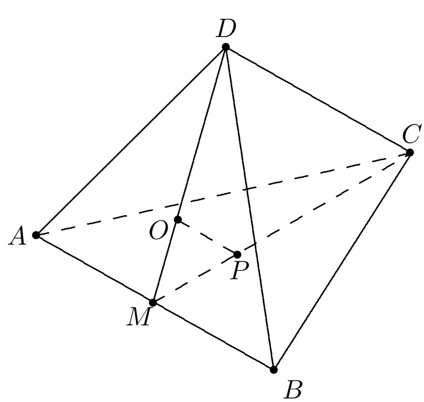
   An integer is divisible by 8 if and only if the number formed by the rightmost three digits is divisible by 8. The greatest integer with the desired property is formed by choosing 9876543 as the seven leftmost digits and finding the arrangement of 012 that yields the greatest multiple of 8, assuming that such an arrangement exists. Checking the 6 permutations of 012 yields 120 as the sole multiple of 8, so $N = 9876543120$, and its remainder when divided by 1000 is 120.

3. (Answer: 192)

   There are three choices for the first letter and two choices for each subsequent letter, so there are $3 \cdot 2^{n-1}$ n-letter good words. Substitute $n = 7$ to find there are $3 \cdot 2^6 = 192$ seven-letter good words.

4. (Answer: 028)

   Let O and P be the centers of faces DAB and ABC, respectively, of regular tetrahedron $ABCD$. Both $\overrightarrow{DO}$ and $\overrightarrow{CP}$ intersect $\overline{AB}$ at its midpoint M. Since $\dfrac{MO}{MD} = \dfrac{MP}{MC} = \dfrac{1}{3}$, triangles MOP and MDC are similar, and $OP = (1/3)DC$. Because the tetrahedra are similar, the ratio of their volumes is the cube of the ratio of a pair of corresponding sides, namely, $(1/3)^3 = 1/27$, so $m + n = 28$.



5. (Answer: 216)

   Let $\overline{AB}$ be a diameter of the circular face of the wedge formed by the first cut, and let $\overline{AC}$ be the longest chord across the elliptical face of the wedge formed by the second cut. Then $\triangle ABC$ is an isosceles right triangle and $BC = 12$ inches. If a third cut were made through the point C on the log and perpendicular to the axis of the cylinder, then a second wedge, congruent to the original, would be formed, and the two wedges would fit together to form a right circular cylinder with radius $AB/2 = 6$ inches and height BC. Thus, the volume of the wedge is $\frac{1}{2}\pi \cdot 6^2 \cdot 12 = 216\pi$, and $n = 216$.

6. (Answer: 112)

   Let M be the midpoint of $\overline{BC}$, let M' be the reflection of M in G, and let Q and R be the points where $\overline{BC}$ meets $\overline{A'C'}$ and $\overline{A'B'}$, respectively. Note that since M is on $\overline{BC}$, M' is on $\overline{B'C'}$. Because a 180° rotation maps each line that does not contain the center of the rotation to a parallel line, $\overline{BC}$ is parallel to $\overline{B'C'}$, and $\triangle A'RQ$ is similar to $\triangle A'B'C'$. Recall that medians of a triangle trisect each other to obtain

$$M'G = MG = (1/3)AM, \text{ so } A'M = AM' = (1/3)AM = (1/3)A'M'.$$

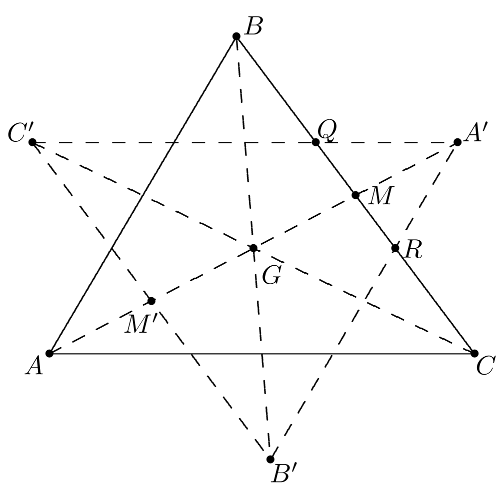
   Thus the similarity ratio between triangles $A'RQ$ and $A'B'C'$ is $1/3$, and

$$[A'RQ] = (1/9)[A'B'C'] = (1/9)[ABC].$$

   Similarly, the area of each of the two small triangles with vertices at B' and C', respectively, is $1/9$ that of $\triangle ABC$. The desired area is therefore

$$[ABC] + 3(1/9)[ABC] = (4/3)[ABC].$$

   Use Heron's formula, $K = \sqrt{s(s-a)(s-b)(s-c)}$, to find $[ABC] = \sqrt{21 \cdot 7 \cdot 6 \cdot 8} = 84$. The desired area is then $(4/3) \cdot 84 = 112$.



7. (Answer: 400)

Let O be the point of intersection of diagonals $\overline{AC}$ and $\overline{BD}$, and E the point of intersection of $\overline{AC}$ and the circumcircle of $\triangle ABD$. Extend $\overline{DB}$ to meet the circumcircle of $\triangle ACD$ at F. From the Power-of-a-Point Theorem, we have

$$AO \cdot OE = BO \cdot OD \text{ and } DO \cdot OF = AO \cdot OC.$$

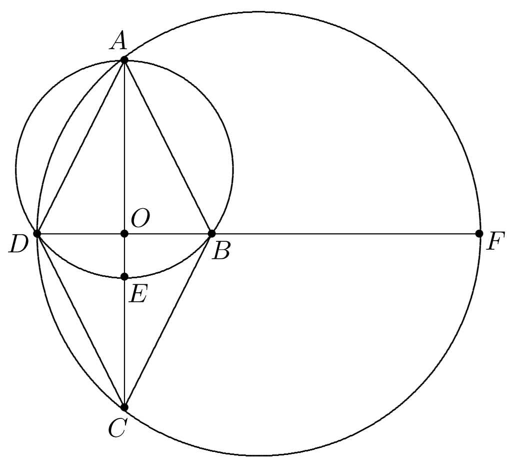
Let $AC = 2m$ and $BD = 2n$. Because $\overline{AE}$ is a diameter of the circumcircle of $\triangle ABD$, and $\overline{DF}$ is a diameter of the circumcircle of $\triangle ACD$, the above equalities can be rewritten as

$$m(25 - m) = n^2 \quad \text{and} \quad n(50 - n) = m^2,$$

or

$$25m = m^2 + n^2 \quad \text{and} \quad 50n = m^2 + n^2.$$

Therefore $m = 2n$. It follows that $50n = 5n^2$, so $n = 10$ and $m = 20$. Thus $[ABCD] = (1/2)AC \cdot BD = 2mn = 400$.



OR

Let R_1 and R_2 be the circumradii of triangles ABD and ACD, respectively. Because $\overline{BO}$ is the altitude to the hypotenuse of right $\triangle ABE$, $AB^2 = AO \cdot AE$. Similarly, in right $\triangle DAF$, $AB^2 = DA^2 = DO \cdot DF$, so $AO \cdot AE = DO \cdot DF$. Thus

$$\frac{AO}{DO} = \frac{DF}{AE} = \frac{R_2}{R_1} = 2.$$

Also, from right $\triangle ADE$, $2 = \dfrac{AO}{DO} = \dfrac{DO}{OE}$. Then

$$25 = 2R_1 = AE = AO + OE = 2 \cdot DO + \frac{1}{2}DO = \frac{5}{2}DO,$$

and $DO = 10$, $AO = 20$, so $[ABCD] = 400$.

OR

Let s be the length of a side of the rhombus, and let $\alpha = \angle BAC$. Then $AO = s\cos\alpha$, and $BO = s\sin\alpha$, so $[ABCD] = 4[ABO] = 2s^2 \sin\alpha\cos\alpha = s^2\sin 2\alpha$. Apply the Extended Law of Sines (In any $\triangle ABC$ with $AB = c$, $BC = a$, $CA = b$, and circumradius R, $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C} = 2R$) in $\triangle ABD$ and $\triangle ACD$ to obtain $s = 2R_1\sin(90^\circ - \alpha) = 2R_1\cos\alpha$, and $s = 2R_2\sin\alpha$. Thus $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha} = \dfrac{R_1}{R_2} = \dfrac{1}{2}$. Also, $s^2 = 4R_1R_2\cos\alpha\sin\alpha = 2R_1R_2\sin 2\alpha$. But $\sin 2\alpha = 2\cdot\dfrac{1}{\sqrt{5}}\cdot\dfrac{2}{\sqrt{5}} = \dfrac{4}{5}$, from which $[ABCD] = 2R_1R_2\sin^2 2\alpha = 2\cdot\frac{25}{2}\cdot 25\cdot\frac{16}{25} = 400$.

OR

Let $AB = s$, $AO = m$, and $BO = n$, and use the fact that the product of the lengths of the sides of a triangle is four times the product of its area and its circumradius to obtain $4[ABD]R_1 = s \cdot s \cdot 2n$ and $4[ACD]R_2 = s \cdot s \cdot 2m$. Since $[ABD] = [ACD]$, conclude that $\dfrac{1}{2} = \dfrac{R_1}{R_2} = \dfrac{n}{m}$, and proceed as above.

8. (Answer: 348)

The nth term of an arithmetic sequence has the form $a_n = pn+q$, so the product of corresponding terms of two arithmetic sequences is a quadratic expression, $s_n = an^2 + bn + c$. Letting $n = 0$, 1, and 2 produces the equations $c = 1440$, $a + b + c = 1716$, and $4a + 2b + c = 1848$, whose common solution is $a = -72$, $b = 348$, and $c = 1440$. Thus the eighth term is $s_7 = -72\cdot 7^2 + 348\cdot 7 + 1440 = 348$. Note that $s_n = -72n^2 + 348n + 1440 = -12(2n - 15)(3n + 8)$ can be used to generate pairs of arithmetic sequences with the desired products, such as $\{180, 156, 132, \ldots\}$ and $\{8, 11, 14, \ldots\}$.

9. (Answer: 006)

Apply the division algorithm for polynomials to obtain

$$P(x) = Q(x)(x^2 + 1) + x^2 - x + 1.$$

Therefore

$$\sum_{i=1}^{4} P(z_i) = \sum_{i=1}^{4} z_i^2 - \sum_{i=1}^{4} z_i + 4 = \left(\sum_{i=1}^{4} z_i\right)^2 - 2\sum_{i<j} z_i z_j - \sum_{i=1}^{4} z_i + 4.$$

Use the formulas for sum and product of the roots to obtain $\sum_{i=1}^{4} P(z_i) = 1 + 2 - 1 + 4 = 6$.

OR

Since, for each root w of $Q(x) = 0$, we have $w^4 - w^3 - w^2 - 1 = 0$, conclude that $w^4 - w^3 = w^2 + 1$, and then $w^6 - w^5 = w^4 + w^2 = w^3 + 2w^2 + 1$. Thus $P(w) = w^3 + 2w^2 + 1 - w^3 - w^2 - w = w^2 - w + 1$. Therefore

$$\sum_{i=1}^{4} P(z_i) = \sum_{i=1}^{4} z_i^2 - \sum_{i=1}^{4} z_i + 4,$$

and, as above, $\sum_{i=1}^{4} P(z_i) = 6$.

10. (Answer: 156)

Let x represent the smaller of the two integers. Then $\sqrt{x} + \sqrt{x + 60} = \sqrt{y}$, and $x + x + 60 + 2\sqrt{x(x + 60)} = y$. Thus $x(x + 60) = z^2$ for some positive integer z. It follows that

$$\begin{aligned} x^2 + 60x &= z^2, \\ x^2 + 60x + 900 &= z^2 + 900, \\ (x + 30)^2 - z^2 &= 900, \quad \text{and} \\ (x + 30 + z)(x + 30 - z) &= 900. \end{aligned}$$

Thus $(x+30+z)$ and $(x+30-z)$ are factors of 900 with $(x+30+z) > (x+30-z)$, and they are both even because their sum and product are even. Note that each pair of even factors of 900 can be found by doubling factor-pairs of 225, so the possible values of $(x+30+z, x+30-z)$ are $(450, 2)$, $(150, 6)$, $(90, 10)$, and $(50, 18)$. Each of these pairs yields a value for x which is 30 less than half their sum. These values are 196, 48, 20, and 4. When $x = 196$ or 4, then $\sqrt{x} + \sqrt{x+60}$ is an integer. When $x = 48$, we obtain $\sqrt{48} + \sqrt{108} = \sqrt{300}$, and when $x = 20$, we obtain $\sqrt{20} + \sqrt{80} = \sqrt{180}$. Thus the desired maximum sum is $48 + 108 = 156$.

OR

Let x represent the smaller of the two integers. Then $\sqrt{x} + \sqrt{x+60} = \sqrt{y}$, and $x + x + 60 + 2\sqrt{x(x+60)} = y$. Thus $x(x+60) = z^2$ for some positive integer z. Let d be the greatest common divisor of x and $x+60$. Then $x = dm$ and $x + 60 = dn$, where m and n are relatively prime. Because $dm \cdot dn = z^2$, there are relatively prime positive integers p and q such that $m = p^2$ and $n = q^2$. Now $d(q^2 - p^2) = 60$. Note that p and q cannot both be odd, else $q^2 - p^2$ would be divisible by 8; and they cannot both be even because they are relatively prime. Therefore p and q are of opposite parity, and $q^2 - p^2$ is odd, which implies that $q^2 - p^2 = 1, 3, 5$, or 15. But $q^2 - p^2$ cannot be 1, and if $q^2 - p^2$ were 15, then d would be 4, and x and $x+60$ would be squares. Thus $q^2 - p^2 = 3$ or 5, and $(q+p, q-p) = (3, 1)$ or $(5, 1)$, and then $(q, p) = (2, 1)$ or $(3, 2)$. This yields $(x+60, x) = (2^2 \cdot 20, 1^2 \cdot 20) = (80, 20)$ or $(x+60, x) = (3^2 \cdot 12, 2^2 \cdot 12) = (108, 48)$, so the requested maximum sum is $108 + 48 = 156$.

OR

Let a and b represent the two integers, with $a > b$. Then $a - b = 60$, and $\sqrt{a} + \sqrt{b} = \sqrt{c}$, where c is an integer that is not a square. Dividing yields $\sqrt{a} - \sqrt{b} = 60/\sqrt{c}$. Adding these last two equations yields

$$2\sqrt{a} = \sqrt{c} + \frac{60}{\sqrt{c}}, \quad \text{so}$$
$$2\sqrt{ac} = c + 60.$$

Therefore $\sqrt{ac}$ is an integer, so c is even, as is ac, which implies $\sqrt{ac}$ is even. Hence c is a multiple of 4, so there is a positive non-square integer d such that $c = 4d$. Then

$$a = \frac{(c+60)^2}{4c} = \frac{(4d+60)^2}{16d} = \frac{(d+15)^2}{d} = \frac{d^2 + 30d + 225}{d} = d + \frac{225}{d} + 30.$$

Thus d is a non-square divisor of 225, so the possible values of d are 3, 5, 15, 45, and 75. The maximum value of a, which occurs when $d = 3$ or $d = 75$, is

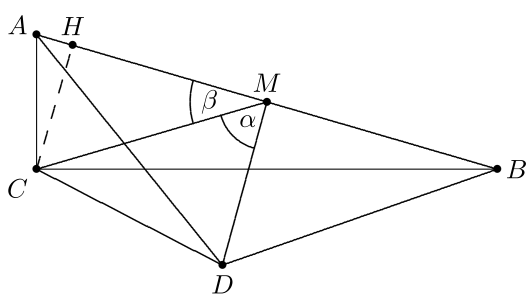
$3+75+30=108$, so the maximum value of b is $108-60=48$, and the requested maximum sum is $48+108=156$.

11. (Answer: 578)

The desired area is given by $(1/2)\cdot CM\cdot DM\cdot\sin\alpha$, where $\alpha=\angle CMD$. The length $AB=\sqrt{7^2+24^2}=25$, and, since $\overline{CM}$ is the median to the hypotenuse of $\triangle ABC$, $CM=25/2$. Because $\overline{DM}$ is both the altitude and median to side AB in $\triangle ABD$, $DM=5\sqrt{11}/2$ by the Pythagorean Theorem. To compute $\sin\alpha$, let $\angle AMC=\beta$, and note that $\angle AMC$ and $\angle CMD$ are complementary, so $\cos\beta=\sin\alpha$. Apply the Law of Cosines in $\triangle AMC$ to obtain

$$\cos\beta=\frac{(\frac{25}{2})^2+(\frac{25}{2})^2-7^2}{2\cdot\frac{25}{2}\cdot\frac{25}{2}}=\frac{527}{625}.$$

The area of $\triangle CMD$ is $(1/2)\cdot CM\cdot DM\cdot\sin\alpha=\frac{1}{2}\cdot\frac{25}{2}\cdot\frac{5\sqrt{11}}{2}\cdot\frac{527}{625}=\frac{527\sqrt{11}}{40}$, and $m+n+p=527+11+40=578$.



OR

Let $\overline{CH}$ be the altitude to hypotenuse $\overline{AB}$. Triangles CDM and HDM share side $\overline{DM}$, and because $\overline{DM}\parallel\overline{CH}$, $[CDM]=[HDM]=(1/2)HM\cdot DM$. Note that $DM=\sqrt{AD^2-AM^2}=\frac{5\sqrt{11}}{2}$, and that $AC^2=AH\cdot AB$. Then $AH=49/25$, and $HM=(25/2)-(49/25)=527/50$. Thus $[CDM]=\frac{1}{2}\cdot\frac{527}{50}\cdot\frac{5\sqrt{11}}{2}=\frac{527}{40}\sqrt{11}$.

OR

Denote the vector $\overrightarrow{CD}$ by $\vec{d}$ and the vector $\overrightarrow{CM}$ by $\vec{m}$. Then $\vec{m}=(12,7/2,0)$, from which $\vec{d}=\big(12-(7/25)k,7/2-(24/25)k,0\big)$, where $k=(5\sqrt{11})/2$. This can be simplified to obtain $\vec{d}=\big(12-(7\sqrt{11}/10),7/2-(24\sqrt{11}/10),0\big)$. The area of $\triangle CDM$ is therefore $(1/2)|\vec{m}\times\vec{d}|=(527/40)\sqrt{11}$.

12. (Answer: 134)

Let t be the number of members of the committee, n_k be the number of votes for candidate k, and let p_k be the percentage of votes for candidate k for $k = 1, 2, \ldots, 27$. We have

$$n_k \geq p_k + 1 = \frac{100n_k}{t} + 1.$$

Adding these 27 inequalities yields $t \geq 127$. Solving for n_k gives $n_k \geq \dfrac{t}{t-100}$, and, since n_k is an integer, we obtain

$$n_k \geq \left\lceil \frac{t}{t-100} \right\rceil,$$

where the notation $\lceil x \rceil$ denotes the least integer that is greater than or equal to x. The last inequality is satisfied for all $k = 1, 2, \ldots, 27$ if and only if it is satisfied by the smallest n_k, say n_1. Since $t \geq 27n_1$, we obtain

$$t \geq 27 \left\lceil \frac{t}{t-100} \right\rceil \tag{1}$$

and our problem reduces to finding the smallest possible integer $t \geq 127$ that satisfies the inequality (1). If $\frac{t}{t-100} > 4$, that is, $t \leq 133$, then $27 \left\lceil \frac{t}{t-100} \right\rceil \geq 27 \cdot 5 = 135$ so that the inequality (1) is not satisfied. Thus 134 is the least possible number of members in the committee. Note that when $t = 134$, an election in which 1 candidate receives 30 votes and the remaining 26 candidates receive 4 votes each satisfies the conditions of the problem.

OR

Let t be the number of members of the committee, and let m be the least number of votes that any candidate received. It is clear that $m \neq 0$ and $m \neq 1$. If $m = 2$, then $2 \geq 1 + 100(2/t)$, so $t \geq 200$. Similarly, if $m = 3$, then $3 \geq 1 + 100(3/t)$, and $t \geq 150$; and if $m = 4$, then $4 \geq 1 + 100(4/t)$, so $t \geq 134$. When $m \geq 5$, $t \geq 27 \cdot 5 = 135$. Thus $t \geq 134$. Verify that t can be 134 by noting that the votes may be distributed so that 1 candidate receives 30 votes and the remaining 26 candidates receive 4 votes each.

13. (Answer: 683)

If the bug is at the starting vertex after move n, the probability is 1 that it will move to a non-starting vertex on move $n+1$. If the bug is not at the starting vertex after move n, the probability is $1/2$ that it will move back to its starting vertex on move $n+1$, and the probability is $1/2$ that it will move to another non-starting starting vertex on move $n+1$. Let p_n be the probability that the bug is at the starting vertex after move n. Then $p_{n+1} = 0 \cdot p_n + \frac{1}{2}(1-p_n) = -\frac{1}{2}p_n + \frac{1}{2}$. This implies that $p_{n+1} - \frac{1}{3} = -\frac{1}{2}(p_n - \frac{1}{3})$. Since $p_0 - \frac{1}{3} = 1 - \frac{1}{3} = 2/3$, conclude that $p_n - \frac{1}{3} = \frac{2}{3} \cdot (-\frac{1}{2})^n$. Therefore

$$p_n = \frac{2}{3} \cdot \left(-\frac{1}{2}\right)^n + \frac{1}{3} = \frac{1 + (-1)^n \frac{1}{2^{n-1}}}{3} = \frac{2^{n-1} + (-1)^n}{3 \cdot 2^{n-1}}.$$

Substitute 10 for n to find that $p_{10} = 171/512$, and $m+n$ is 683.

OR

A 10-step path can be represented by a 10-letter sequence consisting of only A's and B's, where A represents a move in the clockwise direction and B represents a move in the counterclockwise direction. Where the path ends depends on the number of A's and B's, not on their arrangement. Let x be the number of A's, and let y be the number of B's. Note that the bug will be home if and only if $x - y$ is a multiple of 3. After 10 moves, $x + y = 10$. Then $2x = 10 + 3k$ for some integer k, and so $x = 5 + 3j$ for some integer j. Thus the number of A's must be 2, 5, or 8, and the desired probability is

$$\frac{\binom{10}{2} + \binom{10}{5} + \binom{10}{8}}{2^{10}} = \frac{171}{512}.$$

OR

Let X be the bug's starting vertex, and let Y and Z be the other two vertices. Let x_k, y_k, and z_k be the probabilities that the bug is at vertex X, Y, and Z, respectively, at move k, for $k \geq 0$. Then $x_{k+1} = .5y_k + .5z_k$, $y_{k+1} = .5x_k + .5z_k$, and $z_{k+1} = .5x_k + .5y_k$. This can be written as

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ .5 & .5 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}.$$

Thus

$$\begin{bmatrix} x_{10} \\ y_{10} \\ z_{10} \end{bmatrix} = \begin{bmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ .5 & .5 & 0 \end{bmatrix}^{10} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and $x_{10} = 171/512$.

14. (Answer: 051)

Let the x-coordinates of C, D, E, and F be c, d, e, and f, respectively. Note that the y-coordinate of C is not 4, since, if it were, the fact that $AB = BC$ would imply that A, B, and C are collinear or that c is 0. Therefore $F = (f, 4)$. Since $\overline{AF}$ and $\overline{CD}$ are both parallel and congruent, $C = (c, 6)$ and $D = (d, 10)$, and then $E = (e, 8)$. Because the y-coordinates of B, C, and D are 2, 6, and 10, respectively, and $BC = CD$, conclude that $b = d$. Since $\overline{AB}$ and $\overline{DE}$ are both parallel and congruent, $e = 0$. Let a denote the side-length of the hexagon. Then $f^2+16 = AF^2 = a^2 = AB^2 = b^2+4$. Apply the Law of Cosines in $\triangle ABF$ to obtain $3a^2 = BF^2 = (b-f)^2 + 4$. Without loss of generality, assume $b > 0$. Then $f < 0$ and $b = \sqrt{a^2-4}$, $f = -\sqrt{a^2-16}$, and $b - f = \sqrt{3a^2-4}$. Now

$$\begin{aligned}\sqrt{a^2-4}+\sqrt{a^2-16} &= \sqrt{3a^2-4}, \quad \text{so} \\ 2a^2-20+2\sqrt{(a^2-4)(a^2-16)} &= 3a^2-4, \quad \text{and} \\ 2\sqrt{(a^2-4)(a^2-16)} &= a^2+16.\end{aligned}$$

Squaring again and simplifying yields $a^2 = 112/3$, so $b = 10/\sqrt{3}$ and $f = -8/\sqrt{3}$. Hence $A = (0,0)$, $B = (10/\sqrt{3}, 2)$, $C = (6\sqrt{3}, 6)$, $D = (10/\sqrt{3}, 10)$, $E = (0, 8)$, $F = (-8/\sqrt{3}, 4)$. Thus $[ABCDEF] = [ABDE]+2[AEF] = b\cdot AE + (-f)\cdot AE = 8(b-f) = 48\sqrt{3}$, so $m + n = 51$.

OR

Let α denote the measure of the acute angle formed by $\overline{AB}$ and the x-axis. Then the measure of the acute angle formed by $\overline{AF}$ and the x-axis is $60° - \alpha$. Note that $a \sin\alpha = 2$, so

$$\begin{aligned}4 &= a\sin(60° - \alpha) \\ &= a\frac{\sqrt{3}}{2}\cos\alpha - a\cdot\frac{1}{2}\sin\alpha \\ &= a\frac{\sqrt{3}}{2}\cos\alpha - 1.\end{aligned}$$

Thus $a\sqrt{3}\cos\alpha = 10$, and $b = a\cos\alpha = 10/\sqrt{3}$. Then $a^2 = b^2+4 = 112/3$, and $f^2 = a^2 - 16 = 64/3$. Also, $(c-b)^2 = a^2-16 = 64/3$, so $c = b + 8/\sqrt{3} = 6\sqrt{3}$; $c - d = 0 - f = 8/\sqrt{3}$, so $d = 10/\sqrt{3}$; and $e - f = c - b = 8/\sqrt{3}$, so $e = 0$. Proceed as above to obtain $[ABCDEF] = 48\sqrt{3}$.

15. (Answer: 015)

Note that

$$P(x) = x + 2x^2 + 3x^3 + \cdots + 24x^{24} + 23x^{25} + 22x^{26} + \cdots + 2x^{46} + x^{47},$$

and

$$xP(x) = \quad x^2 + 2x^3 + 3x^4 + \cdots \quad + 24x^{25} + 23x^{26} + \quad \cdots + \quad 2x^{47} + x^{48},$$

so

$$
\begin{aligned}
(1-x)P(x) &= x + x^2 + \cdots + x^{24} - (x^{25} + x^{26} + \cdots + x^{47} + x^{48}) \\
&= (1 - x^{24})(x + x^2 + \cdots + x^{24}).
\end{aligned}
$$

Then, for $x \neq 1$,

$$
\begin{aligned}
P(x) &= \frac{x^{24}-1}{x-1}\, x\,(1 + x + \cdots + x^{23}) \\
&= x\left(\frac{x^{24}-1}{x-1}\right)^2 . \qquad (*)
\end{aligned}
$$

One zero of $P(x)$ is 0, which does not contribute to the requested sum. The remaining zeros of $P(x)$ are the same as those of $(x^{24}-1)^2$, excluding 1. Because $(x^{24}-1)^2$ and $x^{24}-1$ have the same distinct zeros, the remaining zeros of $P(x)$ can be expressed as $z_k = \operatorname{cis} 15k^\circ$ for $k = 1, 2, 3, \cdots, 23$. The squares of the zeros are therefore of the form $\operatorname{cis} 30k^\circ$, and the requested sum is

$$
\sum_{k=1}^{23} |\sin 30k^\circ| = 4 \sum_{k=1}^{5} |\sin 30k^\circ| = 4\left(2 \cdot (1/2) + 2 \cdot (\sqrt{3}/2) + 1\right) = 8 + 4\sqrt{3}.
$$

Thus $m + n + p = 15$.

Note: the expression $(*)$ can also be obtained using the identity

$$
(1+x+x^2+\cdots+x^n)^2 = 1+2x+3x^2+\cdots+(n+1)x^n+\cdots+3x^{2n-2}+2x^{2n-1}+x^{2n}.
$$

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MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

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AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Tuesday, March 23, 2004

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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http://www.unl.edu/amc/d-publication/publication.html

1. (Answer: 217)

   Let the digits of n, read from left to right, be a, $a-1$, $a-2$, and $a-3$, respectively, where a is an integer between 3 and 9, inclusive. Then $n = 1000a + 100(a-1) + 10(a-2) + a - 3 = 1111a - 123 = 37(30a-4) + (a+25)$, where $0 \le a + 25 < 37$. Thus the requested sum is

$$\sum_{a=3}^{9}(a+25) = \left(\sum_{a=3}^{9} a\right) + 175 = 42 + 175 = 217.$$

OR

There are seven such four-digit integers, the smallest of which is 3210, whose remainder when divided by 37 is 28. The seven integers form an arithmetic sequence with common difference 1111, whose remainder when divided by 37 is 1, so the sum of the remainders is $28+29+30+31+32+33+34 = 7 \cdot 31 = 217$.

2. (Answer: 201)

   Let the smallest elements of $\mathcal{A}$ and $\mathcal{B}$ be $(n+1)$ and $(k+1)$, respectively. Then

$$2m = (n+1) + (n+2) + \cdots + (n+m) = mn + \frac{1}{2}\cdot m(m+1), \quad \text{and}$$
$$m = (k+1) + (k+2) + \cdots + (k+2m) = 2km + \frac{1}{2}\cdot 2m(2m+1).$$

The second equation implies that $k + m = 0$. Substitute this into $|k + 2m - (n+m)| = 99$ to obtain $n = \pm 99$. Now simplify the first equation to obtain $2 = n + (m+1)/2$, and substitute $n = \pm 99$. This yields $m = -195$ or $m = 201$. Because $m > 0$, $m = 201$.

OR

The mean of the elements in $\mathcal{A}$ is 2, and the mean of the elements in $\mathcal{B}$ is $1/2$. Because the mean of each of these sets equals its median, and the median of $\mathcal{A}$ is an integer, m is odd. Thus $\mathcal{A} = \{2 - \frac{m-1}{2}, \ldots, 2, \ldots, 2 + \frac{m-1}{2}\}$, and $\mathcal{B} = \{-m+1, \ldots, 0, 1, \ldots, m\}$. Therefore $|2 + \frac{m-1}{2} - m| = 99$, which yields $|\frac{3-m}{2}| = 99$, so $|3 - m| = 198$. Because $m > 0$, $m = 201$.

3. (Answer: 241)

The total number of diagonals and edges is $\binom{26}{2} = 325$, and there are $12 \cdot 2 = 24$ face diagonals, so P has $325 - 60 - 24 = 241$ space diagonals. One such polyhedron can be obtained by gluing two dodecahedral pyramids onto the 12-sided faces of a dodecahedral prism.

Note that one can determine that there are 60 edges as follows. The 24 triangles contribute $3 \cdot 24 = 72$ edges, and the 12 quadrilaterals contribute $4 \cdot 12 = 48$ edges. Because each edge is in two faces, there are $\frac{1}{2}(72 + 48) = 60$ edges.

4. (Answer: 086)

Let $\overline{PQ}$ be a line segment in set $\mathcal{S}$ that is not a side of the square, and let M be the midpoint of $\overline{PQ}$. Let A be the vertex of the square that is on both the side that contains P and the side that contains Q. Because $\overline{AM}$ is the median to the hypotenuse of right $\triangle PAQ$, $AM = (1/2) \cdot PQ = (1/2) \cdot 2 = 1$. Thus every midpoint is 1 unit from a vertex of the square, and the set of all the midpoints forms four quarter-circles of radius 1 and with centers at the vertices of the square. The area of the region bounded by the four arcs is $4 - 4 \cdot (\pi/4) = 4 - \pi$, so $100k = 100(4 - 3.14) = 86$.

OR

Place a coordinate system so that the vertices of the square are at $(0,0)$, $(2,0)$, $(2,2)$, and $(0,2)$. When the segment's vertices are on the sides that contain $(0,0)$, its endpoints' coordinates can be represented as $(a,0)$ and $(0,b)$. Let the coordinates of the midpoint of the segment be (x,y). Then $(x,y) = (a/2, b/2)$ and $a^2 + b^2 = 4$. Thus $x^2 + y^2 = (a/2)^2 + (b/2)^2 = 1$, and the midpoints of these segments form a quarter-circle with radius 1 centered at the origin. The set of all the midpoints forms four quarter-circles, and the area of the region bounded by the four arcs is $4 - 4 \cdot (\pi/4) = 4 - \pi$, so $100k = 100(4 - 3.14) = 86$.

5. (Answer: 849)

Let Beta's scores be a out of b on day one and c out of d on day two, so that $0 < a/b < 8/15$, $0 < c/d < 7/10$, and $b + d = 500$. Then $(15/8)a < b$ and $(10/7)c < d$, so $(15/8)a + (10/7)c < b + d = 500$, and $21a + 16c < 5600$. Beta's two-day success ratio is greatest when $a + c$ is greatest. Let $M = a + c$ and subtract $16M$ from both sides of the last inequality to obtain $5a < 5600 - 16M$. Because $a > 0$, conclude that $5600 - 16M > 0$, and $M < 350$. When $M = 349$, $5a < 16$, so $a \leq 3$. If $a = 3$, then $b \geq 6$, but then $d \leq 494$ and $c = 346$ so $c/d \geq 346/494 > 7/10$. Notice that when $a = 2$ and $b = 4$, then $a/b < 8/15$ and

$c/d = 347/496 < 7/10$. Thus Beta's maximum possible two-day success ratio is $349/500$, so $m + n = 849$.

OR

Let M be the total number of points scored by Beta in the two days. Notice first that $M < 350$, because 350 is 70% of 500, and Beta's success ratio is less than 70% on each day of the competition. Notice next that $M = 349$ is possible, because Beta could score 1 point out of 2 attempted on the first day, and 348 out of 498 attempted on the second day. Thus $m = 349$, $n = 500$, and $m+n = 849$.

Note that Beta's two-day success ratio can be greater than Alpha's while Beta's success ratio is less on each day. This is an example of Simpson's Paradox.

6. (Answer: 882)

To find the number of snakelike numbers that have four different digits, distinguish two cases, depending on whether or not 0 is among the chosen digits. For the case where 0 is not among the chosen digits, first consider only the digits 1, 2, 3, and 4. There are exactly 5 snakelike numbers with these digits: 1423, 1324, 2314, 2413, and 3412. There are $\binom{9}{4} = 126$ ways to choose four non-zero digits and five ways to arrange each such set for a total of 630 numbers. In the other case, there are $\binom{9}{3} = 84$ ways to choose three digits to go with 0, and three ways to arrange each set of four digits, because the snakelike numbers with the digits 0, 1, 2, and 3 would correspond to the list above, but with the first two entries deleted. There are $84 \cdot 3 = 252$ such numbers. Thus there are $630 + 252 = 882$ four-digit snakelike numbers with distinct digits.

7. (Answer: 588)

Each of the x^2-terms in the expansion of the product is obtained by multiplying the x-terms from two of the 15 factors of the product. The coefficient of the x^2-term is therefore the sum of the products of each pair of numbers in the set $\{-1, 2, -3, \ldots, 14, -15\}$. Note that, in general,

$$(a_1 + a_2 + \cdots + a_n)^2 = a_1^2 + a_2^2 + \cdots + a_n^2 + 2 \cdot \left(\sum_{1 \le i < j \le n} a_i a_j \right).$$

Thus

$$C = \sum_{1 \le i < j \le 15} (-1)^i i (-1)^j j = \frac{1}{2}\left(\left(\sum_{k=1}^{15}(-1)^k k\right)^2 - \sum_{k=1}^{15} k^2\right)$$
$$= \frac{1}{2}\left((-8)^2 - \frac{15(15+1)(2\cdot 15+1)}{6}\right) = -588.$$

Hence $|C| = 588$.

OR

Note that

$$\begin{aligned} f(x) &= (1-x)(1+2x)(1-3x)\dots(1-15x) \\ &= 1 + (-1+2-3+\cdots-15)x + Cx^2 + \cdots \\ &= 1 - 8x + Cx^2 + \cdots . \end{aligned}$$

Thus $f(-x) = 1 + 8x + Cx^2 - \cdots$.
But $f(-x) = (1+x)(1-2x)(1+3x)\dots(1+15x)$, so

$$\begin{aligned} f(x)f(-x) &= (1-x^2)(1-4x^2)(1-9x^2)\dots(1-225x^2) \\ &= 1 - (1^2+2^2+3^2+\cdots+15^2)x^2 + \cdots . \end{aligned}$$

Also $f(x)f(-x) = (1-8x+Cx^2+\cdots)(1+8x+Cx^2-\cdots) = 1+(2C-64)x^2+\cdots$.
Thus $2C - 64 = -(1^2+2^2+3^3+\cdots+15^2)$, and, as above, $|C| = 588$.

8. (Answer: 199)

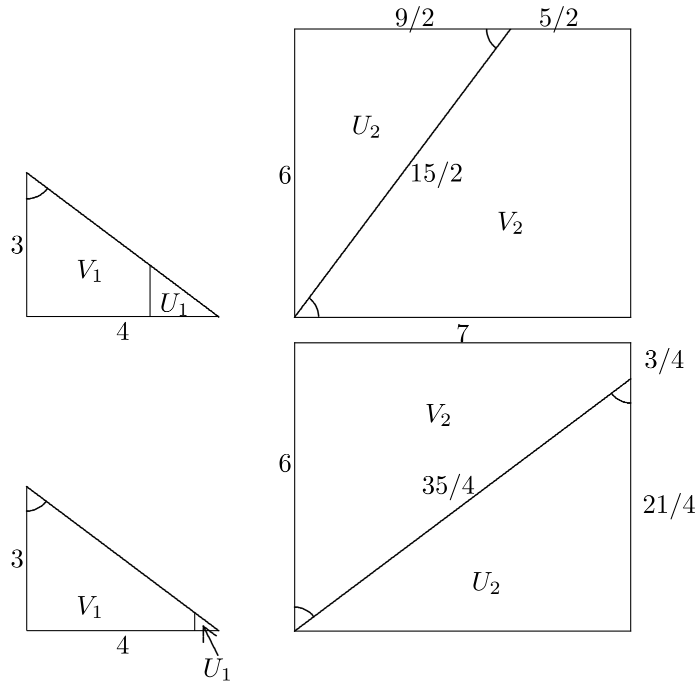
Let $\mathcal{C}$ be the circle determined by P_1, P_2, and P_3. Because the path turns counterclockwise at an angle of less than $180°$ at P_2 and P_3, P_1 and P_4 must be on the same side of line P_2P_3. Note that $\triangle P_1P_2P_3 \cong \triangle P_4P_3P_2$, and so $\angle P_2P_1P_3 \cong \angle P_3P_4P_2$. Thus P_4 is on $\mathcal{C}$. Similarly, because P_2, P_3, and P_4 are on $\mathcal{C}$, P_5 must be too, and, in general, $P_1, P_2, \ldots, P_n$ are all on $\mathcal{C}$. The fact that the minor arcs $P_1P_2, P_2P_3, \ldots,$ and P_nP_1 are congruent implies that $P_1, P_2, \ldots,$ and P_n are equally spaced on $\mathcal{C}$.

Thus any regular n-pointed star can be constructed by choosing n equally spaced points on a circle, and numbering them consecutively from 0 to $n-1$. For positive integers $d < n$, the path consisting of line segments whose vertices are numbered $0, d, 2d, \ldots, (n-1)d, 0$ modulo n will be a regular n-pointed star if and only if $2 \leq d \leq n-2$ and d is relatively prime to n. This is because if $d = 1$ or $d = n-1$, the resulting path will be a polygon; and if d is not relatively prime to n, not every vertex will be included in the path. Also, for any choice of d that yields a regular n-pointed star, any two such stars will be similar because a dilation of one of the stars about the center of its circle will yield the other.

Because $1000 = 2^3 \cdot 5^3$, numbers that are relatively prime to 1000 are those that are multiples of neither 2 nor 5. There are $1000/2 = 500$ multiples of 2 that are less than or equal to 1000; there are $1000/5 = 200$ multiples of 5 that are less than or equal to 1000; and there are $1000/10 = 100$ numbers less than or equal to 1000 that are multiples of both 2 and 5. Hence there are $1000 - (500 + 200 - 100) = 400$ numbers that are less than 1000 and relatively prime to 1000, and 398 of them are between 2 and 998, inclusive. Because $d = k$ yields the same path as $d = n - k$ (and also because one of these two paths turns clockwise at each vertex), there are $398/2 = 199$ non-similar regular 1000-pointed stars.

9. (Answer: 035)

Let s_1 be the line segment drawn in $\triangle ABC$, and let s_2 be the line segment drawn in rectangle $DEFG$. To obtain a triangle and a trapezoid, line segment s_2 must pass through exactly one vertex of rectangle $DEFG$. Hence V_2 is a trapezoid with a right angle, and U_2 is a right triangle. Therefore line segment s_1 is parallel to one of the legs of $\triangle ABC$ and, for all placements of s_1, U_1 is similar to $\triangle ABC$. It follows that there are two possibilities for triangle U_2: one in which the sides are 6, 9/2, and 15/2; and the other in which the sides are 7, 21/4, and 35/4. Were s_1 parallel to the side of length 4, trapezoids V_1 and V_2 could not be similar, because the corresponding acute angles in V_1 and V_2 would not be congruent; but when s_1 is parallel to the side of length 3, the angles of trapezoid V_1 are congruent to the corresponding angles of V_2, so it is possible to place segment s_1 so that V_1 is similar to V_2. In the case when the triangle U_2 has sides 6, 9/2, and 15/2, the bases of trapezoid V_2 are 7 and $7 - (9/2) = 5/2$, so its bases, and therefore the bases of V_1, are in the ratio $5 : 14$. Then the area of triangle U_1 is $(5/14)^2 \cdot (1/2) \cdot 3 \cdot 4 = 75/98$. In the case when the triangle U_2 has sides 7, 21/4, and 35/4, the bases of trapezoid V_2 are 6 and $6 - (21/4) = 3/4$, so its bases, and the bases of V_1, are in the ratio $1 : 8$. The area of triangle U_1 is then $(1/8)^2 \cdot (1/2) \cdot 3 \cdot 4 = 3/32$. The minimum value of the area of U_1 is thus $3/32$, and $m + n = 35$.



10. (Answer: 817)

In order for the circle to lie completely within the rectangle, the center of the circle must lie in a rectangle that is $(15 - 2)$ by $(36 - 2)$ or 13 by 34. The requested probability is equal to the probability that the distance from the circle's center to the diagonal $\overline{AC}$ is greater than 1, which equals the probability that the distance from a randomly selected point in the 13-by-34 rectangle to each of the sides of $\triangle ABC$ and $\triangle CDA$ is greater than 1. Let $AB = 36$ and $BC = 15$. Draw the three line segments that are one unit respectively from each of the sides of $\triangle ABC$ and whose endpoints are on the sides. Let E, F, and G be the three points of intersection nearest to A, B, and C, respectively, of the three line segments. Let P be the intersection of $\overrightarrow{EG}$ and $\overline{BC}$, and let G' and P' be the projections of G and P on $\overline{BC}$ and $\overline{AC}$, respectively. Then $FG = BC - CP - PG' - 1$. Notice that $\triangle PP'C \sim \triangle ABC$ and $PP' = 1$, so $CP = AC/AB$. Similarly, $\triangle GG'P \sim \triangle ABC$ and $GG' = 1$, so $PG' = CB/AB$.

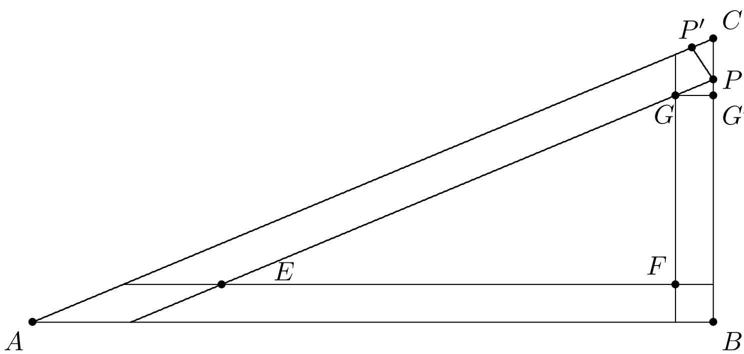
Thus

$$FG = BC - \frac{AC}{AB} - \frac{CB}{AB} - 1.$$

Apply the Pythagorean Theorem to $\triangle ABC$ to obtain $AC = 39$. Substitute these lengths to find that $FG = 25/2$. Notice that $\triangle EFG \sim \triangle ABC$, and their similarity ratio is $(25/2)/15 = 5/6$, so $[EFG] = (25/36)[ABC]$. The requested probability is therefore

$$\frac{2 \cdot \frac{25}{36} \cdot \frac{1}{2} \cdot 15 \cdot 36}{13 \cdot 34} = \frac{375}{442},$$

so $m + n = 817$.



OR

Define E, F, and G as in the previous solution. Each of these three points is equidistant from two sides of $\triangle ABC$, and they are therefore on the angle-bisectors of angles A, B, and C, respectively. These angle-bisectors are also angle-bisectors of $\triangle EFG$ because its sides are parallel to those of $\triangle ABC$. Thus $\triangle ABC$ and $\triangle EFG$ have the same incenter, and the inradius of $\triangle EFG$ is one less than that of $\triangle ABC$. In general, the inradius of a triangle is the area divided by one-half the perimeter, so the inradius of $\triangle ABC$ is 6. The similarity ratio of $\triangle EFG$ to $\triangle ABC$ is the same as the ratio of their inradii, namely $5/6$. Continue as in the previous solution.

OR

Define E, F, G, and G' as in the previous solutions. Notice that $\overline{CG}$ bisects $\angle ACB$ and that $\cos\angle ACB = 5/13$, and so, by the Half-Angle Formula, $\cos\angle GCG' = 3/\sqrt{13}$. Thus, for some x, $CG' = 3x$ and $CG = x\sqrt{13}$. Apply the Pythagorean Theorem in $\triangle CG'G$ to conclude that $(x\sqrt{13})^2 - (3x)^2 = 1$, so $x = 1/2$. Then $CG' = 3x = 3/2$, and $FG = 15 - 1 - 3/2 = 25/2$. Continue as in the first solution.

11. (Answer: 512)

The lateral surface area of a cone with radius R and slant height S can be found by cutting the cone along a slant height and then unrolling it to form a sector of a circle. The sector's arc has length $2\pi R$ and its radius is S, so its area, and the cone's lateral surface area, is $\pi S^2 \cdot \frac{2\pi R}{2\pi S} = \pi RS$. Let r, h, and s represent the radius, height, and slant height of the smaller cone formed by the cut. Then

$$\begin{aligned} k &= \frac{r^2 h}{36 - r^2 h} = \frac{rs}{9 + 15 - rs}, \quad \text{so} \\ \frac{1}{k} &= \frac{36}{r^2 h} - 1 = \frac{24}{rs} - 1. \end{aligned}$$

Thus $3s = 2rh$. Because $r : h : s = 3 : 4 : 5$, let $(r, h, s) = (3x, 4x, 5x)$, and substitute to find that $x = 5/8$, and then that $(r, h, s) = (15/8, 20/8, 25/8)$. The ratio of the volume of $\mathcal{C}$ to that of the large cone is therefore $\left(\frac{15/8}{3}\right)^3 = \frac{125}{512}$, so the ratio of the volumes of $\mathcal{C}$ and $\mathcal{F}$ is $125/(512 - 125) = 125/387$. Thus $m + n = 125 + 512 - 125 = 512$.

12. (Answer: 014)

Because $\lfloor \log_2(\frac{1}{x}) \rfloor = 2k$ for nonnegative integers k, conclude that $2k \le \log_2(\frac{1}{x}) < 2k + 1$, so

$$2^{2k} \le \frac{1}{x} < 2^{2k+1}, \quad \text{and} \quad \frac{1}{2^{2k+1}} < x \le \frac{1}{2^{2k}}\,.$$

Similarly, for nonnegative integers k,

$$\frac{1}{5^{2k+1}} < y \le \frac{1}{5^{2k}}\,.$$

The graph consists of the intersection of two sets of rectangles. The rectangles in one set have vertical sides of length 1 and horizontal sides of lengths $(1 - \frac{1}{2}), (\frac{1}{4} - \frac{1}{8}), (\frac{1}{16} - \frac{1}{32}), \ldots$, and the rectangles in the other set have horizontal sides of length 1 and vertical sides of lengths $(1 - \frac{1}{5}), (\frac{1}{25} - \frac{1}{125}), (\frac{1}{625} - \frac{1}{3125}), \ldots$. The intersection of the two sets of rectangles is also a set of rectangles whose total area is

$$\left[\left(1 - \frac{1}{2}\right) + \left(\frac{1}{4} - \frac{1}{8}\right) + \cdots\right] \cdot \left[\left(1 - \frac{1}{5}\right) + \left(\frac{1}{25} - \frac{1}{125}\right) + \cdots\right] = \frac{2}{3} \cdot \frac{5}{6} = \frac{5}{9},$$

so $m + n = 14$.

13. (Answer: 482)

Note that for $x \neq 1$,

$$\begin{aligned} P(x) &= \left(\frac{x^{18}-1}{x-1}\right)^2 - x^{17} \quad \text{so} \\ (x-1)^2P(x) &= (x^{18}-1)^2 - x^{17}(x-1)^2 \\ &= x^{36} - 2x^{18} + 1 - x^{19} + 2x^{18} - x^{17} \\ &= x^{36} - x^{19} - x^{17} + 1 \\ &= x^{19}(x^{17}-1) - (x^{17}-1) \\ &= (x^{19}-1)(x^{17}-1), \quad \text{and so} \\ P(x) &= \frac{(x^{19}-1)(x^{17}-1)}{(x-1)^2}. \end{aligned}$$

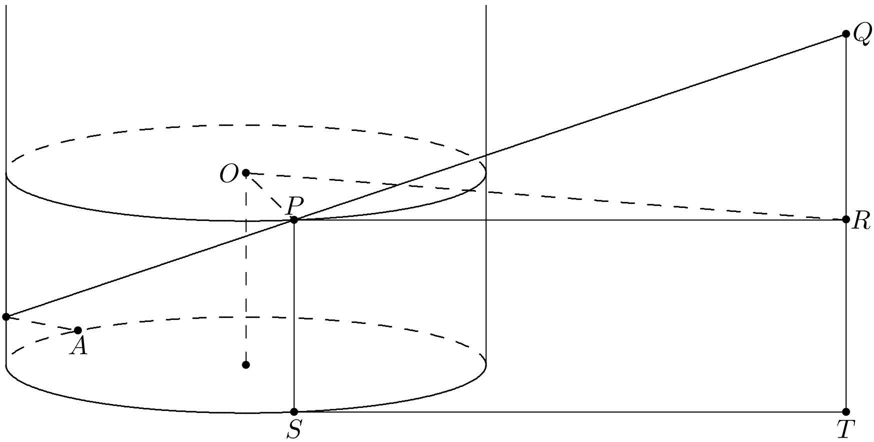
Thus the zeros of $P(x)$ are the 34 complex numbers other than 1 which satisfy $x^{17} = 1$ or $x^{19} = 1$. It follows that $\alpha_1 = 1/19$, $\alpha_2 = 1/17$, $\alpha_3 = 2/19$, $\alpha_4 = 2/17$, and $\alpha_5 = 3/19$, so $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 = 159/323$, and $m + n = 482$.

14. (Answer: 813)

Suppose that the rope is attached to the ground at point A, last touches the tower at point P, and attaches to the unicorn at point Q. Let S and T be on the ground directly below P and Q, respectively. Let O be on the axis of the tower, and let R be directly below Q so that the plane of $\triangle OPR$ is horizontal. Then $\overline{OP}$ is a radius of the tower, so $OP = 8$, and, because Q is 4 feet from the tower, R is too, so $OR = 12$. Also, $\angle OPR$ is a right angle, so $PR = 4\sqrt{5}$. If the tower wall were spread flat in the plane of P, Q, S, and T, then right triangles PQR and AQT would be similar. Because

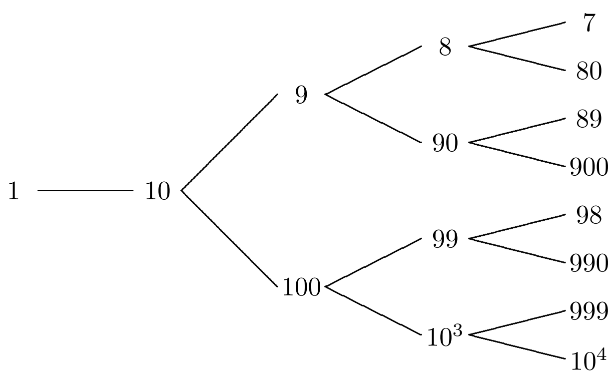
$$\frac{PQ}{PR} = \frac{AQ}{AT} = \frac{20}{\sqrt{20^2 - 4^2}} = \frac{5}{\sqrt{5^2 - 1^2}} = \frac{5}{2\sqrt{6}},$$

where AQ is the rope's length between A and Q and AT is the length of the projection of the rope onto the ground, $PQ = \dfrac{5}{2\sqrt{6}} \cdot 4\sqrt{5} = \dfrac{5\sqrt{30}}{3}$. Then the length of rope touching the tower is $20 - \dfrac{5\sqrt{30}}{3} = \dfrac{60 - \sqrt{750}}{3}$. Thus $a + b + c = 60 + 750 + 3 = 813$.



15. (Answer: 511)

For $i = 1, 2, \ldots$, let $\mathcal{S}_i$ denote the set of positive integers x such that $d(x) = i$. Then, for example, $\mathcal{S}_1 = \{1\}$, $\mathcal{S}_2 = \{10\}$, $\mathcal{S}_3 = \{9, 100\}$, and $\mathcal{S}_4 = \{8, 90, 99, 1000\}$. This can be illustrated in a tree diagram, as shown.



If each vertex in this tree after the first column had two branches, then the 20th column would have 2^{18} vertices; but some vertices have only one branch, namely, the vertex that corresponds to 2 and vertices that correspond to numbers with last digit 1. The vertex that corresponds to 2 is in the 10th column. This causes there to be 2^9 fewer vertices in the 20th column than there would be if each vertex in the tree had two branches. Note that vertices that correspond to numbers with last digit 1 occur 9 columns after vertices that correspond to numbers with last digit 0, except for 10. Thus there will be one 1 in column 12, two 1's in column 13, and in general, 2^{k-12} 1's in column k, for $k = 12, 13, \ldots, 19$. For each of these eight columns, this causes there to be 2^7 fewer vertices in the 20th column than there would be if each vertex in the tree had two branches. Thus m is the number of elements in $\mathcal{S}_{20}$, that is, the number of vertices in column 20, namely,

$$2^{18} - 2^9 - 8 \cdot 2^7 = 2^{18} - 2^9 - 2^{10} = 2^9(2^9 - 1 - 2) = 2^9 \cdot 509,$$

and the sum of the distinct prime factors of m is $2 + 509 = 511$.

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AMERICAN MATHEMATICS COMPETITIONS

22nd Annual *(Alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 592)

   Without loss of generality, let the radius of the circle be 2. The radii to the endpoints of the chord, along with the chord, form an isosceles triangle with vertex angle 120°. The area of the larger of the two regions is thus $2/3$ that of the circle plus the area of the isosceles triangle, and the area of the smaller of the two regions is thus $1/3$ that of the circle minus the area of the isosceles triangle. The requested ratio is therefore $\dfrac{\frac{2}{3}\cdot 4\pi + \sqrt{3}}{\frac{1}{3}\cdot 4\pi - \sqrt{3}} = \dfrac{8\pi + 3\sqrt{3}}{4\pi - 3\sqrt{3}}$, so $abcdef = 8 \cdot 3 \cdot 3 \cdot 4 \cdot 3 \cdot 3 = 2592$, and the requested remainder is 592.

2. (Answer: 441)

   In order for Terry and Mary to get the same color combination, they must select all red candies or all blue candies, or they must each select one of each color. The probability of getting all red candies is $\dfrac{\binom{10}{2}\binom{8}{2}}{\binom{20}{2}\binom{18}{2}} = \dfrac{10 \cdot 9 \cdot 8 \cdot 7}{20 \cdot 19 \cdot 18 \cdot 17}$. The probability of getting all blue candies is the same. The probability that they each select one of each color is $\dfrac{10^2 \cdot 9^2}{\binom{20}{2}\binom{18}{2}} = \dfrac{10^2 \cdot 9^2 \cdot 4}{20 \cdot 19 \cdot 18 \cdot 17}$. Thus the probability of getting the same combination is

   $$2 \cdot \frac{10 \cdot 9 \cdot 8 \cdot 7}{20 \cdot 19 \cdot 18 \cdot 17} + \frac{10^2 \cdot 9^2 \cdot 4}{20 \cdot 19 \cdot 18 \cdot 17} = \frac{10 \cdot 9 \cdot 8 \cdot (14 + 45)}{20 \cdot 19 \cdot 18 \cdot 17} = \frac{2 \cdot 59}{19 \cdot 17} = \frac{118}{323},$$

   and $m + n = 441$.

3. (Answer: 384)

   Let the dimensions of the block be p cm by q cm by r cm. The invisible cubes form a rectangular solid whose dimensions are $p-1$, $q-1$, and $r-1$. Thus $(p-1)(q-1)(r-1) = 231$. There are only five ways to write 231 as a product of three positive integers:

   $$231 = 3 \cdot 7 \cdot 11 = 1 \cdot 3 \cdot 77 = 1 \cdot 7 \cdot 33 = 1 \cdot 11 \cdot 21 = 1 \cdot 1 \cdot 231$$

   The corresponding blocks are $4 \times 8 \times 12$, $2 \times 4 \times 78$, $2 \times 8 \times 34$, $2 \times 12 \times 22$, and $2 \times 2 \times 232$. Their volumes are 384, 624, 544, 528, and 928, respectively. Thus the smallest possible value of N is 384.

4. (Answer: 927)

   There are 99 numbers with the desired property that are less than 100. Three-digit numbers with the property must have decimal representations of the form aaa, aab, aba, or abb, where a and b are digits with $a \geq 1$ and $a \neq b$. There are

9 of the first type and $9 \cdot 9 = 81$ of each of the other three. Four-digit numbers with the property must have decimal representations of the form *aaaa*, *aaab*, *aaba*, *aabb*, *abaa*, *abab*, *abba*, or *abbb*. There are 9 of the first type and 81 of each of the other seven. Thus there are a total of $99 + 9 + 3 \cdot 81 + 9 + 7 \cdot 81 = 927$ numbers with the desired property.

OR

Count the number of positive integers less than 10,000 that contain at least 3 distinct digits. There are $9 \cdot 9 \cdot 8$ such 3-digit integers. The number of 4-digit integers that contain exactly 3 distinct digits is $\binom{4}{2} \cdot 9 \cdot 9 \cdot 8$ because there are $\binom{4}{2}$ choices for the positions of the two digits that are the same, 9 choices for the digit that appears in the first place, and 9 and 8 choices for the two other digits, respectively. The number of 4-digit integers that contain exactly 4 distinct digits is $9 \cdot 9 \cdot 8 \cdot 7$. Thus there are $9 \cdot 9 \cdot 8 + 6 \cdot 9 \cdot 9 \cdot 8 + 9 \cdot 9 \cdot 8 \cdot 7 = 14 \cdot 9 \cdot 9 \cdot 8 = 9072$ positive integers less than 10,000 that contain at least 3 distinct digits, and there are $9999 - 9072 = 927$ integers with the desired property.

5. (Answer: 766)

Choose a unit of time so that the job is scheduled to be completed in 4 of these units. The first quarter was completed in 1 time unit. For the second quarter of the work, there were only 9/10 as many workers as in the first quarter, so it was completed in 10/9 units. For the third quarter, there were only 8/10 as many workers as in the first quarter, so it was completed in 5/4 units. This leaves $4 - (1 + 10/9 + 5/4) = 23/36$ units to complete the final quarter. To finish the job on schedule, the number of workers that are needed is at least 36/23 of the number of workers needed in the first quarter, or (36/23)1000 which is between 1565 and 1566. There are 800 workers at the end of the third quarter, so a minimum of $1566 - 800 = 766$ additional workers must be hired.

6. (Answer: 408)

Let the first monkey take $8x$ bananas from the pile, keeping $6x$ and giving x to each of the others. Let the second monkey take $8y$ bananas from the pile, keeping $2y$ and giving $3y$ to each of the others. Let the third monkey take $24z$ bananas from the pile, keeping $2z$ and giving $11z$ to each of the others. The total number of bananas is $8x + 8y + 24z$. The given ratios imply that $6x + 3y + 11z = 3(x + 3y + 2z)$ and $x + 2y + 11z = 2(x + 3y + 2z)$. Simplify these equations to obtain $3x + 5z = 6y$ and $7z = x + 4y$. Eliminate x to obtain $9y = 13z$. Then $y = 13n$ and $z = 9n$, where n is a positive integer. Substitute to find that $x = 11n$. Thus, the least possible values for x, y and z are 11, 13 and 9, respectively, and the least possible total is $8 \cdot 11 + 8 \cdot 13 + 24 \cdot 9 = 408$.

7. (Answer: 293)

Let $\overline{B'C'}$ and $\overline{CD}$ intersect at H. Note that $B'E = BE = 17$. Apply the Pythagorean Theorem to $\triangle EAB'$ to obtain $AB' = 15$. Because $\angle C'$ and $\angle C'B'E$ are right angles, $\triangle B'AE \sim \triangle HDB' \sim \triangle HC'F$, so the lengths of the sides of each triangle are in the ratio $8 : 15 : 17$. Now $C'F = CF = 3$ implies that $FH = (17/8)3 = 51/8$ and $DH = 25 - (3 + 51/8) = 125/8$. Then $B'D = (8/15)(125/8) = 25/3$. Thus $AD = 70/3$, and the perimeter of $ABCD$ is

$$2 \cdot 25 + 2 \cdot \frac{70}{3} = \frac{290}{3},$$

so $m + n = 290 + 3 = 293$.

OR

Notice first that $B'E = BE = 17$. Apply the Pythagorean Theorem to $\triangle EAB'$ to obtain $AB' = 15$. Draw $\overline{FG}$ parallel to $\overline{CB}$, with G on $\overline{AB}$. Notice that $GE = 17 - 3 = 14$. Because points on the crease $\overline{EF}$ are equidistant from B and B', it follows that $\overline{EF}$ is perpendicular to $\overline{BB'}$, and hence that triangles EGF and $B'AB$ are similar. In particular, $\dfrac{FG}{BA} = \dfrac{GE}{AB'}$. This yields $FG = 70/3$, and the perimeter of $ABCD$ is therefore $290/3$.

8. (Answer: 054)

A positive integer N is a divisor of 2004^{2004} if and only if $N = 2^i 3^j 167^k$ with $0 \le i \le 4008$, $0 \le j \le 2004$, and $0 \le k \le 2004$. Such a number has exactly 2004 positive integer divisors if and only if $(i+1)(j+1)(k+1) = 2004$. Thus the number of values of N meeting the required conditions is equal to the number of ordered triples of positive integers whose product is 2004. Each of the unordered triples $\{1002, 2, 1\}$, $\{668, 3, 1\}$, $\{501, 4, 1\}$, $\{334, 6, 1\}$, $\{334, 3, 2\}$, $\{167, 12, 1\}$, $\{167, 6, 2\}$, and $\{167, 4, 3\}$ can be ordered in 6 possible ways, and the triples $\{2004, 1, 1\}$ and $\{501, 2, 2\}$ can each be ordered in 3 possible ways, so the total is $8 \cdot 6 + 2 \cdot 3 = 54$.

OR

Begin as above. Then, to find the number of ordered triples of positive integers whose product is 2004, represent the triples as $(2^{a_1} \cdot 3^{b_1} \cdot 167^{c_1}, 2^{a_2} \cdot 3^{b_2} \cdot 167^{c_2}, 2^{a_3} \cdot 3^{b_3} \cdot 167^{c_3})$, where $a_1 + a_2 + a_3 = 2$, $b_1 + b_2 + b_3 = 1$, and $c_1 + c_2 + c_3 = 1$, and the a_i's, b_i's, and c_i's are nonnegative integers. The number of solutions of $a_1 + a_2 + a_3 = 2$ is $\binom{4}{2}$ because each solution corresponds to an arrangement of two objects and two dividers. Similarly, the number of solutions of both $b_1 + b_2 + b_3 = 1$ and $c_1 + c_2 + c_3 = 1$ is $\binom{3}{1}$, so the total number of triples is $\binom{4}{2}\binom{3}{1}\binom{3}{1} = 6 \cdot 3 \cdot 3 = 54$.

9. (Answer: 973)

The terms in the sequence are $1, r, r^2, r(2r-1), (2r-1)^2, (2r-1)(3r-2), (3r-2)^2, \ldots$. Assuming that the pattern continues, the ninth term is $(4r-3)^2$ and the tenth term is $(4r-3)(5r-4)$. Thus $(4r-3)^2+(4r-3)(5r-4) = 646$. This leads to $(36r+125)(r-5) = 0$. Because the terms are positive, $r = 5$. Substitute to find that $a_n = (2n-1)^2$ when n is odd, and that $a_n = (2n-3)(2n+1)$ when n is even. The least odd-numbered term greater than 1000 is therefore $a_{17} = 33^2 = 1089$, and $a_{16} = 29 \cdot 33 = 957 < 1000$. The desired value of $n + a_n$ is $957 + 16 = 973$.

The pattern referred to above is

$$\begin{aligned} a_{2n} &= [(n-1)r - (n-2)][nr - (n-1)], \\ a_{2n+1} &= [nr - (n-1)]^2. \end{aligned}$$

This pattern has been verified for the first few positive integral values of n. The above equations imply that

$$\begin{aligned} a_{2n+2} &= 2[nr-(n-1)]^2 - [(n-1)r-(n-2)][nr-(n-1)] \\ &= [nr-(n-1)][2nr-2(n-1)-(n-1)r+(n-2)] \\ &= [nr-(n-1)][(n+1)r-n], \quad \text{and} \\ a_{2n+3} &= \frac{[nr-(n-1)]^2[(n+1)r-n]^2}{[nr-(n-1)]^2} \\ &= [(n+1)r-n]^2. \end{aligned}$$

The above argument, along with the fact that the pattern holds for $n = 1$ and $n = 2$, implies that it holds for all positive integers n.

10. (Answer: 913)

An element of $\mathcal{S}$ has the form 2^a+2^b, where $0 \le a \le 39$, $0 \le b \le 39$, and $a \ne b$, so $\mathcal{S}$ has $\binom{40}{2} = 780$ elements. Without loss of generality, assume $a < b$. Note that 9 divides $2^a + 2^b = 2^a(2^{b-a}+1)$ if and only if 9 divides $2^{b-a}+1$, that is, when $2^{b-a} \equiv 8 \pmod 9$. Because $2^1, 2^2, 2^3, 2^4, 2^5, 2^6$, and $2^7 \equiv 2, 4, 8, 7, 5, 1$, and $2 \pmod 9$, respectively, conclude that $2^{b-a} \equiv 8 \pmod 9$ when $b - a = 6k - 3$ for positive integers k. But $b - a = 6k - 3$ implies that $0 \le a \le 39 - (6k-3)$, so there are $40 - (6k-3) = 43 - 6k$ ordered pairs (a, b) that satisfy $b - a = 6k - 3$. Because $6k - 3 \le 39$, conclude that $1 \le k \le 7$. The number of multiples of 9 in $\mathcal{S}$ is therefore $\displaystyle\sum_{k=1}^{7}(43-6k) = 7 \cdot 43 - 6 \cdot 7 \cdot 8/2 = 7(43-24) = 133$. Thus the probability that an element of $\mathcal{S}$ is divisible by 9 is $133/780$, so $p + q = 913$.

OR

There are $\binom{40}{2} = 780$ such numbers, which can be viewed as strings of length 40 containing 38 0's and two 1's. Express these strings in base-8 by partitioning them into 13 groups of 3 starting from the right and one group of 1, and then expressing each 3-digit binary group as a digit between 0 and 7. Because $9 = 11_8$, a divisibility test similar to the one for 11 in base 10 can be used. Let $(a_d a_{d-1} \ldots a_0)_b$ represent a $(d+1)$-digit number in base b. Then

$$\begin{aligned}(a_d a_{d-1} \ldots a_0)_b &= \sum_{k=0}^{d} a_k b^k \\ &\equiv \sum_{k=0}^{d} a_k (-1)^k \pmod{b+1}.\end{aligned}$$

Thus a base-8 number is divisible by 9 if and only if the sum of the digits in the even-numbered positions differs from the sum of the digits in the odd-numbered positions by a multiple of 9. Because the given base-8 string has at most two nonzero digits, and its greatest digit is at most 110_2 or 6, the two sums must differ by 0. There are two cases. If the first (leftmost) digit in the base-2 string is 1, then the other 1 must be in an odd-numbered group of 3 that has the form 001. There are seven such numbers. In the second case, if the first digit in the base-2 string is 0, one of the 1's must be in an even-numbered group of 3, the other must be in an odd-numbered group, and they must be in the same relative position within each group, that is, both 100, both 010, or both 001. There are $7 \cdot 6 \cdot 3 = 126$ such numbers. Thus the required probability is $(7+126)/780 = 133/780$.

11. (Answer: 625)

Use the Pythagorean Theorem to conclude that the distance from the vertex of the cone to a point on the circumference of the base is

$$\sqrt{\left(200\sqrt{7}\right)^2 + 600^2} = 200\sqrt{\left(\sqrt{7}\right)^2 + 3^2} = 800.$$

Cut the cone along the line through the vertex of the cone and the starting point of the fly, and flatten the resulting figure into a sector of a circle with radius 800. Because the circumference of this circle is 1600π and the length of the sector's arc is 1200π, the measure of the sector's central angle is $270°$. The angle determined by the radius of the sector on which the fly starts and the radius on which the fly stops is $135°$. Use the Law of Cosines to conclude that the least distance the fly could have crawled between the start and end positions is

$$\begin{aligned}&\sqrt{\left(375\sqrt{2}\right)^2 + 125^2 - 2 \cdot 375\sqrt{2} \cdot 125 \cdot \cos 135°} \\ &\qquad = 125\sqrt{\left(3\sqrt{2}\right)^2 + 1 + 2 \cdot 3\sqrt{2} \cdot \left(\sqrt{2}/2\right)}\end{aligned}$$

which can be simplified to $125\sqrt{25} = 625$.

12. (Answer: 134)

Let E be the midpoint of $\overline{AB}$, F be the midpoint of $\overline{CD}$, x be the radius of the inner circle, and G be the center of that circle. Then $\overline{GE} \perp \overline{AB}$. Because the point of tangency of the circles centered at A and G is on $\overline{AG}$, $AG = x + 3$. Use the Pythagorean Theorem in $\triangle AEG$ to obtain $GE = \sqrt{x^2 + 6x}$. Similarly, find that $FG = \sqrt{x^2 + 4x}$. Because the height of the trapezoid is $\sqrt{24}$, conclude that

$$\begin{aligned} \sqrt{x^2+6x} + \sqrt{x^2+4x} &= \sqrt{24}, \quad \text{so} \\ \sqrt{x^2+6x} &= \sqrt{24} - \sqrt{x^2+4x}, \\ x^2 + 6x &= 24 + x^2 + 4x - 2\sqrt{24}\sqrt{x^2+4x}, \quad \text{and} \\ \sqrt{24(x^2+4x)} &= 12 - x. \end{aligned}$$

This yields $23x^2 + 120x - 144 = 0$, whose positive root is $x = \dfrac{-60 + 48\sqrt{3}}{23}$.

Thus $k + m + n + p = 60 + 48 + 3 + 23 = 134$.

13. (Answer: 484)

Let $\overline{AD}$ intersect $\overline{CE}$ at F. Extend $\overline{BA}$ through A to R so that $\overline{BR} \cong \overline{CE}$, and extend $\overline{BC}$ through C to P so that $\overline{BP} \cong \overline{AD}$. Then create parallelogram $PBRQ$ by drawing lines through D and E parallel to $\overline{AB}$ and $\overline{BC}$, respectively, with Q the intersection of the two lines. Apply the Law of Cosines to triangle ABC to obtain $AC = 7$. Now

$$\frac{RA}{AB} = \frac{EF}{FC} = \frac{DE}{AC} = \frac{15}{7}, \quad \text{so} \quad \frac{RB}{AB} = \frac{22}{7}.$$

Similarly, $\dfrac{PB}{CB} = \dfrac{22}{7}$. Thus parallelogram $ABCF$ is similar to parallelogram $RBPQ$. Let $K = [ABCF]$. Then $[RBPQ] = (22/7)^2 K$. Also,

$$\begin{aligned} [EBD] &= [RBPQ] - \frac{1}{2}\left([BCER] + [ABPD] + [DFEQ]\right) \\ &= [RBPQ] - \frac{1}{2}\left([RBPQ] + [ABCF]\right) \\ &= \frac{1}{2}\left([RBPQ] - [ABCF]\right). \end{aligned}$$

Thus

$$\frac{[ABC]}{[EBD]} = \frac{\frac{1}{2}K}{\frac{1}{2}\left[(\frac{22}{7})^2 K - K\right]} = \frac{1}{\frac{484}{49} - 1} = \frac{49}{435},$$

and $m + n = 484$.

OR

Apply the Law of Cosines to triangle ABC to obtain $AC = 7$. Let $\overline{AD}$ intersect $\overline{CE}$ at F. Then $ABCF$ is a parallelogram, which implies that $[ABC] = [BCF] = [CFA] = [FAB]$. Let $ED/AC = r = 15/7$. Since $\overline{AC} \parallel \overline{DE}$, conclude $EF/FC = FD/AF = r$. Hence

$$\begin{aligned}\frac{[EBD]}{[ABC]} &= \frac{[BFE]}{[ABC]} + \frac{[EFD]}{[ABC]} + \frac{[DFB]}{[ABC]} = \frac{[BFE]}{[BCF]} + \frac{[EFD]}{[CFA]} + \frac{[DFB]}{[FAB]} \\ &= r + r^2 + r = r^2 + 2r = 435/49,\end{aligned}$$

implying that $m/n = 49/435$ and $m + n = 484$.

OR

Apply the Law of Cosines to triangle ABC to obtain $AC = 7$. Let $\overline{AD}$ and $\overline{CE}$ intersect at F. Then $ABCF$ is a parallelogram, which implies that $\triangle ABC \cong \triangle CFA$. Note that triangles AFC and DFE are similar. Let the altitudes from B and F to $\overline{AC}$ each have length h. Then the length of the altitude from F to $\overline{ED}$ is $15h/7$. Thus

$$\frac{[ABC]}{[EBD]} = \frac{\frac{1}{2} \cdot 7h}{\frac{1}{2} \cdot 15(h + h + \frac{15}{7}h)} = \frac{7 \cdot 7}{15 \cdot 29} = \frac{49}{435},$$

so $m + n = 484$.

14. (Answer: 108)

To simplify, replace all the 7's with 1's, that is, divide all the numbers in the sum by 7. The desired values of n are the same as the values of n for which $+$ signs can be inserted in a string of n 1's to obtain a sum of 1000. The result will be a sum of x 1's, y 11's, and z 111's, where x, y, and z are nonnegative integers, $x + 11y + 111z = 1000$, and $x + 2y + 3z = n$. Subtract to find that $9y + 108z = 1000 - n$, so $n = 1000 - 9(y + 12z)$. There cannot be more than 1000 1's in a string whose sum is 1000, and the least number of 1's occurs when the string consists of nine 111's and one 1. Therefore $28 \le n \le 1000$, and so $0 \le y + 12z \le 108$. Thus the number of values of n is the number of possible integer values of $y + 12z$ between 0 and 108, inclusive, subject to the condition that

$$11y + 111z \le 1000. \qquad (1)$$

Note that (1) is equivalent to $11y \le 990 - 110z + 10 - z$, and therefore to $y \le 90 - 10z + (10 - z)/11$. Then use the fact that $0 \le z \le 9$ to conclude that

(1) is equivalent to $y \le 90 - 10z$, and therefore to $y + 12z \le 90 + 2z$. The fact that y is nonnegative means that $y + 12z \ge 12z$. Thus an ordered pair (y, z) of integers satisfies

$$12z \le y + 12z \le 90 + 2z \tag{2}$$

if and only if it satisfies (1) and $y \ge 0$. Hence when $z = 0$, $y + 12z$ can have any integer value between 0 and 90, inclusive; when $z = 7$, $y + 12z$ can have any integer value between 84 and 104, inclusive; and when $z = 8$, $y + 12z$ can have any integer value between 96 and 106, inclusive. But $y + 12z > 106$ only if $90 + 2z > 106$, that is, when $z = 9$; and when $z = 9$, (2) implies that $y + 12z = 108$. Thus for nonnegative integers y and z, $y + 12z$ can have any integer value between 0 and 108 except for 107, so there are 108 possible values for n.

OR

To simplify, replace all the 7's with 1's, that is, divide all the numbers in the sum by 7. The desired values of n are the same as the values of n for which $+$ signs can be inserted in a string of n 1's to obtain a sum of 1000. Because the sum is congruent to n modulo 9 and $1000 \equiv 1 \pmod 9$, it follows that $n \equiv 1 \pmod 9$. Also, $n \le 1000$. There are $\lfloor 1000/9 \rfloor + 1 = 112$ positive integers that satisfy both conditions, namely, 1, 10, 19, 28, 37, 46, ..., 1000. For $n = 1$, 10, or 19, the greatest sum that is less than or equal to 1000 is $6 \cdot 111 + 1 = 677$. Thus $n \ge 28$, so there are at most $112 - 3 = 109$ possible values of n, and these values are contained in $\mathcal{S} = \{28, 37, 46, \ldots, 1000\}$. It will be shown that all elements of $\mathcal{S}$ except 37 are possible.

First note that 28 is possible because $9 \cdot 111 + 1 \cdot 1 = 1000$, while 37 is not possible because when $n = 37$, the greatest sum that is at most 1000 is $8 \cdot 111 + 6 \cdot 11 + 1 \cdot 1 = 955$. All other elements of $\mathcal{S}$ are possible because if any element n of $\mathcal{S}$ between 46 and 991 is possible, then $(n + 9)$ must be too. To see this, consider two cases, the case where the sum has no 11's and the case where the sum has at least one 11.

If the sum has no 11's, it must have at least one 1. If it has exactly one 1, there must be nine 111's and $n = 28$. Thus, for $n \ge 46$, the sum has more than one 1, so it must have at least $1000 - 8 \cdot 111 = 112$ 1's, and, for $n < 1000$, at least one 111. To show that if n is possible, then $(n + 9)$ is possible, replace a 111 with $1 + 1 + 1$, replace eleven $(1 + 1)$'s with eleven 11's, and include nine new 1's as $+1$'s. The sum remains 1000.

If the sum has at least one 11, replace an 11 with $1 + 1$, and include nine new 1's as $+1$'s.

Now note that 46 is possible because $8 \cdot 111 + 10 \cdot 11 + 2 \cdot 1 = 1000$, and so all elements of $\mathcal{S}$ except 37 are possible. Thus there are 108 possible values for n.

15. (Answer: 593)

Number the squares from left to right, starting with 0 for the leftmost square, and ending with 1023 for the rightmost square. The 942nd square is thus initially numbered 941. Represent the position of a square after f folds as an ordered triple (p, h, f), where p is the position of the square starting from the left, starting with 0 as the leftmost position, h is the number of paper levels below the square, and f is the number of folds that have been made. For example, the ordered triple that initially describes square number 941 is $(941, 0, 0)$. The first 0 indicates that at the start there are no squares under this one, and the second 0 indicates that no folds have been made.

Note that the function F, defined below, describes the position of a square after $(f + 1)$ folds:

$$F(p,h,f) = \begin{cases} (p, h, f+1) & \text{for } 0 \le p \le 2^{10-f-1} - 1 \\ (2^{10-f} - 1 - p, 2^f + (2^f - 1 - h), f+1) & \\ & \text{for } 2^{10-f-1} \le p \le 2^{10-f} - 1 \end{cases}$$

The top line in the definition indicates that squares on the left half of the strip do not change their position or height as a result of a fold. The second line indicates that, as a result of a fold, the position of a square on the right half of the strip is reflected about the center line of the strip, and that the stack of squares in that position is inverted and placed on the top of the stack that was already in that position's reflection.

Because of the powers of 2 in the definition of F, evaluating F can be made easier if the position and height are expressed in base two. In particular, after f folds, the strip has length 2^{10-f}, so the positions 0 through $2^{10-f} - 1$ are represented by all possible binary strings of $10 - f$ digits. In this representation, $0 \le p \le 2^{10-f-1} - 1$ if and only if the leading digit is 0, and $2^{10-f-1} \le p \le 2^{10-f} - 1$ if and only if the leading digit is 1. In the former case, the new position, now represented by the string of length $10-f-1$, is obtained by deleting the leading 0. In the latter case, the new position $2^{10-f} - 1 - p$ is obtained by truncating the leading 1 and for the remaining digits, changing each 0 to a 1 and each 1 to a 0. Likewise, in this latter case, the new height is $2^f + (2^f - 1 - h)$. When $f \ge 1$, the new height is obtained in the case $2^{10-f-1} \le p \le 2^{10-f} - 1$ by taking the f-digit binary string representing the height, changing each 1 to a 0 and each 0 to a 1, and then appending a 1 on the left. In the case $0 \le p \le 2^{10-f-1} - 1$, the new $(f + 1)$-digit string representing the new height is obtained by appending a 0 to the left of the string.

With these conditions, square number 941 is initially described by $(1110101101, 0, 0)$. In the display below, an arrow is used to denote an application of F. For the first fold

$$(1110101101, 0, 0) \to (001010010, 1, 1),$$

indicating that after the first fold, square 941 is in position $001010010_2 = 82$, and there is one layer under this square. Continue to obtain $(001010010, 1, 1) \rightarrow (01010010, 01, 2) \rightarrow (1010010, 001, 3) \rightarrow (101101, 1110, 4) \rightarrow (10010, 10001, 5) \rightarrow (1101, 101110, 6) \rightarrow (010, 1010001, 7) \rightarrow (10, 01010001, 8) \rightarrow (1, 110101110, 9) \rightarrow (0, 1001010001, 10)$. After 10 folds, the number of layers under square 941 is $1001010001_2 = 593$.

OR

If a square is to the left of the center after n folds, its positions counting from the left and the bottom do not change after $(n+1)$ folds. Otherwise, its positions counting from the right and bottom after n folds become its positions counting from the left and top after $(n+1)$ folds. Also, after n folds the sum of the positions of each square counting from the left and right is $2^{10-n}+1$, and the sum of the positions counting from the bottom and top is 2^n+1. The position of the 942nd square can be described in the table below.

Folds	Position Counting From Left	Position Counting From Right	Position Counting From Bottom	Position Counting From Top
0	942	83	1	1
1	83	430	2	1
2	83	174	2	3
3	83	46	2	7
4	46	19	15	2
5	19	14	18	15
6	14	3	47	18
7	3	6	82	47
8	3	2	82	175
9	2	1	431	82
10	1	1	594	431

Thus there are 593 squares below the final position of the 942nd square.

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The Mathematical Association of America
American Mathematics Competitions

23rd Annual

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Tuesday, March 8, 2005

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:

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The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

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1. (Answer: 942)

   Let r be the radius of each of the six congruent circles, and let A and B be the centers of two adjacent circles. Join the centers of adjacent circles to form a regular hexagon with side $2r$. Let O be the center of $\mathcal{C}$. Draw the radii of $\mathcal{C}$ that contain A and B. Triangle ABO is equilateral, so $OA = OB = 2r$. Because each of the two radii contains the point where the smaller circle is tangent to $\mathcal{C}$, the radius of $\mathcal{C}$ is $3r$, and $K = \pi\left((3r)^2 - 6r^2\right) = 3\pi r^2$. The radius of $\mathcal{C}$ is 30, so $r = 10$, $K = 300\pi$, and $\lfloor K \rfloor = 942$.

2. (Answer: 012)

   Suppose that the nth term of the sequence S_k is 2005. Then $1+(n-1)k = 2005$ so $k(n-1) = 2004 = 2^2 \cdot 3 \cdot 167$. The ordered pairs $(k, n-1)$ of positive integers that satisfy the last equation are $(1, 2004)$, $(2, 1002)$, $(3, 668)$, $(4, 501)$, $(6, 334)$, $(12, 167)$, $(167, 12)$, $(334, 6)$, $(501, 4)$, $(668, 3)$, $(1002, 2)$, and $(2004, 1)$. Thus the requested number of values is 12. Note that the number of divisors of $2^2 \cdot 3 \cdot 167$ can also be found to be $(2+1)(1+1)(1+1) = 12$ by using the formula for the number of divisors.

3. (Answer: 109)

   There are two types of integers n that have three proper divisors. If $n = pq$, where p and q are distinct primes, then the three proper divisors of n are 1, p, and q; and if $n = p^3$, where p is a prime, then the three proper divisors of n are 1, p, and p^2. Because there are 15 prime numbers less than 50, there are $\binom{15}{2} = 105$ integers of the first type. There are 4 integers of the second type because 2, 3, 5, and 7 are the only primes with squares less than 50. Thus there are $105 + 4 = 109$ integers that meet the given conditions.

4. (Answer: 294)

   Let the square formation have s rows and s columns, and let the rectangular formation have x columns and $(x+7)$ rows. Then $x(x+7) = s^2 + 5$, so $x^2 + 7x - (s^2 + 5) = 0$. Because x is a positive integer, $x = \left(-7 + \sqrt{4s^2 + 69}\right)/2$, and there must be a positive integer k for which $k^2 = 4s^2 + 69$. Then $69 = k^2 - 4s^2 = (k+2s)(k-2s)$. Therefore $(k+2s, k-2s) = (69, 1)$ or $(23, 3)$. Thus $(k, s) = (35, 17)$ or $(13, 5)$, and the maximum number of members this band can have is $17^2 + 5 = 294$.

5. (Answer: 630)

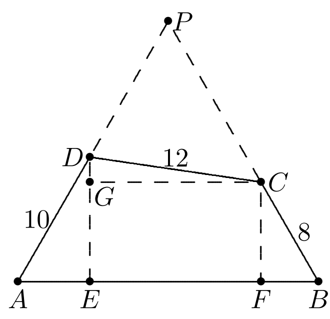
   First consider the orientation of the coins. Label each coin U or D depending upon whether it is face up or face down, respectively. Then for each arrangement of the coins, there is a corresponding string consisting of a total of eight U's and D's that is formed by listing each coin's label starting from the bottom of the stack. An arrangement in which no two adjacent coins are face to face corresponds to such a string that does not contain UD. Thus the first U in the string must have no D's after it. The first U may appear in any of eight positions or not at all, for a total of nine allowable strings. For each of these nine strings, there are $\binom{8}{4}$ ways to pick the positions for the four gold coins, and the positions of the silver coins are then determined. Thus there are $9 \cdot \binom{8}{4} = 630$ arrangements that satisfy Robert's rules of order.

6. (Answer: 045)

The given equation is equivalent to $x^4 - 4x^3 + 6x^2 - 4x + 1 = 2006$, that is, $(x-1)^4 = 2006$. Thus $(x-1)^2 = \pm\sqrt{2006}$, and $x - 1 = \pm\sqrt[4]{2006}$ or $\pm i\sqrt[4]{2006}$. Therefore the four solutions to the given equation are $1 \pm \sqrt[4]{2006}$ and $1 \pm i\sqrt[4]{2006}$. Then $P = (1 + i\sqrt[4]{2006})(1 - i\sqrt[4]{2006}) = 1 + \sqrt{2006}$, so $\lfloor P \rfloor = 45$.

7. (Answer: 150)

Draw lines containing D and C that are perpendicular to $\overline{AB}$ at E and F, respectively. Then $AE = 5$, $DE = 5\sqrt{3}$, $BF = 4$, and $CF = 4\sqrt{3}$. Now draw a line containing C that is perpendicular to $\overline{DE}$ at G. Because $EFCG$ is a rectangle, $GE = CF = 4\sqrt{3}$, so $DG = DE - GE = \sqrt{3}$. Apply the Pythagorean Theorem to $\triangle DGC$ to find that $\sqrt{141} = GC = EF$. Then $AB = AE + EF + FB = 9 + \sqrt{141}$, and $p + q = 150$.



OR

Let P be the intersection of $\overrightarrow{AD}$ and $\overrightarrow{BC}$, and let $AD = a$, $AB = b$, $BC = c$, $CD = d$, $DP = x$, and $PC = y$. Then $\triangle ABP$ is equilateral, and $x + a = y + c = b$. Apply the Law of Cosines to $\triangle DCP$ to obtain $x^2 + y^2 - xy = d^2$, and then substitute to get $(b-a)^2 + (b-c)^2 - (b-a)(b-c) = d^2$. Expand and simplify to get

$$a^2 + b^2 + c^2 = d^2 + ab + bc + ac.$$

For the given quadrilateral, this yields $10^2 + b^2 + 8^2 = 12^2 + 10b + 8b + 80$, and then $b^2 - 18b - 60 = 0$, whose positive solution is $9 + \sqrt{141}$. Thus $p + q = 150$.

8. (Answer: 113)

Let $y = 2^{111x}$. The given equation is equivalent to $(1/4)y^3 + 4y = 2y^2 + 1$, which can be simplified to $y^3 - 8y^2 + 16y - 4 = 0$. Since the roots of the given equation are real, the roots of the last equation must be positive. Let the roots of the given equation be x_1, x_2, and x_3, and let the roots of the equation in y be y_1, y_2, and y_3. Then $x_1 + x_2 + x_3 = (1/111)(\log_2 y_1 + \log_2 y_2 + \log_2 y_3) = (1/111)\log_2(y_1y_2y_3) = (1/111)\log_2 4 = 2/111$, and $m + n = 113$.

Note: It can be verified that $y^3 - 8y^2 + 16y - 4 = 0$ has three positive roots by sketching a graph.

9. (Answer: 074)

A cube can be oriented in 24 ways because each of the six faces can be on top and each of the top face's four edges can be at the front. There are eight corner cubes in the large cube. For the corner cubes, six orientations will expose three orange faces. This is because there are two sets of three orange faces that can be exposed. For each such set, each of the three orange faces can appear in a given position, and the positions of the other two are then determined. Thus the probability that all corner cubes expose three orange faces is $(6/24)^8 = (1/4)^8$. For cubes at the center of an edge, there are 10 orientations that expose two orange faces. This is because there are five sets of two orange faces that share an edge, and each such set can appear in two orientations. The probability that all 12 of these edge cubes expose two orange faces is $(10/24)^{12} = (5/12)^{12}$. A cube that is in the center of a face can have any of the four orange faces outward in four orientations, and thus there is a probability of $(16/24)^6 = (2/3)^6$ that each center cube exposes an orange face. Thus the probability that the entire surface of the larger cube is orange is

$$\left(\frac{1}{4}\right)^8 \cdot \left(\frac{5}{12}\right)^{12} \cdot \left(\frac{2}{3}\right)^6 = \frac{5^{12}}{2^{34} \cdot 3^{18}},$$

and $a + b + c + p + q + r = 12 + 34 + 18 + 5 + 2 + 3 = 74$.

OR

The large cube contains eight corner unit cubes, twelve unit cubes at the center of an edge, and six unit cubes at the center of a face. All visible faces of a unit cube are orange if and only if the shared edge of its two unpainted faces, except perhaps for an endpoint, is in the interior of the large cube. The number of edges interior to the large cube is three for a corner cube, five for a cube at the center of an edge, and eight for a cube at the center of a face. Thus the probability that the entire surface of the large cube is orange is

$$\left(\frac{3}{12}\right)^8 \cdot \left(\frac{5}{12}\right)^{12} \cdot \left(\frac{8}{12}\right)^6 = \frac{5^{12}}{2^{34} \cdot 3^{18}},$$

and, as above, $a + b + c + p + q + r = 74$.

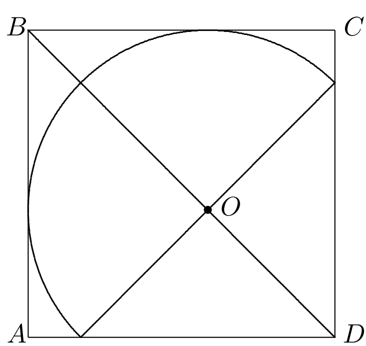
10. (Answer: 047)

Let l be the line containing the median to side $\overline{BC}$. Then l must contain the midpoint of $\overline{BC}$, which is $((12+23)/2,(19+20)/2)=(35/2,39/2)$. Since l has the form $y=-5x+b$, substitute to find that $b=107$. Thus the coordinates of A are $(p,-5p+107)$. Now compute p using the fact that the area of the triangle with coordinates $(0,0)$, (x_1,y_1), and (x_2,y_2) is the absolute value of $(1/2)\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$. To use this formula, translate the point $(12,19)$ to the origin, and, to preserve area, translate points A and C to $A'=(p-12,-5p+107-19)=(p-12,-5p+88)$ and $C'=(23-12,20-19)=(11,1)$, respectively. Apply the above formula to obtain $(1/2)|(p-12)\cdot 1-(-5p+88)\cdot 11|=70$, which yields $|56p-980|=140$. Thus $p=15$ or $p=20$, and the corresponding values of q are 32 and 7, respectively. The largest possible value of $p+q$ is 47.

OR

Let M be the midpoint of $\overline{BC}$. The coordinates of M are $(35/2,39/2)$. An equation of line AM is $y=-5x+107$, so the coordinates of A can be represented as $(p,-5p+107)$. Line BC has equation $x=11y-197$ or, equivalently, $x-11y+197=0$, so the distance from A to line BC is $\dfrac{|p-11(-5p+107)+197|}{\sqrt{1^2+11^2}}=\dfrac{|56p-980|}{\sqrt{122}}$. The length of $\overline{BC}$ is $\sqrt{1^2+11^2}$, so $70=[\triangle ABC]=\dfrac{1}{2}\sqrt{122}\cdot\dfrac{|56p-980|}{\sqrt{122}}$. Solve to obtain $p=20$ or $p=15$, so $p+q=p-5p+107=-4p+107$. Thus $p+q=27$ or 47, and the maximum value of $p+q$ is 47.

11. (Answer: 544)

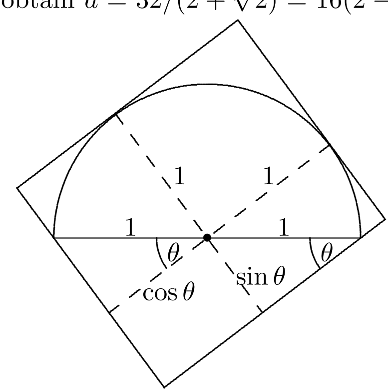


Any semicircle that is contained in a square can be translated to yield a semicircle that is inside the square and tangent to two adjacent sides of the square. Name the square $ABCD$, and, without loss of generality, consider semicircles tangent to $\overline{AB}$ and $\overline{BC}$. The center O of any such semicircle is equidistant from $\overline{AB}$ and $\overline{BC}$ and therefore must lie on diagonal $\overline{BD}$. Let $\mathcal{S}$ be the circle determined by the semicircle. The placement of O that yields the largest semicircle is the point at which the intersection of $\mathcal{S}$ with the square region is a semicircle. This is because if O were placed closer to B, the radius of $\mathcal{S}$ would be smaller; and if O were placed farther from B, the intersection of $\mathcal{S}$ and the square region would be an arc of less than 180°, and so no semicircle centered at O would fit in the square. Because $\overline{BD}$ is a symmetry line for the desired semicircle and the square, $\overline{BD}$ is the perpendicular bisector of the diameter joining the endpoints of the semicircle.

Let the radius of the largest semicircle be r. Then the distance from O to $\overline{AD}$ is $r/\sqrt{2}$. The sum of the distances from O to $\overline{AD}$ and $\overline{BC}$ is 8, so the diameter of the largest semicircle that fits in the square is $2r = \dfrac{2 \cdot 8}{1 + \frac{1}{\sqrt{2}}} = 16(2 - \sqrt{2}) = 32 - \sqrt{512}$. Thus $m + n = 32 + 512 = 544$.

OR

Consider a related problem: find the least possible side-length of a square that contains a semicircle for which the diameter is fixed. Let the orientation of the square with respect to the semicircle be such that the sides of the square and the diameter of the semicircle determine angles of θ and $(\pi/2 - \theta)$, with $0 \le \theta \le \pi/2$. Without loss of generality, assume the radius of the semicircle is 1. The opposite sides of the square are parallel, and, in general, if a pair of parallel lines touch a semicircle, then one will be tangent to its arc and one will contain an endpoint of the diameter. The diagram shows two perpendicular pairs of parallel lines with all four lines touching the semicircle. The distance between the lines in each pair is as small as possible because each line of the pair touches the semicircle. Thus the greater of these two distances is the minimal side-length of a square in this orientation that contains the semicircle. Because the distances between the pairs of parallel lines are $1 + \cos\theta$ and $1 + \sin\theta$, the minimal side-length of a square in this orientation that contains the semicircle is $\max\{1 + \cos\theta, 1 + \sin\theta\}$. For θ between 0 and $\pi/2$, this length is minimum when $\theta = \pi/4$, so the minimum length for any orientation of the square is $1 + \sqrt{2}/2 = (2 + \sqrt{2})/2$. To find the maximum value of d, solve the proportion $\dfrac{d}{2} = \dfrac{8}{(2 + \sqrt{2})/2}$ to obtain $d = 32/(2 + \sqrt{2}) = 16(2 - \sqrt{2}) = 32 - \sqrt{512}$. Thus $m + m = 544$.



12. (Answer: 025)

If d is a divisor of n, then so is $\frac{n}{d}$. Thus the number of divisors of n must be even unless, for some d, $d = \frac{n}{d}$, that is, $n = d^2$. Hence $\tau(n)$ is odd if and only if n is a square. Therefore, as n increases, $S(n)$ changes parity only when n is a square. Thus $S(n)$ is odd for $1^2 \le n \le 2^2 - 1$, even for $2^2 \le n \le 3^2 - 1$, odd for $3^2 \le n \le 4^2 - 1$, and so on. Consequently

$$\begin{aligned} a &= (2^2 - 1 - 1^2 + 1) + (4^2 - 1 - 3^2 + 1) + (6^2 - 1 - 5^2 + 1) + \cdots + (44^2 - 1 - 43^2 + 1) \\ &= (2^2 - 1^2) + (4^2 - 3^2) + (6^2 - 5^2) + \cdots + (44^2 - 43^2) \\ &= (2+1)(2-1) + (4+3)(4-3) + (6+5)(6-5) + \cdots + (44+43)(44-43) \\ &= 1 + 2 + 3 + \cdots + 44 = 44 \cdot 45/2 = 990. \end{aligned}$$

Then $b = 2005 - 990 = 1015$, so $|a - b| = 25$.

13. (Answer: 083)

Let $P(a,\, b)$ be the number of permissible paths from $(0,\, 0)$ to $(a,\, b)$, and define $P(0,\, 0) = 1$. If $a = 0$ or $b = 0$, then $P(a,\, b) = 1$. If $a > 0$ and $b > 0$, then the particle can reach $(a,\, b)$ from any of the points $(a-1,\, b)$, $(a-1,\, b-1)$, $(a, b-1)$. Also, if the particle is at $(a-1,\, b)$ and next moves to $(a,\, b)$, then the particle must have entered the row containing these points on a diagonal path, not a vertical one, because otherwise one of the moves to the right towards $(a,\, b)$ would make a right angle. Thus the number of ways to travel to $(a-1,\, b)$ in such a way that the path continuing to $(a,\, b)$ is permissible is

$$P(0, b-1) + P(1, b-1) + \cdots + P(a-2,\, b-1).$$

A similar statement holds for paths the particle can take to $(a,\, b-1)$ that result in a permissible path to $(a,\, b)$. Thus,

$$P(a,\, b) = \left(\sum_{i=0}^{a-2} P(i,\, b-1)\right) + P(a-1,\, b-1) + \left(\sum_{j=0}^{b-2} P(a-1,\, j)\right).$$

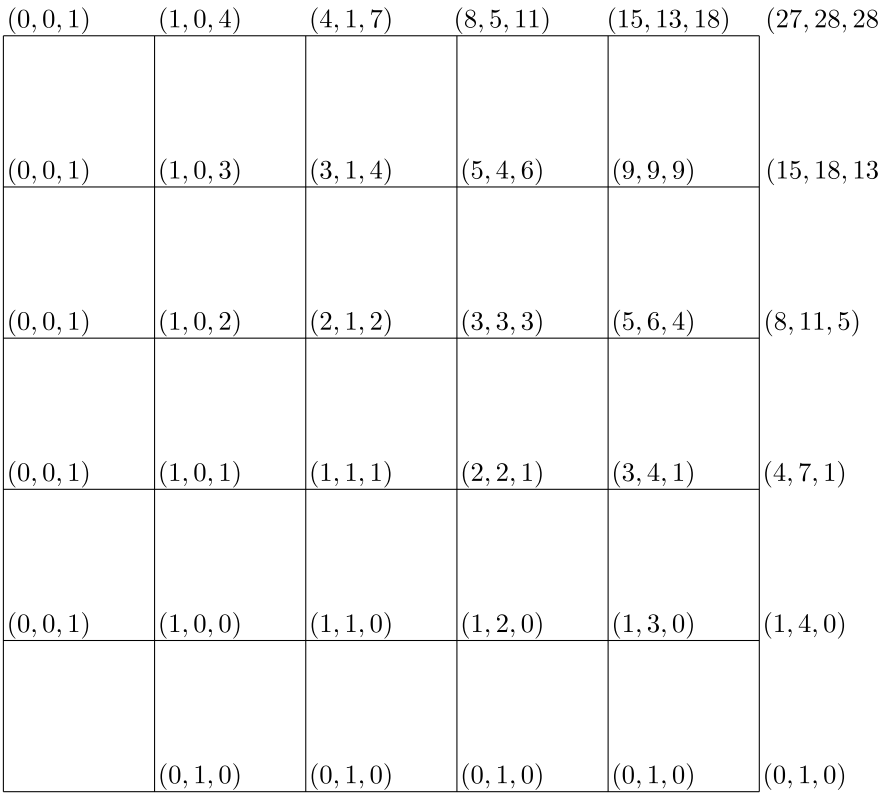
This is simply the sum of the number of permissible paths from the origin to points on the top or right side of the rectangle with vertices $(0,\, 0)$, $(a-1,\, 0)$, $(a-1,\, b-1)$, $(0,\, b-1)$. With this realization, calculate the number of permissible paths to each lattice point as shown in the grid below, to find that there are 83 permissible paths.

1	5	12	24	46	83
1	4	8	15	27	46
1	3	5	9	15	24
1	2	3	5	8	12
1	1	2	3	4	5
1	1	1	1	1	1

OR

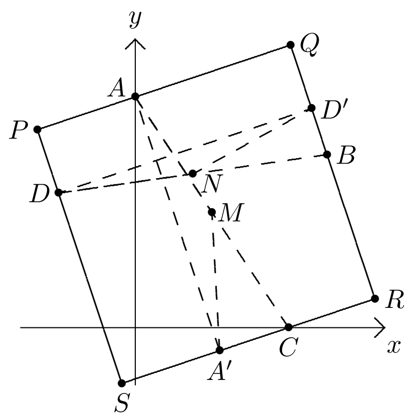
Label each vertex with an ordered triple whose first component represents the number of paths that end at that vertex with a diagonal step, whose second component represents the number of paths that end at that vertex with a step to the right, and whose third component represents the number of paths that end at that vertex with a step up. Begin by labeling the vertices with the triples

$(0,0,1)$, $(0,1,0)$, and $(1,0,0)$ as shown. For the other vertices, the first component at a vertex is the sum of the three components at the vertex diagonally below it and to the left, the second component at a vertex is the sum of the first two components at the vertex directly to its left, and the third component at a vertex is the sum of the first and third components at the vertex directly below it. Use these relationships to complete the labeling of the grid. The requested number of paths is the sum of the components in the upper right vertex, that is, $27+28+28=83$.



14. (Answer: 936)

Because $\overline{AC}$ and $\overline{BD}$ intersect, A and C must be on opposite sides of the square, as must B and D. Name the vertices of the square P, Q, R, and S so that A is on $\overline{PQ}$, B is on $\overline{QR}$, C is on $\overline{RS}$, and D is on $\overline{SP}$. Let h and v represent the horizontal and vertical change, respectively, from P to Q. Then A', the projection of A onto $\overline{RS}$, has coordinates $(0+v, 12-h)$. Let M be the midpoint of $\overline{AC}$. Then M has coordinates $(4,6)$ and $MA = MA'$, so $4^2+6^2 = (v-4)^2+(h-6)^2$. Similarly, D', the projection of D onto $\overline{QR}$, has coordinates $(-4+h, 7+v)$, N, the midpoint of $\overline{BD}$, has coordinates $(3,8)$, and $ND = ND'$, so $7^2+1^2 = (h-7)^2+(v-1)^2$. The two equations imply that $12h+8v = h^2+v^2 = 14h+2v$, and so $h = 3v$. Then $12h+8v = h^2+v^2$ yields $36v+8v = (3v)^2+v^2$, so $v = 44/10$. Thus $K = h^2+v^2 = 10v^2 = 10(44^2/10^2) = 1936/10$, so $10K = 1936$, and the requested remainder is 936.



OR

Let m be the slope of the side of the square containing B. The line containing this side has equation $y-9 = m(x-10)$ or $mx-y+(9-10m) = 0$. Similarly, the line containing the side containing C has equation $y = (-1/m)(x-8)$ or $x+my-8 = 0$. Because the distance from D to the first line is equal to the distance from A to the second line,

$$\frac{|-4m-7+9-10m|}{\sqrt{m^2+1}} = \frac{|12m-8|}{\sqrt{m^2+1}}.$$

Solve to obtain $m = 5/13$ or $m = -3$. For the square obtained with the first slope, some of the points are on extended sides of the square. This is because A and C are on opposite sides of the line with slope $5/13$ that contains B. Thus $m = -3$. Then $K = \dfrac{(12m-8)^2}{m^2+1} = 44^2/10$, so $10K = 1936$, and the requested remainder is 936.

Query: For four arbitrary points A, B, C, D in the plane, what are the necessary and sufficient conditions that a unique square $\mathcal{S}$ exists?

15. (Answer: 038)

Let M be the midpoint of $\overline{AB}$, and let S and N be the points where median $\overline{CM}$ meets the incircle, with S between C and N. Let $\overline{AC}$ and $\overline{AB}$ touch the incircle at R and T, respectively. Assume, without loss of generality, that T is between A and M. Then $AR = AT$. Use the Power-of-a-Point Theorem to conclude that

$$MT^2 = MN \cdot MS \quad \text{and} \quad CR^2 = CS \cdot CN.$$

Because $CS = SN = MN$, conclude that $CR = MT$, and

$$AC = AR + CR = AT + MT = AM = \frac{1}{2}AB = 10.$$

Let $s = (1/2)(AB + BC + CA)$. Then $AT = s - BC$, and

$$MT = MA - AT = \frac{1}{2}AB - s + BC = \frac{BC - AC}{2} = \frac{BC - 10}{2}.$$

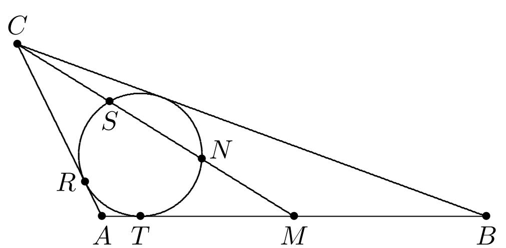
But $MT^2 = MN \cdot MS = (2/9)CM^2$, so $\dfrac{BC - 10}{2} = CM \cdot \dfrac{\sqrt{2}}{3}$. Hence

$$CM = \frac{3}{2\sqrt{2}} \cdot (BC - 10).$$

Apply the Law of Cosines to triangles AMC and ABC to obtain

$$\frac{10^2 + 10^2 - CM^2}{2 \cdot 10 \cdot 10} = \cos A = \frac{10^2 + 20^2 - BC^2}{2 \cdot 10 \cdot 20}.$$

Then $BC^2 = 100 + 2 \cdot CM^2$, so $BC^2 = 100 + (9/4)(BC - 10)^2$. The solutions of this equation are 26 and 10, but $BC > AB - AC = 10$. It follows that $BC = 26$, and then that $CM = 12\sqrt{2}$. The length of the altitude from A in isosceles $\triangle AMC$ is therefore $2\sqrt{7}$. Thus $[ABC] = 2[AMC] = 24\sqrt{14}$, and $m + n = 38$.



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American Mathematics Competitions

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SOLUTIONS PAMPHLET

Tuesday, March 22, 2005

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The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

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207 Corbin Place, Brooklyn, NY 11235 USA

1. (Answer: 013)

   The conditions of the problem imply that $\binom{n}{6} = 6\binom{n}{3}$, so $\dfrac{n!}{6!(n-6)!} = 6 \cdot \dfrac{n!}{3!(n-3)!}$. Then $\dfrac{(n-3)!}{(n-6)!} = 6!$, so $(n-3)(n-4)(n-5) = 720 = 10 \cdot 9 \cdot 8$. Thus $n = 13$ is a solution, and because $(n-3)(n-4)(n-5)$ is increasing for $n \geq 5$, conclude that 13 is the only solution for $n \geq 5$.

2. (Answer: 079)

   The probability that the first bag contains one of each of the three types of rolls is
   $(9/9)(6/8)(3/7) = 9/28$. The probability that the second bag will then contain one of each is $(6/6)(4/5)(2/4) = 2/5$. If the first two bags have a complete selection, then the last bag must too. Thus the probability that all three breakfasts have a complete selection is $(9/28)(2/5) = 9/70$, and $m + n = 9 + 70 = 79$.

3. (Answer: 802)

   Let a be the first term and r the ratio of the original series, and let $S = 2005$. Then $\dfrac{a}{1-r} = S$ and $\dfrac{a^2}{1-r^2} = 10S$. Factor to obtain $10S = \left(\dfrac{a}{1-r}\right)\left(\dfrac{a}{1+r}\right) = S \cdot \dfrac{a}{1+r}$. Then $10 = \dfrac{a}{1+r}$ and $S = \dfrac{a}{1-r}$ imply that $S(1-r) = 10(1+r)$, so $r = \dfrac{S-10}{S+10} = 1995/2015 = 399/403$, and $m + n = 802$.

4. (Answer: 435)

   Note that $10^{10} = 2^{10}5^{10}$, so it has $11 \cdot 11 = 121$ divisors. Similarly, $15^7 = 3^7 \cdot 5^7$, so it has $8 \cdot 8 = 64$ divisors, and $18^{11} = 2^{11}3^{22}$, so it has $12 \cdot 23 = 276$ divisors. There are 8 divisors of both 10^{10} and 15^7, namely those numbers that are divisors of 5^7; there are 11 divisors of both 10^{10} and 18^{11}, namely those numbers that are divisors of 2^{10}; and there are 8 divisors of both 15^7 and 18^{11}, namely those numbers that are divisors of 3^7. There is only one divisor of all three. Therefore, the Inclusion-Exclusion Principle implies that the number of divisors of at least one of the numbers is $(121 + 64 + 276) - (8 + 11 + 8) + 1 = 435$.

5. (Answer: 054)

   Let $x = \log_a b$. Because $\log_b a = 1/\log_a b$, the given equation can be written as $x + (6/x) = 5$, and because $x \neq 0$, this is equivalent to $x^2 - 5x + 6 = 0$, whose solutions are 2 and 3. If $2 = x = \log_a b$, then $a^2 = b$. Now $44^2 = 1936$ and $45^2 = 2025$, so there are $44 - 1 = 43$ ordered pairs (a, b) such that $a^2 = b$ and

a and b satisfy the given conditions. If $3 = x = \log_a b$, then $a^3 = b$. Because $12^3 = 1728$ and $13^3 = 2197$, there are $12 - 1 = 11$ ordered pairs (a, b) such that $a^3 = b$ and a and b satisfy the given conditions. Thus there are $43 + 11 = 54$ of the requested ordered pairs.

6. (Answer: 392)

Note that, after the restacking, all the cards from pile B occupy even-numbered positions and their order is reversed. Similarly, all the cards from pile A will be placed in odd-numbered positions, and their order is also reversed. A card in position i for $1 \le i \le n$ will be moved to position $2(n-i)+1$ in the restacking, and, for $n < i \le 2n$, the card will be moved to position $2(2n-i)+2$. For a card to remain in the 131st position, it must be in pile A. Then $131 = 2(n-131)+1$, and $2n = 392$. Note that the stack is magical because cards number 131 and 262 retain their original positions.

7. (Answer: 125)

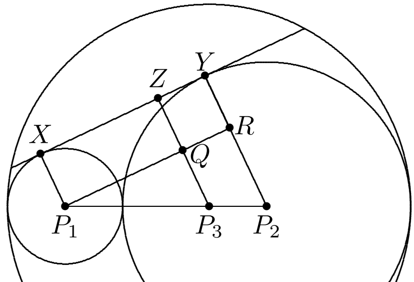
Let $y = \sqrt[16]{5}$. Then

$$\begin{aligned} x = \frac{4}{(y^8+1)(y^4+1)(y^2+1)(y+1)} &= \frac{4(y-1)}{(y^8+1)(y^4+1)(y^2+1)(y+1)(y-1)} \\ &= \frac{4(y-1)}{y^{16}-1} = \frac{4(y-1)}{5-1} = y-1. \end{aligned}$$

Thus $(x+1)^{48} = y^{48} = 5^3 = 125$.

8. (Answer: 405)

The radius of $\mathcal{C}_3$ is 14. Let P_1, P_2, and P_3 be the centers of $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, respectively. Draw perpendiculars from P_1, P_2, and P_3 to the external tangent of $\mathcal{C}_1$ and $\mathcal{C}_2$ intersecting it at X, Y, and Z, respectively, so that $\overline{P_1X}$, $\overline{P_2Y}$, $\overline{P_3Z}$ are parallel, with $P_1X = 4$ and $P_2Y = 10$. From P_1, draw a line parallel to $\overline{XY}$ intersecting $\overline{P_3Z}$ and $\overline{P_2Y}$ at Q and R, respectively. Note that P_1XYR is a rectangle and that right triangles P_1P_2R and P_1P_3Q are similar. Then $P_3Z = P_3Q + QZ = (10/14)\cdot 6 + 4 = 58/7$. Because Z is the midpoint of the chord, the chord's length is $2\sqrt{14^2 - (58/7)^2} = 8\sqrt{390}/7$, and $m + n + p = 8 + 390 + 7 = 405$.



9. (Answer: 250)

Note that

$$\begin{aligned}(\sin t + i\cos t)^n &= \left[\cos\left(\frac{\pi}{2} - t\right) + i\sin\left(\frac{\pi}{2} - t\right)\right]^n \\ &= \cos n\left(\frac{\pi}{2} - t\right) + i\sin n\left(\frac{\pi}{2} - t\right) \\ &= \cos\left(\frac{n\pi}{2} - nt\right) + i\sin\left(\frac{n\pi}{2} - nt\right),\end{aligned}$$

and that $\sin nt + i\cos nt = \cos\left(\frac{\pi}{2} - nt\right) + i\sin\left(\frac{\pi}{2} - nt\right)$. Thus the given condition is equivalent to

$$\cos\left(\frac{n\pi}{2} - nt\right) = \cos\left(\frac{\pi}{2} - nt\right) \quad \text{and} \quad \sin\left(\frac{n\pi}{2} - nt\right) = \sin\left(\frac{\pi}{2} - nt\right).$$

In general, $\cos\alpha = \cos\beta$ and $\sin\alpha = \sin\beta$ if and only if $\alpha - \beta = 2\pi k$. Thus

$$\frac{n\pi}{2} - nt - \frac{\pi}{2} + nt = 2\pi k,$$

which yields $n = 4k + 1$. Because $1 \le n \le 1000$, conclude that $0 \le k \le 249$, so there are 250 values of n that satisfy the given conditions.

OR

Observe that

$$\begin{aligned}&(\sin t + i\cos t)^n = [i(\cos t - i\sin t)]^n = i^n(\cos nt - i\sin nt), \quad \text{and that} \\ &\sin nt + i\cos nt = i(\cos nt - i\sin nt).\end{aligned}$$

Thus the given equation is equivalent to $i^n(\cos nt - i\sin nt) = i(\cos nt - i\sin nt)$. This is true for all real t when $i^n = i$. Thus n must be 1 more than a multiple of 4, so there are 250 values of n that satisfy the given conditions.

10. (Answer: 011)

Without loss of generality, let the edges of $\mathcal{O}$ have length 1. Note that $\mathcal{O}$ can be formed by adjoining two square pyramids at their bases. Consider an altitude from a vertex of one of these pyramids to the square base. This altitude is also a leg of a right triangle whose other leg joins the center of the square and a vertex of the square, and whose hypotenuse is an edge of $\mathcal{O}$. The length of the altitude is therefore $\sqrt{1^2 - \left(\sqrt{2}/2\right)^2} = \sqrt{2}/2$, and so the volume of $\mathcal{O}$ is $2 \cdot (1/3)(1^2)(\sqrt{2}/2) = \sqrt{2}/3$. To find the volume of $\mathcal{C}$, consider a triangle, one of whose vertices P is a vertex of $\mathcal{O}$ not on the square and whose other two vertices, Q and R, are midpoints of opposite sides of the square. The line segment that joins the centers of the faces containing $\overline{PQ}$ and $\overline{PR}$, respectively, is a diagonal of a face of $\mathcal{C}$. Because these centers are two-thirds of the way from P to Q and from P to R, respectively, the length of the face diagonal joining them is two-thirds of QR. But $QR = 1$, so the length of each of the edges of $\mathcal{C}$ is $(2/3)/\sqrt{2} = \sqrt{2}/3$. Hence the volume of $\mathcal{C}$ is $(\sqrt{2}/3)^3 = 2\sqrt{2}/27$. The requested ratio is thus $(\sqrt{2}/3)/(2\sqrt{2}/27) = 9/2$, so $m + n = 11$.

OR

The six vertices of $\mathcal{O}$ are equidistant from its center, and the diagonals that join the three pairs of opposite vertices are mutually perpendicular. Without loss of generality, let the length of each of these three diagonals be 2. It is possible to place a coordinate system so that the coordinates of the vertices of $\mathcal{O}$ are $(1, 0, 0)$, $(-1, 0, 0)$, $(0, 1, 0)$, $(0, -1, 0)$, $(0, 0, 1)$, and $(0, 0, -1)$. Because $\mathcal{O}$ is composed of two square pyramids, its volume is $2(1/3)(\sqrt{2})^2 \cdot 1 = 4/3$. The vertices of $\mathcal{C}$ are the centroids of the faces of $\mathcal{O}$, so the coordinates of the vertices of $\mathcal{C}$ are $(\pm 1/3, \pm 1/3, \pm 1/3)$. Thus the length of each of the edges of $\mathcal{C}$ is $2/3$, and the volume of $\mathcal{C}$ is $8/27$. The ratio of the volumes is $(4/3)/(8/27) = 9/2$, so $m + n = 11$.

11. (Answer: 889)

For $1 \le k \le m - 1$, we have $a_{k+1}a_k = a_k a_{k-1} - 3$. Let $b_k = a_k a_{k-1}$ for $1 \le k \le m$. Then $b_1 = a_1 a_0 = 72 \cdot 37 = 3 \cdot 8 \cdot 3 \cdot 37 = 3 \cdot 888$ and $b_{k+1} = b_k - 3$. Hence $b_{889} = 0$ and $b_k > 0$ for $1 \le k \le 888$. Thus $a_{889} = 0$ and $m = 889$.

12. (Answer: 307)

Let G be the midpoint of $\overline{AB}$, let $\alpha = m\angle EOG$, and let $\beta = m\angle FOG$. Then $OG = 450$, $EG = 450 \tan\alpha$, $FG = 450 \tan\beta$, and $\alpha + \beta = 45^\circ$. Therefore $450(\tan\alpha + \tan\beta) = 400$, so $\tan\alpha + \tan\beta = 8/9$. Notice that $\tan\beta = \tan(45^\circ - \alpha) = \dfrac{1 - \tan\alpha}{1 + \tan\alpha}$. Hence $\tan\alpha + \dfrac{1 - \tan\alpha}{1 + \tan\alpha} = \dfrac{8}{9}$. Simplify to obtain $9\tan^2\alpha - 8\tan\alpha + 1 = 0$, and conclude that $\{\tan\alpha, \tan\beta\} = \{(4 \pm \sqrt{7})/9\}$. Because $BF > AE$, conclude that $EG > FG$, and so $\alpha > \beta$. Then $\tan\alpha = (4 + \sqrt{7})/9$ and $\tan\beta = (4 - \sqrt{7})/9$. Thus

$$\begin{aligned} BF = BG - FG = 450 - 450\tan\beta = 450\left(1 - \frac{4 - \sqrt{7}}{9}\right) &= 450\left(\frac{5 + \sqrt{7}}{9}\right) \\ &= 250 + 50\sqrt{7}, \end{aligned}$$

so $p+q+r=307$.

OR

Draw $\overline{AO}$ and $\overline{BO}$. Then $m\angle OAB = 45° = m\angle EOF$, and $m\angle OEF = m\angle OAB + m\angle AOE = 45° + m\angle AOE = m\angle AOF$. Therefore $\triangle AFO \sim \triangle BOE$, so $\dfrac{AO}{AF} = \dfrac{BE}{BO}$. Let $BF = x$. Then $AF = 900 - x$ and $BE = 400 + x$. Thus

$$\begin{aligned}
&\frac{450\sqrt{2}}{900-x} = \frac{400+x}{450\sqrt{2}}, \quad \text{which yields}\\
&2 \cdot 450^2 = 360000 + 500x - x^2, \quad \text{and then}\\
&x^2 - 500x + 45000 = 0.
\end{aligned}$$

Use the Quadratic Formula to obtain $x = 250 \pm 50\sqrt{7}$. Recall that $BF > AE$, and so $x > (900-400)/2 = 250$. Then $BF = x = 250+50\sqrt{7}$, and $p+q+r = 307$.

13. (Answer: 418)

Let $S(x) = P(x) - x - 3$. Because $S(17) = -10$ and $S(24) = -10$,

$$S(x) = -10 + (x-17)(x-24)Q(x)$$

for some polynomial $Q(x)$ with integer coefficients. If n is an integer such that $P(n) = n+3$, then $S(n) = 0$, and $(n-17)(n-24)Q(n) = 10$. Thus the integers $n-17$ and $n-24$ are divisors of 10 that differ by 7. The only such pairs are $(2,-5)$ and $(5,-2)$. This yields $\{n_1, n_2\} = \{19, 22\}$, hence $n_1 \cdot n_2 = 418$. An example of a polynomial that satisfies the conditions of the problem is $P(x) = x - 7 - (x-17)(x-24)$.

14. (Answer: 463)

Let $m\angle BAE = \alpha = m\angle CAD$, and let $\beta = m\angle EAD$. Then

$$\frac{BD}{DC} = \frac{[ABD]}{[ADC]} = \frac{(1/2)AB \cdot AD \sin BAD}{(1/2)AD \cdot AC \sin CAD} = \frac{AB}{AC} \cdot \frac{\sin(\alpha+\beta)}{\sin\alpha}.$$

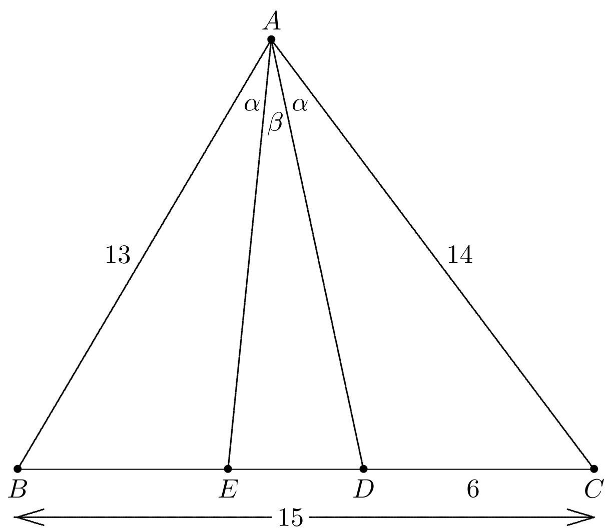
Similarly,

$$\frac{BE}{EC} = \frac{AB}{AC} \cdot \frac{\sin BAE}{\sin CAE} = \frac{AB}{AC} \cdot \frac{\sin\alpha}{\sin(\alpha+\beta)},$$

and so

$$\frac{BE}{EC} = \frac{AB^2 \cdot DC}{AC^2 \cdot BD}.$$

Substituting the given values yields $BE/EC = (13^2 \cdot 6)/(14^2 \cdot 9) = 169/294$. Therefore $BE = (15 \cdot 169)/(169 + 294) = (3 \cdot 5 \cdot 13^2)/463$. Because none of 3, 5, and 13 divides 463, $q = 463$.



15. (Answer: 169)

Complete the square to obtain $(x+5)^2+(y-12)^2 = 256$ and $(x-5)^2+(y-12)^2 = 16$ for ω_1 and ω_2, respectively. Hence ω_1 is centered at $F_1(-5, 12)$ with radius 16, and ω_2 is centered at $F_2(5, 12)$ with radius 4. Let P be the center of the third circle, and let r be its radius. Then $PF_1 = 16 - r$ and $PF_2 = 4 + r$. Thus P is on the ellipse with foci F_1, F_2 and $PF_1 + PF_2 = 20$. Therefore the coordinates of P satisfy

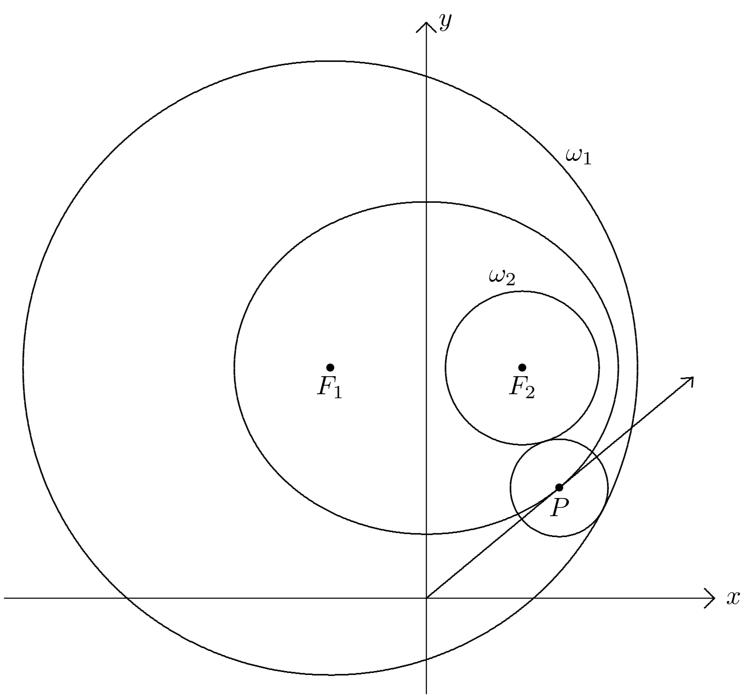
$$\frac{x^2}{100} + \frac{(y-12)^2}{75} = 1,$$

which is equivalent to $3x^2 + 4y^2 - 96y + 576 = 300$. Because P is on the line with equation $y = ax$, conclude that the x-coordinate of P satisifies

$$(3 + 4a^2)x^2 - 96ax + 276 = 0.$$

In order for P to exist, the discriminant of the above quadratic equation must be nonnegative, that is, $(-96a)^2 - 4 \cdot 276 \cdot (4a^2 + 3) \geq 0$. Thus $a^2 \geq 69/100$, so $m^2 = 69/100$, and $p + q = 169$.

Note that a attains its minimum when the line with equation $y = ax$ is tangent to the ellipse.



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AMERICAN MATHEMATICS COMPETITIONS

24th Annual

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Tuesday, March 7, 2006

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:

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The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

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1. (Answer: 084)
Because $AB^2 + BC^2 = AC^2$ and $AC^2 + CD^2 = DA^2$, it follows that $DA^2 = AB^2 + BC^2 + CD^2 = 18^2 + 21^2 + 14^2 = 961$, so $DA = 31$. Then the perimeter of $ABCD$ is $18 + 21 + 14 + 31 = 84$.

2. (Answer: 901)
The least possible value of S is $1 + 2 + 3 + \cdots + 90 = 4095$, and the greatest possible value is $11 + 12 + 13 + \cdots + 100 = 4995$. Furthermore, every integer between 4096 and 4994, inclusive, is a possible value of S. To see this, let $\mathcal{A}$ be a 90-element subset the sum of whose elements is S, and let k be the smallest element of $\mathcal{A}$ such that $k + 1$ is not an element of $\mathcal{A}$. Because $S \leq 4994$, conclude that $k \neq 100$. Therefore, for every 90-element subset with sum S, where $S \leq 4994$, a 90-element subset with sum $S + 1$ can be obtained by replacing k by $k + 1$. Thus there are $4995 - 4095 + 1 = 901$ possible values of S.

3. (Answer: 725)
The desired integer has at least two digits. Let d be its leftmost digit, and let n be the integer that results when d is deleted. Then for some positive integer p, $10^p \cdot d + n = 29n$, and so $10^p \cdot d = 28n$. Therefore 7 is a divisor of d, and because $1 \leq d \leq 9$, it follows that $d = 7$. Hence $10^p = 4n$, so $n = \dfrac{10^p}{4} = \dfrac{100 \cdot 10^{p-2}}{4} = 25 \cdot 10^{p-2}$. Thus every positive integer with the desired property must be of the form $7 \cdot 10^p + 25 \cdot 10^{p-2} = 10^{p-2}(7 \cdot 10^2 + 25) = 725 \cdot 10^{p-2}$ for some $p \geq 2$. The smallest such integer is 725.

OR

The directions for the AIME imply that the desired integer has at most three digits. Because it also has at least two digits, it is of the form abd or cd, where a, b, c, and d are digits, and a and c are positive. Thus $bd \cdot 29 = abd$ or $d \cdot 29 = cd$. Note that no values of c and d satisfy $d \cdot 29 = cd$, and that d must be 0 or 5. Thus $b0 \cdot 29 = ab0$ or $b5 \cdot 29 = ab5$. But $b0 \cdot 29 = ab0$ implies $b \cdot 29 = ab$, which is not satisfied by any values of a and b. Now $b5 \cdot 29 = ab5$ implies that $b5 < 1000/29 < 35$, and so $b = 1$ or $b = 2$. Because $15 \cdot 29 = 435$ and $25 \cdot 29 = 725$, conclude that the desired integer is 725.

4. (Answer: 124)
Let $P = 1!\,2!\,3!\,4! \cdots 99!\,100!$. Then N is equal to the number of factors of 5 in P. For any positive integer k, the number of factors of 5 is the same for $(5k)!$, $(5k+1)!$, $(5k+2)!$, $(5k+3)!$, and $(5k+4)!$. The number of factors of 5 in $(5k)!$ is 1 more than the number of factors of 5 in $(5k-1)!$ if $5k$ is not a multiple of 25; and the number of factors of 5 in $(5k)!$ is 2 more than the number of factors of 5 in $(5k-1)!$ if $5k$ is a multiple of 25 but not 125.

Thus

$N = 4\cdot 0+5(1+2+3+4+6+7+8+9+10+12+13+14+15+16+18+19+20+21+22)+24.$

This sum is equal to

$$5 \cdot \left(\frac{22 \cdot 23}{2} - (5 + 11 + 17) \right) + 24 = 1124,$$

so the required remainder is 124.

OR

Let $P = 1!\,2!\,3!\,4! \cdots 99!\,100!$. Then N is equal to the number of factors of 5 in P. When the factorials in P are expanded, 5 appears 96 times (in $5!, 6!, \ldots, 100!$), 10 appears 91 times, and, in general, n appears $101 - n$ times. Every appearance of a multiple of 5 yields a factor of 5, and every appearance of a multiple of 25 yields an additional factor of 5. The number of multiples of 5 is $96 + 91 + 86 + \cdots + 1 = 970$, and the number of multiples of 25 is $76 + 51 + 26 + 1 = 154$, so P ends in $970 + 154 = 1124$ zeros. The required remainder is 124.

5. (Answer: 936)
Expand $(a\sqrt{2} + b\sqrt{3} + c\sqrt{5})^2$ to obtain $2a^2 + 3b^2 + 5c^2 + 2ab\sqrt{6} + 2ac\sqrt{10} + 2bc\sqrt{15}$, and conclude that $2a^2 + 3b^2 + 5c^2 = 2006, 2ab = 104$, $2ac = 468$, and $2bc = 144$. Therefore $ab = 52 = 2^2 \cdot 13$, $ac = 234 = 2 \cdot 3^2 \cdot 13$, and $bc = 72 = 2^3 \cdot 3^2$. Then $a^2b^2c^2 = ab \cdot ac \cdot bc = 2^6 \cdot 3^4 \cdot 13^2$, and $abc = 2^3 \cdot 3^2 \cdot 13 = 936$.
Note that $a = \dfrac{abc}{bc} = \dfrac{2^3 \cdot 3^2 \cdot 13}{2^3 \cdot 3^2} = 13$ and similarly that $b = 4$ and $c = 18$. These values yield $2a^2 + 3b^2 + 5c^2 = 2006$, as required.

6. (Answer: 360)
For any digit x, let x' denote the digit $9 - x$. If $0.\overline{abc}$ is an element of $\mathcal{S}$, then $0.\overline{a'b'c'}$ is also in $\mathcal{S}$, is not equal to $0.\overline{abc}$, and

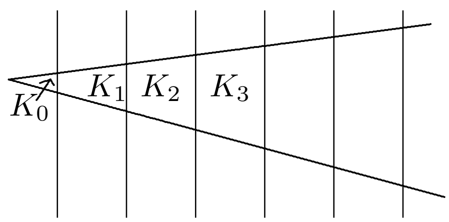
$$0.\overline{abc} + 0.\overline{a'b'c'} = 0.\overline{999} = 1.$$

It follows that $\mathcal{S}$ can be partitioned into pairs so that the elements of each pair add to 1. Because $\mathcal{S}$ has $10 \cdot 9 \cdot 8 = 720$ elements, the sum of the elements of $\mathcal{S}$ is $\frac{1}{2} \cdot 720 = 360$.

OR

Recall that $0.\overline{abc} = abc/999$, so the requested sum is 1/999 times the sum of all numbers of the form abc. To find the sum of their units digits, notice that each of the digits 0 through 9 appears $720/10 = 72$ times, and conclude that their sum is $72(0+1+2+\cdots+9) = 72\cdot 45$. Similarly, the sum of the tens digits and the sum of the hundreds digits are both equal to $72 \cdot 45$. The requested sum is therefore $72 \cdot 45(100+10+1)/999 = 360$.

7. (Answer: 408)

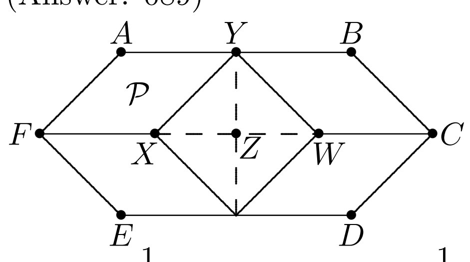


Without loss of generality, choose a unit of length equal to the distance between two adjacent parallel lines. Let x be the distance from the vertex of the angle to the closest of the parallel lines that intersect the angle. Denote the areas of the regions bounded by the parallel lines and the angle by $K_0, K_1, K_2, \ldots$, as shown. Then for $m > 0$ and $n > 0$,

$$\begin{aligned}\frac{K_m}{K_0} &= \frac{K_0+K_1+K_2+\cdots+K_m}{K_0} - \frac{K_0+K_1+K_2+\cdots+K_{m-1}}{K_0}\\ &= \left(\frac{x+m}{x}\right)^2 - \left(\frac{x+m-1}{x}\right)^2 = \frac{2x+2m-1}{x^2}, \quad \text{so}\\ \frac{K_m}{K_n} &= \frac{K_m/K_0}{K_n/K_0} = \frac{2x+2m-1}{2x+2n-1}.\end{aligned}$$

Thus the ratio of the area of region $\mathcal{C}$ to the area of region $\mathcal{B}$ is $\dfrac{K_4}{K_2} = \dfrac{2x+7}{2x+3}$, and so $\dfrac{2x+7}{2x+3} = \dfrac{11}{5}$. Solve this equation to obtain $x = 1/6$. Also, the ratio of the area of region $\mathcal{D}$ to the area of region $\mathcal{A}$ is $\dfrac{K_6}{K_0} = \dfrac{2x+11}{x^2}$. Substitute 1/6 for x to find that the requested ratio is 408.

8. (Answer: 089)



Draw the diagonals of rhombus $\mathcal{T}$. Let Z be their point of intersection, and let X and Y be the shared vertices of rhombuses $\mathcal{P}$ and $\mathcal{T}$, with Y on $\overline{AB}$. Let $YZ = x$ and $XY = z$. Consequently $\sqrt{2006} = [\mathcal{P}] = FX \cdot YZ = zx$, so $z = \sqrt{2006}/x$. Thus

$$K = \frac{1}{2}(2\cdot YZ)(2\cdot XZ) = \frac{1}{2}(2x)\left(2\sqrt{z^2-x^2}\right) = 2x\sqrt{\frac{2006}{x^2} - x^2} = \sqrt{8024-4x^4}.$$

There are $\left\lfloor \sqrt{8023} \right\rfloor = 89$ positive values of x that yield a positive square for the radicand, so there are 89 possible values for K.

OR

Define X, Y, and Z as in the first solution, and let W be the shared vertex of rhombuses $\mathcal{Q}$ and $\mathcal{T}$, $W \neq Y$, let $\alpha = m\angle AYX$, let $\beta = m\angle XYW$, and let z be the length of the sides of the rhombuses. Then $\beta + 2\alpha = 180°$, and the area of each of the four rhombuses is $z^2 \sin\alpha = \sqrt{2006}$. Therefore

$$K = z^2 \sin\beta = z^2 \sin 2\alpha = 2z^2 \sin\alpha\cos\alpha = 2\sqrt{2006}\,\cos\alpha.$$

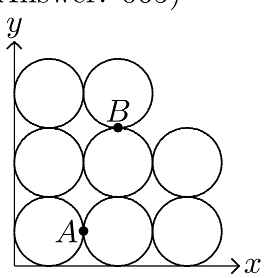
Thus $1 \le K < 2\sqrt{2006} = \sqrt{8024}$, so $1 \le K \le 89$, and there are 89 possible values for K.

9. (Answer: 046)
Note that

$$\log a_1 + \log a_2 + \cdots + \log a_{12} = \log(a_1 a_2 \cdots a_{12}) = \log\left(a \cdot ar \cdot \cdots \cdot ar^{11}\right) = \log\left(a^{12} r^{66}\right),$$

where the base of the logarithms is 8. Therefore $a^{12}r^{66} = 8^{2006} = 2^{3\cdot 2006}$, so $a^2 r^{11} = 2^{1003}$. Because a and r are positive integers, each must be a factor of 2^{1003}. Thus $a = 2^x$ and $r = 2^y$ for nonnegative integers x and y. Hence $2x + 11y = 1003$, and each ordered pair (a, r) corresponds to exactly one ordered pair (x, y) that satisfies this equation. Because $2x$ is even and 1003 is odd, y must be odd, so y has the form $2k-1$, where k is a positive integer. Then $1003 = 2x + 11y = 2x + 22k - 11$, so $x = 507 - 11k$. Therefore $507 - 11k \ge 0$, and so $1 \le k \le \lfloor 507/11 \rfloor = 46$. Thus there are 46 possible ordered pairs (a, r).

10. (Answer: 065)



Such a line is unique. This is because when a line with equation $y = 3x + d$ intersects $\mathcal{R}$, then as d decreases, the area of the part of $\mathcal{R}$ above the line is strictly increasing and the area of the part of $\mathcal{R}$ below the line is strictly decreasing. The symmetry point of the two circles that touch at $A(1, 1/2)$ is A, and so any line passing through A divides their region into two regions of equal area. Similarly, any line passing through $B(3/2, 2)$ divides the region consisting of the two circular regions that touch at B into two regions of equal area. Of the remaining four circles, two of them are on either side of line AB. Thus line AB divides $\mathcal{R}$ into two regions of equal area. Because the slope of line AB is 3, line AB is line ℓ, and it has equation $y - (1/2) = 3(x - 1)$ or $6x = 2y + 5$. Then $a^2 + b^2 + c^2 = 36 + 4 + 25 = 65$.

11. (Answer: 458)
Let $S(n)$ be the number of permissible towers that can be constructed from n cubes, one each with edge-length k for $1 \le k \le n$. Observe that $S(1) = 1$ and $S(2) = 2$. For $n \ge 2$, a tower of $n+1$ cubes can be constructed from any tower of n cubes by inserting the cube with edge-length $n+1$ in one of three positions: on the bottom, on top of the cube with edge-length n, or on top of the cube with edge-length $n-1$. Thus, from each tower of n cubes, three different towers of $n+1$ cubes can be constructed. Also, different towers of n cubes lead to different towers of $n+1$ cubes, and each tower of $n+1$ cubes becomes a permissible tower of n cubes when the cube with edge-length $n+1$ is removed. Hence, for $n \ge 2$, $S(n+1) = 3S(n)$. Because $S(2) = 2$, it follows that $S(n) = 2 \cdot 3^{n-2}$ for $n \ge 2$. Hence $T = S(8) = 2 \cdot 3^6 = 1458$, and the requested remainder is 458.

12. (Answer: 906)
The given equation implies that

$$\begin{aligned}\cos^3 3x + \cos^3 5x &= (2\cos 4x \cos x)^3\\ &= (\cos(4x+x) + \cos(4x-x))^3\\ &= (\cos 5x + \cos 3x)^3.\end{aligned}$$

Let $y = \cos 3x$ and $z = \cos 5x$. Then $y^3 + z^3 = (y+z)^3$. Expand and simplify to obtain $0 = 3yz(y+z)$. Thus $y = 0$ or $z = 0$ or $y + z = 0$, that is, $\cos 3x = 0$ or $\cos 5x = 0$ or $\cos 5x + \cos 3x = 0$. The solutions to the first equation are of the form $x = 30 + 60j$, where j is an integer; the second equation has solutions of the form $x = 18 + 36k$, where k is an integer. The third equation is equivalent to $\cos 4x \cos x = 0$, so its solutions are of the form $x = 22\frac{1}{2} + 45m$ and $x = 90 + 180n$, where m and n are integers. The solutions in the interval $100 < x < 200$ are 150, 126, 162, 198, $112\frac{1}{2}$, and $157\frac{1}{2}$, and their sum is 906.

13. (Answer: 899)
Note that S_n is defined as the sum of the greatest powers of 2 that divide the 2^{n-1} consecutive even numbers $2, 4, 6, \ldots, 2^n$. Of these, 2^{n-2} are divisible by 2 but not 4, 2^{n-3} are divisible by 4 but not $8, \ldots, 2^0$ are divisible by 2^{n-1} but not 2^n, and the only number not accounted for is 2^n. Thus

$$S_n = 2 \cdot 2^{n-2} + 2^2 \cdot 2^{n-3} + \cdots + 2^{n-1} \cdot 2^0 + 2^n = (n+1)2^{n-1}.$$

In order for $S_n = 2^{n-1}(n+1)$ to be a perfect square, n must be odd, because if n were even, then the prime factorization of S_n would have an odd number of factors of 2. Because n is odd, $n+1$ must be a square, and because $n+1$ is even, $n+1$ must be the square of an even integer. The greatest $n < 1000$ that is 1 less than the square of an even integer is $30^2 - 1 = 899$.

14. (Answer: 183)
The feet of the unbroken tripod are the vertices of an equilateral triangle ABC, and the foot of the perpendicular from the top to the plane of this triangle is at the center of the triangle. By the Pythagorean Theorem, the distance from the center to each vertex of the triangle is 3. Place a coordinate system so that the coordinates of the top T are (0, 0, 4) and the coordinates of A, B, and C are (3, 0, 0), $(-3/2, 3\sqrt{3}/2, 0)$, and $(-3/2, -3\sqrt{3}/2, 0)$, respectively. Let the break point A' be on $\overline{TA}$. Then $TA' : A'A = 4 : 1$. Thus the coordinates of A' are

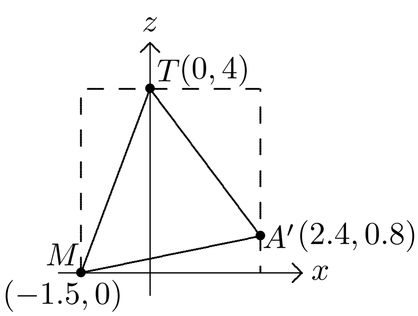
$$\frac{4}{5}(3,\ 0,\ 0) + \frac{1}{5}(0,\ 0,\ 4) = (12/5,\ 0,\ 4/5).$$

Note that the coordinates of M, the midpoint of $\overline{BC}$, are $(-3/2, 0, 0)$. The perpendicular from T to the plane of $\triangle A'BC$ will intersect this plane at a point on $\overline{MA'}$. This segment lies in the xz-plane and has equation $8x - 39z + 12 = 0$ in this plane. Then h is the distance from T to line MA', and is equal to

$$\frac{|\,8\cdot 0 - 39\cdot 4 + 12|}{\sqrt{8^2 + (-39)^2}} = \frac{144}{\sqrt{1585}},$$

so $\lfloor m + \sqrt{n} \rfloor = 144 + 39 = 183$.

OR



Place a coordinate system as in the first solution. Note that $\triangle A'MT$ is in the xz-plane. In this plane, circumscribe a rectangle around $\triangle A'MT$ with its sides parallel to the axes. Then

$$[A'MT] = 4(3.9) - \tfrac{1}{2}(4)(1.5) - \tfrac{1}{2}(3.2)(2.4) - \tfrac{1}{2}(3.9)(0.8) = 7.2.$$

Thus $h = \dfrac{2[A'MT]}{A'M} = \dfrac{14.4}{\sqrt{15.85}} = \dfrac{144}{\sqrt{1585}}$, so $\lfloor m + \sqrt{n} \rfloor = 144 + 39 = 183$.

OR

The feet of the unbroken tripod are the vertices of an equilateral triangle ABC, and the foot of the perpendicular from the vertex to the plane of this triangle is at the center of the triangle. Let T be the top of the tripod, let O be the center of $\triangle ABC$, let A' be the break point on $\overline{TA}$, and let M be the midpoint of $\overline{BC}$. Apply the Pythagorean Theorem to $\triangle TOA$ to conclude that $OA = 3$. Therefore $\triangle ABC$ has sides of length $3\sqrt{3}$. Notice that A', M, and T are all equidistant from B and C, so the plane determined by $\triangle TA'M$ is perpendicular to $\overline{BC}$, and so h is the length of the altitude from T in $\triangle TA'M$. Because

$$\frac{1}{2}A'M \cdot h = [TA'M] = \frac{1}{2}A'T \cdot TM \sin\angle A'TM,$$

it follows that

$$h = \frac{A'T \cdot TM \sin\angle A'TM}{A'M}.$$

The length of $\overline{A'T}$ is 4, and $TM = \sqrt{TB^2 - BM^2} = \sqrt{25 - (27/4)} = \sqrt{73}/2$. To find $A'M$, note that $A'M^2 = A'C^2 - CM^2$, and that $A'C^2 = A'T^2 + TC^2 - 2A'T \cdot TC \cos\angle A'TC$. But $\cos\angle A'TC = \cos\angle ATC = \dfrac{5^2 + 5^2 - (3\sqrt{3})^2}{2 \cdot 5 \cdot 5} = \dfrac{23}{50}$,
so $A'C^2 = 4^2 + 5^2 - 2 \cdot 4 \cdot 5 \cdot (23/50) = 113/5$, and $A'M = \sqrt{\dfrac{113}{5} - \dfrac{27}{4}} = \sqrt{\dfrac{317}{20}}$.
Now $\cos\angle A'TM = \dfrac{16 + 73/4 - 317/20}{2 \cdot 4 \cdot \sqrt{73}/2} = \dfrac{23}{5\sqrt{73}}$,
so $\sin^2\angle A'TM = 1 - \dfrac{23^2}{25 \cdot 73}$, and $\sin\angle A'TM = \dfrac{36}{5\sqrt{73}}$. Thus

$$h = \frac{4 \cdot \frac{\sqrt{73}}{2} \cdot \frac{36}{5\sqrt{73}}}{\sqrt{\frac{317}{20}}} = \frac{144}{\sqrt{1585}},$$

so $\lfloor m + \sqrt{n} \rfloor = 144 + 39 = 183$.

15. (Answer: 027)
The condition $|x_k| = |x_{k-1}+3|$ is equivalent to $x_k^2 = (x_{k-1}+3)^2$. Thus

$$\sum_{k=1}^{n+1} x_k^2 = \sum_{k=1}^{n+1}(x_{k-1}+3)^2 = \sum_{k=0}^{n}(x_k+3)^2 = \left(\sum_{k=0}^{n} x_k^2\right) + \left(6\sum_{k=0}^{n} x_k\right) + 9(n+1), \quad \text{so}$$

$$x_{n+1}^2 = \sum_{k=1}^{n+1} x_k^2 - \sum_{k=0}^{n} x_k^2 = \left(6\sum_{k=0}^{n} x_k\right) + 9(n+1), \quad \text{and}$$

$$\sum_{k=0}^{n} x_k = \frac{1}{6}\left[x_{n+1}^2 - 9(n+1)\right].$$

Therefore $\left|\sum_{k=1}^{2006} x_k\right| = \frac{1}{6}\left|x_{2007}^2 - 18063\right|$. Notice that x_k is a multiple of 3 for all k, and that x_k and k have the same parity. The requested sum will be a minimum when $|x_{2007}^2 - 18063|$ is a minimum, that is, when x_{2007} is the multiple of 3 whose square is as close as possible to 18063. Check odd multiples of 3, and find that $129^2 < 16900$, $141^2 > 19600$, and $135^2 = 18225$. The requested minimum is therefore $\frac{1}{6}|135^2 - 18063| = 27$, provided there exists a sequence that satisfies the given conditions and for which $x_{2007} = 135$. An example of such a sequence is

$$x_k = \begin{cases} 3k & \text{for } k \le 45, \\ -138 & \text{for } k > 45 \text{ and } k \text{ even}, \\ 135 & \text{for } k > 45 \text{ and } k \text{ odd}. \end{cases}$$

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THE MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

24th Annual *(Alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Wednesday, March 22, 2006

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:
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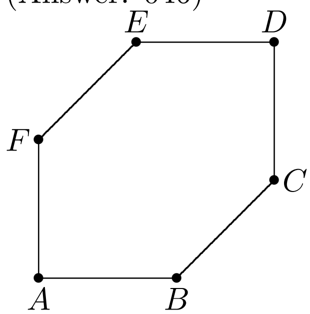
1. (Answer: 046)



Because $\angle B$, $\angle C$, $\angle E$, and $\angle F$ are congruent, the degree-measure of each of them is $\dfrac{720 - 2 \cdot 90}{4} = 135$. Lines BF and CE divide the hexagonal region into two right triangles and a rectangle. Let $AB = x$. Then $BF = x\sqrt{2}$. Thus

$2116(\sqrt{2}+1) = [ABCDEF] = 2 \cdot \frac{1}{2}x^2 + x \cdot x\sqrt{2} = x^2(1+\sqrt{2})$,

so $x^2 = 2116$, and $x = 46$.

2. (Answer: 893)
The Triangle Inequality yields

$$\log n < \log 75 + \log 12 = \log 900, \quad \text{and}$$
$$\log n > \log 75 - \log 12 = \log(25/4).$$

Therefore $25/4 < n < 900$, and so $7 \le n \le 899$. Hence there are $899 - 7 + 1 = 893$ possible values of n.

3. (Answer: 049)
Of the first 100 positive odd integers, $1, 3, 5, \ldots, 199$,

33 of them, namely $3, 9, 15, \ldots, 195 = 3(2 \cdot 33 - 1)$, are divisible by 3;
11 of them, namely $9, 27, 45, \ldots, 189 = 9(2 \cdot 11 - 1)$, are divisible by 9;
4 of them, namely $27, 81, 135, 189 = 27(2 \cdot 4 - 1)$, are divisible by 27; and
1 of them, namely 81, is divisible by 81.

Therefore $k = 33 + 11 + 4 + 1 = 49$.

OR

Note that
$$P = \frac{200!}{2 \cdot 4 \cdot \cdots \cdot 200} = \frac{200!}{2^{100} \cdot 100!}.$$
The number of factors of 3 in the numerator is

$$\lfloor 200/3 \rfloor + \lfloor 200/3^2 \rfloor + \lfloor 200/3^3 \rfloor + \lfloor 200/3^4 \rfloor = 66 + 22 + 7 + 2 = 97,$$

and the number of factors of 3 in the denominator is

$$\lfloor 100/3 \rfloor + \lfloor 100/3^2 \rfloor + \lfloor 100/3^3 \rfloor + \lfloor 100/3^4 \rfloor = 33 + 11 + 3 + 1 = 48.$$

Therefore $k = 97 - 48 = 49$.

4. (Answer: 462)
Because a_6 is less than each of the other 11 numbers, $a_6 = 1$. Choose any five numbers of the remaining 11 to fill the first five positions. Their order is then uniquely determined. The order of the remaining six numbers which fill the last six positions is also uniquely determined. Thus the number of such permutations is the number of choices for the first five numbers, which is $\binom{11}{5} = 462$.

5. (Answer: 029)
Let $p(a, b)$ denote the probability of obtaining a on the first die and b on the second. Then the probability of obtaining a sum of 7 is

$$p(1,6) + p(2,5) + p(3,4) + p(4,3) + p(5,2) + p(6,1).$$

Let the probability of obtaining face F be $(1/6) + x$. Then the probability of obtaining the face opposite face F is $(1/6) - x$. Therefore

$$\begin{aligned}\frac{47}{288} &= 4\left(\frac{1}{6}\right)^2 + 2\left(\frac{1}{6} + x\right)\left(\frac{1}{6} - x\right)\\ &= \frac{4}{36} + 2\left(\frac{1}{36} - x^2\right)\\ &= \frac{1}{6} - 2x^2.\end{aligned}$$

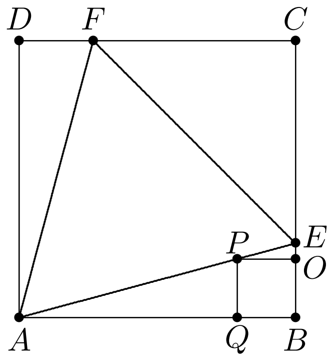
Then $2x^2 = 1/288$, and so $x = 1/24$. The probability of obtaining face F is therefore $(1/6) + (1/24) = 5/24$, and $m + n = 29$.

6. (Answer: 012)
Let $CF = x$. Then, because $\triangle ADF \cong \triangle ABE$, it follows that $DF = BE = 1 - x$, and $CE = x$. Hence $2x^2 = EF^2 = AE^2 = (1-x)^2 + 1$, and so $x = \sqrt{3} - 1$. Let P and Q be the vertices of the smaller square that are on $\overline{AE}$ and $\overline{AB}$, respectively. Then

$$\begin{aligned}\frac{AB - PQ}{PQ} &= \frac{AB - BQ}{PQ} = \frac{AQ}{PQ} = \frac{AB}{BE}, \quad \text{so}\\ \frac{AB}{PQ} &= 1 + \frac{AB}{BE}, \quad \text{and}\\ \frac{1}{PQ} &= \frac{1}{AB} + \frac{1}{BE}.\end{aligned}$$

Thus $\dfrac{1}{PQ} = 1 + \dfrac{1}{1 - (\sqrt{3} - 1)} = 1 + \dfrac{1}{2 - \sqrt{3}} = 1 + 2 + \sqrt{3} = 3 + \sqrt{3}$. Consequently $PQ = \dfrac{1}{3 + \sqrt{3}} = \dfrac{3 - \sqrt{3}}{6}$, and $a + b + c = 12$.

OR



Let $BOPQ$ be the smaller square, where Q is between A and B, and let $BQ = y$. Then $QP = y$, and $AQ = y \tan 75°$. Thus $1 = AB = AQ + QB = y \tan 75° + y$, so $y = \dfrac{1}{1 + \tan 75°}$. But $\tan 75° = \tan(45° + 30°) = \dfrac{1 + (1/\sqrt{3})}{1 - (1/\sqrt{3})} = 2 + \sqrt{3}$. Therefore $y = \dfrac{1}{3 + \sqrt{3}} = \dfrac{3 - \sqrt{3}}{6}$.

OR

Place a coordinate system so that A is the origin, and the coordinates of B, C, and D are $(1, 0)$, $(1, 1)$, and $(0, 1)$, respectively. Let $BE = p$. Then, as in the first solution, $p = 1 - (\sqrt{3} - 1) = 2 - \sqrt{3}$. Hence line AE has slope $2 - \sqrt{3}$ and contains the origin. Thus line AE has equation $y = (2 - \sqrt{3})x$. Let q be the length of a side of the smaller square. Then one vertex of that square has coordinates $(1 - q, q)$ and is on line AE. Therefore $q = (2 - \sqrt{3})(1 - q)$, which yields $q = (3 - \sqrt{3})/6$.

7. (Answer: 738)
Count the number of such ordered pairs with $a < b$. If a is a one-digit number, then b's digits are 9, 9, and $10 - a$. There are 9 choices for a in this case. In the case where a is a two-digit number, represent the digits of a as t and u. Then b's digits are 9, $9 - t$, and $10 - u$. Because $t \neq 0$ and $t \neq 9$, there are 8 choices for t and 9 choices for u, and so there are 72 choices for a. In the case where a is a three-digit number, represent the digits of a as h, t, and u. Then b's digits are $9 - h$, $9 - t$, and $10 - u$. Because $h = 1, 2, 3$, or 4, there are $4 \cdot 8 \cdot 9 = 288$ choices for a.

Thus the number of pairs with $a < b$ is $9 + 72 + 288 = 369$. Because each such pair can be reversed to give another allowable pair with $b < a$, there are $2 \cdot 369 = 738$ pairs.

OR

Count the number of forbidden pairs, that is, pairs in which a or b has a zero digit. If a or b has units digit 0, then both do, and the given equation reduces to $r + s = 100$, where $a = 10r$ and $b = 10s$. Thus, in this case, there are 99 forbidden pairs.

In the case where neither a nor b has units digit 0, then exactly one of them must be of the form $h0u$, where neither h nor u is 0. There are $9 \cdot 9 = 81$ such values of a and 81 such values for b for a total of 162 forbidden pairs in this case. Therefore the total number of forbidden pairs is $99 + 162 = 261$, and there are $999 - 261 = 738$ of the requested pairs.

8. (Answer: 336)
Because any permutation of the vertices of a large triangle can be obtained by rotation or reflection, the coloring of the large triangle is determined by which set of three colors is used for the corner triangles and the color that is used for the center triangle. If the three corner triangles are the same color, there are six possible sets of colors for them. If exactly two of the corner triangles are the same color, there are $6 \cdot 5 = 30$ possible sets of colors. If the three corner triangles are different colors, there are $\binom{6}{3} = 20$ possible sets of colors. Therefore there are $6 + 30 + 20 = 56$ sets of colors for the corner triangles. Because there are six choices for the color of the center triangle, there are $6 \cdot 56 = 336$ distinguishable triangles.

9. (Answer: 027)
Let O_1, O_2, and O_3 be the centers of $\mathcal{C}_1$, $\mathcal{C}_2$, and $\mathcal{C}_3$, respectively, let A and B be the points where t_1 is tangent to $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, and let D and E be the points where t_2 is tangent to $\mathcal{C}_2$ and $\mathcal{C}_3$, respectively. Radii $\overline{O_1A}$ and $\overline{O_2B}$ are perpendicular to line AB. Let P be the intersection of $\overline{AB}$ and $\overline{O_1O_2}$. Then $\triangle O_1AP \sim \triangle O_2BP$ with similarity ratio $1:2$. Therefore $O_1P = 4$ and $O_2P = 8$, so $PB = \sqrt{8^2 - 2^2} = 2\sqrt{15}$. The slope of line t_1 is equal to $\tan\angle BPO_2 = 1/\sqrt{15}$, so line t_1 has equation $y = (1/\sqrt{15})(x - 4)$. Similarly, let Q be the intersection of $\overline{DE}$ and $\overline{O_2O_3}$, and conclude that $O_2Q = 4$ and $O_3Q = 8$, and then that $DQ = \sqrt{4^2 - 2^2} = 2\sqrt{3}$. The slope of line t_2 is equal to $\tan\angle DQO_3 = -\tan\angle DQO_2 = -1/\sqrt{3}$, so line t_2 has equation $y = (-1/\sqrt{3})(x - 16)$. Now $(1/\sqrt{15})(x - 4) = (-1/\sqrt{3})(x - 16)$ implies $x - 4 = -\sqrt{5}(x - 16)$, so $x = \dfrac{16\sqrt{5} + 4}{\sqrt{5} + 1} = 19 - 3\sqrt{5}$, and $p + q + r = 27$.

10. (Answer: 831)
Each team has to play six games in all. Team A and team B each has 5 more games to play, and they do not play against each other, for a total of $2^5 \cdot 2^5$ possible outcomes. For team A to finish with more points, it has to win at least as many games as team B does. The number of outcomes in which the two teams win the same number of games is

$$\binom{5}{0}^2 + \binom{5}{1}^2 + \binom{5}{2}^2 + \binom{5}{3}^2 + \binom{5}{4}^2 + \binom{5}{5}^2 = 252.$$

Of the remaining $1024 - 252 = 772$ outcomes, A wins more than B in half of them. Thus the requested probability is $\dfrac{252 + (772/2)}{1024} = \dfrac{319}{512}$, and $m + n = 831$.

OR

Team A and team B each has 5 more games to play. For $1 \le k \le 5$, the kth game for team A and the kth game for team B will change the difference between the scores of team A and team B by $+1$, 0, or -1, and a change of 0 is twice as likely as each of the other changes. Hence consider the coefficients of the generating function $g(x) = (x^{-1} + 2 + x)^5$, and find the sum of all the coefficients of terms of the form x^k, where $k \ge 0$. Note that

$$g(x) = \frac{(1+2x+x^2)^5}{x^5} = \frac{(1+x)^{10}}{x^5}.$$

Thus the sum is

$$\binom{10}{5} + \binom{10}{6} + \cdots + \binom{10}{10} = \frac{1}{2}\left(\sum_{i=0}^{10}\binom{10}{i} + \binom{10}{5}\right) = \frac{1}{2}\left(2^{10} + \binom{10}{5}\right) = 638,$$

so the requested probability is $638/4^5 = 319/512$, and $m + n = 831$.

11. (Answer: 834)
Because $a_{k+3} - a_{k+2} = a_{k+1} + a_k$ for all positive integers k, conclude that

$$\sum_{k=1}^{n}(a_{k+3} - a_{k+2}) = \sum_{k=1}^{n}(a_{k+1} + a_k).$$

Let $S_n = \sum_{k=1}^{n} a_k$. Notice that $\sum_{k=1}^{n}(a_{k+3} - a_{k+2})$ telescopes to $a_{n+3} - a_3$, and that $\sum_{k=1}^{n}(a_{k+1} + a_k) = (S_n - a_1 + a_{n+1}) + S_n$. Therefore $a_{n+3} - a_3 = S_n - a_1 + a_{n+1} + S_n$, so $S_n = (1/2)(a_{n+3} - a_{n+1}) = (1/2)(a_{n+2} + a_n)$, and in particular $S_{28} = (1/2)(a_{30} + a_{28})$. Thus the last three digits of the sum are the same as those of $(1/2)(3361 + 0307)$, namely 834, and the requested remainder is 834.

12. (Answer: 865)
Notice that $\angle GCB \cong \angle GAB$ and $\angle CAG \cong \angle CBG$ because each pair of angles intercepts the same arc. Also $\angle CAG \cong \angle AFD$ because $\overline{AE} \parallel \overline{DF}$. Thus $\triangle AFD \sim \triangle CBG$, and $[CBG] = t^2[AFD]$, where t is the similarity ratio. Because $m\angle ADF = 120^\circ$, $[AFD] = (1/2)AD \cdot DF \cdot \sin 120^\circ = 143\sqrt{3}/4$. The length of each side of $\triangle ABC$ is $2\sqrt{3}$. The Law of Cosines implies that $AF^2 = 13^2 + 11^2 - 2 \cdot 13 \cdot 11(-1/2) = 433$, so $AF = \sqrt{433}$. Therefore $t = \dfrac{BC}{AF} = \dfrac{2\sqrt{3}}{\sqrt{433}}$, so

$$[CBG] = t^2 \cdot [AFD] = \left(\frac{2\sqrt{3}}{\sqrt{433}}\right)^2 \cdot \frac{143\sqrt{3}}{4} = \frac{429\sqrt{3}}{433},$$

and $p + q + r = 865$.

OR

Let $\alpha = m\angle DAF$, and let $\beta = m\angle EAF$. Then $m\angle BCG = \alpha$ and $m\angle CBG = \beta$. Note that $[BGC] = (1/2)BC \cdot CG \sin\alpha$, and apply the Law of Sines in $\triangle BGC$ to conclude that $\dfrac{BC}{\sin 120^\circ} = \dfrac{CG}{\sin\beta}$. Then $[BGC] = \dfrac{1}{2} \cdot BC \cdot \dfrac{BC\sin\beta}{\sin 120^\circ}\sin\alpha = \dfrac{BC^2\sin\alpha\sin\beta}{\sqrt{3}}$. Use the Law of Cosines in $\triangle AEF$ to conclude that $AF = \sqrt{433}$, and use the Law of Sines to conclude that $\dfrac{\sqrt{433}}{\sin 120^\circ} = \dfrac{13}{\sin\beta}$, so $\sin\beta = \dfrac{13\sqrt{3}}{2\sqrt{433}}$. The Law of Sines implies that $\sin\alpha = \sin\angle AFE = \dfrac{11\sqrt{3}}{2\sqrt{433}}$. Thus

$$[BGC] = \frac{\left(2\sqrt{3}\right)^2\left(\frac{11\sqrt{3}}{2\sqrt{433}}\right)\left(\frac{13\sqrt{3}}{2\sqrt{433}}\right)}{\sqrt{3}} = \frac{429\sqrt{3}}{433}.$$

13. (Answer: 015)

Recall that the sum of the first m positive odd integers is m^2. Thus if N is equal to the sum of the $(k+1)$th through mth positive odd integers, then $N = m^2 - k^2 = (m-k)(m+k)$. Let $a = m-k$, and let $b = m+k$. Note that $a \le b$, and a and b have the same parity. Thus N is either odd or a multiple of 4. Conversely, if $N = ab$, where a and b are positive integers with the same parity and $a \le b$, then $N = m^2 - k^2$, where $m = (b+a)/2$ and $k = (b-a)/2$, and it follows that N is the sum of the $(k+1)$th through mth odd integers. Thus there is a one-to-one correspondence between the sets of consecutive positive odd integers whose sum is N and the ordered pairs (a, b) of positive integers such that a and b are of the same parity, $ab = N$, and $a \le b$.

First consider the case where N is odd. All the divisors of N have the same parity because they are all odd. Since five pairs of positive integers have product N, N must have either 9 or 10 divisors. N must therefore have the form p^8, p^9, p^2q^2, or pq^4, where p and q are distinct odd primes. But N cannot have the form p^8 or p^9, because that would imply that $N \ge 3^8 > 1000$. If N has the form p^2q^2, then $pq \le 31$ because $N < 1000$, and there are two possible values of N, namely $3^2 \cdot 5^2$ and $3^2 \cdot 7^2$. If N has the form pq^4, then N must be $5 \cdot 3^4$, $7 \cdot 3^4$, or $11 \cdot 3^4$.

In the case where N is even, $N = ab$, where $a = 2a'$ and $b = 2b'$ for positive integers a' and b'. In this case, N has five pairs of divisors of the same parity if and only if $N/4$ has 9 or 10 divisors. Count the number of positive integers less than 250 that are of the previously mentioned forms, except that now p or q can be 2. There are no integers less than 250 that are of the form p^8 or p^9; there are four such integers of the form p^2q^2, namely, $2^2 \cdot 3^2$, $2^2 \cdot 5^2$, $2^2 \cdot 7^2$, and $3^2 \cdot 5^2$; and there are six such integers of the form pq^4, namely, $3 \cdot 2^4$, $5 \cdot 2^4$, $7 \cdot 2^4$, $11 \cdot 2^4$, $13 \cdot 2^4$, and $2 \cdot 3^4$.

Thus there are a total of $2 + 3 + 4 + 6 = 15$ possible values for N.

14. (Answer: 063)

Each of the 10^n integers from 0 to $10^n - 1$, inclusive, can be written as an n-digit string, using leading 0's as necessary. Imagine these strings written one beneath the other to form a table of digits with n columns and 10^n rows. Each column contains an equal number of digits of each type, so there are $(1/10) \cdot 10^n$ digits of each type in each column, and there are $(n/10) \cdot 10^n = n \cdot 10^{n-1}$ digits of each type in the table. Therefore

$$S_n = 1 + \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}\right) n \cdot 10^{n-1}.$$

The sum S_n is not an integer when $n = 1, 2$, or 3, and when $n \geq 4$,

$$\left(\frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{5} + \frac{1}{8}\right) n \cdot 10^{n-1} \quad \text{and} \quad \left(\frac{1}{3} + \frac{1}{6}\right) n \cdot 10^{n-1} = \frac{1}{2} n \cdot 10^{n-1}$$

are integers. Thus S_n is an integer when

$$\left(\frac{1}{7} + \frac{1}{9}\right) n \cdot 10^{n-1} = \frac{16n}{63} \cdot 10^{n-1}$$

is an integer. Because $16 \cdot 10^{n-1}$ and 63 are relatively prime, the smallest value of n for which S_n is an integer is 63.

15. (Answer: 009)

The radicals on the right side of the first equation are reminiscent of the Pythagorean Theorem. Each radical represents the length of a leg of a right triangle whose other leg has length 1/4, and whose hypotenuse has length y or z. Adjoin these two triangles along the leg of length 1/4 to create $\triangle XYZ$ with $x = YZ$, $y = ZX$, and $z = XY$, and with altitude to side $\overline{YZ}$ of length 1/4. Because of similar considerations in the other two equations, let the lengths of the altitudes to sides $\overline{XZ}$ and $\overline{XY}$ be 1/5 and 1/6, respectively. In the $\triangle XYZ$ thus created, the lengths x, y, and z of the sides satisfy the given equations, provided the altitudes of $\triangle XYZ$ are inside the triangle, that is, provided $\triangle XYZ$ is acute.

In general, a triangle the lengths of whose sides are a, b, and c is acute if and only if $a^2 + b^2 > c^2$, where $a \leq b \leq c$. Denote the area of the triangle by K and the lengths of the altitudes to the sides of lengths a, b, and c by h_a, h_b, and h_c, respectively. Then $K = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$, so the condition $a^2+b^2 > c^2$ is equivalent to $(1/h_a)^2+(1/h_b)^2 > (1/h_c)^2$, where $1/h_a \leq 1/h_b \leq 1/h_c$. Thus $\triangle XYZ$ is acute because $4^2 + 5^2 > 6^2$.

Let K be the area of $\triangle XYZ$. Then $x = 8K$, $y = 10K$, and $z = 12K$, so $x+y+z = 30K$. Apply Heron's Formula to obtain $K^2 = 15K \cdot 7K \cdot 5K \cdot 3K$. Because $K > 0$, conclude that $K = 1/(15\sqrt{7})$. Then $x + y + z = 30K = 2/\sqrt{7}$, so $m + n = 9$.

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THE MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

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AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Tuesday, March 13, 2007

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:

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1. (Answer: 083)

   Because $24 = 3 \cdot 2^3$, a square is divisible by 24 if and only if it is divisible by $3^2 \cdot 2^4 = 144$. Furthermore, a perfect square N^2 less than 10^6 is a multiple of 144 if and only if N is a multiple of 12 less than 10^3. Because 996 is the largest multiple of 12 less than 10^3, there are $\frac{996}{12} = 83$ such positive integers less than 10^3 and 83 positive perfect squares which are multiples of 24.

2. (Answer: 052)

   Let t be Al's travel time. Then $t-2$ is Bob's time, and $t-4$ is Cy's time, and $t \geq 4$. If Cy is in the middle, then $10(t-2) - 8(t-4) = 8(t-4) - 6t$, which has no solution. If Bob is in the middle, then $10(t-2) - 8(t-4) = 6t - 10(t-2)$, which has solution $t = 4/3$. But $t \geq 4$, so this is impossible. If Al is in the middle, then $6t - 8(t-4) = 10(t-2) - 6t$, which has solution $t = 26/3$. In this case, Al is 52 feet from the start and is $44/3$ feet from both Bob and Cy. Thus the required distance is 52.

3. (Answer: 015)

   The complex number $z = 9 + bi$, so $z^2 = (81 - b^2) + 18bi$ and $z^3 = (729 - 27b^2) + (243b - b^3)i$. These two numbers have the same imaginary part, so $243b - b^3 = 18b$. Because b is not zero, $243 - b^2 = 18$, and $b = 15$.

4. (Answer: 105)

   All four positions will be collinear if and only if the difference in the number of revolutions made by each pair of planets is an integer multiple of $\dfrac{1}{2}$. When the outermost planet has made r revolutions, the middle and innermost planets will have made $\dfrac{140r}{84} = \dfrac{5}{3}r$ and $\dfrac{140r}{60} = \dfrac{7}{3}r$ revolutions, respectively. Thus it is necessary and sufficient that $\dfrac{2}{3}r = \dfrac{1}{2}k$ for some integer k, so the smallest positive solution for r is $\dfrac{3}{4}$. Hence $n = \dfrac{3}{4} \cdot 140 = 105$.

5. (Answer: 539)

   Note that a temperature T converts back to T if and only if $T+9$ converts back to $T+9$. Thus it is only necessary to examine nine consecutive temperatures. It is easy to show that 32 converts back to 32, 33 and 34 both convert back to 34, 35 and 36 both convert back to 36, 37 and 38 both convert back to 37, and 39 and 40 both convert back to 39. Hence out of every nine consecutive temperatures starting with 32, five are converted correctly and four are not. For $32 \leq T < 32 + 9 \cdot 107 = 995$. There are $107 \cdot 5 = 535$ temperatures that are converted correctly. The remaining six temperatures $995, 996, \ldots, 1000$ behave like $32, 33, \ldots, 37$, so four of the remaining six temperatures are converted correctly. Thus there is a total of $535 + 4 = 539$ temperatures.

OR

Because one Fahrenheit degree is 5/9 of a Celsius degree, every integer Celsius temperature is the conversion of either one or two Fahrenheit temperatures (nine Fahrenheit temperatures are being converted to only five Celsius temperatures) and converts back to one of those temperatures. The Fahrenheit temperatures 32 and 1000 convert to 0 and 538, respectively, which convert back to 32 and 1000. Therefore there are 539 Fahrenheit temperatures with the required property, corresponding to the integer Celsius temperatures from 0 to 538.

6. (Answer: 169)

The set of move sequences can be partitioned into two subsets, namely the set of move sequences that contain 26, and the set of move sequences that do not contain 26. If a move sequence does not contain 26, then it must contain the subsequence 24, 27. To count the number of such sequences, note that there are six ways for the frog to get to 24 (five ways that go through 13, and one way that does not go through 13), and there are four ways for the frog to get from 27 to 39, since any of 27, 30, 33, 36 can be the second to last element of a move sequence. Thus there are 24 move sequences that do not contain 26. If a move sequence contains 26, then it either contains 13 or it does not. If such a move sequence does not contain 13, then it must contain the subsequence 12, 15, and in this case, there is one way for the frog to get from 0 to 12, and there are 4 ways for it to get from 15 to 26. If a move sequence contains 13 and 26, then there are five ways for the frog to get from 0 to 13, and there are are five ways for it to get from 13 to 26. Thus there are $1 \cdot 4 + 5 \cdot 5 = 29$ ways for the frog to get from 0 to 26, and there are five ways for the frog to get from 26 to 39, for a total of 145 move sequences that contain 26. Thus there are a total of $24 + 145 = 169$ move sequences for the frog.

OR

There are five sequences from 0 to 13, five from 13 to 26, and five from 26 to 39. There are four sequences from 0 to 26 not containing 13 and four sequences from 13 to 39 not containing 26. Finally, there are four sequences from 0 to 39 containing neither 13 nor 26. Thus there are $5 \cdot 5 \cdot 5 + 2 \cdot 4 \cdot 5 + 4 = 169$ sequences in all.

7. (Answer: 477)

First note that

$$\lceil x\rceil - \lfloor x\rfloor = \begin{cases} 1, & \text{if } x \text{ is not an integer} \\ 0, & \text{if } x \text{ is an integer} \end{cases}.$$

Thus for any positive integer k,

$$\lceil \log_{\sqrt{2}} k\rceil - \lfloor \log_{\sqrt{2}} k\rfloor = \begin{cases} 1, & \text{if } k \text{ not an integer power of } \sqrt{2} \\ 0, & \text{if } k \text{ an integer power of } \sqrt{2} \end{cases}.$$

The integers k, $1 \le k \le 1000$, that are integer powers of $\sqrt{2}$ are described by $k = 2^j$, $0 \le j \le 9$. Thus

$$\sum_{k=1}^{1000} k(\lceil \log_{\sqrt{2}} k\rceil - \lfloor \log_{\sqrt{2}} k\rfloor) = \sum_{k=1}^{1000} k - \sum_{j=0}^{9} 2^j = \frac{1000 \cdot 1001}{2} - 1023 = 499477.$$

The requested remainder is 477.

8. (Answer: 030)

Because $P(x)$ has three roots, if $Q_1(x) = x^2 + (k-29)x - k$ and $Q_2(x) = 2x^2 + (2k-43)x + k$ are both factors of $P(x)$, then they must have a common root r. Then $Q_1(r) = Q_2(r) = 0$, and $mQ_1(r) + nQ_2(r) = 0$, for any two constants m and n. Taking $m = 2$ and $n = -1$ yields the equation $15r + 3k = 0$, so $r = \frac{-k}{5}$. Thus $Q_1(r) = \frac{k^2}{25} - (k-29)\left(\frac{k}{5}\right) - k = 0$, which is equivalent to $4k^2 - 120k = 0$, whose roots are $k = 30$ and 0. When $k = 30$, $Q_1(x) = x^2 + x - 30$ and $Q_2(x) = 2x^2 + 17x + 30$, and both polynomials are factors of $P(x) = (x+6)(x-5)(2x+5)$. Thus the requested value of k is 30.

9. (Answer: 737)

Let T_1 and T_2 be the points of tangency on AB so that $\overline{O_1T_1}$ and $\overline{O_2T_2}$ are radii of length r. Let circles O_1 and O_2 be tangent to each other at point T_3, let circle O_1 be tangent to the extension of $\overline{AC}$ at E_1, and let circle O_2 be tangent to the extension of $\overline{BC}$ at E_2. Let D_1 be the foot of the perpendicular from O_1 to $\overline{BC}$, let D_2 be the foot of the perpendicular from O_2 to $\overline{AC}$, and let $\overline{O_1D_1}$ and $\overline{O_2D_2}$ intersect at P. Let r be the radius of O_1 and O_2. Note that $O_1T_3 = O_2T_3 = r$, $CD_1 = O_1E_1 = r$, and $CD_2 = O_2E_2 = r$. Because triangles ABC and O_1O_2P are similar, $\dfrac{O_1O_2}{AB} = \dfrac{O_1P}{AC} = \dfrac{O_2P}{BC}$. Because $O_1O_2 = 2r$ and $AB = \sqrt{30^2 + 16^2} = 34$, $O_1P = \dfrac{30r}{17}$ and $O_2P = \dfrac{16r}{17}$. By equal tangents, $AE_1 = AT_1 = x$ and $BE_2 = BT_2 = y$. Thus

$$AB = 34 = AT_1 + T_1T_2 + BT_2$$

$$= AT_1 + O_1O_2 + BT_2$$
$$= x + 2r + y.$$

Also, $O_1P = E_1D_2$, so $\frac{30r}{17} = AD_2 + AE_1 = 30 - r + x$, and $O_2P = E_2D_1$, so $\frac{16r}{17} = BD_1 + BE_2 = 16 - r + y$. Adding these two equations produces $\frac{46r}{17} = 46 - 2r + x + y$. Substituting $x + y = 34 - 2r$ yields $\frac{46r}{17} = 80 - 4r$. Thus $r = \frac{680}{57}$, and $p + q = 737$.

OR

Draw $\overline{O_1A}$ and $\overline{O_2B}$ and note that these segments bisect the external angles of the triangle at A and B, respectively. Thus $\angle T_1O_1A = \dfrac{1}{2}(\angle A)$ and $T_1A = r\tan\angle T_1O_1A = r\tan\dfrac{1}{2}(\angle A)$. Similarly $T_2B = r\tan\dfrac{1}{2}(\angle B)$. By the half-angle identity for tangent,

$$\tan\left(\frac{1}{2}\angle A\right) = \frac{\sin\angle A}{\cos\angle A + 1} = \frac{16/34}{(30/34) + 1} = \frac{1}{4},$$

and similarly $\tan\left(\frac{1}{2}(\angle B)\right) = \frac{3}{5}$. Then $34 = AB = AT_1 + T_1T_2 + T_2B = r/4 + 2r + 3r/5 = 57r/20$ and $r = 680/57$. Thus $p + q = 737$.

10. (Answer: 860)

There are $\binom{6}{3}$ ways to shade three squares in the first column. Given a shading scheme for the first column, consider the ways the second, third, and fourth columns can then be shaded. If the shaded squares in the second column are in the same rows as those of the first column, then the shading pattern for the last two columns is uniquely determined. Thus there are $\binom{6}{3} = 20$ shadings in which the first two columns have squares in the same rows shaded. Next shade the second column so that two of the shaded squares are in the same rows as shaded squares in the first column. Given a shading of the first column, there are $\binom{3}{2}$ ways to choose the two shaded rows in common, then $\binom{3}{1}$ ways to choose the third shaded square in this column. This leaves two rows with no shaded squares, so the squares in these rows must be shaded in the third and fourth columns. There are also two rows with one shaded square each, one of these in the first column and one in the second. For the first of these rows the square can be shaded in the same row in the third or fourth column, for two choices, and this uniquely determines the shading in the second of these rows. Thus there are $\binom{6}{3} \cdot \binom{3}{2} \cdot \binom{3}{1} \cdot 2 = 360$ shadings in which the first two columns have two shaded rows in common. Next shade the second column so that only one of the shaded squares is in the same row as a shaded square in the first column. The row containing the shaded square in both columns can be selected in $\binom{3}{1}$ ways and the other two shaded squares in the second column in $\binom{3}{2}$ ways. This leaves one row with no shaded squares, so the squares in this row in the third and fourth columns must be shaded. There are four rows each with one square shaded in the first two columns. Two of these rows must be shaded in the third column, and two in the fourth. This can be done in $\binom{4}{2}$ ways. Thus there are $\binom{6}{3} \cdot \binom{3}{1} \cdot \binom{3}{2} \cdot \binom{4}{2} = 1080$ shadings in which the first two columns have one shaded row in common. Finally, if the first two columns have no shaded row in common, then the shading of the second column is uniquely determined. The three squares to be shaded in the third column can be selected in $\binom{6}{3}$ ways, and the shading for the fourth column is then uniquely determined. There are $\binom{6}{3} \cdot \binom{6}{3} = 400$ shading patterns in which the first two columns have no shaded row in common. Thus the total number of shadings is $20 + 360 + 1080 + 400 = 1860$, and the requested remainder is 860.

OR

The three shaded squares in the first column can be chosen in $\binom{6}{3} = 20$ ways. For $0 \le k \le 3$, there are $\binom{3}{k}\binom{3}{3-k}$ ways to choose the shaded squares in the second column so that k rows have two shaded squares. No shaded square in the third column can be in one of these rows. In each case there are also k rows with no shaded squares, and each of these rows must contain a shaded square in the third column. The remaining $3 - k$ shaded squares in the third column must be chosen from the remaining $6 - 2k$ rows. Thus there are $\binom{6-2k}{3-k}$ ways to choose the shaded squares in the third column. In each case, the shaded squares in the fourth column are uniquely determined. Therefore $N = 20\sum_{k=0}^{3}\binom{3}{k}\binom{3}{3-k}\binom{6-2k}{3-k} = 20(20 + 54 + 18 + 1) = 1860$, and the requested remainder is 860.

11. (Answer: 955)

First, if k is a nonnegative integer, and n is a positive integer then $\sqrt{n^2+k} < n + \frac{1}{2} \iff n^2 + k < n^2 + n + \frac{1}{4} \iff k < n + \frac{1}{4} \iff k = 0, 1, 2, \ldots, n$. Similarly, $n - \frac{1}{2} < \sqrt{n^2-k} \iff n^2 - n + \frac{1}{4} < n^2 - k \iff k < n - \frac{1}{4}$. So if k is a positive integer, the second inequality is satisfied when $k = 1, 2, 3, \ldots, n-1$. Thus a positive integer n is the value of $b(p)$ precisely when p is one of the $(n+1)+(n-1) = 2n$ integers $n^2-(n-1)$, $n^2-(n-2)$, …, n^2-1, n^2, n^2+1, …, n^2+n.

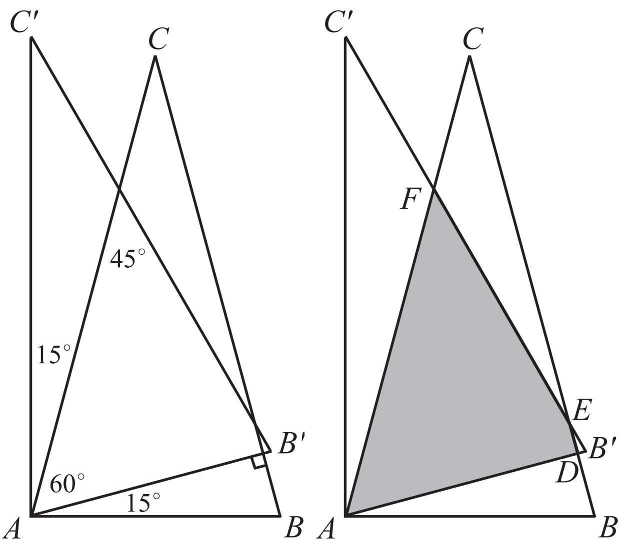
Next, observe that $44^2 = 1936 < 1936 + 44 = 1980 < 2007 < 45^2 = 2025$. Therefore each integer $n = 1, 2, 3, \ldots, 44$ contributes $n \cdot 2n = 2n^2$ to the sum. Consequently,

$$S = \sum_{p=1}^{2007} b(p) = \sum_{p=1}^{1980} b(p) + \sum_{p=1981}^{2007} b(p) = \left(\sum_{n=1}^{44} 2n^2\right) + 27 \cdot 45$$
$$= 2\left(\frac{44 \cdot 45 \cdot 89}{6}\right) + 1215 = 59955.$$

Thus the requested remainder is 955.

12. (Answer: 875)

Let $[XYZ]$ represent the area of triangle XYZ.



For future use, $\sin 75° = \cos 15° = (\sqrt{6}+\sqrt{2})/4$. Let B' and C' be the images of B and C respectively under the given rotation. Let D denote the point at which $\overline{BC}$ intersects $\overline{AB'}$, let E denote the point at which $\overline{BC}$ intersects $\overline{B'C'}$, and let F denote the point at which $\overline{AC}$ intersects $\overline{B'C'}$. Then the region common to the two triangles (shaded in the figure on the right) is $ADEF$, and its area is $[ADEF] = [AB'F] - [EB'D]$. Note that $\angle B + \angle B'AB = 75° + 15° = 90°$ implies $\overline{AB'} \perp \overline{BC}$. Because $AB' = AB = 20$, the Law of Sines applied to $\triangle B'FA$ gives $B'F = 20 \sin 60° / \sin 45° = 20\sqrt{3/2} = 10\sqrt{6}$, and thus
$[AB'F] = \frac{1}{2} \cdot 20 \cdot 10\sqrt{6} \sin 75° = 100\sqrt{6}\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right) = 50(3+\sqrt{3})$.

Note that $B'D = 20(1 - \cos 15°)$, $BD = 20 \sin 15°$, and $\triangle EB'D \sim \triangle ABD$. Because $[ABD] = \frac{1}{2} \cdot 20 \cos 15° \cdot 20 \sin 15° = 100 \sin 30° = 50$, it follows that $[EB'D] = 50(\frac{1-\cos 15^\circ}{\sin 15^\circ})^2$.

Using $\cos^2 15^\circ = \frac{1+\cos 30^\circ}{2} = \frac{2+\sqrt{3}}{4}$ and $\sin^2 15^\circ = \frac{1-\cos 30^\circ}{2} = \frac{2-\sqrt{3}}{4}$ yields

$$\begin{aligned}\left(\frac{1-\cos 15^\circ}{\sin 15^\circ}\right)^2 &= \frac{1-2\cos 15^\circ+\cos^2 15^\circ}{\sin^2 15^\circ}\\ &= \frac{1-(\sqrt{6}+\sqrt{2})/2+(2+\sqrt{3})/4}{(2-\sqrt{3})/4}\\ &= (6+\sqrt{3}-2\sqrt{6}-2\sqrt{2})(2+\sqrt{3})\\ &= 15+8\sqrt{3}-6\sqrt{6}-10\sqrt{2}.\end{aligned}$$

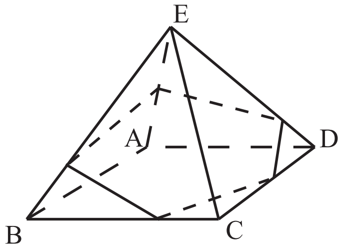
Finally,

$$\begin{aligned}[ADEF] &= [AB'F]-[EB'D]\\ &= 50(3+\sqrt{3})-50(15+8\sqrt{3}-6\sqrt{6}-10\sqrt{2})\\ &= 50(10\sqrt{2}-7\sqrt{3}+6\sqrt{6}-12),\end{aligned}$$

so $(p-q+r-s)/2 = 25(10+7+6+12) = 875$.

13. (Answer: 80)

Place the pyramid on a coordinate system with A at $(0,0,0)$, B at $(4,0,0)$, C at $(4,4,0)$, D at $(0,4,0)$ and with E at $(2,2,2\sqrt{2})$. Let R, S, and T be the midpoints of $\overline{AE}$, $\overline{BC}$, and $\overline{CD}$ respectively. The coordinates of R, S, and T are respectively $(1,1,\sqrt{2})$, $(4,2,0)$ and $(2,4,0)$. The equation of the plane containing R, S, and T is $x+y+2\sqrt{2}z=6$. Points on $\overline{BE}$ have coordinates of the form $(4-t,t,t\sqrt{2})$, and points on $\overline{DE}$ have coordinates of the form $(t,4-t,t\sqrt{2})$. Let U and V be the points of intersection of the plane with $\overline{BE}$ and $\overline{DE}$ respectively. Substituting into the equation of the plane yields $t=\dfrac{1}{2}$ and $\left(\dfrac{7}{2},\dfrac{1}{2},\dfrac{\sqrt{2}}{2}\right)$ for U, and $t=\dfrac{1}{2}$ and $\left(\dfrac{1}{2},\dfrac{7}{2},\dfrac{\sqrt{2}}{2}\right)$ for V. Then $RU=RV=\sqrt{7}$, $US=VT=\sqrt{3}$ and $ST=2\sqrt{2}$. Note also that $UV=3\sqrt{2}$. Thus the pentagon formed by the intersection of the plane and the pyramid can be partitioned into isosceles triangle RUV and isosceles trapezoid $USTV$ with areas of $3\sqrt{5}/2$ and $5\sqrt{5}/2$ respectively. Therefore the total area is $4\sqrt{5}$ or $\sqrt{80}$, and $p=80$.



14. (Answer: 224)

The fact that the equation $a_{n+1}a_{n-1} = a_n^2 + 2007$ holds for $n \geq 2$ implies that $a_n a_{n-2} = a_{n-1}^2 + 2007$ for $n \geq 3$. Subtracting the second equation from the first one yields $a_{n+1}a_{n-1} - a_n a_{n-2} = a_n^2 - a_{n-1}^2$, or $a_{n+1}a_{n-1} + a_{n-1}^2 = a_n a_{n-2} + a_n^2$. Dividing the last equation by $a_{n-1}a_n$ and simplifying produces $\frac{a_{n+1}+a_{n-1}}{a_n} = \frac{a_n+a_{n-2}}{a_{n-1}}$. This equation shows that $\frac{a_{n+1}+a_{n-1}}{a_n}$ is constant for $n \geq 2$. Because $a_3 a_1 = a_2^2 + 2007$, $a_3 = 2016/3 = 672$. Thus $\frac{a_{n+1}+a_{n-1}}{a_n} = \frac{672+3}{3} = 225$, and $a_{n+1} = 225a_n - a_{n-1}$ for $n \geq 2$. Note that $a_3 = 672 > 3 = a_2$. Furthermore, if $a_n > a_{n-1}$, then $a_{n+1}a_{n-1} = a_n^2 + 2007$ implies that

$$a_{n+1} = \frac{a_n^2}{a_{n-1}} + \frac{2007}{a_{n-1}} = a_n\left(\frac{a_n}{a_{n-1}}\right) + \frac{2007}{a_{n-1}} > a_n + \frac{2007}{a_{n-1}} > a_n.$$

Thus by mathematical induction, $a_n > a_{n-1}$ for all $n \geq 3$. Therefore the recurrence $a_{n+1} = 225a_n - a_{n-1}$ implies that $a_{n+1} > 225a_n - a_n = 224a_n$ and therefore $a_n \geq 2007$ for $n \geq 4$. Finding a_{n+1} from $a_{n+1}a_{n-1} = a_n^2 + 2007$ and substituting into $225 = \frac{a_{n+1}+a_{n-1}}{a_n}$ shows that $\frac{a_n^2+a_{n-1}^2}{a_n a_{n-1}} = 225 - \frac{2007}{a_n a_{n-1}}$. Thus the largest integer less than or equal to the original fraction is 224.

15. (Answer: 989)

In the following, let $[XYZ]$ = the area of triangle XYZ.
In general, let $AB = s$, $FA = a$, $DC = c$, $EF = x$, $FD = y$, and $AE = t$. Then $EC = s - t$. In triangle AEF, angle A is 60°, and so $[AEF] = \frac{1}{2} \cdot \sin 60^\circ \cdot AE \cdot AF = \frac{\sqrt{3}}{4}at$. Similarly, $[BDF] = \frac{\sqrt{3}}{4}(s-a)(s-c)$, $[CDE] = \frac{\sqrt{3}}{4}c(s-t)$, and $[ABC] = \frac{\sqrt{3}}{4}s^2$. It follows that

$$\begin{aligned}[DEF] &= [ABC] - [AEF] - [BDF] - [CDE] \\ &= \frac{\sqrt{3}}{4}[s^2 - at - (s-a)(s-c) - c(s-t)] \\ &= \frac{\sqrt{3}}{4}[a(s-t) + ct - ac].\end{aligned}$$

The given conditions then imply that $56 = 5(s-t) + 2t - 10$, or $5(s-t) + 2t = 66$. Because $\angle A = \angle DEF = 60^\circ$, it follows that $\angle AEF + \angle AFE = 120^\circ = \angle AEF +$

$\angle CED$, implying that $\angle AFE = \angle CED$. Note also that $\angle C = \angle A$. Thus $\triangle AEF \sim \triangle CDE$. Consequently, $\frac{AE}{AF} = \frac{CD}{CE}$, or $\frac{t}{5} = \frac{2}{s-t}$. Thus $t(s-t) = 10$ or $s - t = \frac{10}{t}$. Substituting this into the equation $5(s-t) + 2t = 66$ gives $5 \cdot 10 + 2t^2 = 66t$. Solving this quadratic equation gives $t = \frac{33 \pm \sqrt{989}}{2}$, and hence $s = t + \frac{10}{t} = \frac{231}{10} \pm \frac{3}{10}\sqrt{989}$. Repeated applications of the Law of Cosines show that both values of s produce valid triangles. Thus $r = 989$.

THE MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

25th Annual *(Alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Wednesday, March 28, 2007

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:
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The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:
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1. (Answer: 372)

   If a sequence contains no more than one 0, there are $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$ sequences formed from the characters A, I, M, E, 2, 0, and 7. If a sequence contains two 0's, the 0's can be placed in $\binom{5}{2} = 10$ ways, the remaining characters can be chosen in $\binom{6}{3} = 20$ ways, and those remaining characters can be arranged in $3! = 6$ ways, for a total of $10 \cdot 20 \cdot 6 = 1200$ sequences. Thus $N = 2520 + 1200 = 3720$, and $\frac{N}{10} = 372$.

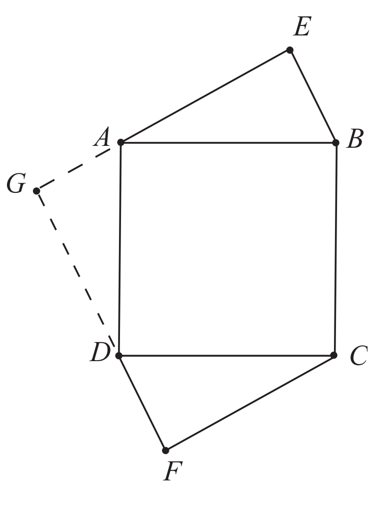
2. (Answer: 200)

   For any such ordered triple (a, b, c), because a is a factor of $b + c$, a is also a factor of 100. Thus a is an element of $\{1, 2, 4, 5, 10, 20, 25\}$, and $\dfrac{b}{a}$ and $\dfrac{c}{a}$ are positive integers for which $\dfrac{b}{a} + \dfrac{c}{a} = \dfrac{100}{a} - 1$ (Note that if $a = 50$ or 100, then at least one of b and c is zero). Because $\dfrac{b}{a}$ and $\dfrac{c}{a}$ are positive integers, there are (for each choice of a) $\dfrac{100}{a} - 2$ pairs $\dfrac{b}{a}$ and $\dfrac{c}{a}$. Thus there are

   $$\frac{100}{1} + \frac{100}{2} + \frac{100}{4} + \frac{100}{5} + \frac{100}{10} + \frac{100}{20} + \frac{100}{25} - 2 \cdot 7 = 214 - 14 = 200 \text{ such triples.}$$

3. (Answer: 578)

   Extend $\overline{AE}$ past A and $\overline{DF}$ past D to meet at G. Note that $\angle ADG = 90^\circ - \angle CDF = \angle DCF = \angle BAE$ and $\angle DAG = 90^\circ - \angle BAE = \angle ABE$. Thus $\triangle AGD \cong \triangle BEA$. Therefore $EG = FG = 17$, and because $\angle EGF$ is a right angle, $EF^2 = 2 \cdot 17^2 = 578$.



4. (Answer: 450)

The fact that 60 workers produce 240 widgets and 300 whoosits in two hours implies that 100 workers produce 400 widgets and 500 whoosits in two hours, or 200 widgets and 250 whoosits in one hour. Let a be the time required for a worker to produce a widget, and let b be the time required for a worker to produce a whoosit. Then $300a + 200b = 200a + 250b$, which is equivalent to $b = 2a$. In three hours, 50 workers produce 300 widgets and 375 whoosits, so $150a + mb = 300a + 375b$ and $150a + 2ma = 300a + 750a$. Solving the last equation yields $m = 450$.

5. (Answer: 888)

The graph passes through the points $(0, 9), (24\frac{7}{9}, 8), (49\frac{5}{9}, 7), \ldots, (198\frac{2}{9}, 1), (223, 0)$, where each decrease of 1 unit in y results in an increase of $24\frac{7}{9}$ units in x. Therefore the region can be divided into rectangles with dimensions 8×24, 7×25, 6×25, 5×25, 4×24, 3×25, 2×25, and 1×25, for a total of 888 squares.

OR

Consider the rectangle with vertices $(0, 0)$, $(0, 9)$, $(223, 0)$, and $(223, 9)$. There are 2007 1 by 1 squares within this rectangle. The diagonal from $(0, 0)$ to $(223, 9)$ crosses exactly one of these squares between $x = n$ and $x = n+1$ for most of the 223 possible values of n. There are exactly 8 values of n for which the diagonal crosses one of the horizontal lines $y = m (1 \le m \le 8)$, and for these values the diagonal crosses two squares. The diagonal never passes through any corners, because 9 and 223 are relatively prime and $9 \cdot 223 = 2007$. Thus, out of the 2007 squares, $223 + 8$ of them are crossed by the diagonal, leaving 1776 squares untouched. Half of these, or 888 of them, lie below the diagonal.

6. (Answer: 640)

Let $a_1 a_2 a_3 \cdots a_k$ be the decimal representation of a parity-monotonic integer. It is not difficult to check that for each fixed a_{i+1}, there are four choices for a_i; for example, if $a_{i+1} = 8$ or 9, then $a_i \in \{1, 3, 5, 7\}$; if $a_{i+1} = 4$ or 5, then $a_i \in \{1, 3, 6, 8\}$, and so on. There are 10 choices for the digit a_k, and 4 choices for each of the remaining digits. Hence there are $4^{k-1} \cdot 10$ k-digit parity-monotonic integers, and the number of four-digit parity-monotonic integers is $4^3 \cdot 10 = 640$.

7. (Answer: 553)

Because $k \le \sqrt[3]{n_i} < k+1$, it follows that $k^3 \le n_i < (k+1)^3 = k^3 + 3k^2 + 3k + 1$. Because k is a divisor of n_i, there are $3k + 4$ possible values for n_i, namely $k^3, k^3 + k, \ldots, k^3 + 3k^2 + 3k$. Hence $3k + 4 = 70$ and $k = 22$. The desired maximum is $\dfrac{k^3 + 3k^2 + 3k}{k} = k^2 + 3k + 3 = 553$.

8. (Answer: 896)

   Let h be the number of 4 unit line segments and v be the number of 5 unit line segments. Then $4h+5v=2007$. Each pair of adjacent 4 unit line segments and each pair of adjacent 5 unit line segments determine one basic rectangle. Thus the number of basic rectangles determined is $B=(h-1)(v-1)$. To simplify the work, make the substitutions $x=h-1$ and $y=v-1$. The problem is now to maximize $B=xy$ subject to $4x+5y=1998$, where x, y are integers. Solve the second equation for y to obtain

$$y=\frac{1998}{5}-\frac{4}{5}x,$$

   and substitute into $B=xy$ to obtain

$$B=x\left(\frac{1998}{5}-\frac{4}{5}x\right).$$

   The graph of this equation is a parabola with x intercepts 0 and 999/2. The vertex of the parabola is halfway between the intercepts, at $x=999/4$. This is the point at which B assumes its maximum. However, this corresponds to a nonintegral value of x (and hence h). From $4x+5y=1998$ both x and y are integers if and only if $x\equiv 2 \pmod 5$. The nearest such integer to $999/4=249.75$ is $x=252$. Then $y=198$, and this gives the maximal value for B for which both x and y are integers. This maximal value for B is $252\cdot 198=49896$, and the requested remainder is 896.

9. (Answer: 259)

   Let G and H be the points where the inscribed circle of triangle BEF is tangent to $\overline{BE}$ and $\overline{BF}$, respectively. Let $x=EP$, $y=BG$, and $z=FH$. Then by equal tangents, $EG=x$, $BH=y$, and $FP=z$. Note that $y+z=BF=BC-CF=364$, and $x+y=\sqrt{63^2+84^2}=\sqrt{21^2(3^2+4^2)}=105$. Also note that $\triangle BEF\cong\triangle DFE$, so $FQ=x$. Thus $PQ=FP-FQ=z-x=(y+z)-(x+y)=364-105=259$.

OR

   Let x, y, and z be defined as in the first solution, and let O be the foot of the perpendicular from E to $\overline{BF}$. Applying the Pythagorean Theorem to triangle EOF yields $EF=\sqrt{EO^2+FO^2}=\sqrt{63^2+(448-2\cdot 84)^2}=\sqrt{7^2(9^2+40^2)}=7\cdot 41=287$. Thus $z+x=EF=287$, and $x+y=105$ and $y+z=364$, as shown in the first solution. Adding these three equations together and dividing by 2 yields $x+y+z=378$. Thus $x=378-364=14$, $y=378-287=91$, and $z=378-105=273$. Therefore $PQ=z-x=273-14=259$.

10. (Answer: 710)

There are 2^6 subsets of S, and for $0 \le k \le 6$, there are $\binom{6}{k}$ subsets with k elements, so the probability that A has k elements is $\binom{6}{k}/2^6$. If A has k elements, there are 2^k subsets of S contained in A and 2^{6-k} subsets contained in $S-A$. One of these, the empty set, is contained in both A and $S-A$, so the probability that B is contained in either A or $S-A$ is $\dfrac{2^k+2^{6-k}-1}{2^6}$. The requested probability is therefore

$$\sum_{k=0}^{6} \frac{\binom{6}{k}}{2^6} \cdot \frac{2^k+2^{6-k}-1}{2^6} = \frac{1}{2^{12}}\left(\sum_{k=0}^{6} 2^k \binom{6}{k} + \sum_{k=0}^{6} 2^{6-k}\binom{6}{k} - \sum_{k=0}^{6}\binom{6}{k}\right)$$

$$= \frac{1}{2^{12}}\left(2\sum_{k=0}^{6} 2^k\binom{6}{k} - \sum_{k=0}^{6}\binom{6}{k}\right) = \frac{1}{2^{12}}\left(2\cdot 3^6 - 2^6\right) = \frac{3^6-2^5}{2^{11}} = \frac{697}{2^{11}}.$$

Thus $m+n+r = 697+2+11 = 710$.

OR

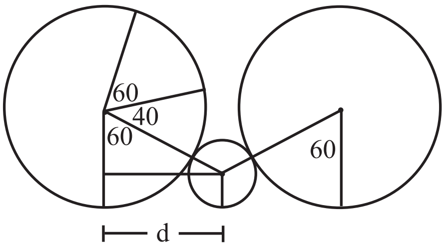
Let $S = \{1,2,3,4,5,6\}$. With $S_1 = A$ and $S_2 = B$, let M be the 6×2 matrix in which $m_{ij} = 1$ if $i \in S_j$ and 0 otherwise. There are 2^{12} such matrices. Observe that $B \subseteq A$ precisely when each row of M is 11, 10, or 00. There are 3^6 such matrices. Similarly, $B \subseteq S-A$ precisely when each row of M is 01, 00, or 10, and again there are 3^6 such matrices. The intersection of these two types of matrices are those in which each row is 00 or 10, and there are 2^6 such matrices. The requested probability is therefore

$$\frac{3^6+3^6-2^6}{2^{12}} = \frac{2\cdot 3^6-2^6}{2^{12}} = \frac{3^6-2^5}{2^{11}} = \frac{697}{2^{11}}, \text{ as before.}$$

11. (Answer: 179)

Let the larger tube roll until it is tangent to the smaller tube as shown in the diagram. At that point, the centers of the tubes are a horizontal distance d apart, where d is one leg of a right triangle with hypotenuse $72+24=96$ and other leg $72-24=48$. It follows that the triangle is a 30 - 60 - 90 triangle with a $30°$ angle at the center of the smaller tube, d is equal to $48\sqrt{3}$, and the larger tube rests on a point of its circumference $60°$ from the point of its circumference where it is tangent to the smaller tube.

When the larger tube finishes rolling over the smaller tube, it is tangent to the smaller tube on the other side of the smaller tube. Its center has moved a horizontal distance of $2d = 96\sqrt{3}$. It has rolled over an arc of $180° - 2(30°) = 120°$ of the smaller tube, and thus it has rolled over an arc of one-third of $120°$, or $40°$ of the larger tube. The point of its circumference where the larger tube rests after rolling over the smaller tube is $60° + 40° + 60° = 160°$ from the point where it rested before rolling over the smaller tube. Thus the larger tube has rolled over an arc of $160°$ while moving horizontally a distance of $96\sqrt{3}$. When the larger tube completes one revolution, it has rolled horizontally by rolling through $360° - 160° = 200°$ of arc and moving a distance of $96\sqrt{3}$. Therefore the total horizontal distance covered is $\frac{200}{360} \cdot 72 \cdot 2\pi + 96\sqrt{3} = 80\pi + 96\sqrt{3}$. Thus $a + b + c = 80 + 96 + 3 = 179$.



12. (Answer: 091)

The sequence is geometric, so there exist numbers a and r such that $x_n = ar^n$. It follows that

$$308 = \sum_{n=0}^{7} \log_3(x_n) = \sum_{n=0}^{7} \log_3(ar^n) = \sum_{n=0}^{7} [\log_3(a) + n\log_3(r)] =$$

$$8\log_3(a) + \left(\sum_{n=0}^{7} n\right)\log_3(r) = 8\log_3(a) + 28\log_3(r).$$

Thus $2\log_3(a) + 7\log_3(r) = 77$. Furthermore,

$$\log_3\left(\sum_{n=0}^{7} x_n\right) = \log_3\left(\sum_{n=0}^{7} ar^n\right) = \log_3\left(a \cdot \frac{r^8 - 1}{r - 1}\right) = \log_3\left(ar^7 \cdot \frac{1 - \frac{1}{r^8}}{1 - \frac{1}{r}}\right) =$$

$$\log_3(a) + 7\log_3(r) + \log_3\left(\frac{1 - \frac{1}{r^8}}{1 - \frac{1}{r}}\right),$$

which is between 56 and 57.

Because the terms are all integral powers of 3, it follows that a and r must be powers of 3. Also, the sequence is increasing, so r is at least 3. Therefore

$$1 = \frac{1 - \frac{1}{r}}{1 - \frac{1}{r}} < \frac{1 - \frac{1}{r^8}}{1 - \frac{1}{r}} < \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} < 3, \text{ and } 0 < \log_3\left(\frac{1 - \frac{1}{r^8}}{1 - \frac{1}{r}}\right) < 1.$$

Also note that since a and r are powers of 3, $\log_3(a) + 7\log_3(r)$ is an integer and therefore must equal 56. Thus $\log_3(a) + 7\log_3(r) = 56$. The two equations $\log_3(a)+7\log_3(r) = 56$ and $2\log_3(a)+7\log_3(r) = 77$ have the solution $\log_3(a) = 21$ and $\log_3(r) = 5$.

It follows that $\log_3(x_{14}) = \log_3(ar^{14}) = \log_3(a) + 14\log_3(r) = 21 + 14 \cdot 5 = 91$.

13. (Answer: 640)

Number the rows from bottom to top, with the bottom row numbered 0 and the top row numbered 10. Let the entries in row 0, from left to right, be $x_0, x_1, \ldots, x_{10}$, where each x_k is 0 or 1. In row k (the kth row from the bottom) label the squares from left to right by $0, 1, \ldots, 10-k$. It can then be shown by induction that the number in row k and square j, $0 \le j \le 10-k$, is

$$\binom{k}{0}x_{j+0} + \binom{k}{1}x_{j+1} + \cdots + \binom{k}{k}x_{j+k} = \sum_{i=0}^{k}\binom{k}{i}x_{j+i}.$$

Thus the entry in the top square (in row 10) is

$$\sum_{i=0}^{10}\binom{10}{i}x_i.$$

It is easy to check that $\binom{10}{k}$ is a multiple of 3 for $2 \le k \le 8$. Thus

$$\sum_{i=0}^{10}\binom{10}{i}x_i \equiv x_0 + \binom{10}{1}x_1 + \binom{10}{9}x_9 + x_{10} \equiv x_0 + x_1 + x_9 + x_{10} \pmod{3}.$$

For this last expression to be a multiple of 3, either $x_0 = x_1 = x_9 = x_{10} = 0$ or three of these four numbers are 1 and the fourth is 0. Thus there are five choices of x_0, x_1, x_9, x_{10} that make the sum a multiple of 3. Furthermore, each of $x_2, x_3, x_4, \ldots, x_8$ can be either 0 or 1, so these 7 values can be assigned in 2^7 ways. Thus there are $5 \cdot 2^7 = 640$ initial distributions that result in the number in the top square being a multiple of 3.

14. (Answer: 676)

If the leading term of $f(x)$ is ax^m, then the leading term of $f(x)f(2x^2) = ax^m \cdot a(2x^2)^m = 2^m a^2 x^{3m}$, and the leading term of $f(2x^3 + x) = 2^m a x^{3m}$. Hence $2^m a^2 = 2^m a$, and $a = 1$. Because $f(0) = 1$, the product of all the roots of $f(x)$ is ± 1. If $f(\lambda) = 0$, then $f(2\lambda^3 + \lambda) = 0$. Assume that there exists a root λ with $|\lambda| \ne 1$. Then there must be such a root λ_1 with $|\lambda_1| > 1$. Then $|2\lambda^3 + \lambda| \ge 2|\lambda|^3 - |\lambda| > 2|\lambda| - |\lambda| = |\lambda|$. But then $f(x)$ would have infinitely many roots, given by $\lambda_{k+1} = 2\lambda_k^3 + \lambda_k$, for $k \ge 1$. Therefore $|\lambda| = 1$ for all of the roots of the polynomial. Thus $\lambda\overline{\lambda} = 1$, and $(2\lambda^3 + \lambda)\overline{(2\lambda^3 + \lambda)} = 1$. Solving these equations simultaneously for $\lambda = a + bi$ yields $a = 0$, $b^2 = 1$, and so $\lambda^2 = -1$. Because the polynomial has real coefficients, the polynomial must have the form $f(x) = (1 + x^2)^n$ for some integer $n \ge 1$. The condition $f(2) + f(3) = 125$ implies $n = 2$, giving $f(5) = 676$.

15. (Answer: 389)

Let O_A, O_B, and O_C be the centers of ω_A, ω_B, and ω_C, respectively. Then $\overline{O_AO_B} \parallel \overline{AB}$, $\overline{O_BO_C} \parallel \overline{BC}$, and $\overline{O_CO_A} \parallel \overline{CA}$. Also, the lines AO_A, BO_B, and CO_C are concurrent at I, the incenter of triangle ABC, and therefore there is a dilation $\mathcal{D}$ centered at I that sends triangle $O_AO_BO_C$ to triangle ABC. Let R and r be the circumradius and inradius of triangle ABC, respectively, and let R_1 and r_1 be the circumradius and inradius of triangle $O_AO_BO_C$, respectively. Then $\dfrac{R}{r} = \dfrac{R_1}{r_1}$. By Heron's formula,

$$\begin{aligned}[ABC] &= \frac{\sqrt{(13+14+15)(13+14-15)(14+15-13)(15+13-14)}}{4} \\ &= \frac{r(13+14+15)}{2} = \frac{13 \cdot 14 \cdot 15}{4R},\end{aligned}$$

implying that $r = 4$ and $R = \frac{65}{8}$. Let x be the radius of ω. Because I is the center of $\mathcal{D}$, $r_1 = r - x$. Let S be the center of ω. Then S is equidistant from O_A, O_B, and O_C, that is, S is the circumcenter of triangle $O_AO_BO_C$. Thus $R_1 = SO_A = 2x$. Therefore

$$\frac{2x}{4-x} = \frac{2x}{r-x} = \frac{R_1}{r_1} = \frac{R}{r} = \frac{65}{32}.$$

Solving the last equation gives $x = \dfrac{260}{129}$, and $m + n = 389$.

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THE MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

26th Annual

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Tuesday, March 18, 2008

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:

American Mathematics Competitions
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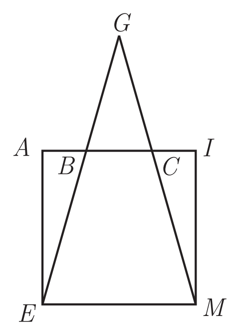
The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

Steve Blasberg, AIME Chair
San Jose, CA 95129 USA

1. (Answer: 252)

   Suppose at the beginning there are x students at the party. Then $0.6x$ students are girls, and $0.4x$ students like to dance. When the 20 boys join the party, there are $0.58(x+20)$ girls at the party, which must equal the original number of girls at the party. Thus $0.58(x+20) = 0.6x$, which implies $0.58 \cdot 20 = (0.6 - 0.58)x = 0.02x$, and $x = 580$. Thus the current number of students who like to dance is $0.40 \cdot 580 + 20 = 252$.

2. (Answer: 025)



   Let h be the required altitude, and let B and C be the points of intersection of $\overline{AI}$ with $\overline{GE}$ and $\overline{GM}$, respectively. Then the fact that $\triangle GEM$ is similar to $\triangle GBC$ implies that $\frac{h-10}{h} = \frac{BC}{10}$. Thus $BC = \frac{10h-100}{h}$, and the area common to GEM and $AIME$ equals $\frac{1}{2} \cdot 10h - \frac{1}{2}(\frac{10h-100}{h})(h-10) = 80$. This equation reduces to $5h^2 - 5(h-10)^2 = 80h$, or $100h - 500 = 80h$. The required length is then 25.

3. (Answer: 314)

   Let $b =$ Ed and Sue's biking rate, let $j =$ their jogging rate, and let $s =$ their swimming rate. Then

$$2b + 3j + 4s = 74$$
$$4b + 2j + 3s = 91.$$

   Adding -2 times the first equation to the second equation and solving for j yields $j = \frac{57}{4} - \frac{5}{4}s$. The only ordered pairs (s, j) that satisfy this equation in which both s and j are positive integers are $(1, 13)$, $(5, 8)$, and $(9, 3)$. However, subtracting the first equation from the second equation and solving for b yields $b = \frac{s+j+17}{2}$, and only the ordered pair $(5, 8)$ produces an integer value (of 15) for b. Thus the sum of the squares of Ed's rates is $15^2 + 8^2 + 5^2 = 314$.

4. (Answer: 080)

Observe that if x is a positive integer, then

$$(x+42)^2 = x^2+84x+1764 < x^2+84x+2008 < x^2+90x+2025 = (x+45)^2.$$

Because $x^2 + 84x + 2008$ is a square, it is either $(x + 43)^2$ or $(x + 44)^2$. In the first case, $2x = 159$, which is impossible, and in the second case, $4x = 72$, which implies $x = 18$. In that case, $y = x + 44 = 62$, and $x + y = 18 + 62 = 80$.

OR

Complete the square in x to find that $x^2 + 84x + 1764 = (x + 42)^2 = y^2 - 244$. Letting $v = x + 42$, the condition $y^2 - v^2 = 244$ is equivalent to $(y - v)(y + v) = 2 \cdot 2 \cdot 61$, whose only positive integer solutions are given by $y - v = 2$ and $y + v = 2 \cdot 61$. Thus the only two perfect squares that differ by 244 are 60^2 and 62^2. Hence $x = 60 - 42 = 18$, and $y = 62$.

5. (Answer: 014)

The slant height of the cone is $\sqrt{r^2 + h^2}$. This is the radius of the circle described by the cone as it rolls on the table. The circumference of this circle is $2\pi\sqrt{r^2 + h^2}$. The circumference of the base of the cone is $2\pi r$. The cone makes 17 complete rotations in rolling back to its original position, and so

$$17 = \frac{2\pi\sqrt{r^2 + h^2}}{2\pi r} = \sqrt{1 + \left(\frac{h}{r}\right)^2}.$$

Thus

$$\frac{h}{r} = \sqrt{17^2 - 1} = 12\sqrt{2},$$

and $m + n = 14$.

6. (Answer: 017)

Note that row r is an arithmetic progression with common difference 2^r. It can be shown by induction that the first element of row r is $r(2^{r-1})$, and thus the nth element of row r is $r(2^{r-1})+(n-1)2^r = 2^{r-1}(r+2n-2)$, for $1 \le n \le 51 - r$. For the element $2^{r-1}(r + 2n - 2)$ to be a multiple of 67, it must be true that $r + 2n - 2$ is a multiple of 67, or that $2n \equiv 2 - r \pmod{67}$. Multiplying both sides of this congruence by 34 yields $68n \equiv 68 - 34r \pmod{67}$, and the fact that $68 \equiv 1 \pmod{67}$ and $-34 \equiv 33 \pmod{67}$ produces $n \equiv 1 + 33r \pmod{67}$. Now in odd-numbered rows, the equations $r = 2k - 1$ and $n \equiv 1 + 33r \equiv 66k - 32 \equiv 35 - k \pmod{67}$ yield $(k, r, n) = (1, 1, 34), (2, 3, 33), (3, 5, 32), \ldots, (17, 33, 18)$. Note that $k > 17$ does not produce any solutions, because row $2k - 1$ has fewer than $35 - k$ entries for $k > 17$. In even-numbered rows, the equations

$r = 2k$ and $n \equiv 1 + 33r \equiv 66k + 1 \equiv 68 - k \pmod{67}$ yield $(k, r, n) = (1, 2, 67), (2, 4, 66), \ldots, (25, 50, 43)$, none of which correspond to entries in the array, so there are no solutions in the even-numbered rows. Thus there are 17 multiples of 67 in the array.

OR

Create a reduced array by dividing each new row by the largest common factor of the elements in the row. The first few rows of this reduced array will be

$$\begin{array}{ccccccccccccccccccc}
1 & & 3 & & 5 & & 7 & & \ldots & & 93 & & 95 & & 97 & & 99 \\
& 1 & & 2 & & 3 & & 4 & \ldots & 46 & & 47 & & 48 & & 49 & \\
& & 3 & & 5 & & 7 & & \ldots & & 93 & & 95 & & 97 & & \\
& & & 2 & & 3 & & 4 & \ldots & 46 & & 47 & & 48 & & & \\
& & & & 5 & & 7 & & \ldots & & 93 & & 95. & & & &
\end{array}$$

Note that in row n, if $n = 2k + 1$, then the elements of row $n + 1$ will have the form $[(2i + 1) + (2i + 3)]/4 = i + 1$. If $n = 2k$, then the elements of row $n + 1$ will have the form $[i + (i + 1)] = 2i + 1$. Thus each odd-numbered row consists of the consecutive odd integers from n to $100 - n$, and each even-numbered row consists of the consecutive integers from $n/2$ to $50 - n/2$. Furthermore, because 67 is not divisible by 4, the entries in the reduced array which are multiples of 67 correspond exactly to the entries in the original array which are multiples of 67. The additive definition of the array combined with the distributive property of multiplication ensures this. Because all elements of the reduced array are between 1 and 99, the only elements of the array divisible by 67 are those equal to 67. All elements of the even-numbered row n are at most $50 - n/2$, so even-numbered rows contain no multiples of 67, and elements of the odd-numbered row n are at most $100 - n$, so the odd-numbered rows up to 33 contain exactly one 67. Thus the reduced array contains 17 entries equal to 67, and the original array contains 17 multiples of 67.

7. (Answer: 708)

Note that $1^2 = 1 \in S_0$. All sets S_i contain a perfect square unless i is so large that, for some integer a, $a^2 \in S_j$ with $j < i$, and $(a + 1)^2 \in S_k$ with $k > i$. This implies that $(a + 1)^2 - a^2 > 100$, and hence $a \geq 50$. But $50^2 = 2500$, which is a member of S_{25}, so all sets $S_0, S_1, S_2, \ldots, S_{25}$ contain at least one perfect square. Furthermore, for all $i > 25$, each set S_i which contains a perfect square can only contain one such square. The largest value in any of the given set is 99999, and $316^2 < 99999 < 317^2$. Disregarding the first 50 squares dealt with above, there are $316 - 50 = 266$ perfect squares, each the sole perfect square member of one of the 974 sets

$S_{26}, S_{27}, S_{28}, \ldots, S_{999}$. Thus there are $974 - 266$ sets without a perfect square, and the answer is 708.

8. (Answer: 047)

   Applying the addition formula for tangent to $\tan(\arctan x + \arctan y)$ results in the formula $\arctan x + \arctan y = \arctan \frac{x+y}{1-xy}$, which is valid for $0 < xy < 1$. Thus $\arctan \frac{1}{3} + \arctan \frac{1}{4} + \arctan \frac{1}{5} = \arctan \frac{7}{11} + \arctan \frac{1}{5} = \arctan \frac{23}{24}$. The left-hand side of the original equation is therefore equivalent to $\arctan \frac{23n+24}{24n-23}$. Because this must equal $\arctan 1$, it follows that $n = 47$.

9. (Answer: 190)

   Because each crate has three different possible heights, there are 3^{10} equally likely ways to stack the ten crates. Suppose a stack of crates is 41 ft tall. Let x, y, and z be the number of crates with heights 3, 4, and 6 ft, respectively. Then $x+y+z = 10$ and $3x+4y+6z = 41$. This system is equivalent to the system $x - 2z = -1$ and $y + 3z = 11$, or $x = 2z - 1$, $y = 11 - 3z$. Because $x \geq 0$ and $y \geq 0$, $z \in \{1, 2, 3\}$, with each value of z yielding a different solution (x, y, z):

   $$(1, 8, 1), (3, 5, 2), (5, 2, 3).$$

   For each ordered triple (x, y, z) there are $\frac{10!}{x!y!z!}$ equally likely ways to stack the crates. Thus the number of ways to stack the crates is $\frac{10!}{1!8!1!} + \frac{10!}{3!5!2!} + \frac{10!}{5!2!3!} = 5130$. The desired probability is $\frac{5130}{3^{10}} = \frac{190}{3^7}$, so the required value of m is 190.

   Note: Another way to find the ordered triples which solve the two given equations is to notice that a stack of 10 crates must be at least 30 ft tall. Each y crate adds 1 additional foot to the height, and each z crate adds 3 additional feet. The stack requires 11 more feet than the minimum of 30 feet, so $y + 3z = 11$.

10. (Answer: 032)

    Because $30\sqrt{7} = DE \leq DA + AE = DA + 10\sqrt{7}$, it follows that $DA \geq 20\sqrt{7}$. Let $\theta = \angle DCA$. Applying the Law of Sines to $\triangle DCA$ yields $\frac{10\sqrt{21}}{\sin(\pi/3)} = \frac{DA}{\sin\theta} \geq \frac{20\sqrt{7}}{\sin\theta}$, which implies $\sin\theta \geq 1$. Then θ must be $\frac{\pi}{2}$, $DA = 20\sqrt{7}$, and point E lies on the extension of side $\overline{DA}$. Applying the Pythagorean Theorem to $\triangle DCA$ yields $DC = \sqrt{DA^2 - CA^2} = 10\sqrt{7}$. Then $DF = DC \cdot \cos\frac{\pi}{3} = 5\sqrt{7}$. Therefore $EF = DE - DF = 30\sqrt{7} - 5\sqrt{7} = 25\sqrt{7}$, and $m + n = 32$.

11. (Answer: 172)

    Let a_n and b_n be the number of permissible sequences of length n beginning with A and B, respectively, and let $x_n = a_n + b_n$ be the total number of permissible sequences of length n. Any permissible sequence of length $n+2$ that begins with A must begin with AA followed by a permissible sequence of length n, so that $a_{n+2} = x_n$ for $n \geq 1$. Any permissible sequence of length $n+2$ that begins with B must begin with a single B followed by a permissible sequence of length $n+1$ beginning with A, or else it must begin with BB followed by a permissible sequence of length n beginning with B. Thus $b_{n+2} = a_{n+1} + b_n$ for $n \geq 1$, and hence $b_{n+2} = a_{n+1} + x_n - a_n = x_n + x_{n-1} - x_{n-2}$. Because $b_{n+2} = a_{n+1} + b_n$, it follows that

    $$x_n + x_{n-1} - x_{n-2} = x_{n-1} + (x_{n-2} + x_{n-3} - x_{n-4}),$$

    and so $x_n = 2x_{n-2} + x_{n-3} - x_{n-4}$ for $n \geq 5$. Note that the permissible sequences of length at most 4 are B, AA, AAB, BAA, BBB, $AAAA$, and $BAAB$. Thus $x_1 = x_2 = 1$, $x_3 = 3$, and $x_4 = 2$. Applying these results to the recursion given above produces the sequence 1, 1, 3, 2, 6, 6, 11, 16, 22, 37, 49, 80, 113, and 172, and the required value is 172.

12. (Answer: 375)

    Let s be the speed of the cars in kilometers per hour. Then the number of car lengths between consecutive cars is $\lceil s/15 \rceil$(the least integer $\geq \frac{s}{15}$), so the distance between consecutive cars is $4\lceil s/15 \rceil$ meters. Thus the distance from the front of one car to the front of the next is $d = 4\lceil s/15 \rceil + 4$ meters. In one hour each car travels $1000s$ meters. Let the interval from the front of one car to the front of the next be called a *gap*. Then the number N of gaps that pass the eye in one hour is

    $$\frac{1000s}{d} = \frac{1000s}{4\lceil s/15 \rceil + 4} = \frac{250s}{\lceil s/15 \rceil + 1}.$$

    Let $\lceil s/15 \rceil = \frac{s}{15} + \varepsilon$, where $0 \leq \varepsilon < 1$. Then

    $$N = \frac{250s}{\frac{s}{15} + \varepsilon + 1} = \frac{3750}{1 + \frac{15\varepsilon + 15}{s}}.$$

    The quantity $\dfrac{15\varepsilon + 15}{s}$ is positive and approaches zero as s increases. Thus $N \leq 3750$, and by taking s sufficiently large, N can be made as close to 3750 as desired. Therefore, at a high enough speed, 3749.9 gaps will pass the eye in one hour. Assuming that at the start of the hour the eye is exactly even with the front of a car, it will be passed by one car in each of the ensuing 3749 gaps, and by one additional car at the beginning of the last 0.9 of a gap. Thus the eye will be passed by 3750 cars. The quotient when 3750 is divided by 10 is 375.

Note: With a little more calculation, it can be shown that the minimum speed for which 3750 cars pass the eye is $s = 56250$ kph. Do not try this at home!

13. (Answer: 040)

Applying the first five conditions in order yields $a_0 = 0$, and then $a_1 + a_3 + a_6 = 0, a_1 - a_3 + a_6 = 0, a_2 + a_5 + a_9 = 0, a_2 - a_5 + a_9 = 0$, which imply that $a_3 = a_5 = 0$ and $a_6 = -a_1, a_9 = -a_2$. Thus $p(x, y) = a_1(x - x^3) + a_2(y - y^3) + a_4xy + a_7x^2y + a_8xy^2$. Similarly, the next two conditions imply that $a_8 = 0$ and $a_7 = -a_4$, so that $p(x, y) = a_1(x - x^3) + a_2(y - y^3) + a_4(xy - x^2y)$. The last condition implies that $-6a_1 - 6a_2 - 4a_4 = 0$, so that $a_4 = -\frac{3}{2}(a_1 + a_2)$. Thus $p(x, y) = a_1(x - x^3 - \frac{3}{2}(xy - x^2y)) + a_2(y - y^3 - \frac{3}{2}(xy - x^2y))$. If $p(r, s) = 0$ for every such polynomial, then

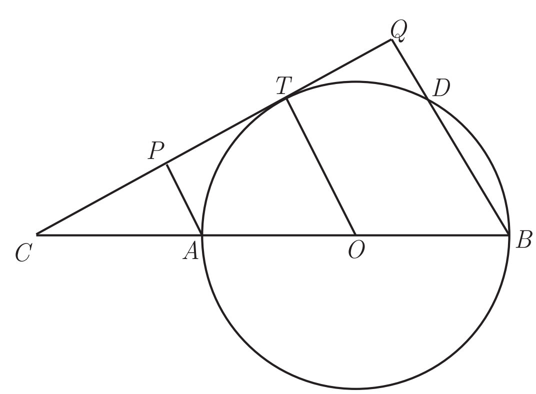
$$0 = r - r^3 - \frac{3}{2}(rs - r^2s) = \frac{1}{2}r(r - 1)(3s - 2r - 2), \text{ and}$$
$$0 = s - s^3 - \frac{3}{2}(rs - r^2s) = \frac{1}{2}s(2 - 2s^2 - 3r + 3r^2).$$

The solutions to the first equation are $r = 0$ or 1, or $s = \frac{2}{3}(r + 1)$. Substituting $r = 0$ or 1 into the second equation implies that $s = 0$, 1, or -1. If $s = 0$ and $3s - 2r - 2 = 0$, then $r = -1$. These solutions represent the first seven points. Finally, if $s = 0$ and $s = \frac{2}{3}(r + 1)$, the second equation reduces to

$$0 = 2 - 2s^2 - 3r + 3r^2 = 2 - 2 \cdot \frac{4}{9}(r^2 + 2r + 1) - 3r + 3r^2 = \frac{10 - 43r + 19r^2}{9}.$$

Because $10 - 43r + 19r^2 = (2 - r)(5 - 19r)$, it follows that $r = 2$ or $r = \frac{5}{19}$. In the first case $s = 2$, which produces the point $(2, 2)$. The second case yields $s = \frac{16}{19}$. Thus $a + b + c = 5 + 16 + 19 = 40$.

14. (Answer: 432)



Let O be the center of the circle, and let Q be the foot of the perpendicular from B to line CT. Segment BQ meets the circle again (other than at B) at D. Since $\overline{AB}$ is a diameter, $\angle ADB = 90^\circ$ and $ADQP$ is a rectangle. Then

$$\begin{aligned} BP^2 &= PQ^2 + BQ^2 = AD^2 + BQ^2 = AB^2 - BD^2 + BQ^2 \\ &= AB^2 + (BQ - BD)(BQ + BD) = AB^2 + DQ(BQ + BD). \end{aligned}$$

Note that $ABQP$ is a trapezoid with $\overline{OT}$ as its midline. Hence $AB = 2OT = AP + BQ = DQ + BQ$. Consequently

$$\begin{aligned} BQ + BD &= BQ + BQ - DQ = 2BQ - DQ = 2(BQ + DQ) - 3DQ \\ &= 2AB - 3DQ. \end{aligned}$$

Combining the above shows that

$$BP^2 = AB^2 + DQ(2AB - 3DQ) = AB^2 + \frac{4 \cdot (3DQ)(2AB - 3DQ)}{12}.$$

For real numbers x and y, $4xy \le (x + y)^2$. Hence

$$BP^2 \le AB^2 + \frac{(3DQ + 2AB - 3DQ)^2}{12} = \frac{4AB^2}{3} = \frac{1296}{3}, \text{ which equals } 432.$$

This maximum may be obtained by setting $3DQ = 2AB - 3DQ$, or $3PA = 3DQ = AB = 2OT$. It follows that $\frac{AC}{CO} = \frac{PA}{OT} = \frac{2}{3}$, implying that $CA = AB$.

OR

Let $\angle ABQ = \alpha$. Let R be on $\overline{BQ}$ such that $\overline{OR} \perp \overline{BQ}$. Let r be the radius of the circle. To maximize BP^2, observe that $BP^2 = PQ^2 + BQ^2$, and from $\triangle BRO$ it follows that $OR = r \sin\alpha$ and $BR = r\cos\alpha$. Thus $PQ = 2r\sin\alpha$, and $BQ = r + r\cos\alpha$. Then

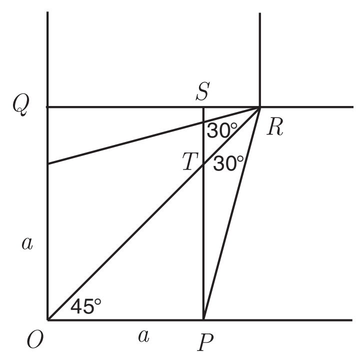
$$\begin{aligned} PB^2 &= (2r\sin\alpha)^2 + (r + r\cos\alpha)^2 \\ &= r^2 + r^2(4\sin^2\alpha + 1 + 2\cos\alpha + \cos^2\alpha) \\ &= r^2(5 - 3\cos^2\alpha + 2\cos\alpha) \\ &= -3r^2\left(-\frac{5}{3} + \cos^2\alpha - \frac{2}{3}\cos\alpha\right) \\ &= -3r^2\left(\left(\cos\alpha - \frac{1}{3}\right)^2 - \frac{16}{9}\right). \end{aligned}$$

The maximum for PB^2 is thus attained when $\cos\alpha = \frac{1}{3}$ and has the value $9^2(16/3) = 432$.

Query: Segment BP reaches its maximum length when lines AD, BP, and OT are concurrent. Why?

15. (Answer: 871)

Place the paper in the Cartesian plane with a corner at the origin and two of the sides on the positive coordinate axes. Label the origin as O. Let P be the point on the x-axis from which the cut starts, R be the point of intersection of the two cuts made near this corner, Q the foot of the perpendicular from R to the y-axis, S the foot of the perpendicular from P to $\overline{QR}$, and T the point of intersection of $\overline{PS}$ and $\overline{OR}$ (see the accompanying figure).



Let $OP = a = \sqrt{17}$. When the left and bottom edges of the paper are folded up to create two sides of the tray, the cut edges will meet, by symmetry, above the line $y = x$, that is, above segment OR. As the bottom edge folds up, point P traces a circular arc, with the arc centered at S and of radius SP. When the cut edges of the two sides meet, P will be at the point P' directly above T. Because $\overline{P'T}$ is perpendicular to the xy-plane, $P'T$ is the height of the tray.
Observe that $\angle ROP = 45°$. Applying the Law of Sines in $\triangle OPR$ yields

$$PR = \sin 45° \frac{OP}{\sin 30°} = a\sqrt{2}.$$

Then in right triangle PRS,

$$\begin{aligned} SP &= PR \sin 75° = PR \sin(45° + 30°) \\ &= PR(\sin 45° \cos 30° + \sin 30° \cos 45°) \\ &= a\left(\frac{\sqrt{3}+1}{2}\right). \end{aligned}$$

It then follows that

$$ST = SP - PT = a\left(\frac{\sqrt{3}+1}{2}\right) - a = a\left(\frac{\sqrt{3}-1}{2}\right).$$

Next apply the Pythagorean Theorem to right triangle $P'ST$ to find that

$$P'T^2 = P'S^2 - ST^2 = PS^2 - ST^2 = a^2\left(\frac{\sqrt{3}+1}{2}\right)^2 - a^2\left(\frac{\sqrt{3}-1}{2}\right)^2 = a^2\sqrt{3},$$

so $P'T = a\sqrt[4]{3}$. Hence the height of the tray is $\sqrt{17}\sqrt[4]{3} = \sqrt[4]{867}$, and so $m + n = 4 + 867 = 871$.

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The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

Steve Blasberg, AIME Chair
San Jose, CA 95129 USA

1. (Answer: 100)

   Reordering the sum shows that

   $N = (100^2 - 98^2) + (99^2 - 97^2) + (96^2 - 94^2) + \cdots + (4^2 - 2^2) + (3^2 - 1^2),$

   which equals

$$\begin{aligned}
&2 \cdot 198 + 2 \cdot 196 + 2 \cdot 190 + 2 \cdot 188 + \cdots + 2 \cdot 6 + 2 \cdot 4 \\
&= 4(99 + 98 + 95 + 94 + \cdots + 3 + 2) \\
&= 4(99 + 95 + 91 + \cdots + 3) + 4(98 + 94 + 90 + \cdots + 2) \\
&= 4\left[\frac{25(99+3)}{2}\right] + 4\left[\frac{25(98+2)}{2}\right] \\
&= 100 \cdot (51 + 50) \\
&= 10100,
\end{aligned}$$

   and the required remainder is 100.

2. (Answer: 620)

   Suppose Rudolph bikes at r miles per minute. He takes 49 five-minute breaks in reaching the 50-mile mark, so his total time in minutes is $\frac{50}{r} + 49 \cdot 5 = \frac{50}{r} + 245$. Jennifer bikes at $\frac{3}{4}r$ miles per minute and takes 24 five-minute breaks in reaching the 50-mile mark, so her total time in minutes is $\frac{50}{0.75r} + 24 \cdot 5 = \frac{200}{3r} + 120$. Setting these two times equal gives $\frac{50}{3r} = 125$, and hence $r = \frac{2}{15}$. This yields a total time of $\frac{50}{2/15} + 245 = 620$ minutes.

3. (Answer: 729)

   Let a, b, and c be the dimensions of the cheese after ten slices have been cut off, giving a volume of abc cu cm. Because each slice shortens one of the dimensions of the cheese by 1 cm, $a+b+c = (10+13+14) - 10 = 27$. By the Arithmetic-Geometric Mean Inequality, the product of a set of positive numbers with a given sum is greatest when the numbers are equal, so the remaining cheese has maximum volume when $a = b = c = 9$. The volume is then $9^3 = 729$ cu cm.

   Query: What happens if $a + b + c$ is not divisible by 3?

   OR

   The volume of the remaining cheese is greatest when it forms a cube. This can be accomplished by taking one slice from the 10 cm dimension, 4 slices from the 13 cm dimension, and 5 slices from the 14 cm dimension for a total of $1 + 4 + 5 = 10$ slices. The remaining cheese is then 9 cm by 9 cm by 9 cm for a volume of 729 cu cm.

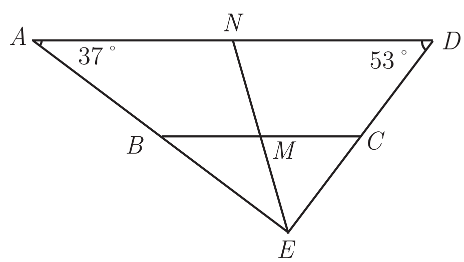
4. (Answer: 021)

Every positive integer has a unique base-3 representation, which for 2008 is 2202101_3. Note that $2 \cdot 3^k = (3 + (-1)) \cdot 3^k = 3^{k+1} + (-1) \cdot 3^k$, so that

$$\begin{aligned}
&2202101_3 \\
=\,&2 \cdot 3^6 + 2 \cdot 3^5 + 2 \cdot 3^3 + 1 \cdot 3^2 + 1 \cdot 3^0 \\
=\,&(3^7 + (-1) \cdot 3^6) + (3^6 + (-1) \cdot 3^5) + (3^4 + (-1) \cdot 3^3) + 1 \cdot 3^2 + 1 \cdot 3^0 \\
=\,&1 \cdot 3^7 + (-1) \cdot 3^5 + 1 \cdot 3^4 + (-1) \cdot 3^3 + 1 \cdot 3^2 + 1 \cdot 3^0,
\end{aligned}$$

and $n_1 + n_2 + \cdots + n_r = 7 + 5 + 4 + 3 + 2 + 0 = 21$.

5. (Answer: 504)



Extend leg $\overline{AB}$ past B and leg $\overline{CD}$ past C, and let E be the point of intersection of these extensions. Then because $\frac{BM}{AN} = \frac{CM}{DN}$, line MN must pass through point E. But $\angle A = 37^\circ$ and $\angle D = 53^\circ$ implies that $\angle AED = 90^\circ$. Thus $\triangle EDA$ is a right triangle with median $\overline{EN}$, and $\triangle EBC$ is a right triangle with median $\overline{EM}$. The median to the hypotenuse in any right triangle is half the hypotenuse, so $EN = \frac{2008}{2} = 1004$, $EM = \frac{1000}{2} = 500$, and $MN = EN - EM = 504$.

6. (Answer: 561)

Observe that if $x_n = x_{n-1} + \dfrac{x_{n-1}^2}{x_{n-2}}$ and $y_n = \dfrac{x_n}{x_{n-1}}$, then $y_n = 1 + y_{n-1}$, so

$$\frac{x_n}{x_0} = \frac{x_n}{x_{n-1}} \cdot \frac{x_{n-1}}{x_{n-2}} \cdots \frac{x_1}{x_0} = y_1 y_2 \cdots y_n = y_1(y_1+1)\cdots(y_1+n-1).$$

In particular, for the first sequence, $y_1 = \frac{a_1}{a_0} = 1$, and so $a_n = n!$. Similarly, for the second sequence, $y_1 = \frac{b_1}{b_0} = 3$, and so $b_n = \frac{1}{2}(n+2)!$. The required ratio is then $\frac{34!/2}{32!} = 17 \cdot 33 = 561$.

7. (Answer: 753)

   Because the equation is cubic and there is no x^2 term, the sum of the roots is 0; that is, $r + s + t = 0$. Therefore,

   $$(r+s)^3 + (s+t)^3 + (t+r)^3 = (-t)^3 + (-r)^3 + (-s)^3 = -(r^3 + s^3 + t^3).$$

   Because r is a root, $8r^3 + 1001r + 2008 = 0$, and similarly for s and t. Therefore, $8(r^3 + s^3 + t^3) + 1001(r + s + t) + 3 \cdot 2008 = 0$, and

   $$r^3 + s^3 + t^3 = \frac{1001(r+s+t) + 3 \cdot 2008}{-8} = \frac{3 \cdot 2008}{-8} = -753,$$

   and hence $(r+s)^3 + (s+t)^3 + (t+r)^3 = -(r^3 + s^3 + t^3) = 753$.

8. (Answer: 251)

   The product-to-sum formula for $2\cos(k^2 a)\sin(ka)$ yields $\sin(k^2 a + ka) - \sin(k^2 a - ka)$, which equals $\sin(k(k+1)a) - \sin((k-1)ka)$. Thus the given sum becomes

   $$\begin{aligned}\sin(2 \cdot 1a) - \sin(0) + \sin(3 \cdot 2a) - \sin(2 \cdot 1a) + \sin(4 \cdot 3a) - \sin(3 \cdot 2a) + \cdots \\ + \sin(n(n+1)a) - \sin((n-1)na).\end{aligned}$$

   This is a telescoping sum, which simplifies to $\sin(n(n+1)a) - \sin(0) = \sin(n(n+1)a)$. But $\sin(x)$ is an integer only when x is an integer multiple of $\pi/2$, so $n(n+1)$ must be an integer multiple of $1004 = 2^2 \cdot 251$. Thus either n or $n+1$ is a multiple of 251, because 251 is prime. The smallest such n is 250, but $250 \cdot 251$ is not a multiple of 1004. The next smallest such n is 251, and $251 \cdot 252$ is a multiple of 1004. Hence the smallest such n is 251.

9. (Answer: 019)

   Let the coordinate plane be the complex plane, and let z_k be the complex number that represents the position of the particle after k moves. Multiplying a complex number by cis θ corresponds to a rotation of θ about the origin, and adding 10 to a complex number corresponds to a horizontal translation of 10 units to the right. Thus $z_0 = 5$, and $z_{k+1} = \omega z_k + 10$, where $\omega = \text{cis}\ (\pi/4)$, for $k \geq 0$. Then

   $$\begin{aligned} z_1 &= 5\omega + 10, \\ z_2 &= \omega(5\omega + 10) + 10 = 5\omega^2 + 10\omega + 10, \\ z_3 &= \omega(5\omega^2 + 10\omega + 10) + 10 = 5\omega^3 + 10\omega^2 + 10\omega + 10, \end{aligned}$$

   and, in general,

   $$z_k = 5\omega^k + 10(\omega^{k-1} + \omega^{k-2} + \cdots + 1).$$

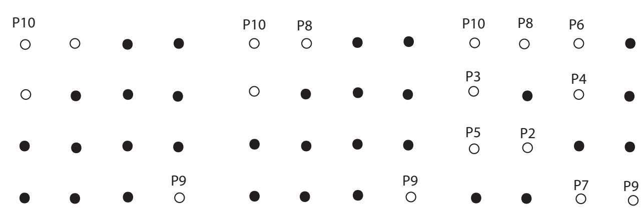
In particular, $z_{150} = 5\omega^{150} + 10(\omega^{149} + \omega^{148} + \cdots + 1)$. Note that $\omega^8 = 1$ and $\omega^{k+4} = \text{cis } ((k+4)\pi/4) = \text{cis } (k\pi/4 + \pi) = -\text{cis } (k\pi/4) = -\omega^k$. Applying the second equality repeatedly shows that $z_{150} = 5\omega^{150} + 10(\omega^{149} + \omega^{148} + \cdots + 1) = 5\omega^6 + 10(-\omega + (-1) + \omega^3 + \omega^2 + \omega + 1) = 5\omega^6 + 10(\omega^3 + \omega^2)$. The last expression equals

$$\begin{aligned} & 5\text{cis } (3\pi/2) + 10(\text{cis } (3\pi/4) + \text{cis } (\pi/2)) \\ = & 5(-i) + 10\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i + i\right) \\ = & -5\sqrt{2} + (5\sqrt{2} + 5)i \end{aligned}$$

Thus $(p, q) = (-5\sqrt{2}, 5\sqrt{2}+5)$, and $|p|+|q| = 10\sqrt{2}+5$. The required value is therefore the greatest integer less than or equal to $10 \cdot 1.414 + 5 = 19.14$, which is 19.

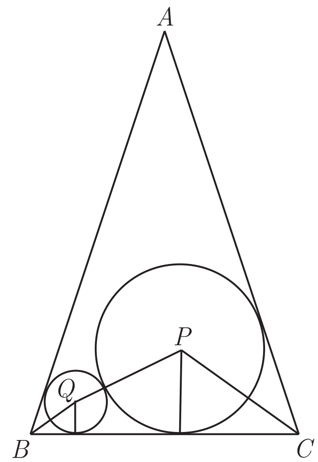
10. (Answer: 240)

The following argument shows that $m = 10$ and $r = 24$. Note that the square of any distance between distinct points in the array has the form $a^2 + b^2$ for some integers a and b in the set $\{0, 1, 2, 3\}$ with a and b not both equal to 0. There are 9 such possible values, namely, 1, 2, 4, 5, 8, 9, 10, 13, and 18. Hence a growing path can consist of at most 10 points. It remains to show that 10 points can be so chosen, and that there are 24 such paths. Let the points of these paths be labeled $P_1, P_2, \ldots, P_{10}$. First, P_9P_{10} must be $\sqrt{18}$; that is, P_9 and P_{10} must be a pair of opposite corners in the array. There are 4 equivalent ways to label P_9 and P_{10}. Without loss of generality, assume that P_9 and P_{10} are the bottom right corner and upper left corner, respectively. Then there are 2 symmetrical positions for P_8, both neighboring P_{10}. (See the left-hand diagram below.) Assume that P_8 is beside P_{10}, as shown in the middle diagram below.



Note that the bottom left corner cannot be P_7, because otherwise $P_6 = P_9$ or $P_6 = P_{10}$. The positions of points P_7, P_6, P_5, P_4, P_3, and P_2 are then fixed. (See the right-hand diagram above.) Finally, there are 3 possible positions for P_1. Hence $r = 4 \cdot 2 \cdot 3 = 24$, and $mr = 240$.

11. (Answer: 254)



Let q be the radius of circle Q. The perpendicular from A to $\overline{BC}$ has length $4\sqrt{25^2 - 7^2} = 4 \cdot 24$, and so $\sin B = \sin C = \frac{24}{25}$. Thus

$$\tan \frac{C}{2} = \tan \frac{B}{2} = \frac{\sin B}{1 + \cos B} = \frac{3}{4}.$$

Place the figure in a Cartesian coordinate system with $B = (0, 0)$, $C = (56, 0)$, and $A = (28, 96)$. Note that circles P and Q are both tangent to $\overline{BC}$, and their centers P and Q lie on the angle bisectors of angles C and B, respectively. Thus $P = (56 - 4 \cdot 16/3, 16)$ and $Q = (4q/3, q)$. Also, $PQ = q + 16$, and so

$$PQ^2 = \left(\frac{4(q + 16)}{3} - 56\right)^2 + (q - 16)^2 = (q + 16)^2,$$

which yields $(4q - 104)^2 = 576q$, and thus $q^2 - 88q + 676 = 0$. The roots of this equation are $44 \pm 6\sqrt{35}$, but $44 + 6\sqrt{35}$ is impossible because it exceeds the base of $\triangle ABC$. Hence $q = 44 - 6\sqrt{35}$, and the requested number is $44 + 6 \cdot 35 = 254$.

12. (Answer: 310)

For convenience, label the flagpoles 1 and 2, and denote by G_n the number of flag arrangements in which flagpole 1 has n green flags. If either flagpole contains all the green flags, then it must also contain at least 8 blue flags to act as separators. The flagpole with no green flags must therefore contain either 1 or 2 blue flags. Hence, $G_0 = G_9 = 11$, because there are 10 possible positions in which to place 1 blue flag on the flagpole with all the green flags (otherwise, the flagpole with no green flags contains 2 blue flags). If $0 < n < 9$, then a flagpole that has n green flags must contain at least $n-1$ blue flags, and the other flagpole has $9-n$ green flags and must contain at least $8-n$ blue flags. Thus in any flag arrangement where each flagpole contains at least one green flag, 7 of the blue flags have fixed positions relative to the green flags, and there are 3 remaining blue flags that can be freely distributed. If flagpole 1 has n green flags, then there are $n+1$ possible positions in which 1 blue flag can be placed on that flagpole, and there are $10-n$ possible positions in which 1 blue flag can be placed on flagpole 2. Thus, for $0 < n < 9$, after placing the green flags and 7 of the blue flags, there remain 3 blue flags with 11 distinguishable locations. This is a standard problem of distributing 3 indistinguishable objects among 11 distinguishable bins. Because 11 bins require 10 separators to divide them, the problem is equivalent to choosing 3 locations out of 13 (the 3 flags and the 10 separators). Thus the number of ways to place the blue flags is $\binom{13}{3}$. The desired number of flag arrangements is then

$$N = \sum_{i=0}^{9} G_i = 2 \cdot 11 + 8\binom{13}{3} = 2310,$$

and the required remainder is 310.

13. (Answer: 029)

One pair of vertices lies at $\frac{1}{2} \pm \frac{1}{2\sqrt{3}}i$. Express points on the line segment determined by these two vertices in the form $z = \frac{1}{2} + yi$, where y is real and $|y| \leq \frac{1}{2\sqrt{3}}$. Reciprocals of points on this line segment are then of the form $\dfrac{\frac{1}{2} - yi}{\frac{1}{4} + y^2}$ with $|y| \leq \frac{1}{2\sqrt{3}}$. Because

$$\left|\frac{1}{z} - 1\right|^2 = \left|\frac{\frac{1}{2} - yi}{\frac{1}{4} + y^2} - \frac{\frac{1}{4} + y^2}{\frac{1}{4} + y^2}\right|^2 = \frac{(\frac{1}{4} - y^2)^2 + y^2}{(\frac{1}{4} + y^2)^2} = \frac{(\frac{1}{4} + y^2)^2}{(\frac{1}{4} + y^2)^2} = 1,$$

the curve traced by the reciprocals of complex numbers on this line segment is an arc of a circle centered at 1 with radius 1, running from $\frac{3}{2} - \frac{\sqrt{3}}{2}i$ to $\frac{3}{2} + \frac{\sqrt{3}}{2}i$. The region enclosed by this arc and the lines from the origin to the endpoints can be partitioned into a 120-degree sector of the disk centered at 1 with radius 1, together with two triangles, each of base 1 and height $\frac{\sqrt{3}}{2}$. Thus it has area $\frac{\pi}{3} + \frac{\sqrt{3}}{2}$. Multiply by six to find that the total area is $2\pi + 3\sqrt{3} = 2\pi + \sqrt{27}$. Thus $a + b = 29$.

Query: The above argument shows that the reciprocals of the points on a line not through the origin fall on a circle through the origin. What happens to a line through the origin?

OR

Because pairs of parallel sides are one unit apart, one side of the hexagon lies along the line $x = \text{Re } z = \frac{1}{2}$. If $w = u + vi = \frac{1}{z}$, where z is on this line, then

$$\frac{1}{2}\left(\frac{1}{w} + \frac{1}{\overline{w}}\right) = \frac{1}{2},$$

which is equivalent to $w+\overline{w} = w\overline{w}$. Hence $u^2+v^2 = 2u$, or $(u-1)^2+v^2 = 1$, which is the equation of a circle of radius 1 centered at 1. Because $z = 1$ is mapped to $w = 1$, it follows that $\text{Re } z > \frac{1}{2}$ is mapped to the interior of this circle. By symmetry, the five remaining half-planes whose union produces R are mapped to corresponding disks. Thus the points that are in the image of the half-plane $\text{Re } z > \frac{1}{2}$ but none of the other five half-planes belong to the disk $(u-1)^2+v^2 < 1$ and the region bounded by the two rays $v = \pm\frac{u}{\sqrt{3}}$. The resulting set can be partitioned into a circular sector with radius 1 and central angle $\frac{2\pi}{3}$ and two isosceles triangles with equal sides of length 1 and vertex angle $\frac{2\pi}{3}$. There are six such nonoverlapping congruent figures forming S. It follows that the area of S is $6(\frac{\pi}{3} + \frac{\sqrt{3}}{2}) = 2\pi + \sqrt{27}$. Thus $a + b = 29$.

14. (Answer: 007)

Let $ABCD$ be a rectangle such that $AB = CD = a$, and $BC = DA = b$. Let E and F be points on the sides $\overline{AB}$ and $\overline{BC}$ respectively, such that $AE = x$ and $CF = y$. Solving the given system of equations is thus equivalent to requiring that $\triangle DEF$ be equilateral. Let $\angle ADE = \alpha$. Then $\angle FDC = 30^\circ - \alpha$, $\tan\alpha = \frac{x}{b}$, and $\angle EDF = 60^\circ$. Thus

$$\frac{y}{a} = \tan(30^\circ - \alpha) = \frac{\frac{1}{\sqrt{3}} - \frac{x}{b}}{1 + \frac{1}{\sqrt{3}} \cdot \frac{x}{b}} = \frac{b - x\sqrt{3}}{b\sqrt{3} + x}. \qquad (1)$$

Squaring and adding 1 yields $\dfrac{4(x^2 + b^2)}{(b\sqrt{3} + x)^2} = \dfrac{y^2 + a^2}{a^2} = \dfrac{x^2 + b^2}{a^2}$. Thus $4a^2 = (b\sqrt{3} + x)^2$, and $x \geq 0$ implies that $x = 2a - b\sqrt{3}$, which is a positive real number because $a \geq b$. Equation (1) shows that $y \geq 0$ if and only if $b - x\sqrt{3} = b - (2a - b\sqrt{3})\sqrt{3} = 4b - 2a\sqrt{3} \geq 0$. It follows that $\frac{a}{b} \leq \frac{2}{\sqrt{3}}$, and so $\rho = \frac{2}{\sqrt{3}}$. Hence $\rho^2 = \frac{4}{3}$, and $m + n = 7$. This value of ρ is achieved when $a = 2, b = \sqrt{3}, x = 1$, and $y = 0$.

15. (Answer: 181)

Let m be an integer such that $(m + 1)^3 - m^3 = n^2$, which implies that $3(2m+1)^2 = (2n-1)(2n+1)$. Because $2n-1$ and $2n+1$ are consecutive odd integers, they are relatively prime. Thus 3 can only divide one of $2n - 1$ and $2n + 1$. Therefore either $2n - 1 = 3k^2$ and $2n + 1 = j^2$, or $2n - 1 = k^2$ and $2n + 1 = 3j^2$. The first case implies that $j^2 - 3k^2 = 2$, which can be shown to be impossible by examining the equation modulo 3. The second case implies that $4n = 3j^2 + k^2$ and $3j^2 - k^2 = 2$. Let $k = 2a+1$ for some integer a. Then $3j^2 = k^2+2 = (2a+1)^2+2$. Therefore $4n = (2a+1)^2 + (2a+1)^2 + 2 = 8a^2 + 8a + 4$, or $n = 2a^2 + 2a + 1$. Furthermore, because $2n + 79 = d^2$ for some integer d, then $2n + 79 = d^2 = 2(2a^2 + 2a + 1) + 79 = 4a^2 + 4a + 81$. This equation is equivalent to $80 = d^2 - (4a^2 + 4a + 1) = d^2 - (2a+1)^2 = (d-2a-1)(d+2a+1)$. Both factors on the right side are of the same parity, so they both must be even. Then the two factors on the right are either $(2, 40)$, $(4, 20)$, or $(8, 10)$, and $(d, a) = (21, 9)$, $(12, \frac{7}{2})$, $(9, 0)$. The first solution gives $n = \frac{441-79}{2} = 181$, and the last solution gives $n = \frac{81-79}{2} = 1$. Thus the largest such n is 181 (with $m = 104$).

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AMERICAN MATHEMATICS COMPETITIONS

27th Annual

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME I)

SOLUTIONS PAMPHLET

Tuesday, March 17, 2009

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:

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The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

Steve Blasberg, AIME Chair
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1. (Answer: 840)

   For a 3-digit sequence to be geometric, there are numbers a and r such that the terms of the sequence are a, ar, ar^2. The largest geometric number must have $a \leq 9$. Because both ar and ar^2 must be digits less than 9, r must be a fraction less than 1 with a denominator whose square divides a. For $a = 9$, the largest such fraction is $\frac{2}{3}$, and so the largest geometric number is 964. The smallest geometric number must have $a \geq 1$. Because both ar and ar^2 must be digits greater than 1, r must be at least 2 and so the smallest geometric number is 124. Thus the required difference is $964 - 124 = 840$.

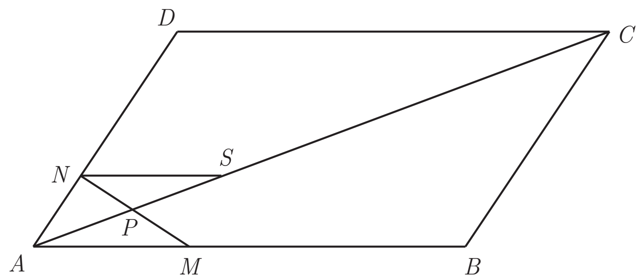
2. (Answer: 697)

   Let $z = a + bi$. Then $z = a + bi = (z + n)4i = -4b + 4i(a + n)$. Thus $a = -4b$ and $b = 4(a + n) = 4(n - 4b)$. Solving the last equation for n yields $n = \dfrac{b}{4} + 4b = \dfrac{164}{4} + 4 \cdot 164$, so $n = 697$.

3. (Answer: 011)

   The conditions of the problem imply that $\binom{8}{3}p^3(1-p)^5 = \frac{1}{25}\binom{8}{5}p^5(1-p)^3$, and hence $(1-p)^2 = \frac{1}{25}p^2$, so that $1-p = \frac{1}{5}p$. Thus $p = \frac{5}{6}$, and $m+n = 11$.

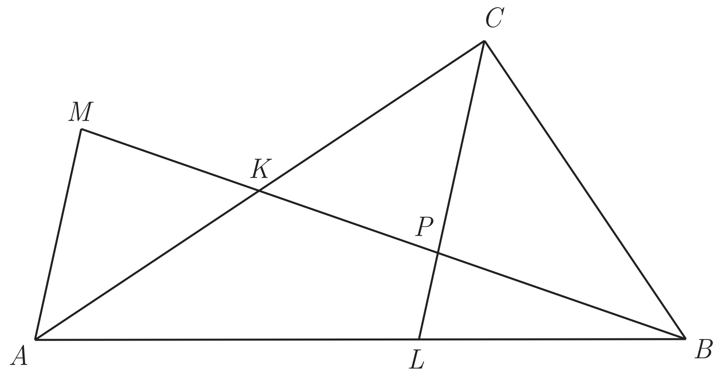
4. (Answer: 177)



   Let point S be on $\overline{AC}$ such that $\overline{NS}$ is parallel to $\overline{AB}$. Because $\triangle ASN$ is similar to $\triangle ACD$, $\dfrac{AS}{AC} = \dfrac{AP + PS}{AC} = \dfrac{AN}{AD} = \dfrac{17}{2009}$. Because $\triangle PSN$ is similar to $\triangle PAM$, $\dfrac{PS}{AP} = \dfrac{SN}{AM} = \dfrac{\frac{17}{2009}CD}{\frac{17}{1000}AB} = \dfrac{1000}{2009}$, and so $\dfrac{PS}{AP} + 1 = \dfrac{3009}{2009}$. Hence $\dfrac{\frac{17}{2009}AC}{AP} = \dfrac{3009}{2009}$, and $\dfrac{AC}{AP} = 177$.

5. (Answer: 072)

Because the diagonals of $APCM$ bisect each other, $APCM$ is a parallelogram. Thus $\overline{AM}$ is parallel to $\overline{CP}$. Because $\triangle ABM$ is similar to $\triangle LBP$, $\frac{AM}{LP} = \frac{AB}{BL} = 1 + \frac{AL}{BL}$. Apply the Angle Bisector Theorem in triangle ABC to obtain $\frac{AL}{BL} = \frac{AC}{BC}$. Therefore $\frac{AM}{LP} = 1 + \frac{AC}{BC}$, and $LP = \frac{AM \cdot BC}{AC+BC}$. Thus $LP = \dfrac{180 \cdot 300}{450 + 300} = 72$.



6. (Answer: 412)

For a positive integer k, consider the problem of counting the number of integers N such that $x^{\lfloor x \rfloor} = N$ has a solution with $\lfloor x \rfloor = k$. Then $x = \sqrt[k]{N}$, and because $k \le x < k+1$, it follows that $k^k \le x^k \le (k+1)^k - 1$. Thus there are $(k+1)^k - k^k$ possible integer values of N for which the equation $x^k = N$ has a solution. Because $5^4 < 1000$ and $5^5 > 1000$, the desired number of values of N is $\displaystyle\sum_{k=1}^{4}[(k+1)^k - k^k] = 1 + 5 + 37 + 369 = 412$.

7. (Answer: 041)

Rearranging the given equation and taking the logarithm base 5 of both sides yields

$$a_{n+1} - a_n = \log_5(3n+5) - \log_5(3n+2).$$

Successively substituting $n = 1, 2, 3, \ldots$ and adding the resulting equations produces $a_{n+1} - 1 = \log_5(3n+5) - 1$. Thus the closed form for the sequence is $a_n = \log_5(3n+2)$, which is an integer only when $3n+2$ is a positive integer power of 5. The least positive integer power of 5 greater than 1 of the form $3k+2$ is $5^3 = 125 = 3 \cdot 41 + 2$, so $k = 41$.

8. (Answer: 398)

For $n \geq 1$, let T_n denote the sum of all positive differences of all pairs of elements of the set $\{2^0, 2^1, 2^2, \ldots, 2^n\}$. Given two elements in this set, if neither equals 2^n, then the difference of these elements contributes to the sum T_{n-1}. Thus

$$\begin{aligned} T_n &= T_{n-1} + \left(2^n - 2^{n-1}\right) + \left(2^n - 2^{n-2}\right) + \cdots + \left(2^n - 2^0\right) \\ &= T_{n-1} + n \cdot 2^n - (2^n - 1). \end{aligned}$$

By applying this recursion repeatedly, it follows that

$$\begin{aligned} T_n &= \sum_{k=1}^{n} (k \cdot 2^k - 2^k + 1) \\ &= \sum_{k=1}^{n} k \cdot 2^k - \sum_{k=1}^{n} 2^k + \sum_{k=1}^{n} 1 \\ &= \left(\sum_{k=1}^{n} k \cdot 2^k \right) - (2^{n+1} - 2) + n \\ &= \left(\sum_{k=1}^{n} \left(\sum_{i=k}^{n} 2^i \right) \right) - (2^{n+1} - 2) + n \\ &= \left(\sum_{k=1}^{n} \frac{2^k (2^{n-k+1} - 1)}{2 - 1} \right) - (2^{n+1} - 2) + n \\ &= \left(\sum_{k=1}^{n} (2^{n+1} - 2^k) \right) - (2^{n+1} - 2) + n \\ &= n \cdot 2^{n+1} - (2^{n+1} - 2) - (2^{n+1} - 2) + n \\ &= (n - 2) 2^{n+1} + n + 4. \end{aligned}$$

Setting $n = 10$ gives $T_{10} = 2^{14} + 14 = 16398$. Thus the required remainder is 398.

9. (Answer: 420)

The number of possible orderings of the given seven digits is $\dfrac{7!}{4!\,3!} = 35$. These 35 orderings correspond to 35 seven-digit numbers, and the digits of each number can be subdivided to represent a unique combination of guesses for A, B, and C. Thus, for a given ordering, the number of guesses it represents is the number of ways to subdivide the seven-digit number into three nonempty sequences, each with no more than four digits. These subdivisions have possible lengths $1|2|4$, $2|2|3$, $1|3|3$, and their permutations. The first subdivision can be ordered in 6 ways and the second and third in 3 ways each, for a total of 12 possible subdivisions. Thus the total number of guesses is $35 \cdot 12$ or 420.

10. (Answer: 346)

Each acceptable seating arrangement can be specified in two steps. The first step is to assign a planet to each chair according to the committee rules. The second step is to assign an individual from the appropriate planet to each seat. Because the committee members from each planet can be seated in any of 5! ways, the second step can be completed in $(5!)^3$ ways. Thus N is the number of ways in which the first step can be completed.

In clockwise order around the table, every group of one or more Martians seated together must be followed by a group of one or more Venusians and then a group of one or more Earthlings. Thus the possible assignments of planets to chairs are in one-to-one correspondence with all sequences of positive integers $m_1, v_1, e_1, \ldots, m_k, v_k, e_k$ with $1 \le k \le 5$ and $m_1 + \cdots + m_k = v_1 + \cdots + v_k = e_1 + \cdots + e_k = 5$. For each k, the number of ordered k-tuples $(m_1, \ldots, m_k)$ with $m_1 + \cdots + m_k = 5$ is $\binom{4}{k-1}$ as are the numbers of ordered k-tuples $(v_1, \ldots, v_k)$ with $v_1 + \cdots + v_k = 5$ and $(e_1, \ldots, e_k)$ with $e_1 + \cdots + e_k = 5$. Hence the number of possible assignments of planets to chairs is

$$N = \sum_{k=1}^{5} \binom{4}{k-1}^3 = 1^3 + 4^3 + 6^3 + 4^3 + 1^3 = 346.$$

11. (Answer: 600)

First note that the distance from $(0,0)$ to the line $41x + y = 2009$ is

$$\frac{|41 \cdot 0 + 0 - 2009|}{\sqrt{41^2 + 1^2}} = \frac{2009}{29\sqrt{2}},$$

and that this distance is the altitude of any of the triangles under consideration. Thus such a triangle has integer area if and only if its base is an even multiple of $29\sqrt{2}$. There are 50 points with nonnegative integer coefficients on the given line, namely, $(0, 2009)$, $(1, 1968)$, $(2, 1927)$,..., $(49, 0)$, and the distance between any two consecutive points is $29\sqrt{2}$. Thus a triangle has positive integer area if and only if the base contains 3, 5, 7,..., or 49 of these points, with the two outermost points being vertices of the triangle. The number of bases with one of these possibilities is

$$48 + 46 + 44 + \cdots + 2 = \frac{24 \cdot 50}{2} = 600.$$

OR

Assume that the coordinates of P and Q are (x_0, y_0) and $(x_0 + k, y_0 - 41k)$, where x_0 and y_0 are nonnegative integers such that $41x_0 + y_0 = 2009$, and

k is a positive integer. Then the area of $\triangle OPQ$ is the absolute value of

$$\frac{1}{2}\begin{vmatrix} x_0+k & y_0-41k & 1 \\ x_0 & y_0 & 1 \\ 0 & 0 & 1 \end{vmatrix} = \left|\frac{1}{2}(x_0y_0+ky_0-x_0y_0+41kx_0)\right|$$
$$= \left|\frac{1}{2}k(41x_0+y_0)\right| = \left|\frac{1}{2}\cdot 2009k\right|.$$

Thus the area is an integer if and only if k is a positive even integer. The points P_i with coordinates $(i, 2009-41i)$, $0 \le i \le 49$, represent exactly the points with nonnegative integer coordinates that lie on the line with equation $41x+y=2009$. There are 50 such points. The pairs of points (P_i, P_j) with $j-i$ even and $j>i$ are in one-to-one correspondence with the triangles OPQ having integer area. Thus $j-i=2p$, $1 \le p \le 24$ and for each possible value of p, there are $50-2p$ pairs of points (P_i, P_j) meeting the conditions that P_i and P_j are points on $41x+y=2009$ with $j-i$ even and $j>i$. Thus the number of such pairs and the number of triangles OPQ with integer area is

$$\sum_{p=1}^{24}(50-2p) = \sum_{q=1}^{24} 2q = 2\cdot\frac{24\cdot 25}{2} = 600.$$

12. (Answer: 011)

Let E and F be the points of tangency on $\overline{AI}$ and $\overline{BI}$, respectively. Let $IE=IF=x$, $AE=AD=y$, $BD=BF=z$, $r=$ the radius of the circle ω, $CD=h$, and k be the area of triangle ABI. Then $h=\sqrt{yz}$, and so $r=\frac{1}{2}\sqrt{yz}$. Let s be the semiperimeter of $\triangle ABI$, so that $s=x+y+z$. On one hand $k=sr=\frac{1}{2}(x+y+z)\sqrt{yz}$, and on the other hand, by Heron's Formula, $k=\sqrt{(x+y+z)xyz}$. Equating these two expressions and simplifying yields $4x=x+y+z$, or $4x=x+AB$. Thus $x=\frac{AB}{3}$ and $2s=2\cdot\frac{AB}{3}+2\cdot AB=\frac{8}{3}\cdot AB$. Hence $m+n=8+3=11$.

13. (Answer: 090)

The definition gives

$$a_3(a_2+1)=a_1+2009, \quad a_4(a_3+1)=a_2+2009, \quad a_5(a_4+1)=a_3+2009.$$

Subtracting adjacent equations yields $a_3-a_1=(a_3+1)(a_4-a_2)$ and $a_4-a_2=(a_4+1)(a_5-a_3)$. Suppose that $a_3-a_1\ne 0$. Then $a_4-a_2\ne 0$, $a_5-a_3\ne 0$, and so on. Because $|a_{n+2}+1|\ge 2$, it follows that $0<|a_{n+3}-a_{n+1}|=\frac{|a_{n+2}-a_n|}{|a_{n+2}+1|}<|a_{n+2}-a_n|$, that is, $|a_3-a_1|>|a_4-a_2|>|a_5-a_3|>\cdots$, which is a contradiction. Therefore $a_{n+2}-a_n=0$ for all $n\ge 1$, which implies that all terms with an odd index are equal, and all terms with an even index are equal. Thus as long as a_1 and a_2 are integers, all the terms are integers. The definition of the sequence then implies that $a_1 = a_3 = \frac{a_1+2009}{a_2+1}$, giving $a_1a_2 = 2009 = 7^2 \cdot 41$. The minimum value of $a_1 + a_2$ occurs when $\{a_1, a_2\} = \{41, 49\}$, which has a sum of 90.

14. (Answer: 905)

For $j = 1, 2, 3, 4$, let m_j be the number of a_i's that are equal to j. Then

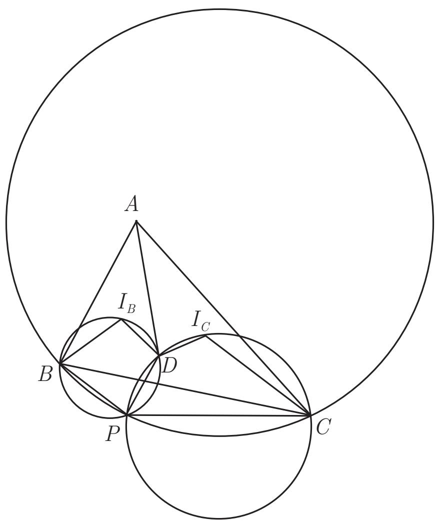
$$m_1 + m_2 + m_3 + m_4 = 350,$$
$$S_1 = m_1 + 2m_2 + 3m_3 + 4m_4 = 513, \text{ and}$$
$$S_4 = m_1 + 2^4m_2 + 3^4m_3 + 4^4m_4 = 4745.$$

Subtracting the first equation from the second, then the first from the third yields

$$m_2 + 2m_3 + 3m_4 = 163, \text{ and}$$
$$15m_2 + 80m_3 + 255m_4 = 4395.$$

Now subtracting 15 times the first of these equations from the second yields $50m_3 + 210m_4 = 1950$ or $5m_3 + 21m_4 = 195$. Thus m_4 must be a nonnegative multiple of 5, and so m_4 must be either 0 or 5. If $m_4 = 0$, then the m_j's must be $(226, 85, 39, 0)$, and if $m_4 = 5$, then the m_j's must be $(215, 112, 18, 5)$. The first set of values results in $S_2 = 1^2 \cdot 226 + 2^2 \cdot 85 + 3^2 \cdot 39 + 4^2 \cdot 0 = 917$, and the second set of values results in $S_2 = 1^2 \cdot 215 + 2^2 \cdot 112 + 3^2 \cdot 18 + 4^2 \cdot 5 = 905$. Thus the minimum value is 905.

15. (Answer: 150)



Note that

$$\angle BI_BD = \angle I_BBA + \angle BAD + \angle ADI_B = \angle BAD + \frac{\angle DBA}{2} + \frac{\angle ADB}{2}$$

and

$$\angle CI_CD = \angle I_CDA + \angle DAC + \angle ACI_C = \angle DAC + \frac{\angle CDA}{2} + \frac{\angle ACD}{2}.$$

Because $\angle BAD + \angle DAC = \angle BAC$ and $\angle ADB + \angle CDA = 180^\circ$, it follows that

$$\begin{aligned} \angle BI_BD + \angle CI_CD &= \angle BAC + \frac{\angle ACD}{2} + \frac{\angle DBA}{2} + 90^\circ \\ &= 180^\circ + \frac{\angle BAC}{2}. \end{aligned} \tag{1}$$

The points P and I_B must lie on opposite sides of $\overline{BC}$, and BI_BDP and CI_CDP are convex cyclic quadrilaterals. If P and I_B were on the same side, then both BI_BPD and CI_CPD would be convex. It would then follow by (1) and the fact that quadrilaterals BI_BPD and CI_CPD are cyclic that

$$\angle BPC = \angle BPD + \angle DPC = \angle BI_BD + \angle CI_CD = 180^\circ + \frac{\angle BAC}{2} > 180^\circ,$$

which is impossible.

By (1),

$$\begin{aligned} \angle BPC &= \angle BPD + \angle DPC = 180^\circ - \angle BI_BD + 180^\circ - \angle CI_CD \\ &= 180^\circ - \frac{\angle BAC}{2}. \end{aligned}$$

Therefore, $\angle BPC$ is constant, and so P lies on the arc of a circle passing through B and C.

The Law of Cosines yields $\cos\angle BAC = \frac{10^2+16^2-14^2}{2\cdot 10\cdot 16} = \frac{1}{2}$, and so $\angle BAC = 60^\circ$. Hence $\angle BPC = 150^\circ$, and the minor arc subtended by the chord BC measures 60°. Thus the radius of the circle is equal to $BC = 14$. The maximum area of $\triangle BPC$ occurs when $BP = PC$. Applying the Law of Cosines to $\triangle BPC$ with $BP = PC = x$ yields $14^2 = 2x^2 + x^2\sqrt{3}$, so $x^2 = \dfrac{196}{2+\sqrt{3}} = 196(2-\sqrt{3})$. The area of this triangle is $\dfrac{1}{2}x^2 \sin 150^\circ = 98 - 49\sqrt{3}$, and so $a + b + c = 150$.

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American Mathematics Competitions

27th Annual *(Alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME II)

SOLUTIONS PAMPHLET

Wednesday, April 1, 2009

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

Steve Blasberg, AIME Chair
San Jose, CA 95129 USA

1. 1,2,3,. . .,4, . . . , 5
   $1, 2, 3, 4, \ldots, 5$

   (Answer: 114)

   Bill must have used $164 - 130 = 34$ more ounces of red paint than blue paint and $188 - 130 = 58$ more ounces of white paint than blue paint. It follows that it took $34 + 58 = 92$ ounces of paint to paint the pink stripe, and therefore 92 ounces to paint each stripe. Thus $130 - 92 = 38$ ounces of blue paint were left, and a total of $3 \cdot 38 = 114$ ounces of paint were left.

2. (Answer: 469)

   It follows from the properties of exponents that

$$\begin{aligned}
&a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2} \\
&= (a^{\log_3 7})^{\log_3 7} + (b^{\log_7 11})^{\log_7 11} + (c^{\log_{11} 25})^{\log_{11} 25} \\
&= 27^{\log_3 7} + 49^{\log_7 11} + \sqrt{11}^{\log_{11} 25} \\
&= 3^{3\log_3 7} + 7^{2\log_7 11} + 11^{\frac{1}{2}\cdot\log_{11} 25} \\
&= 7^3 + 11^2 + \sqrt{25} = 343 + 121 + 5 = 469.
\end{aligned}$$

3. (Answer: 141)

   Because $\angle EBA$ and $\angle ACB$ are both complementary to $\angle EBC$, the angles EBA and ACB are equal, and right triangles BAE and CBA are similar. Thus

$$\frac{AE}{AB} = \frac{AB}{BC} = \frac{AB}{2AE}.$$

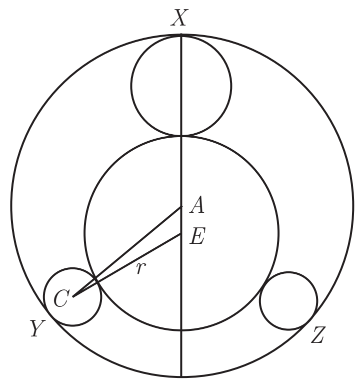
   Hence $2 \cdot AE^2 = AB^2$ and $AD = 2 \cdot AE = 2 \cdot \frac{100}{\sqrt{2}} = 100\sqrt{2}$. Because $141 < 100\sqrt{2} < 142$, the requested answer is 141.

4. (Answer: 089)

   Let c be the number of children in the contest, and let g be the average number of grapes eaten by each contestant. Then g is an integer, and $c \cdot g = 2009 = 7^2 \cdot 41$. Furthermore, the number of grapes eaten by the child in last place is $g - (c - 1) \geq 0$, so $c \leq g + 1$. Therefore the possible choices for the ordered pair (c, g) are $(1, 2009)$, $(7, 287)$, and $(41, 49)$. The value of $n = g + (c - 1)$ is minimized when $(c, g) = (41, 49)$, and the minimum value is $41 + 49 - 1 = 89$.

5. (Answer: 032)

   Let circles B, C, and D be tangent to circle A at points X, Y, and Z, respectively. Circle E has radius r. Because circles C and D are congruent, point E is on the diameter of circle A through point X. Note that $AC = 8$, $AE = r - 4$, $CE = r + 2$, and $\angle CAE = 60^\circ$. Therefore by the Law of Cosines, it follows that $(r-4)^2 + 8^2 - 2(r-4)\cdot 8 \cdot \cos 60^\circ = (r+2)^2$. Expanding and simplifying yields $r^2 - 8r + 16 + 64 - 8(r-4) = r^2 + 4r + 4$. Solving for r gives $r = \frac{27}{5}$. The requested answer is $27 + 5 = 32$.



6. (Answer: 750)

Let A be the number of ways in which 5 distinct numbers can be selected from the set of the first 14 natural numbers, and let B be the number of ways in which 5 distinct numbers, no two of which are consecutive, can be selected from the same set. Then $m = A - B$. Because $A = \binom{14}{5}$, the problem is reduced to finding B.
Consider the natural numbers $1 \le a_1 < a_2 < a_3 < a_4 < a_5 \le 14$. If no two of them are consecutive, the numbers $b_1 = a_1$, $b_2 = a_2 - 1$, $b_3 = a_3 - 2$, $b_4 = a_4 - 3$, and $b_5 = a_5 - 4$ are distinct numbers from the interval $[1, 10]$. Conversely, if $b_1 < b_2 < b_3 < b_4 < b_5$ are distinct natural numbers from the interval $[1, 10]$, then $a_1 = b_1$, $a_2 = b_2 + 1$, $a_3 = b_3 + 2$, $a_4 = b_4 + 3$, and $a_5 = b_5 + 4$ are from the interval $[1, 14]$, and no two of them are consecutive. Therefore counting B is the same as counting the number of ways of choosing 5 distinct numbers from the set of the first 10 natural numbers. Thus $B = \binom{10}{5}$. Hence $m = A - B = \binom{14}{5} - \binom{10}{5} = 2002 - 252 = 1750$ and the answer is 750.

7. (Answer: 401)

The fact that $\displaystyle\binom{2m}{m} = \frac{2^m m!(2m-1)!!}{m!m!} = \frac{2^m(2m-1)!!}{m!}$ is an integer implies that if p^a is a power of an odd prime dividing $(2m)!!$, then p^a divides $m!$ and hence $(2m-1)!!$. Thus any odd prime powers which divide the denominator in any given term in the sum divide the numerator as well. Therefore, when reduced to lowest terms, the denominator of the mth term in the sum is the highest power of 2 that divides $(2m)!!$. Hence $2^a b$ is the highest power of 2 dividing $(2\cdot 2009)!! = 4018!!$. Also note that $b = 1$ because b is odd and $2^a b$ is a power of 2. Because $4018!! = 2^{2009} \cdot 2009!$, the highest power of 2 that divides $4018!!$ is

$$2009 + \left\lfloor \frac{2009}{2} \right\rfloor + \left\lfloor \frac{2009}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{2009}{2^{10}} \right\rfloor =$$
$$2009 + 1004 + 502 + 251 + 125 + 62 + 31 + 15 + 7 + 3 + 1 = 4010.$$

Thus the requested answer is $\dfrac{4010 \cdot 1}{10} = 401$.

8. (Answer: 041)

The probability that Dave or Linda rolls a die k times to get the first six is the probability that there are $k-1$ rolls which are not six followed by one roll of six, which is $p_k = \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)$. The probability that Dave will need one roll and Linda will need one or two rolls is then $p_1(p_1+p_2)$. The probability that Dave will need $k > 1$ rolls and Linda will need $k-1$, k, or $k+1$ rolls is then $p_k(p_{k-1}+p_k+p_{k+1})$. It follows that the desired probability is $p_1(p_1+p_2) + \displaystyle\sum_{k=2}^{\infty} p_k(p_{k-1}+p_k+p_{k+1})$. This is

$$\begin{aligned}
&\frac{1}{6}\cdot\left(\frac{1}{6}+\frac{5}{6}\cdot\frac{1}{6}\right) \\
&\qquad + \sum_{k=2}^{\infty}\left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)\left(\left(\frac{5}{6}\right)^{k-2}\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^{k}\left(\frac{1}{6}\right)\right) \\
&= \left(\frac{1}{6}\right)\cdot\left(\frac{6}{36}+\frac{5}{36}\right) \\
&\qquad + \sum_{k=2}^{\infty}\left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{k-2}\left(\frac{1}{6}\right)\left(1+\left(\frac{5}{6}\right)+\left(\frac{5}{6}\right)^{2}\right) \\
&= \frac{11}{6^3}+\frac{91}{6^4}\cdot\frac{\frac{5}{6}}{1-\left(\frac{5}{6}\right)^2} \\
&= \frac{11}{6^3}+\frac{91}{6^3}\cdot\frac{5}{11}=\frac{576}{6^3\cdot 11}=\frac{8}{33}.
\end{aligned}$$

Thus the final answer is $8 + 33 = 41$.

9. (Answer: 000)

Let (a, b, c) be a nonnegative integer solution to $4x+3y+2z = 2000$. Then $(a+1, b+1, c+1)$ is a positive integer solution to $4x+3y+2z = 2009$. Conversely, if (a, b, c) is a positive integer solution to $4x+3y+2z = 2009$, then $(a-1, b-1, c-1)$ is a nonnegative integer solution to $4x+3y+2z = 2000$. This establishes a one-to-one correspondence between the nonnegative integer solutions to $4x+3y+2z = 2000$ and the positive integer solutions to $4x+3y+2z = 2009$. The difference $m-n$ is therefore the number of nonnegative integer solutions to $4x+3y+2z = 2000$ in which $xyz = 0$. If $x = 0$, then $3y = 2(1000-z)$ and it follows that z must be one more than a nonnegative multiple of 3, and $z \leq 1000$. Thus the possible values of z are 1, 4,..., 1000, and this is a total of 334 values. Similarly, there are 501 solutions with $y = 0$, and 167 solutions with $z = 0$. Note that the solutions $(0, 0, 1000)$ and $(500, 0, 0)$ are each counted twice, so the total number of nonnegative integer solutions to $4x+3y+2z = 2000$ in which $xyz = 0$ is $334+501+167-2 = 1000$, and the requested remainder is 0.

(Note: Using a computer algebra system, one can verify that $m = 83834$ and $n = 82834$.)

10. (Answer: 096)

Note that $\triangle ABC$ has a right angle at B. Place $\triangle ABC$ on the coordinate plane with A at the origin, B at $(5, 0)$, C at $(5, 12)$, and D at a point (x, y). Let point E be at $(10, 0)$; then $\angle ACB$ equals $\angle ECB$. This means that point D lies on the line segment CE. Then $\tan \angle CEB = \frac{12}{5} = \frac{y}{10-x}$, so $5y = 120 - 12x$. Thus

$$\tan(\angle BAD) = \tan\left(\frac{\angle CAB}{2}\right) = \frac{\sin(\angle CAB)}{1+\cos(\angle CAB)} = \frac{12/13}{1+5/13} = \frac{12}{18} = \frac{y}{x}.$$

It follows that $12x = 18y$. Combining this with $5y = 120-12x$ yields $y = \frac{120}{23}$ and $x = \frac{180}{23}$. The distance from A to D is then $\sqrt{\left(\frac{180}{23}\right)^2 + \left(\frac{120}{23}\right)^2} = \frac{60}{23}\sqrt{3^2+2^2} = \frac{60\sqrt{13}}{23}$. The requested answer is then $60+23+13 = 96$.

11. (Answer: 125)

The inequality $|\log m - \log k| < \log n$ is equivalent to $-\log n < \log m - \log k < \log n$, which is equivalent to $\log \frac{m}{n} < \log k < \log mn$. Write $m = nq+r$, where q is a positive integer and r is an integer with $0 \leq r < n$. The inequality then becomes

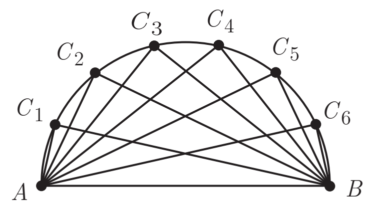
$$\log\left(q+\frac{r}{n}\right) < \log k < \log(n(nq+r)).$$

There are $n(nq+r)-q-1$ possible values of k, namely, $q+1$, $q+2$,..., $n(nq+r)-1$. By the given condition, $n(nq+r)-q-1 = 50$ or $(n^2-1)q+nr = 51$. The potential values of n are 2, 3,..., 7. The only solutions are $(n, q, r) = (2, 17, 0)$ and $(3, 6, 1)$. Hence $(m, n) = (34, 2)$ or $(19, 3)$, and $mn = 68$ or 57. Thus the sum is 125.

12. (Answer: 803)

Let the pairs be (a_i, b_i) for $i = 1, 2, 3, \ldots, k$, and set $S = \sum_{i=1}^{k}(a_i + b_i)$. From the given conditions, it follows that $1 + 2 + \cdots + 2k \le S \le 2009 + 2008 + \cdots + (2010 - k)$, giving $k(2k+1) \le \frac{1}{2}k(4019 - k)$. Solving this inequality for k yields $k \le \frac{4017}{5}$, and therefore k cannot exceed 803. This value of 803 can be achieved by choosing pairs $(1, 1207)$, $(2, 1208), \ldots$, $(401, 1607)$, $(402, 805), \ldots$, $(803, 1206)$.

13. (Answer: 672)



Draw the semicircular arc in the complex plane so that A is at -2 and B is at 2. This arc is then half the circle of radius 2 centered at 0 and the twelve given chords are congruent to the twelve chords $\overline{AC_1}$, $\overline{AC_2}$, $\ldots$, $\overline{AC_6}$, $\overline{AC_7}, \ldots$, $\overline{AC_{11}}$, $\overline{AC_{12}}$, where C_7, $C_8, \ldots$, C_{12} are the reflections of C_1, $C_2, \ldots$, C_6 in the real axis. The 14 points A, B, C_1, $C_2, \ldots$, C_6, C_7, $C_8, \ldots$, C_{12} are then the 14 fourteenth roots of 2^{14}, all satisfying the equation $z^{14} = 2^{14}$. The chord from point z to 2 has the same length as the modulus of the complex number $w = z - 2$. These complex numbers w each satisfy the equation $(w+2)^{14} = 2^{14}$. The product of the lengths of the original twelve chords and AB is the same as the modulus of the product of the roots of the equation $\dfrac{(w+2)^{14} - 2^{14}}{w} = 0$. The product of the roots is equal to the constant term in the fraction when written as a polynomial. According to the Binomial Theorem, this constant term is $\binom{14}{13}2^{13} = 14 \cdot 2^{13}$. This product equals the required product times AB, which equals 4, so n is $\dfrac{14 \cdot 2^{13}}{4} = 14 \cdot 2^{11} = 28672$, and the requested remainder is 672.

14. (Answer: 983)

The recursion formula is equivalent to $a_{n+1} = 2\left(\dfrac{4}{5}a_n + \dfrac{3}{5}\sqrt{4^n - a_n^2}\right)$, which resembles a sum-of-angles trigonometric identity with $a_n = 2^n \sin\alpha$.

Let $\theta = \sin^{-1}(\frac{3}{5})$. Then $\cos\theta = \frac{4}{5}$ and

$$\begin{aligned} a_{n+1} &= 2\left(2^n \cos\theta \sin\alpha + \sin\theta\sqrt{4^n(1-\sin^2\alpha)}\right) \\ &= \begin{cases} 2^{n+1}\sin(\alpha+\theta), & \text{if } \cos\alpha > 0 \\ 2^{n+1}\sin(\alpha-\theta), & \text{if } \cos\alpha < 0. \end{cases} \end{aligned}$$

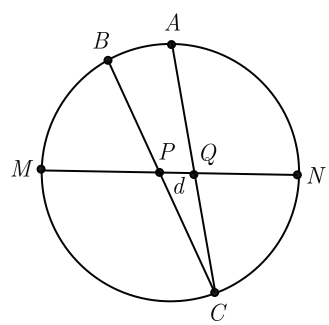
Because $\cos 45^\circ < \frac{4}{5} < \cos 30^\circ$, it follows that $30^\circ < \theta < 45^\circ$. Hence the angle increases by θ until it reaches 3θ, after which it oscillates between 2θ and 3θ. Thus

$$a_n = \begin{cases} 2^n \sin(n\theta), & \text{if } n = 0, 1, 2 \\ 2^n \sin(2\theta), & \text{if } n > 2 \text{ and even} \\ 2^n \sin(3\theta), & \text{if } n > 1 \text{ and odd.} \end{cases}$$

Thus $a_{10} = 2^{10}\sin 2\theta = 1024 \cdot \frac{24}{25} = \frac{24576}{25}$, and the requested answer is 983.

15. (Answer: 014)

Let $[XYZ]$ represent the area of $\triangle XYZ$.



Let $\overline{BC}$ and $\overline{AC}$ intersect $\overline{MN}$ at points P and Q respectively, and let $\frac{CM}{CN} = x$. Then

$$\frac{MP}{NP} = \frac{[BMC]}{[BNC]} = \frac{BM \cdot CM \sin\angle BMC}{BN \cdot CN \sin\angle BNC} = \frac{3x}{4}.$$

Because $NP = 1 - MP$, it follows that $MP = \frac{3x}{3x+4}$. Similarly,

$$\frac{MQ}{NQ} = \frac{[AMC]}{[ANC]} = \frac{AM \cdot CM \sin\angle AMC}{AN \cdot CN \sin\angle ANC} = x,$$

giving $MQ = \frac{x}{x+1}$. The fact that $MQ - MP = d$ implies that $\frac{x}{x+1} - \frac{3x}{3x+4} = d$, or equivalently, $3dx^2 + (7d-1)x + 4d = 0$. Because the discriminant of this equation, which is $d^2 - 14d + 1$, must be nonnegative and $0 < d < 1$, the largest value of d is $7 - 4\sqrt{3}$, and $r + s + t = 14$.

Mathematical Association of America
American Mathematics Competitions

28th Annual

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME I)

SOLUTIONS PAMPHLET

Tuesday, March 16, 2010

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

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American Mathematics Competitions
University of Nebraska, P.O. Box 81606
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The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

Steve Blasberg, AIME Chair
San Jose, CA 95129 USA

1. (Answer: 107)

   The prime factorization of 2010^2 is $2^2 \cdot 3^2 \cdot 5^2 \cdot 67^2$. Thus there are $(2+1)^4 = 81$ positive divisors of 2010^2. A perfect square divisor of 2010^2 has the form $2^w 3^x 5^y 67^z$, where each of w, x, y, z is either 0 or 2. Thus there are a total of $2^4 = 16$ perfect square divisors of 2010^2. The requested probability is therefore

   $$p = \frac{16 \cdot (81 - 16)}{\binom{81}{2}} = \frac{16 \cdot 65}{81 \cdot 40} = \frac{26}{81},$$

   and $m + n = 26 + 81 = 107$.

2. (Answer: 109)

   Let N be the product in the problem. Then

   $$N \equiv 9 \cdot 99(-1)^{997} \equiv -891 \equiv 109 \pmod{1000}.$$

   Thus the desired remainder is 109.

3. (Answer: 529)

   The conditions imply that

   $$\left(\frac{3}{4}x\right)^x = x^{\frac{3}{4}x},$$

   and hence $\pm\frac{3}{4}x = x^{\frac{3}{4}}$, or $\pm\frac{3}{4} = x^{-\frac{1}{4}}$. Thus $x = \left(\pm\frac{4}{3}\right)^4 = \frac{256}{81}$, and $y = \frac{64}{27}$. Then

   $$x + y = \frac{256}{81} + \frac{64}{27} = \frac{256 + 192}{81} = \frac{448}{81},$$

   and the requested sum is 529.

   Note: This problem goes back to Euler. For the history, see the paper by Bennett and Reznick: "Positive rational solutions to $x^y = y^{mx}$: a number-theoretic excursion", *Amer. Math. Monthly*, 111 (2004), 13–21. The positive rational solutions to $x^y = y^x$ are precisely

   $$\{(x_n, y_n)\} = \left\{\left(\left(1 + \frac{1}{n}\right)^n, \left(1 + \frac{1}{n}\right)^{n+1}\right)\right\}$$

   for positive integers n.

4. (Answer: 515)

Let $p(h)$ be the probability that Jackie flips h heads. Then

$$\begin{aligned} p(0) &= \left(\frac{1}{2}\right)^2 \cdot \frac{3}{7} = \frac{3}{28}, \\ p(1) &= 2 \cdot \frac{1}{4} \cdot \frac{3}{7} + \frac{1}{4} \cdot \frac{4}{7} = \frac{5}{14}, \\ p(2) &= \frac{1}{4} \cdot \frac{3}{7} + 2 \cdot \frac{1}{4} \cdot \frac{4}{7} = \frac{11}{28}, \text{ and} \\ p(3) &= \frac{1}{4} \cdot \frac{4}{7} = \frac{1}{7}. \end{aligned}$$

The probability that Jackie and Phil flip exactly the same number of heads is $[p(0)]^2 + [p(1)]^2 + [p(2)]^2 + [p(3)]^2 = \left(\frac{3}{28}\right)^2 + \left(\frac{5}{14}\right)^2 + \left(\frac{11}{28}\right)^2 + \left(\frac{1}{7}\right)^2 = \frac{123}{392}$, and the requested sum is $123 + 392 = 515$.

5. (Answer: 501)

Note that

$$a^2 - b^2 + c^2 - d^2 = (a - b)(a + b) + (c - d)(c + d) = a + b + c + d,$$

and thus $a-b = c-d = 1$. Hence $2010 = a+(a-1)+c+(c-1)$, so $a+c = 1006$. The condition $a > c$ implies that $a \geq 504$, and the condition $c > d$ implies that $c \geq 2$, so that $a \leq 1004$. For each integer k with $0 \leq k \leq 500$, the ordered quadruple $(a, b, c, d) = (504 + k, 503 + k, 502 - k, 501 - k)$ satisfies the required conditions, and thus the number of possible values of a is 501.

6. (Answer: 406)

Completing the square yields

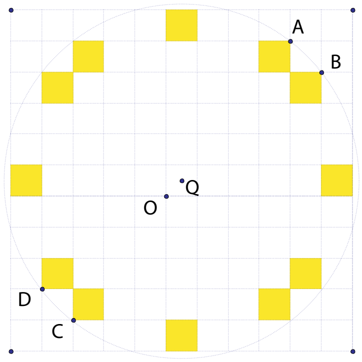
$$(x - 1)^2 + 1 \leq P(x) \leq 2(x - 1)^2 + 1.$$

The left hand and right hand expressions represent parabolas with a vertex at $(1, 1)$, so $P(x)$ must also represent a parabola with vertex at $(1, 1)$. Therefore $P(x) = a(x-1)^2+1$, $P(11) = 100a+1 = 181$, and $a = \frac{9}{5}$. Thus $P(x) = \frac{9}{5}(x - 1)^2 + 1$, and $P(16) = 406$.

7. (Answer: 760)

Let $\mathcal{S} = \{1, 2, 3, 4, 5, 6, 7\}$. If $(\mathcal{A}, \mathcal{B}, \mathcal{C})$ is a minimally intersecting ordered triple of subsets of $\mathcal{S}$, then there exist distinct $x, y, z \in \mathcal{S}$ such that $\mathcal{A} \cap \mathcal{B} = \{x\}$, $\mathcal{B} \cap \mathcal{C} = \{y\}$, and $\mathcal{A} \cap \mathcal{C} = \{z\}$. There are $7 \cdot 6 \cdot 5$ ways to assign values to x, y, z. For each of the four remaining elements of $\mathcal{S}$, there are four possibilities, namely that either the element belongs to exactly one of the three sets $\mathcal{A}, \mathcal{B}$, or $\mathcal{C}$, or it belongs to none of the three sets. Thus there are a total of $7 \cdot 6 \cdot 5 \cdot 4^4$ minimally intersecting ordered triples of sets in which each set in each triple is a subset of $\mathcal{S}$. Thus $N = 7 \cdot 6 \cdot 5 \cdot 4^4 = 210 \cdot 256 = 53760$, and the requested remainder is 760.

8. (Answer: 132)



Suppose $(x, y) \in \mathcal{R}$. Because $\lfloor a \rfloor$ takes integer values and $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 25$, the ordered pairs $(\lfloor x \rfloor, \lfloor y \rfloor)$ must be elements of the set

$$S = \{(\pm 5, 0), (\pm 4, \pm 3), (\pm 3, \pm 4), (0, \pm 5)\}.$$

Thus region $\mathcal{R}$ is a subset of the 12 unit-square regions with lower left corners in S.

Region $\mathcal{R}$ is symmetric about point $Q = (\frac{1}{2}, \frac{1}{2})$. Points $A = (4, 5)$, $B = (5, 4)$, $C = (-3, -4)$, $D = (-4, -3)$ lie on the boundary of region $\mathcal{R}$. Thus $ABCD$ is a rectangle centered at Q. The smallest circle that can be drawn to cover these four points (which are boundary points of $\mathcal{R}$) is the circumcircle of $ABCD$, which has diameter $AC = \sqrt{7^2 + 9^2} = \sqrt{130}$. Substitution of the vertices of the twelve unit square regions into the inequality defining this circumcircle and its interior confirms that this circle does cover region $\mathcal{R}$. Hence the minimum value of r is $\frac{\sqrt{130}}{2}$, and $m + n = 132$.

9. (Answer: 158)

Add xyz to each side of each equation to obtain $x^3 = 2 + xyz$, $y^3 = 6 + xyz$, and $z^3 = 20 + xyz$. Letting $P = xyz$, it follows that $P^3 = (2 + P)(6 + P)(20 + P) = P^3 + 28P^2 + 172P + 240$. Simplifying yields the equation $7P^2 + 43P + 60 = 0$, and thus $P = -\frac{15}{7}$ or $P = -4$. By adding the original three equations, it follows that $x^3 + y^3 + z^3 = 28 + 3P$, and this can be maximized by taking the greater of the two values of P, which is

$-\frac{15}{7}$. This value of P corresponds to the solution $\left(-\frac{1}{\sqrt[3]{7}}, \frac{3}{\sqrt[3]{7}}, \frac{5}{\sqrt[3]{7}}\right)$ of the system. Thus the largest possible value of $x^3+y^3+z^3$ is $28-\frac{45}{7}=\frac{151}{7}$, and $m+n=151+7=158$.

Note: The solution corresponding to $P=-4$ is $\left(-\sqrt[3]{2}, \sqrt[3]{2}, 2\sqrt[3]{2}\right)$.

10. (Answer: 202)

    Write $a_i = 10b_i + c_i$, where b_i, $c_i \in \{0,1,2,\ldots,7,8,9\}$; if b_i and c_i are chosen in this way, they determine a unique acceptable a_i.

    Let $m = b_3 \cdot 10^3 + b_2 \cdot 10^2 + b_1 \cdot 10^1 + b_0 \cdot 10^0$, and $n = c_3 \cdot 10^3 + c_2 \cdot 10^2 + c_1 \cdot 10^1 + c_0 \cdot 10^0$, and write the representation as

$$\begin{aligned} 2010 &= (10b_3+c_3)10^3 + (10b_2+c_2)10^2 + (10b_1+c_1)10^1 + (10b_0+c_0)10^0 \\ &= 10m+n. \end{aligned}$$

    The number of such representations is the number of ways to write 2010 as $10m+n$, where m and n are nonnegative integers. That is, $m \in \{0,1,\ldots,201\}$ and $n = 2010-10m$. Thus $N=202$.

11. (Answer: 365)

    The region $\mathcal{R}$ is a triangular region bounded by the lines $3y-x=15$, $y=x+2$, and $y=-x+18$. The vertices of this triangle are $A = \left(\dfrac{9}{2}, \dfrac{13}{2}\right)$, $B = \left(\dfrac{39}{4}, \dfrac{33}{4}\right)$, and $C=(8,10)$. Let D be the foot of the perpendicular from C to line AB. It can be verified that the coordinates of point D are $(8.7, 7.9)$, and hence D is between A and B. Thus the solid of revolution consists of two right circular cones with heights AD and BD, each having a base radius of CD. The desired volume is therefore $\dfrac{1}{3}\pi \cdot CD^2 \cdot AD + \dfrac{1}{3}\pi \cdot CD^2 \cdot BD = \frac{1}{3}\pi \cdot CD^2 \cdot AB$. Note that

$$AB = \sqrt{\left(\frac{21}{4}\right)^2 + \left(\frac{7}{4}\right)^2} = \frac{7}{4}\sqrt{3^2+1^2} = \frac{7\sqrt{10}}{4} \quad \text{and}$$

$$CD = \sqrt{(8-8.7)^2 + (10-7.9)^2} = \frac{7}{\sqrt{10}}.$$

    Thus the desired volume is $\dfrac{1}{3}\pi \cdot \dfrac{49}{10} \cdot \dfrac{7\sqrt{10}}{4} = \dfrac{343\pi}{12\sqrt{10}}$, and $m+n+p = 343+12+10 = 365$.

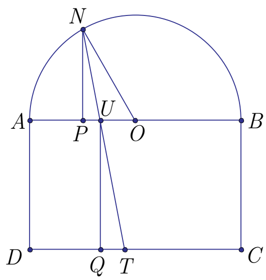
12. (Answer: 243)

    First prove that $m \le 243$. Let $S = \{3,4,\ldots,243\}$ and assume that T and U form a partition of S such that neither of the subsets contains a solution to the given equation. Without loss of generality assume that $3 \in T$. Then necessarily $3^2 \in U$.

If $3^3 \in T$, then because $3^4 = 3 \cdot 3^3 = 3^2 \cdot 3^2$, one of the subsets must contain a solution to the given equation, which contradicts the assumption.

On the other hand, if $3^3 \in U$ then $3^4 \in T$ and thus $3^5 \in U$. This implies that $3^2 \in U$. Then the set U contains a solution to the given equation, which again contradicts the assumption. Thus no such partition exists. If $m \leq 242$, then $S = \{3, 4, \ldots, m\}$ can be partitioned by taking $T = \{3, 4, \ldots, 8, 81, 82, \ldots, m\}$, and $U = \{9, 10, \ldots, 80\}$, and neither of the sets contains a solution to the equation.

13. (Answer: 069)



Let O be the midpoint of segment $\overline{AB}$. Then $AO = (AU + UB)/2 = 126 = AN$ and $UO = 42$. Thus $\triangle AON$ is equilateral and $\angle AON = 60°$, implying that the ratio of the areas of sectors AON and NOB is $1 : 2$. Let Q be the foot of the perpendicular from U to line DC. Because $AU/UB = 1/2$, the ratio of the areas of rectangles $AUQD$ and $UQCB$ is also $1 : 2$. Because line l divides region $\mathcal{R}$ into two parts with area ratio $1 : 2$, it follows that triangles NUO and UQT have the same area. Let P be the foot of the perpendicular from N to line AB. Note that triangles NUP and UTQ are similar, with area ratio equal to

$$\frac{UQ^2}{NP^2} = \frac{\text{Area}(UTQ)}{\text{Area}(NUP)} = \frac{\text{Area}(NOU)}{\text{Area}(NUP)} = \frac{UO}{UP} \text{ or } UQ = NP \cdot \sqrt{\frac{UO}{UP}}.$$

In $\triangle NOP$, $NO = AO = 126$, $\angle NPO = 90°$, and $\angle NOP = 60°$. Hence $NP = 63\sqrt{3}$, $OP = 63$, $UP = OP - UO = 21$, and $\sqrt{\frac{UO}{UP}} = \sqrt{\frac{42}{21}} = \sqrt{2}$. Therefore

$$DA = UQ = NP \cdot \sqrt{\frac{UO}{UP}} = 63\sqrt{6},$$

and $m + n = 69$.

14. (Answer: 109)

Note that $f(1) = 9 \cdot 0 + 90 \cdot 1 + 2 = 92$, and $f(10n) = 100 + f(n)$, so $f(100) = 292$ and $f(1000) = 392$. For $0 \leq j < 900$, $\log_{10}(k(100+j)) \geq 2$. Furthermore $\log_{10}(k(100+j)) \geq 3$ if and only if

$$k \geq \frac{1000}{100+j} = 10 - \frac{10j}{100+j}, \text{ that is, } k \geq 10 - \left\lfloor \frac{10j}{100+j} \right\rfloor.$$

Therefore the number of terms in the sequence having a value of at least 3 is $91 + \left\lfloor \frac{10j}{100+j} \right\rfloor$. Similarly, the number of terms having a value of 4 is $1 + \left\lfloor \frac{100j}{100+j} \right\rfloor$, which implies

$$f(100+j) = 200 + 91 + 1 + \left\lfloor \frac{10j}{100+j} \right\rfloor + \left\lfloor \frac{100j}{100+j} \right\rfloor.$$

Thus the required value of $n = 100 + j$ must satisfy $100j < 9(100+j)$, and therefore $j \leq 9$. It can be verified that $f(109) = 300$, so the answer is 109.

15. (Answer: 045)

Let $\frac{AM}{CM} = k$ and the common altitude of $\triangle AMB$ and $\triangle CMB$ be h. Because the radius of the incircle of triangle equals twice its area divided by its perimeter, the ratio of the areas of two triangles with equal inradii is the same as the ratio of their perimeters. Thus $\frac{12 + AM + BM}{13 + CM + BM} = \frac{\frac{1}{2}AM \cdot h}{\frac{1}{2}CM \cdot h} = k$. Replacing AM by $k \cdot CM$ and solving for BM yields $BM = \frac{13k - 12}{1 - k}$. The fact that $BM > 0$ implies that $\frac{12}{13} < k$. Because $\frac{AM}{CM} = k$ and $AM + CM = 15$, it follows that $CM = \frac{15}{k+1}$ and $AM = \frac{15k}{k+1}$. Applying the Law of Cosines to triangles ABM and BCM and to angles $\angle BMA = \alpha$ and $\angle CMB = \pi - \alpha$ respectively yields

$$12^2 = AM^2 + BM^2 - 2AM \cdot BM \cos\alpha, \text{ and}$$
$$13^2 = BM^2 + CM^2 + 2BM \cdot CM \cos\alpha.$$

Using $AM = k \cdot CM$, multiplying the second equation by k, and adding the two equations yields

$$13^2 k + 12^2 = BM^2(k+1) + AM^2 + CM^2 k.$$

Substituting into the above equation produces

$$169k + 144 = \left(\frac{13k-12}{1-k}\right)^2 (k+1) + \left(\frac{15k}{k+1}\right)^2 + \left(\frac{15}{k+1}\right)^2 k.$$

Simplifying this equation yields $4k(69k^2 - 112k + 44) = 0$. Its solutions are $k = 0$, $k = \dfrac{2}{3}$, and $k = \dfrac{22}{23}$. Because $k > \frac{12}{13}$, only the last solution is valid, and so $p + q = 45$.

MATHEMATICAL ASSOCIATION OF AMERICA
American Mathematics Competitions

28th Annual *(alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME II)

SOLUTIONS PAMPHLET

Wednesday, March 31, 2010

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Steve Blasberg, AIME Chair
San Jose, CA 95129 USA

1. (Answer: 640)

   Because $36 = 4 \cdot 9$ and $\gcd(4, 9) = 1$, it suffices to verify divisibility by 4 and by 9 separately. Because 9 divides N, the sum of the digits of N is divisible by 9. Because the digits of N are even, their sum must be divisible by 18, and hence the set of possible digits of N is $\{0, 4, 6, 8\}$. The maximum value of N formed by these digits is 8640, which is divisible by 4. Thus $N = 8640$, and the requested remainder is 640.

2. (Answer: 281)

   Note that because the square has area 1, the requested probability is equal to the area of the region determined by the given conditions. For $0 < r < 1$, let S_r denote the square concentric with S which has side length r. Every point P inside S except its center lies on the boundary of S_r for exactly one r, and for such a point, the distance $d(P)$ is $\frac{1-r}{2}$. The given inequality is satisfied if P is inside $S_{3/5}$ but outside $S_{1/3}$. This occurs with probability
   $$\frac{9}{25} - \frac{1}{9} = \frac{56}{225},$$
   and the requested sum is 281.

3. (Answer: 150)

   The product K contains nineteen 1's $(2-1, 3-2, 4-3, \ldots, 20-19)$, eighteen 2's $(3-1, 4-2, 5-3, \ldots, 20-18)$, and so forth. Thus $K = 1^{19} \cdot 2^{18} \cdot 3^{17} \cdot 4^{16} \cdots 19^1$. The power of 2 in this product is $2^{18} \cdot 4^{16} \cdot 2^{14} \cdot 8^{12} \cdot 2^{10} \cdot 4^8 \cdot 2^6 \cdot 16^4 \cdot 2^2$. The number of factors of 2 is therefore
   $$1 \cdot 18 + 2 \cdot 16 + 1 \cdot 14 + 3 \cdot 12 + 1 \cdot 10 + 2 \cdot 8 + 1 \cdot 6 + 4 \cdot 4 + 1 \cdot 2 = 150.$$

4. (Answer: 052)

   Let D_i be the event that the original departure gate was i, and N_i be the event that the new gate is i. Then
   $$\begin{aligned}
   &P(\text{distance} \le 400 \text{ ft})\\
   &= \sum_{i=1}^{4} P(D_i)P(N_1 \text{ through } N_{i+4}) + \sum_{i=5}^{8} P(D_i)P(N_{i-4} \text{ through } N_{i+4})\\
   &\quad + \sum_{i=9}^{12} P(D_i)P(N_{i-4} \text{ through } N_{12})\\
   &= 2 \cdot \frac{1}{12}\left(\frac{4}{11} + \frac{5}{11} + \frac{6}{11} + \frac{7}{11}\right) + \frac{1}{12}\left(4 \cdot \frac{8}{11}\right)\\
   &= \frac{11}{33} + \frac{8}{33} = \frac{19}{33},
   \end{aligned}$$

and $m + n = 52$.

5. (Answer: 075)

Let $a = \log_{10} x$, $b = \log_{10} y$, and $c = \log_{10} z$. Take the log of each side of the equation $xyz = 10^{81}$ to obtain $\log_{10} xyz = a+b+c = 81$. Now square each side of this equation to obtain $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc = 81^2$. Note that $(\log_{10} x)(\log_{10} yz) = (\log_{10} x)(\log_{10} y + \log_{10} z) = ab + ac$. Thus $(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2 = 81^2 - 2 \cdot 468 = 5625$, and the answer is $\sqrt{5625} = 75$.

Note: There are an infinite number of values of a, b, and c which satisfy the conditions of the problem, including $a = 69, b = 6 + 6\sqrt{11}$, and $c = 6 - 6\sqrt{11}$.

6. (Answer: 008)

If $ax - b$ is a factor of the given polynomial, then $a = 1$ and b is a root. Thus $n = b^3 + \frac{63}{b}$, which achieves a minimum integer value of 48 when $b = 3$.

On the other hand, suppose

$$x^4 - nx + 63 = (ax^2 + bx + c)(dx^2 + ex + f),$$

where all coefficients are integers. By multiplying both factors by -1 if necessary, it can be assumed that $a > 0$; thus $ad = 1$ implies $a = d = 1$. Equating coefficients for x^3 implies that $b + e = 0$, so

$$\begin{aligned} x^4 - nx + 63 &= x^4 - (c + f - b^2)x^2 - (bc - bf)x + cf \\ &= x^4 + (c + f - b^2)x^2 + b(f - c)x + cf. \end{aligned}$$

The coefficient of x^2 is $c + f - b^2$, and the constant term is $cf = 63$. Thus $c + f = b^2$, and so c and f are positive. The pairs of positive factors of 63 sum to $1 + 63 = 64$, $3 + 21 = 24$, $7 + 9 = 16$, of which only the first and the last are squares. In the first case, $b = \pm 8$, and

$$(x^2 \pm 8x + 63)(x^2 \mp 8x + 1) = x^4 \mp 496x + 63.$$

In the second case, $b = \pm 4$, and

$$(x^2 \pm 4x + 9)(x^2 \mp 4x + 7) = x^4 \mp 8x + 63.$$

Thus the smallest possible value of n in this case is 8, which is less than the value in the previous case and hence the minimum.

7. (Answer: 136)

Let $w = x + yi$, where x and y are real. Then because a is real and the sum of the three roots is $-a$, it follows that $\mathrm{Im}((w+3i)+(w+9i)+(2w-4)) = 0$.

Thus $y+3+y+9+2y=0$, and $y=-3$. Therefore the three roots are $x, x+6i$, and $2x-4-6i$. Because the coefficients of $P(z)$ are real, the non-real roots must occur in conjugate pairs, and so $x=2x-4$ and $x=4$. Thus $P(z)=(z-4)(z-(4+6i))(z-(4-6i))$ and $1+a+b+c=P(1)=(-3)(-3-6i)(-3+6i)=-135$. Thus $|a+b+c|=|-135-1|=136$. Such a polynomial exists: $P(z)=z^3-12z^2+84z-208$ has the zeros $4, 4\pm 6i$, which satisfy the conditions of the problem for $w=4-9i$.

8. (Answer: 772)

Let $|\mathcal{M}|$ represent the number of elements in the set $\mathcal{M}$.

Let $|\mathcal{A}|=k$. Then the first two properties imply that $|\mathcal{B}|=12-k$, and because $\mathcal{A}$ and $\mathcal{B}$ are nonempty, it follows that $k\neq 0$ and $k\neq 12$. The last two properties imply that $k\notin\mathcal{A}$ and $12-k\notin\mathcal{B}$. Thus the first property implies that $k\in\mathcal{B}$ and $12-k\in\mathcal{A}$. Furthermore, k cannot equal 6, because otherwise, $|\mathcal{A}|=|\mathcal{B}|=6$. Thus $6\in\mathcal{A}\cap\mathcal{B}$, which violates the second property. After assigning k to $\mathcal{B}$ and $12-k$ to $\mathcal{A}$, the remaining $k-1$ elements of $\mathcal{A}$ can be chosen in $\binom{10}{k-1}$ ways, and the remaining $11-k$ elements must belong to set $\mathcal{B}$. Thus

$$N=\left(\sum_{k=1}^{11}\binom{10}{k-1}\right)-\binom{10}{6-1}=2^{10}-252=772,$$

and the answer is 772.

9. (Answer: 011)

Without loss of generality, let $AB=2$, and place $ABCDEF$ in the first and second quadrants of the coordinate plane with $A=(0,0)$ and $B=(2,0)$. Then $C=(3,\sqrt{3})$, $E=(0,2\sqrt{3})$, $F=(-1,\sqrt{3})$, $G=(1,0)$, $H=\left(\frac{5}{2},\frac{\sqrt{3}}{2}\right)$, and $L=\left(-\frac{1}{2},\frac{\sqrt{3}}{2}\right)$. Then line AH has equation $y=\frac{\sqrt{3}}{5}x$, line FG has equation $y=\frac{-\sqrt{3}}{2}x+\frac{\sqrt{3}}{2}$, and line EL has equation $y=(3\sqrt{3})x+2\sqrt{3}$. The intersection of lines AH and FG is then $X=\left(\frac{5}{7},\frac{\sqrt{3}}{7}\right)$, and the intersection of lines EL and FG is $Y=\left(\frac{-3}{7},\frac{5\sqrt{3}}{7}\right)$. Then $\overline{XY}$ is a side of the smaller hexagon, and the ratio of the areas is the square of the ratio of the sides, which is

$$\left(\frac{XY}{2}\right)^2=\left(\frac{1}{2}\sqrt{\left(\frac{8}{7}\right)^2+\left(-\frac{4\sqrt{3}}{7}\right)^2}\right)^2=\frac{\frac{64}{49}+\frac{48}{49}}{4}=\frac{112}{196}=\frac{4}{7},$$

so $m+n=11$.

10. (Answer: 163)

If $f(x) = c(x - r_1)(x - r_2)$, the product cr_1r_2 must be $2010 = 2 \cdot 3 \cdot 5 \cdot 67$.

For $1 \le k \le 4$, if c has $4 - k$ prime factors, there are $\binom{4}{k}$ choices for the k prime factors of 2010 that divide r_1r_2. Of these, there are 2^k choices for the factors dividing r_1; the others must divide r_2. The roots of each polynomial obtained in this way are distinct and each possible pair of roots is counted exactly twice. Therefore there are $\binom{4}{k} \cdot 2^{k-1}$ choices for the two roots, up to sign. Furthermore, an even number of $\{c, r_1, r_2\}$ must be negative. This gives $\binom{3}{0} + \binom{3}{2} = 4$ possible assignments of signs for each of the $\sum_{k=1}^{4} \binom{4}{k} \cdot 2^{k-1} = 4 \cdot 1 + 6 \cdot 2 + 4 \cdot 4 + 1 \cdot 8 = 40$ choices of $\{|c|, |r_1|, |r_2|\}$.

Finally, if $|c| = 2010$ then $|r_1| = |r_2| = 1$. There are only three sign assignments that give rise to distinct polynomials in this case, because both cases in which $r_1 = -r_2$ give rise to the same polynomial. Combining this with the prior discussion, there are $4 \cdot 40 + 3 = 163$ such polynomials in all.

11. (Answer: 068)

Note that choosing 5 of the 8 entries to be 1's shows that the number of 3×3 matrices satisfying (1) is $\binom{9}{5}$ or 126. Counting the number of 3×3 matrices satisfying (1) but not (2) and subtracting from 126 will then produce the required answer.

The 3×3 matrices satisfying (1) but not (2) fall into two groups: those 3×3 matrices which have one row, column, or long diagonal of all 1's and one row, column, or long diagonal of all 0's, and those 3×3 matrices which have two rows, columns, or diagonals which are both all 1's or both all 0's. In the first group note that no matrix can have both a row of 1's and a column or diagonal of 0's, both a column of 1's and a row or diagonal of 0's, or both a diagonal of 1's and a row or column of 0's. Thus the only members of the first group are matrices with one row of 1's and one row of 0's or with one column of 1's and one column of 0's. There are 6 ways to choose the row or column of 1's, 2 ways to choose the row or column of 0's, and then 3 ways to fill in the remaining three entries (that is, 110, 101, or 011). Thus the first group has $6 \cdot 2 \cdot 3$ or 36 members. The members of the second group must, by the previous argument, have two overlapping groups (rows, columns, or diagonals) of the same number. But two overlapping groups of 0's would require $3 + 3 - 1 = 5$ 0's, and the matrix only contains four 0's. Thus the second group consists of those 3×3 matrices with both a row and a column of 1's, both a row or column and a diagonal of 1's, or two diagonals of 1's. Because the two groups of 1's require five 1's, the remaining entries in the matrix must all be 0's. There are $3 \cdot 3 = 9$ matrices of the first type, $6 \cdot 2 = 12$ matrices of the second type, and 1 matrix of the third type. Thus the total number of matrices in the second group is $9 + 12 + 1 = 22$ matrices. The number of matrices satisfying (1) but not (2) is therefore $36+22$ or 58. The requested number of T-grids is then $126 - 58 = 68$.

12. (Answer: 338)

Let the length of the base of one of the triangles be $8a$ and let the length of the base of the other triangle be $7a$, for some positive integer a. Because these two triangles have the same area, the lengths of the corresponding altitudes must be $7h$ and $8h$. Because the perimeters of the triangles are equal, it follows that

$$\begin{aligned} 8a + 2\sqrt{16a^2 + 49h^2} &= 7a + 2\sqrt{\frac{49}{4}a^2 + 64h^2}, \text{ or} \\ a + 2\sqrt{16a^2 + 49h^2} &= 2\sqrt{\frac{49}{4}a^2 + 64h^2}. \end{aligned}$$

Squaring both sides of the last equation and simplifying gives

$$\begin{aligned} a^2 + 64a^2 + 196h^2 + 4a\sqrt{16a^2 + 49h^2} &= 49a^2 + 256h^2, \text{ or} \\ a\sqrt{16a^2 + 49h^2} &= 15h^2 - 4a^2. \end{aligned}$$

Squaring both sides of this equation and simplifying yields

$$a^2(16a^2 + 49h^2) = 225h^4 - 120a^2h^2 + 16a^4, \text{ or } 225h^2 = 169a^2.$$

Thus $h = \frac{13a}{15}$, and the common perimeter is

$$8a + 2\sqrt{16a^2 + 49h^2} = 8a + \frac{218a}{15}.$$

Because the perimeter p is an increasing function of a, it must attain its minimum for the smallest acceptable value of a. The triangle are integer-sided, and therefore the value of p must also be an integer. Because 218 and 15 have no common factors, the smallest value of a for which p is an integer is 15. Thus the value requested is

$$8 \cdot 15 + \frac{436 \cdot 15}{15} = 120 + 218 = 338.$$

13. (Answer: 263)

Alex and Dylan are on the same team if Blair and Corey picked cards numbered b and c with either $1 \le b,\ c \le a - 1$ or $a + 10 \le b,\ c \le 52$ from the 50 cards from the deck excluding the cards numbered a and $a + 9$. Thus

$$p(a) = \frac{(a-1)(a-2) + (43-a)(42-a)}{50 \cdot 49} = \frac{a^2 - 44a + 904}{25 \cdot 49}.$$

Because $p(a) \geq \frac{1}{2}$, it follows that

$$p(a) = \frac{a^2 - 44a + 904}{25 \cdot 49} \geq \frac{1}{2},$$

and thus

$$(a-22)^2 + 420 \geq \frac{25 \cdot 49}{2}.$$

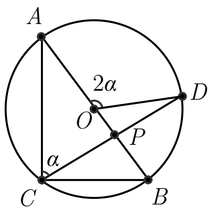
Hence $(a-22)^2 \geq \frac{385}{2}$. Because a is an integer, it follows that $a - 22 \geq 14$ or $a - 22 \leq -14$; that is, $a \geq 36$ or $a \leq 8$. Thus the minimum possible value of $p(a)$ is equal to

$$p(8) = p(36) = \frac{88}{175},$$

and the requested sum is 263.

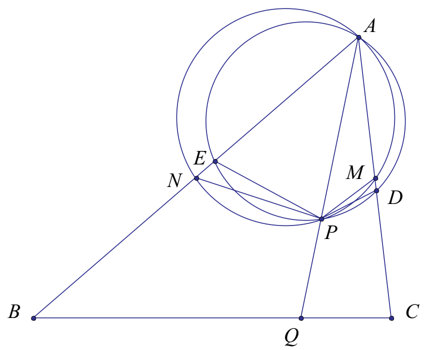
14. (Answer: 007)

Let the circumcircle of $\triangle ABC$ have center at O and radius r, and let $\angle ACP = \alpha$. Extend $\overline{CP}$ to intersect the circle at the point D. Because $\angle AOD = \angle DPB = 2\alpha$, it follows that $DO = DP = r$. Because inscribed angles subtended by the same arc are equal, it follows that $\triangle APD$ and $\triangle CPB$ are similar. Therefore $\dfrac{CP}{BP} = \dfrac{AP}{DP}$ and $\dfrac{CP}{AP} = \dfrac{BP}{DP}$. Thus $\dfrac{CP}{BP} + \dfrac{CP}{AP} = \dfrac{AP}{DP} + \dfrac{BP}{DP} = \dfrac{AB}{DP} = \dfrac{2r}{r} = 2$. Observe that $\angle BAC < 45^\circ$ implies that $AP > BP$. Because $CP = 1$, the previous equation takes the form $\dfrac{1}{4 - AP} + \dfrac{1}{AP} = 2$, giving $2 + \sqrt{2} = AP$. It follows that $BP = 2 - \sqrt{2}$, and so $\dfrac{AP}{BP} = \dfrac{2+\sqrt{2}}{2-\sqrt{2}} = 3 + 2\sqrt{2}$. Hence $p + q + r = 7$.



Note: The existence of such a triangle can be shown by using Stewart's Theorem.

15. (Answer: 218)



The Angle Bisector Theorem implies that E lies on $\overline{AN}$ and D lies on $\overline{MC}$ because $AE/EB = AC/BC < 1$ and $AD/DC = AB/CB > 1$. The Angle Bisector Theorem furthermore implies

$$NE = AN - AE = \frac{AB}{2} - \frac{AC}{AC+BC} \cdot AB = \frac{5}{18}$$

and

$$MD = CM - CD = \frac{AC}{2} - \frac{BC}{BC+BA} \cdot AC = \frac{13}{58}.$$

Because $ANPM$ is cyclic, $\angle ENP = \angle ANP = \angle PMD$. Because $AEPD$ is cyclic, $\angle NEP = 180° - \angle AEP = \angle ADP = \angle MDP$. Because $\angle ENP = \angle PMD$ and $\angle NEP = \angle MDP$, triangles NEP and MDP are similar. Hence

$$\frac{NE}{MD} = \frac{NP}{MP}.$$

Applying the Law of Sines to $\triangle ANP$ and $\triangle AMP$ gives

$$\frac{NE}{MD} = \frac{NP}{MP} = \frac{\sin \angle NAP}{\sin \angle PAM} = \frac{\sin \angle BAQ}{\sin \angle QAC}$$

and thus

$$\frac{\sin \angle BAQ}{\sin \angle QAC} = \frac{(\frac{5}{18})}{(\frac{13}{58})} = \frac{145}{117}.$$

Thus

$$\frac{BQ}{QC} = \frac{\text{Area}(ABQ)}{\text{Area}(ACQ)} = \frac{\frac{1}{2} \cdot AB \cdot AQ \cdot \sin \angle BAQ}{\frac{1}{2} \cdot AC \cdot AQ \cdot \sin \angle QAC} = \frac{15}{13} \cdot \frac{145}{117} = \frac{725}{507},$$

and $m - n = 218$.

MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

29th Annual

AIME I

Solutions Pamphlet

American Invitational Mathematics Examination I Solutions Pamphlet
Thursday, March 17, 2011

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:

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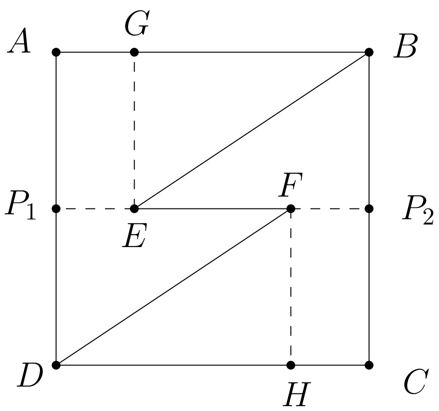
1. (Answer: 085)

   If the contents of all three jars were mixed together, the mixture would be ten liters that is 50% acid. Thus the total amount of acid in the three solutions is $4\cdot 0.45+5\cdot 0.48+0.01k=5$, so $k=80$. When x liters from jar C are added to jar A, the final proportion of acid in jar A is $0.5=\frac{4\cdot 0.45+0.8x}{4+x}$. Solving gives $x=\frac{2}{3}$, and so the requested sum is $80+2+3=85$.

2. (Answer: 036)

   Let line EF intersect $\overline{AD}$ at P_1, and let line EF intersect $\overline{BC}$ at P_2.



Let G be the foot of the perpendicular from E to $\overline{AB}$, and let H be the foot of the perpendicular from F to $\overline{CD}$. Note that $\triangle BEG$ is similar to $\triangle DFH$, thus $\dfrac{EG}{FH}=\dfrac{BE}{DF}$. Therefore $\dfrac{10-FH}{FH}=\dfrac{9}{8}$, and $FH=\dfrac{80}{17}$. Note that

$$\begin{aligned}
EF &= P_1P_2-P_1E-P_2F\\
&= AB-AG-CH\\
&= AB-(AB-BG)-(CD-DH)\\
&= BG+DH-CD\\
&= \frac{9}{8}\cdot DH+DH-CD\\
&= \frac{17}{8}\cdot DH-CD\\
&= \frac{17}{8}\sqrt{DF^2-FH^2}-CD\\
&= \frac{17}{8}\sqrt{8^2-\left(\frac{80}{17}\right)^2}-12\\
&= 3\sqrt{21}-12.
\end{aligned}$$

Thus $m+n+p=36$.

OR

Extend $\overline{AB}$ and $\overline{DC}$ to X and Y respectively, so that $BX=CY=EF$. Then DYX is a right triangle and $EFXB$ is a parallelogram. It follows that $DX=17$, $XY=10$, and $DY=EF+12$. By the Pythagorean Theorem, $(EF+12)^2=17^2-10^2$, and thus $EF=3\sqrt{21}-12$, as before.

3. (Answer: 031)

The equations for L and M are $5x-12y-132=0$ and $12x+5y-90=0$, respectively. Because P lies in the second quadrant in the new coordinate system, it follows that

$$\alpha = \text{negative distance from } P \text{ to } M = \frac{-|12\cdot(-14)+5\cdot 27-90|}{\sqrt{12^2+5^2}} = \frac{-123}{13}, \text{ and}$$
$$\beta = \text{positive distance from } P \text{ to } L = \frac{|5\cdot(-14)-12\cdot 27-132|}{\sqrt{12^2+5^2}} = \frac{526}{13}.$$

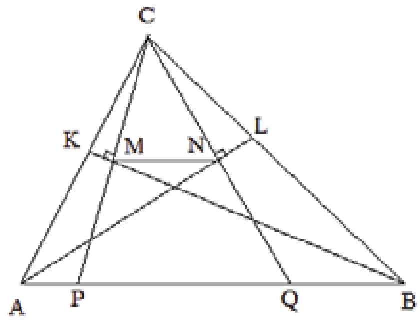
Thus $\alpha+\beta=-\frac{123}{13}+\frac{526}{13}=31$.

4. (Answer: 056)

Extend $\overline{CM}$ and $\overline{CN}$ to meet $\overline{AB}$ at points P and Q, respectively. Triangles BPM and BCM are congruent. Thus M is the midpoint of $\overline{PC}$. Analogously, N is the midpoint of $\overline{QC}$. Hence $\overline{MN}$ is a midline of triangle PQC, and $MN=\frac{PQ}{2}$. Furthermore,

$$PQ=AQ+BP-AB=AC+BC-AB=112.$$

Thus $MN=56$.



5. (Answer: 144)

   Let the nonagon be $ABCDEFGHJ$, and let the digits on the vertices be a, b, c, d, e, f, g, h, and j, respectively. It may be assumed that $a = 1$, so $b + c \equiv 2 \pmod 3$. If $\{b, c\} = \{4, 7\}$, it is impossible for $b + c + d$ to be a multiple of 3. Therefore one of b and c belongs to the set $\{2, 5, 8\}$, and the other belongs to $\{3, 6, 9\}$. It follows that the only possible sequences of digits (mod 3) are $(1, 2, 0, 1, 2, 0, 1, 2, 0)$ and $(1, 0, 2, 1, 0, 2, 1, 0, 2)$. In each of the two sequences, there are 2 possible choices for the ordered pair (d, g) and 6 possible choices for each of the ordered triples (b, e, h) and (c, f, j). Thus the total number of distinguishable acceptable arrangements is $2 \cdot 2 \cdot 6 \cdot 6 = 144$.

6. (Answer: 011)

   Because the vertex of the parabola is $\left(\frac{1}{4}, -\frac{9}{8}\right)$, the equation of the parabola can be written as $y = a\left(x - \frac{1}{4}\right)^2 - \frac{9}{8}$. Note that $a + b + c$ is equal to the value of y at $x = 1$. Hence $a + b + c = a\left(1 - \frac{1}{4}\right)^2 - \frac{9}{8} = \frac{9(a-2)}{16}$, which is given to be an integer. If $\frac{9(a-2)}{16} \le -2$, then $a \le -\frac{14}{9} < 0$. Therefore $\frac{9(a-2)}{16} \ge -1$, which is equivalent to $a \ge \frac{2}{9}$, and thus $p + q = 11$.

7. (Answer: 016)

   The value $m = 1$ does not satisfy the given equation, and thus $m \ge 2$. Subtracting 2011 from both sides produces the equation

   $$\sum_{k=1}^{2011} (m^{x_k} - 1) = m^{x_0} - 2011 = m^{x_0} - 1 - 2010\,.$$

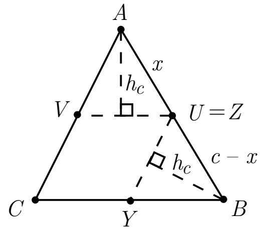
   Because $m - 1$ divides the left side, it must divide the right side, and therefore it divides 2010. Thus if m satisfies the original equation, then $m-1$ must be a factor of 2010. Conversely, suppose that $2010 = (m-1) \cdot n$ for some positive integer n. Let $x_0 = n$, $x_1 = x_2 = \cdots = x_m = 0$, and divide the remaining $(m-1)(n-1)$ numbers into $n-1$ blocks of length $m-1$. Let all x_i's in the rth block equal r, where $1 \le r \le n-1$. Then

   $$\sum_{k=1}^{2011} m^{x_k} = m + (m-1)m + (m-1)m^2 + \cdots + (m-1)m^{n-1} = m^n = m^{x_0}.$$

   Thus the given equation has a solution exactly when $m - 1$ divides 2010. Because $2010 = 2 \cdot 3 \cdot 5 \cdot 67$, there are 16 positive integer factors of 2010 and hence 16 values of m for which the equation has solutions.

8. (Answer: 318)

   Call a table *functional* if its top is parallel to the floor. Let $BC = a$, $CA = b$, and $AB = c$, with $a \le b \le c$. The height of a functional table is the common height of triangles AUV, CXW, and BYZ, where each height is measured to the fold. For a functional table to be of maximum height, two of the folds must intersect on a side of the triangle. If this were not the case, then each fold could be shifted by the same small amount to obtain a functional table with greater height. Thus $Z = U$, $V = W$, or $X = Y$. Assume that $Z = U$. Let the common height of $\triangle AUV$ and $\triangle BUY$ be h_c, and let $AU = x$, so that $BU = BZ = c - x$. Because $\triangle AUV$ is similar to $\triangle ABC$, it follows that $UV = a \cdot \frac{x}{c}$ and $[AUV] = \left(\frac{x}{c}\right)^2 [ABC]$.



Thus

$$h_c = \frac{2[AUV]}{UV} = \frac{2x}{ac} \cdot [ABC].$$

Similarly, by using $\triangle BYZ$, it follows that $h_c = \frac{2[BYZ]}{YZ} = \frac{2(c-x)}{bc} \cdot [ABC]$. Equating these two expressions and solving for x yields $x = \frac{ac}{a+b}$, and hence $h_c = \frac{2[ABC]}{a+b}$. By similar arguments, the heights h_a and h_b that result from $X = Y$ and $V = W$, respectively, are

$$h_a = \frac{2[ABC]}{b+c} \qquad \text{and} \qquad h_b = \frac{2[ABC]}{c+a}.$$

Thus the maximum height h of a functional table is the minimum of h_a, h_b, and h_c, which is $h_a = \frac{2[ABC]}{b+c}$, because if the height were larger than this, some of the folds would intersect.

For the given triangle, $a = 23$, $b = 27$, and $c = 30$, and by Heron's Formula, $[ABC] = 20\sqrt{221}$. Because the two longer sides have lengths 27 and 30, the formula yields

$$h = \frac{2 \cdot 20\sqrt{221}}{27 + 30} = \frac{40\sqrt{221}}{57},$$

and the required sum is $40 + 221 + 57 = 318$.

9. (Answer: 192)

By the change of base formula, $\log_{24 \sin x}(24 \cos x) = \frac{\log_{10}(24 \cos x)}{\log_{10}(24 \sin x)}$. The given equation then becomes $2 \log_{10}(24 \cos x) = 3 \log_{10}(24 \sin x)$. This equation is equivalent to $24^2 \cos^2 x = 24^3 \sin^3 x$, which, after dividing by 24^2 and using the Pythagorean identity for the cosine function yields the equation $24\sin^3 x + \sin^2 x - 1 = 0$. The left side of this equation can be rewritten as

$$24\sin^3 x + \sin^2 x - 3\sin x + 3\sin x - 1 = \sin x(3\sin x - 1)(8\sin x + 3) + (3\sin x - 1),$$

which equals $(3\sin x - 1)(8\sin^2 x + 3\sin x + 1)$. Thus the solutions of the original equation are $\sin x = 1/3$ and two non-real complex values. Then $\cos x = \dfrac{\sqrt{8}}{3}$ and $\cot x = \sqrt{8}$. The required result is $24 \cdot 8 = 192$.

10. (Answer: 503)

Consider the cases of even and odd values of n separately. If n is even, let $n = 2k$, and label the vertices of the polygon consecutively from v_{-k+1} through v_k. Assume without loss of generality that one of the chosen vertices is v_0. If the diametrically opposite vertex v_k is also chosen, the resulting triangle is a right triangle. Among all other choices for the two remaining vertices, the resulting triangle is obtuse if and only if the difference between the indices of the vertices is at most $k - 1$. This will always be the case if both vertices have indices of the same sign. There are $\binom{k-1}{2}$ ways to choose two vertices with positive indices and the same number of ways to choose two vertices with negative indices. If exactly one of the chosen vertices has positive index j with $1 \le j \le k - 1$, then exactly $k - 1 - j$ choices of a vertex with a negative index result in an obtuse triangle. Therefore the number of obtuse triangles with exactly one vertex of positive index is $\displaystyle\sum_{j=1}^{k-1}(k-1-j) = \sum_{i=0}^{k-2} i = \binom{k-1}{2}$, and the total number of obtuse triangles is $3\binom{k-1}{2}$. The total number of possible triangles with one vertex at v_0 is $\binom{2k-1}{2}$, so the probability of an obtuse triangle is

$$\frac{3\binom{k-1}{2}}{\binom{2k-1}{2}} = \frac{3(k-1)(k-2)/2}{(2k-1)(2k-2)/2} = \frac{3(k-2)}{2(2k-1)}\,.$$

Setting the probability equal to $\frac{93}{125}$ gives $k = 188$, and so $n = 376$. If n is odd, let $n = 2k + 1$, and label the vertices of the polygon consecutively from v_{-k} through v_k. The total number of triangles with one vertex at v_0 is $\binom{2k}{2}$. An argument similar to the previous argument shows that the number of obtuse triangles with m remaining vertices of positive index is $\binom{k}{2}$ for each of the three cases $m = 0, 1,$ and 2. The probability of an obtuse triangle is therefore

$$\frac{3\binom{k}{2}}{\binom{2k}{2}} = \frac{3k(k-1)/2}{2k(2k-1)/2} = \frac{3(k-1)}{2(2k-1)}\,.$$

Setting the probability equal to $\frac{93}{125}$ gives $k = 63$, and so $n = 127$. The sum of all possible values of n is $376 + 127 = 503$.

11. (Answer: 007)

The numbers $2^0 = 1, 2^1 = 2$, and $2^2 = 4$ are all elements of R. Because 2^n is divisible by 8 for $n \geq 3$, all other elements of R are multiples of 8. Note that $2^{10} + 1 = 1025 \equiv 0 \pmod{25}$, so $2^{50} + 1 = (2^{10} + 1)(2^{40} - 2^{30} + 2^{20} - 2^{10} + 1) = (2^{10} + 1)((2^{40} - 1) - (2^{30} + 1) + (2^{20} - 1) - (2^{10} + 1) + 5) \equiv 0 \pmod{125}$. Furthermore, $2^n \equiv 0 \pmod 8$ implies $2^n(2^{50} + 1) = 2^{n+50} + 2^n \equiv 0 \pmod{1000}$ for all $n \geq 3$. Because 500 is not in R, the multiples of 8 occur in pairs whose remainder modulo 1000 sum to 1000. Therefore the requested remainder is $1 + 2 + 4 = 7$.

12. (Answer: 594)

Let n be the number of women in the line. For every man to stand next to at least one other man, the men need to be grouped in one of the orders MM-MM-MM, MMMM-MM, MM-MMMM, MMM-MMM, or MMMMMM.

The number of arrangements of the men and women in the line of the form MM-MM-MM can be counted by placing one woman between each of the groups of two men, and then placing the remaining $n - 2$ women in any of the four positions around the men. Thus the number of arrangements of men and women for this grouping is $\binom{n+1}{3} = \frac{(n+1)\cdot n\cdot(n-1)}{6}$.

The number of ways men and women can be arranged so that the men form two separate groups, that is, in the arrangements MMMM-MM, MM-MMMM, or MMM-MMM, can be counted using the same technique by placing one woman between the two groups then placing the remaining $n - 1$ women in any of the three positions around the men. Each of these three groupings results in $\binom{n+1}{2} = \frac{(n+1)\cdot n}{2}$ arrangements.

Finally, the number of arrangements of men and women in the line where all six men stand together is $n + 1$, because there are that many positions in which to place the single group of men among the n women.

Thus the probability that at least four men stand together =

$$\frac{2 \cdot \frac{(n+1)\cdot n}{2} + (n+1)}{\frac{(n+1)\cdot n\cdot(n-1)}{6} + 3 \cdot \frac{(n+1)\cdot n}{2} + (n+1)} = \frac{6n+6}{n^2 + 8n + 6}\,.$$

For this probability to be less than or equal to $\frac{1}{100}$, the quadratic function $f(n) = n^2 - 592n - 594$ must be nonnegative. It follows that the least number of women needed for p not to exceed one percent is the least positive integer n for which $f(n)$ is positive. Because f is a quadratic function which is negative at zero, the least n for which f is positive is the positive integer value at which f changes from negative to positive. Note that $f(593) = 593^2 - 592 \cdot 593 - 594 = 593 \cdot (593 - 592) - 594 = -1 < 0$, but $f(594) = 594^2 - 592 \cdot 594 - 594 = 594 \cdot (594 - 592) - 594 = 594 > 0$. It follows that the answer is 594

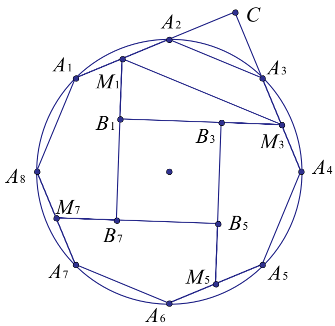
13. (Answer: 330)

Let the vertices of heights 10, 11, and 12 above the plane be labeled B, C, and D, respectively. Note that these vertices are the vertices of an equilateral triangle with side length $10\sqrt{2}$. Set up a coordinate system where the given plane is the graph of $z = -10$, B is at $(0,0,0)$, C is at $(7, 10\sqrt{\frac{3}{2}}, 1)$, and D is at $(14, 0, 2)$. Suppose vertex A is at (x, y, z). Because A is a distance 10 from vertex B, it follows that $x^2+y^2+z^2 = 100$. Because the vectors from B to A, C to A, and D to A are mutually perpendicular, their dot products are all zero, or $(x, y, z)\cdot(x-14, y, z-2) = 0$ and $(x, y, z) \cdot (x-7, y-10\sqrt{\frac{3}{2}}, z-1) = 0$. Thus $14x + 2z = 100$ and $7x + 10\sqrt{\frac{3}{2}}y + z = 100$. These equations yield $7x + z = 50$ and $y = 5\sqrt{\frac{2}{3}}$. Substituting $x = \frac{50-z}{7}$ and $y = 5\sqrt{\frac{2}{3}}$ into $x^2+y^2+z^2 = 100$ and simplifying yields the quadratic equation $3z^2-6z-95 = 0$. This equation has solutions $z = \frac{3\pm\sqrt{294}}{3}$. Because the plane is at $z = -10$ and vertex A lies below the xy plane, it follows that vertex A is at a height $10 + \frac{3-\sqrt{294}}{3} = \frac{33-\sqrt{294}}{3}$. The requested sum is therefore $33 + 294 + 3 = 330$.

Note that the vertex A is at a height of about 5.2845 above the plane.

14. (Answer: 037)

Because this configuration can be scaled without affecting the angles, assume that $A_1A_2 = 2$. Then $M_1A_2 = M_3A_3 = 1$.



In this regular octagon $A_iA_{i+1} \perp A_{i+2}A_{i+3}$ (where $A_{k+8} = A_k$). Consider the 90° counterclockwise rotation centered at the center of the octagon. Under this rotation, $\mathbf{R}_{i+2}$ goes to $\mathbf{R}_i$. Hence by symmetry, $B_1B_3B_5B_7$ is a square and $M_1B_1 = M_3B_3 = M_5B_5 = M_7B_7$. Set $M_1B_1 = a$ and $M_3B_1 = b$. Then

$$b - a = M_3B_1 - M_1B_1 = M_3B_1 - M_3B_3 = B_1B_3 = A_1A_2 = 2.$$

Let C denote the intersection of lines A_1A_2 and A_3A_4. The properties of the regular octagon show that $\angle A_1A_2A_3 = \angle A_2A_3A_4 = 135°$, $\angle M_1CM_3 = 90°$, $A_2C = A_3C = \sqrt{2}$, and $M_1C = M_3C = 1 + \sqrt{2}$. In particular, in the right triangles M_1M_3C and $M_1M_3B_1$,

$$a^2+b^2 = (M_1B_1)^2+(M_3B_1)^2 = (M_1M_3)^2 = (M_1C)^2+(M_3C)^2 = 2(1+\sqrt{2})^2.$$

It follows that

$$(a+b)^2 + 4 = (a+b)^2 + (a-b)^2 = 2(a^2+b^2) = 12 + 8\sqrt{2}$$

or $(a+b)^2 = 8 + 8\sqrt{2}$.

Set $\angle M_1M_3B_1 = \alpha$. Then $\angle A_3M_3B_1 = \angle A_3M_3M_1 + \angle M_1M_3B_1 = 45° + \alpha$. Hence

$$\tan \angle A_3M_3B_1 = \frac{1+\tan\alpha}{1-\tan\alpha} = \frac{a+b}{b-a} = \sqrt{2+2\sqrt{2}}.$$

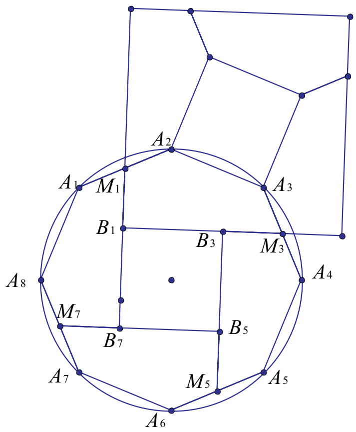
Thus

$$\begin{aligned}
\cos 2\angle A_3M_3M_1 &= 2\cos^2\angle A_3M_3M_1 - 1 = \frac{2}{\sec^2\angle A_3M_3M_1} - 1 \\
&= \frac{2}{\tan^2\angle A_3M_3M_1 + 1} - 1 = \frac{1-\tan^2\angle A_3M_3M_1}{\tan^2\angle A_3M_3M_1 + 1} \\
&= -\frac{1+2\sqrt{2}}{3+2\sqrt{2}} \\
&= -(1+2\sqrt{2})(3-2\sqrt{2}) \\
&= -(-5+4\sqrt{2}) = 5 - \sqrt{32},
\end{aligned}$$

and $5 + 32 = 37$.

Note:

The octagon is dissected into 4 congruent pentagons and one square. These five pieces can be reassembled to form a square. Because $A_1A_2 = B_1B_3$, this square and the octagon have the same area, from which it follows that $(a+b)^2 = (2+2\sqrt{2})^2 - 4$.



15. (Answer: 098)

The integers a, b, and c are roots of $x^3 - 2011x + m$ if and only if $a + b + c = 0$, $ab + ac + bc = -2011$, and $abc = -m$. Assume a, b, and c are roots of $x^3 - 2011x + m$; then $-a, -b$, and $-c$ are roots of $x^3 - 2011x - m$. Moreover, $a + b + c = 0$, so assume without loss of generality that $a \geq b \geq 0$ and $c \leq 0$. Solving for c and substituting in the two other equations yields $a^2 + ab + b^2 = 2011$ and $m = ab(a + b)$. The first equation yields $3b^2 \leq a^2 + ab + b^2 \leq 2011$, that is, $b \leq \lfloor\sqrt{\frac{2011}{3}}\rfloor = 25$, and also $(2a+b)^2 = 4\cdot 2011 - 3b^2$. Thus $4\cdot 2011 - 3b^2$ is a square. Now the quadratic residues modulo 5 are 0, 1, and 4, and $4 \cdot 2011 - 3b^2 \equiv 4 + 2b^2 \pmod 5$, so $4 + 2b^2 \equiv 0$, 1, or 4 $\pmod 5$. The first congruence has no solutions, the second has solutions 1 and 4, and the third has the solution 0. Thus $b \equiv 0$, 1, or 4 $\pmod 5$. Similarly, the quadratic residues modulo 7 are 0, 1, 2, and 4, and $4 \cdot 2011 - 3b^2 \equiv 1 + 4b^2 \pmod 7$, so $1 + 4b^2 \equiv 0$, 1, 2, or 4 $\pmod 7$. The first and the fourth congruences have no solutions, the second has the solution 0, and the third has solutions 3 and 4. Thus $b \equiv 0$, 3, or 4 $\pmod 7$. The only integers $0 \leq b \leq 25$ satisfying these congruences are $0, 4, 10, 11, 14, 21, 24$, and 25. These yield the corresponding values for $4 \cdot 2011 - 3b^2$ of $8044, 7996, 7744, 7681, 7456, 6721, 6316$, and 6169. Note that $6169 = 31 \cdot 199$, $6316 = 2^2 \cdot 1579$, and $6721 = 11 \cdot 13 \cdot 47$, so none of them are squares. Finally, the perfect squares from 86^2 to 90^2 are $7396, 7569, 7744, 7921$, and 8100. Therefore the only b for which $4 \cdot 2011 - 3b^2$ is a perfect square is $b = 10$. Solving for a yields $a = 39$ and consequently $c = -49$. Therefore there are only two such polynomials with the required conditions: $(x-10)(x-39)(x+49)$ and $(x+10)(x+39)(x-49)$. The required sum $|a| + |b| + |c|$ is therefore $10 + 39 + 49 = 98$.

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2011 AIME II

Problem 1

Gary purchased a large beverage, but only drank *m*/*n* of it, where *m* and *n* are relatively prime positive integers. If he had purchased half as much and drunk twice as much, he would have wasted only 2/9 as much beverage. Find *m*+*n*.

Problem 2

On square $ABCD$, point Elies on side $AD$and point Flies on side BC, so that $BE = EF = FD = 30$. Find the area of the square $ABCD$.

Problem 3

The degree measures of the angles in a convex 18-sided polygon form an increasing arithmetic sequence with integer values. Find the degree measure of the smallest angle.

Problem 4

In triangle ABC, AB=(20/11)AC. The angle bisector of angle A intersects BC at point D, and point M is the midpoint of AD. Let P be the point of intersection of AC and the line BM. The ratio of CP to PA can be expresses in the form m/n, where m and n are relatively prime positive integers. Find m+n.

Problem 5

The sum of the first 2011 terms of a geometric sequence is 200. The sum of the first 4022 terms is 380. Find the sum of the first 6033 terms.

Problem 6

Define an ordered quadruple (a, b, c, d) as *interesting* if $1 \leq a < b < c < d \leq 10$, and a+d>b+c. How many interesting ordered quadruples are there?

Problem 7

Ed has five identical green marbles, and a large supply of identical red marbles. He arranges the green marbles and some of the red ones in a row and finds that the number of marbles whose right hand neighbor is the same color as themselves is equal to the number of marbles whose right hand neighbor is the other color. An example of such an arrangement is GGRRRGGRG. Let m be the maximum number of red marbles for which such an arrangement is possible, and let N be the number of ways he can arrange the $m+5$ marbles to satisfy the requirement. Find the remainder when N is divided by 1000.

Problem 8

Let z_1, z_2, z_3, $\dots$, z_{12} be the 12 zeroes of the polynomial $z^{12}-2^{36}$. For each j, let w_j be one of z_j or iz_j. Then the maximum possible value of the real part of $\sum_{j=1}^{12} w_j$ can be written as $m+\sqrt{n}$, where m and n are positive integers. Find $m+n$.

Problem 9

Let x_1, x_2, $\dots$, x_6 be nonnegative real numbers such that $x_1+x_2+x_3+x_4+x_5+x_6=1$, and $x_1x_3x_5+x_2x_4x_6 \ge \frac{1}{540}$. Let p and q be positive relatively prime integers such that $\frac{p}{q}$ is the maximum possible value of $x_1x_2x_3+x_2x_3x_4+x_3x_4x_5+x_4x_5x_6+x_5x_6x_1+x_6x_1x_2$. Find $p+q$.

Problem 10

A circle with center O has radius 25. Chord $\overline{AB}$ of length 30 and chord $\overline{CD}$ of length 14 intersect at point P. The distance between the midpoints of the two chords is 12. The quantity OP^2 can be represented as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when $m+n$ is divided by 1000.

Problem 11

Let M_n be the $n \times n$ matrix with entries as follows: for $1 \le i \le n$, $m_{i,i}=10$; for $1 \le i \le n-1$, $m_{i+1,i}=m_{i,i+1}=3$; all other entries in M_n are zero. Let D_n be the determinant of matrix M_n. Then $\sum_{n=1}^{\infty} \frac{1}{8D_n+1}$ can be represented as $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p+q$. Note: The determinant of the 1×1 matrix $[a]$ is a, and the determinant of the 2×2 matrix $\left[\begin{array}{cc} a & b \\ c & d \end{array}\right] = ad - bc$; for $n \geq 2$, the determinant of an $n\times n$ matrix with first row or first column $a_1 a_2 a_3 \ldots a_n$ is equal to $a_1C_1 - a_2C_2 + a_3C_3 - \cdots + (-1)^{n+1}a_nC_n$, where C_i is the determinant of the $(n-1)\times(n-1)$ matrix formed by eliminating the row and column containing a_i.

Problem 12

Nine delegates, three each from three different countries, randomly select chairs at a round table that seats nine people. Let the probability that each delegate sits next to at least one delegate from another country be $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Problem 13

Point P lies on the diagonal AC of square $ABCD$ with $AP > CP$. Let O_1 and O_2 be the circumcenters of triangles ABP and CDP, respectively. Given that $AB = 12$ and $\angle O_1PO_2 = 120^\circ$, then $AP = \sqrt{a} + \sqrt{b}$, where a and b are positive integers. Find $a+b$.

Problem 14

There are N permutations $(a_1, a_2, \dots, a_{30})$ of $1, 2, \dots, 30$ such that for $m \in \{2, 3, 5\}$, m divides $a_{n+m} - a_n$ for all integers n with $1 \leq n < n+m \leq 30$. Find the remainder when N is divided by 1000.

Problem 15

Let $P(x) = x^2 - 3x - 9$. A real number x is chosen at random from the interval $5 \leq x \leq 15$. The probability that $\lfloor\sqrt{P(x)}\rfloor = \sqrt{P(\lfloor x \rfloor)}$ is equal to

$\dfrac{\sqrt{a}+\sqrt{b}+\sqrt{c}-d}{e}$, where a, b, c, d, and $e$are positive integers, and none of a, b, or $c$is divisible by the square of a prime. Find $a+b+c+d+e$.

Problem 1

Gary purchased a large beverage, but only drank *m*/*n* of it, where *m* and *n* are relatively prime positive integers. If he had purchased half as much and drunk twice as much, he would have wasted only 2/9 as much beverage. Find *m*+*n*.

Solution

Let $x$be the fraction consumed, then $(1-x)$is the fraction wasted. We have $1/2-2x=2/9(1-x)$, or $9-36x=4-4x$, or $32x=5$or $x=5/32$.

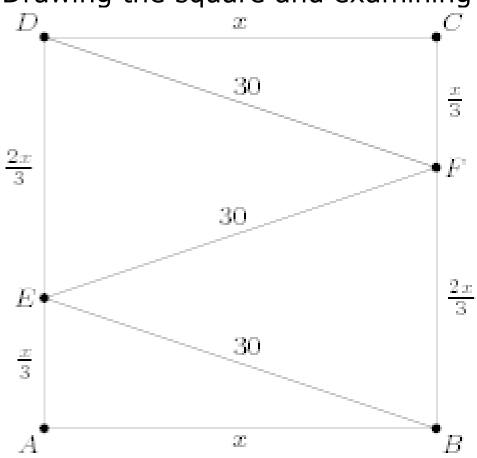
Therefore, $m+n=5+32=\boxed{37.}$

Problem 2

On square ABCD, point E lies on side AD and point F lies on side BC, so that BE=EF=FD=30. Find the area of the square ABCD.

Solution

Drawing the square and examining the given lengths,



you find that the three segments cut the square into three equal horizontal sections. Therefore, (x being the side length),

$\sqrt{x^2+(x/3)^2}=30$, or $x^2+(x/3)^2=900$. Solving for x, we get $x=9\sqrt{10}$, and $x^2=810$.

Area of the square is $\boxed{810.}$

Problem 3

The degree measures of the angles in a convex 18-sided polygon form an increasing arithmetic sequence with integer values. Find the degree measure of the smallest angle.

Solution

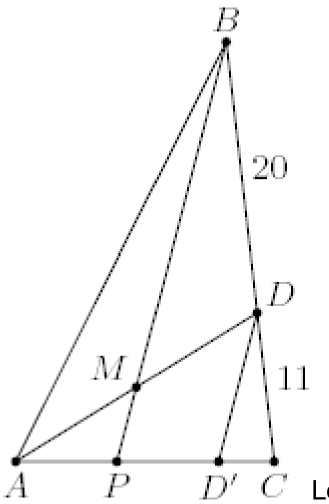
The average angle in an 18-gon is 160°. In an arithmetic sequence the average is the same as the median, so the middle two terms of the sequence average to 160°. Thus for some positive (the sequence is increasing and thus non-constant) integer d, the middle two terms are $(160-d)^\circ$and $(160+d)^\circ$. Since the step is $2d$the last term of the sequence is $(160+17d)^\circ$, which must be less than 180°, since the polygon is convex. This gives $17d<20$, so the only suitable positive integer $d$is 1. The first term is then $(160-17)^\circ=\boxed{143^\circ.}$

Problem 4

In triangle ABC, $AB=\frac{20}{11}AC$. The angle bisector of Aintersects BCat point D, and point $M$is the midpoint of AD. Let Pbe the point of the intersection of $AC$and BM. The ratio of $CP$to PAcan be expressed in the form $\frac{m}{n}$, where m and $n$are relatively prime positive integers. Find $m+n$.

Solutions

Solution 1



Let D' be on $\overline{AC}$ such that $BP \parallel DD'$. It follows that $\triangle BPC \sim \triangle DD'C$, so $\frac{PC}{D'C} = 1 + \frac{BD}{DC} = 1 + \frac{AB}{AC} = \frac{31}{11}$ by the Angle Bisector Theorem. Similarly, we see by the midline theorem that $AP = PD'$. Thus, $\frac{CP}{PA} = \frac{1}{\frac{PD'}{PC}} = \frac{1}{1 - \frac{D'C}{PC}} = \frac{31}{20}$, and $m + n = \boxed{051}$.

Solution 2

Assign mass points as follows: by Angle-Bisector Theorem, $BD/DC = 20/11$, so we assign $m(B) = 11, m(C) = 20, m(D) = 31$. Since $AM = MD$, then $m(A) = 31$, and $\frac{CP}{PA} = \frac{m(A)}{m(C)} = \frac{31}{20}$.

Solution 3

By Menelaus' Theorem on $\triangle ACD$ with transversal PB,

$$1 = \frac{CP}{PA} \cdot \frac{AM}{MD} \cdot \frac{DB}{CB} = \frac{CP}{PA} \cdot \left(\frac{1}{1 + \frac{AC}{AB}} \right) \implies \frac{CP}{PA} = \frac{31}{20}.$$

Problem 5

The sum of the first 2011 terms of a geometric sequence is 200. The sum of the first 4022 terms is 380. Find the sum of the first 6033 terms.

Solution

Since the sum of the first 2011terms is 200, and the sum of the fist 4022terms is 380, the sum of the second 2011terms is 180. This is decreasing from the first 2011, so the common ratio (or whatever the term for what you multiply it by is) is less than one.

Because it is a geometric sequence and the sum of the first 2011 terms is 200, second $2011$is 180, the ratio of the second 2011terms to the first 2011terms is $\frac{9}{10}$. Following the same pattern, the sum of the third 2011terms is $\frac{9}{10} * 180 = 162$.

Thus, $200 + 180 + 162 = 542$

Sum of the first $6033$is $\boxed{542.}$

Problem 6

Define an ordered quadruple (a, b, c, d)as interesting if $1 \le a < b < c < d \le 10$, and $a + d > b + c$. How many interesting ordered quadruples are there?

Solution

Rearranging the inequality we get $d - c > b - a$. Let $e = 11$, then $(a, b - a, c - b, d - c, e - d)$is a partition of 11 into 5 positive integers or equivalently: $(a - 1, b - a - 1, c - b - 1, d - c - 1, e - d - 1)$is a partition of 6 into 5 non-negative integer parts. Via a standard balls and urns argument, the number of ways to partition 6 into 5 non-negative parts is $\binom{6+4}{4} = \binom{10}{4} = 210$.

The interesting quadruples correspond to partitions where the second number is less than the fourth. By symmetry there as many partitions where the fourth is less than the second. So, if $N$is the number of partitions where the second element is equal to the fourth, our answer is $(210 - N)/2$.

We find Nas a sum of 4 cases:

- two parts equal to zero, $\binom{8}{2} = 28$ ways,
- two parts equal to one, $\binom{6}{2} = 15$ ways,

- two parts equal to two, $\binom{4}{2} = 6$ ways,
- two parts equal to three, $\binom{2}{2} = 1$ way.

Therefore, $N = 28 + 15 + 6 + 1 = 50$ and our answer is $(210 - 50)/2 = \boxed{80.}$

Problem 7

Ed has five identical green marbles, and a large supply of identical red marbles. He arranges the green marbles and some of the red ones in a row and finds that the number of marbles whose right hand neighbor is the same color as themselves is equal tot he number of marbles whose right hand neighbor is the other color. An example of such an arrangement is GGRRRGGRG. Let *m* be the maximum number of red marbles for which such an arrangement is possible, and let N be the number of ways he can arrange the *m*+5 marbles to satisfy the requirement. Find the remainder when N is divided by 1000.

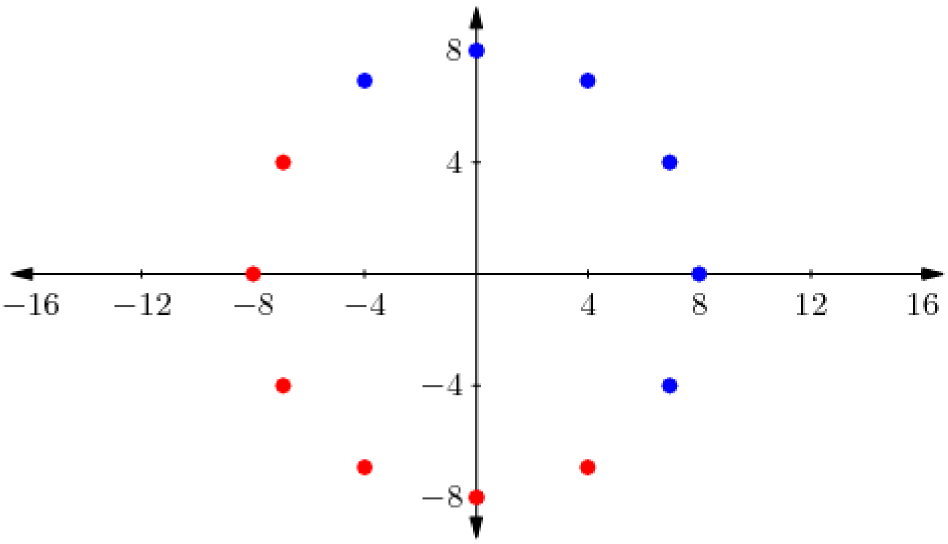
Solution

We are limited by the number of marbles whose right hand neighbor is not the same color as the marble. By surrounding every green marble with red marbles - RGRGRGRGRGR. That's 10 "not the same colors" and 0 "same colors." Now, for every red marble we add, we will add one "same color" pair and keep all 10 "not the same color" pairs. It follows that we can add 10 more red marbles for a total of 16 = *m*. We can place those ten marbles in any of 6 "boxes": To the left of the first green marble, to the right of the first but left of the second, etc. up until to the right of the last. This is a stars-and-bars problem, the solution of which can be found as $\binom{n+k}{k}$ where n is the number of stars and k is the number of bars. There are 10 stars (The unassigned Rs, since each "box" must contain at least one, are not counted here) and 5 "bars," the green marbles. So the answer is $\binom{15}{5} = 3003$, take the remainder when divided by 1000 to get the answer: $\boxed{003.}$

Problem 8

Let z_1, z_2, z_3, $\dots$, z_{12} be the 12 zeroes of the polynomial $z^{12}-2^{36}$. For each j, let w_j be one of z_j or iz_j. Then the maximum possible value of the real part of $\sum_{j=1}^{12} w_j$ can be written as $m+\sqrt{n}$, where m and n are positive integers. Find $m+n$.

Solution



The twelve dots above represent the 12 roots of the equation $z^{12}-2^{36}=0$. If we write $z=a+bi$, then the real part of z is a and the real part of iz is $-b$. The blue dots represent those roots z for which the real part of z is greater than the real part of iz, and the red dots represent those roots z for which the real part of iz is greater than the real part of z. Now, the sum of the real parts of the blue dots is easily seen to be $8+16\cos\frac{\pi}{6}=8+8\sqrt{3}$ and the negative of the sum of the imaginary parts of the red dots is easily seen to also be $8+8\sqrt{3}$. Hence our desired sum is $16+16\sqrt{3}=16+\sqrt{768}$, giving the answer $\boxed{784}$.

Problem 9

Let $x_1, x_2, \dots, x_6$ be non-negative real numbers such that $x_1+x_2+x_3+x_4+x_5+x_6=1$, and $x_1x_3x_5+x_2x_4x_6 \ge \frac{1}{540}$. Let p and q be positive relatively prime integers such that $\frac{p}{q}$ is the maximum possible value of $x_1x_2x_3+x_2x_3x_4+x_3x_4x_5+x_4x_5x_6+x_5x_6x_1+x_6x_1x_2$. Find $p+q$.

Solution

Note that neither the constraint nor the expression we need to maximize involves products $x_i x_j$ with $i - j \equiv 3 \pmod 6$. Factoring out say x_1 and x_4 we see that the constraint is $x_1(x_3x_5) + x_4(x_2x_6) \ge \frac{1}{540}$, while the expression we want to maximize is $x_1(x_2x_3 + x_5x_6 + x_6x_2) + x_4(x_2x_3 + x_5x_6 + x_3x_5)$. Adding the left side of the constraint to the expression we get: $(x_1 + x_4)(x_2x_3 + x_5x_6 + x_6x_2 + x_3x_5) = (x_1 + x_4)(x_2 + x_5)(x_3 + x_6)$. This new expression is the product of three non-negative terms whose sum is equal to 1. By AM-GM this product is at most $\frac{1}{27}$. Since we have added at least $\frac{1}{540}$ the desired maximum is at most $\frac{1}{27} - \frac{1}{540} = \frac{19}{540}$. It is easy to see that this upper bound can in fact be achieved by ensuring that the constraint expression is equal to $\frac{1}{540}$ with $x_1 + x_4 = x_2 + x_5 = x_3 + x_6 = \frac{1}{3}$—for example, by choosing x_1 and x_2 small enough—so our answer is $540 + 19 = \boxed{559}$.

An example is:

$$x_3 = x_6 = \frac{1}{6}$$
$$x_1 = x_2 = \frac{5 - \sqrt{20}}{30}$$
$$x_5 = x_4 = \frac{5 + \sqrt{20}}{30}$$

Another example is

$$x_1 = x_3 = \frac{1}{3}$$
$$x_2 = \frac{19}{60},\ x_5 = \frac{1}{60}$$
$$x_4 = x_6 = 0$$

Problem 10

A circle with center O has radius 25. Chord $\overline{AB}$ of length 30 and chord $\overline{CD}$ of length 14 intersect at point P. The distance between the midpoints of the two chords is 12. The quantity OP^2 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find the remainder when m + n is divided by 1000.

Solution

Let E and F be the midpoints of $\overline{AB}$ and $\overline{CD}$, respectively, such that $\overline{BE}$ intersects $\overline{CF}$.

Since E and F are midpoints, $BE = 15$ and $CF = 7$.

B and C are located on the circumference of the circle, so $OB = OC = 25$.

The line through the midpoint of a chord of a circle and the center of that circle is perpendicular to that chord, so $\triangle OEB$ and $\triangle OFC$ are right triangles (with $\angle OEB$ and $\angle OFC$ being the right angles). By the Pythagorean Theorem, $OE = \sqrt{25^2 - 15^2} = 20$, and $OF = \sqrt{25^2 - 7^2} = 24$.

Let x, a, and b be lengths OP, EP, and FP, respectively. OEP and OFP are also right triangles, so $x^2 = a^2 + 20^2 \to a^2 = x^2 - 400$, and $x^2 = b^2 + 24^2 \to b^2 = x^2 - 576$

We are given that EF has length 12, so, using the Law of Cosines with $\triangle EPF$:

$$12^2 = a^2 + b^2 - 2ab\cos(\angle EPF) = a^2 + b^2 - 2ab\cos(\angle EPO + \angle FPO)$$

Substituting for a and b, and applying the Cosine of Sum formula:

$$144 = (x^2 - 400) + (x^2 - 576) + 2\sqrt{x^2 - 400}\sqrt{x^2 - 576}\,(\cos\angle EPO \cos\angle FPO - \sin\angle EPO\, s$$

$\angle EPO$ and $\angle FPO$ are acute angles in right triangles, so substitute opposite/hypotenuse for sines and adjacent/hypotenuse for cosines:

$$144 = 2x^2 - 976 + 2\sqrt{(x^2 - 400)(x^2 - 576)}\left(\frac{\sqrt{x^2 - 400}}{x}\frac{\sqrt{x^2 - 576}}{x} - \frac{20}{x}\frac{24}{x}\right)$$

Combine terms and multiply both sides by x^2:

$$144x^2 = 2x^4 - 976x^2 - 2(x^2 - 400)(x^2 - 576) + 960\sqrt{(x^2 - 400)(x^2 - 576)}$$

Combine terms again, and divide both sides by 64:

$$13x^2 = 7200 + 15\sqrt{x^4 - 976x^2 + 230400}$$

Square both sides:

$$169x^4 - 187000x^2 + 51,840,000 = 225x^4 - 219600x^2 + 51840000$$

This reduces to $x^2 = \dfrac{4050}{7} = (OP)^2$; (4050 + 7) divided by 1000 has remainder $\boxed{057}$.

Problem 11

Let M_n be the n x n matrix with entries as follows: for $1 \le i \le n$, $m_{i,i} = 10$; for $1 \le i \le n-1$, $m_{i,i+1} = m_{i+1,i} = 3$; all other entries in M_n are zero. Let D_n be the determinant of the matrix M_n. Then $\sum_{n=1}^{\infty} \frac{1}{8D_n+1}$ can be represented as $\frac{p}{q}$ where p and q are relatively prime positive integers. Find p + q.

Note: The determinant of the 1 x 1 matrix $D_1 = \begin{bmatrix} a \end{bmatrix}$ is a, and the determinant of the 2 x 2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is ad-bc; for n $\geq$2, the determinant of an n x n matrix with first row or first column $a_1 a_2 a_3$. . . . a_n is equal to $a_1C_1 - a_2C_2 + a_3C_3 - ... + (-1)^{n+1} a_nC_n$ where C_i is the determinant of the (n-1) x (n-1) matrix found by eliminating the row and column containing a_i.

Solution

$$D_1 = \begin{vmatrix} 10 \end{vmatrix} = 10$$

$$D_2 = \begin{vmatrix} 10 & 3 \\ 3 & 10 \end{vmatrix} = (10)(10) - (3)(3) = 91$$

$$D_3 = \begin{vmatrix} 10 & 3 & 0 \\ 3 & 10 & 3 \\ 0 & 3 & 10 \end{vmatrix}$$

Using the expansionary/recursive definition of determinants (also stated in the problem):

$$D_3 = \begin{vmatrix} 10 & 3 & 0 \\ 3 & 10 & 3 \\ 0 & 3 & 10 \end{vmatrix} = 10\begin{vmatrix} 10 & 3 \\ 3 & 10 \end{vmatrix} - 3\begin{vmatrix} 3 & 3 \\ 0 & 10 \end{vmatrix} + 0\begin{vmatrix} 3 & 10 \\ 0 & 3 \end{vmatrix} = 10D_2 - 9D_1 = 820$$

This pattern repeats because the first element in the first row of M_n is always 10, the second element is always 3, and the rest are always 0. The ten element directly expands to $10D_{n-1}$. The three element expands to 3 times the determinant of the

the matrix formed from omitting the second column and first row from the original matrix. Call this matrix X_n. $X_n$has a first column entirely of zeros except for the first element, which is a three. A property of matrices is that the determinant can be expanded over the rows instead of the columns (still using the recursive definition as given in the problem), and the determinant found will still be the same. Thus, expanding over this first column yields $3D_{n-2} + 0(otherthings) = 3D_{n-2}$. Thus, the $3det(X_n)$expression turns into $9D_{n-2}$. Thus, the equation $D_n = 10D_{n-1} - 9D_{n-2}$holds for all n > 2.

This equation can be rewritten as $D_n = 10(D_{n-1} - D_{n-2}) + D_{n-2}$. This version of the equation involves the difference of successive terms of a recursive sequence. Calculating D_0backwards from the recursive formula and $D_4$from the formula yields $D_0 = 1, D_4 = 7381$. Examining the differences between successive terms, a pattern emerges. $D_0 = 1 = 9^0$

$D_1 - D_0 = 10 - 1 = 9 = 9^1$

$D_2 - D_1 = 91 - 10 = 81 = 9^2$

$D_3 - D_2 = 820 - 91 = 729 = 9^3$

$D_4 - D_3 = 7381 - 820 = 6561 = 9^4$

Thus, $$D_n = D_0 + 9^1 + 9^2 + ... + 9^n = \sum_{i=0}^{n} 9^i = \frac{(1)(9^{n+1} - 1)}{9 - 1} = \frac{9^{n+1} - 1}{8}.$$

Thus, the desired sum is $$\sum_{n=1}^{\infty} \frac{1}{8\frac{9^{n+1}-1}{8} + 1} = \sum_{n=1}^{\infty} \frac{1}{9^{n+1} - 1 + 1} = \sum_{n=1}^{\infty} \frac{1}{9^{n+1}}$$

This is an infinite geometric sequence with first term $\frac{1}{81}$and common ratio $\frac{1}{9}$. Thus, the sum is $$\frac{\frac{1}{81}}{1 - \frac{1}{9}} = \frac{\frac{1}{81}}{\frac{8}{9}} = \frac{9}{(81)(8)} = \frac{1}{(9)(8)} = \frac{1}{72}.$$

Thus, p + q = 1 + 72 = $\boxed{073.}$

Problem 12

Nine delegates, three each from three different countries, randomly select chairs at a round table that seats nine people. Let the probability that each delegate sits next to at least one delegate from another country be $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

Solution

Use complementary probability and PIE.

If we consider the delegates from each country to be indistinguishable and number the chairs, we have $\frac{9!}{(3!)^3}$ total ways to seat the candidates. This comes to: $\frac{9\cdot8\cdot7\cdot6\cdot5\cdot4}{6\cdot6}=6\cdot8\cdot7\cdot5=30\cdot56.$

Of these there are $3\times9\times\frac{6!}{(3!)^2}$ ways to have the candidates of at least some one country sit together. This comes to $\frac{27\cdot6\cdot5\cdot4}{6}=27\cdot20.$

Among these there are $3\times9\times4$ ways for candidates from two countries to each sit together. This comes to $27\cdot4.$

Finally, there are $9\times2=18.$ ways for the candidates from all the countries to sit in three blocks (9 clockwise arrangements, and 9 counter-clockwise arrangements).

So, by PIE, the total count of unwanted arrangements is $27\cdot20-27\cdot4+18=16\cdot27+18=18\cdot25.$

So the fraction $\frac{m}{n}=\frac{30\cdot56-18\cdot25}{30\cdot56}=\frac{56-15}{56}=\frac{41}{56}.$ Thus $m+n=56+41=\boxed{97.}$

Problem

Point P lies on the diagonal AC of square ABCD with AP > CP. Let O_1 and O_2 be the circumcenters of triangles ABP and CDP respectively. Given that AB = 12 and $\angle O_1PO_2=120^\circ$, then $AP=\sqrt{a}+\sqrt{b}$, where a and b are positive integers. Find a + b.

Solution

Denote the midpoint of DC be E and the midpoint of AB be F. Because they are the circumcenters, both Os lie on the perpendicular bisectors of AB and CD and these bisectors go through E and F.

It is given that ${}_1PO_2 = 120^\circ$. Because $O_1P$and $O_1B$are radii of the same circle, the have the same length. This is also true of $O_2P$and O_2D. Because $m\angle CAB = m\angle ACD = 45^\circ$, $m\ \widehat{PD} = m\ \widehat{PB} = 2(45^\circ) = 90^\circ$. Thus, O_1PB and $O_2PD$are isosceles right triangles. Using the given information above and symmetry, $m\angle DPB = 120^\circ$. Because ABP and ADP share one side, have one side with the same length, and one equal angle, they are congruent by SAS. This is also true for triangle CPB and CPD. Because angles APB and APD are equal and they sum to 120 degrees, they are each 60 degrees. Likewise, both angles CPB and CPD have measures of 120 degrees.

Because the interior angles of a triangle add to 180 degrees, angle ABP has measure 75 degrees and angle PDC has measure 15 degrees. Subtracting, it is found that both angles $O_1BF$and O_2DEhave measures of 30 degrees. Thus, both triangles $O_1BF$and $O_2DE$are 30-60-90 right triangles. Because F and E are the midpoints of AB and CD respectively, both FB and DE have lengths of 6. Thus, $DO_2 = BO_1 = 4\sqrt{3}$. Because of 45-45-90 right triangles, $PB = PD = 4\sqrt{6}$.

Now, using Law of Cosines on triangle ABP and letting AP be x,

$$96 = 144 + x^2 - 24x\frac{\sqrt{2}}{2}$$

$$96 = 144 + x^2 - 12x\sqrt{2}$$

$$0 = x^2 - 12x\sqrt{2} + 48$$

Using quadratic formula,

$$x = \frac{12\sqrt{2} \pm \sqrt{288 - (4)(48)}}{2}$$

$$x = \frac{12\sqrt{2} \pm \sqrt{288 - 192}}{2}$$

$$x = \frac{12\sqrt{2} \pm \sqrt{96}}{2}$$

$$x = \frac{2\sqrt{72} \pm 2\sqrt{24}}{2}$$

$$x = \sqrt{72} \pm \sqrt{24}$$

Because it is given that $AP > CP$, $AP > 6\sqrt{2}$, so the minus version of the above equation is too small. Thus, $AP = \sqrt{72} + \sqrt{24}$ and a + b = 24 + 72 = $\boxed{96.}$

Problem 14

There are N permutations $(a_1, a_2, \dots, a_{30})$ of 1, 2, ... , 30 such that for $m \in \{2, 3, 5\}$, m divides $a_{n+m} - a_n$ for all integers n with $1 \le n < n+m \le 30$. Find the remainder when N is divided by 1000.

Solutions

Solution 1

Each position in the 30-position permutation is uniquely defined by an ordered triple (i, j, k). The nth position is defined by this ordered triple where i is n mod 2, j is n mod 3, and k is n mod 5. There are 2 choices for i, 3 for j, and 5 for k, yielding 2*3*5=30 possible triples. Because the least common multiple of 2, 3, and 5 is 30, none of these triples are repeated and all are used. By the conditions of the problem, if i is the same in two different triples, then the two numbers in these positions must be equivalent mod 2. If j is the same, then the two numbers must be equivalent mod 3, and if k is the same, the two numbers must be equivalent mod 5.

The ordered triple (or position) in which the number one can be placed has 2 options for i, 3 for j, and 5 for k, resulting in 30 different positions it can be placed.

The ordered triple where 2 can be placed in somewhat constrained by the placement of the number 1. Because 1 is not equivalent to 2 mod 2, 3, or 5, the i, j, and k in their ordered triples must be different. Thus, for the number 2, there are (2-1) choices for i, (3-1) choices for j, and (5-1) choices for k. Thus, there are 1*2*4=8 possible placements for the number two once the number one is placed.

Because 3 is equivalent to 1 mod 2, it must have the same i as the ordered triple of 1. Because 3 is not equivalent to 1 or 2 mod 3 or 5, it must have different j and k values. Thus, there is 1 choice for i, (2-1) choices for j, and (4-1) choices for k, for a total of 1*1*3=3 choices for the placement of 3.

As above, 4 is even, so it must have the same value of i as 2. It is also 1 mod 3, so it must have the same j value of 2. 4 is not equivalent to 1, 2, or 3 mod 5, so it must have a different k value than that of 1, 2, and 3. Thus, there is 1 choice for i, 1 choice for j, and (3-1) choices for k, yielding a total of 1*1*2=2 possible placements for 4.

5 is odd and is equivalent to 2 mod 3, so it must have the same i value as 1 and the same j value of 2. 5 is not equivalent to 1, 2, 3, or 4 mod 5, so it must have a different k value from 1, 2, 3, and 4. However, 4 different values of k are held by these four numbers, so 5 must hold the one remaining value. Thus, only one possible triple is found for the placement of 5.

All numbers from 6 to 30 are also fixed in this manner. All values of i, j, and k have been used, so every one of these numbers will have a unique triple for placement, as above with the number five. Thus, after 1, 2, 3, and 4 have been placed, the rest of the permutation is fixed.

Thus, N = 30*8*3*2=30*24=1440. Thus, the remainder when N is divided by 1000 $\boxed{440.}$

Solution 2

We observe that the condition on the permutations means that two numbers with indices congruent $\mod m$are themselves congruent $\mod m$for $m \in \{2, 3, 5\}$. Furthermore, suppose that $a_n \equiv k \mod m$.Then, there are $30/m$indices congruent to $n \mod m$,and $30/m$numbers congruent to $k \mod m$,because 2, 3, and 5 are all factors of 30. Therefore, since every index congruent to $n$must contain a number congruent to k,and no number can appear twice in the permutation, only the indices congruent to $n$contain numbers congruent to k.In other words, $a_i \equiv a_j \mod m \iff i \equiv j \mod m$.

This tells us that in a valid permutation, the congruence classes $\mod m$are simply swapped around, and if the set $S$is a congruence class $\mod m$for $m =$2, 3, or 5, the set $\{a_i | i \in S\}$is still a congruence class $\mod m$.Clearly, each valid permutation of the numbers 1 through 30 corresponds to exactly one permutation of the congruence classes modulo 2, 3, and 5. Also, if we choose some permutations of the congruence classes modulo 2, 3, and 5, they correspond to exactly one valid permutation of the numbers 1 through 30. This can be shown as follows: First of all, the choice of permutations of the congruence classes gives us every number in the permutation modulo 2, 3, and 5, so by the Chinese Remainder Theorem, it gives us every number $\bmod 2 \cdot 3 \cdot 5 = 30$.Because the numbers must be between 1 and 30 inclusive, it thus uniquely determines what number goes in each index. Furthermore, two different indices cannot contain the same number. We will prove this by contradiction, calling the indices $a_i$and $a_j$for $i \neq j$.If $a_i = a_j$,then they must have the same residues modulo 2, 3, and 5, and so $i \equiv j$modulo 2, 3, and 5. Again using the Chinese Remainder Theorem, we conclude that $i \equiv j \mod 30$,so because $i$and $j$are both between 1 and 30 inclusive, $i = j$,giving us a contradiction. Therefore, every choice of permutations of the congruence classes modulo 2, 3, and 5 corresponds to exactly one valid permutation of the numbers 1 through 30.

We have now established a bijection between valid permutations of the numbers 1 through 30 and permutations of the congruence classes modulo 2, 3, and 5, so N is equal to the number of permutations of congruence classes. There are always m congruence classes $\bmod m$,so

$N = 2! \cdot 3! \cdot 5! = 2 \cdot 6 \cdot 120 = 1440 \equiv \boxed{440} \mod 1000.$

Problem 15

Let $P(x) = x^2 - 3x - 9$. A real number $x$is chosen at random from the interval $5 \le x \le 15$. The probability that $\lfloor\sqrt{P(x)}\rfloor = \sqrt{P(\lfloor x \rfloor)}$is equal to $\frac{\sqrt{a} + \sqrt{b} + \sqrt{c} - d}{e}$, where a, b, c, d, and $e$are positive integers. Find $a + b + c + d + e$.

Solution

Table of values of $P(x)$:

$P(5) = 1$
$P(6) = 9$
$P(7) = 19$
$P(8) = 31$
$P(9) = 45$
$P(10) = 61$
$P(11) = 79$
$P(12) = 99$
$P(13) = 121$
$P(14) = 145$
$P(15) = 171$

In order for $\lfloor\sqrt{P(x)}\rfloor = \sqrt{P(\lfloor x \rfloor)}$ to hold, $\sqrt{P(\lfloor x \rfloor)}$ must be an integer and hence $P(\lfloor x \rfloor)$must be a perfect square. This limits $x$to $5 < x < 6$or $6 < x < 7$or $13 < x < 14$since, from the table above, those are the only values of $x$for which $P(\lfloor x \rfloor)$is an perfect square. However, in order for $\sqrt{P(x)}$to be rounded down to $P(\lfloor x \rfloor)$, $P(x)$must not be greater than the next perfect square after $P(\lfloor x \rfloor)$(for the said intervals). Note that in all the cases the next value of $P(x)$always passes the next perfect square after $P(\lfloor x \rfloor)$, so in no cases will all values of $x$in the said intervals work. Now, we consider the three difference cases.

Case $5 < x < 6$:

$P(x)$must not be greater than the first perfect square after 1, which is 4. Since $P(x)$is increasing for $x > 5$, we just need to find where $P(x) = 4$and the values that will work will be $5 < x < \text{root}$.

$x^2 - 3x - 9 = 4$
$x = \dfrac{3 + \sqrt{61}}{2}$

So in this case, the only values that will work are $5 < x < \dfrac{3 + \sqrt{61}}{2}$.

Case $6 < x < 7$:

$P(x)$must not be greater than the first perfect square after 9, which is 16.

$x^2 - 3x - 9 = 16$
$x = \dfrac{3 + \sqrt{109}}{2}$

So in this case, the only values that will work are $6 < x < \dfrac{3 + \sqrt{109}}{2}$.

Case $13 < x < 14$:

$P(x)$must not be greater than the first perfect square after 121, which is 144.

$$x^2 - 3x - 9 = 144$$
$$x = \frac{3 + \sqrt{621}}{2}$$

So in this case, the only values that will work are $13 < x < \frac{3 + \sqrt{621}}{2}$.

Now, we find the length of the working intervals and divide it by the length of the total interval, $15 - 5 = 10$:

$$\frac{\left(\frac{3+\sqrt{61}}{2} - 5\right) + \left(\frac{3+\sqrt{109}}{2} - 6\right) + \left(\frac{3+\sqrt{621}}{2} - 13\right)}{10}$$
$$= \frac{\sqrt{61} + \sqrt{109} + \sqrt{621} - 39}{20}$$

So the answer is $61 + 109 + 621 + 39 + 20 = \boxed{850}$.